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Existence and uniqueness results using interactive fuzzy fractional derivative

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Abstract. This study describes the use of an interactive derivative of Caputo to a fuzzy fractional starting value issue. The beginning setting of the equation is represented by a fuzzy subset, and the differentiation is indicated by F-correlated derivative. In first we provide new and essentials theorems related to F-derivative of order $\alpha \in (0,1]$. Secondly we use these theorems to extract the mild solution of the main problem. Thirdly we demonstrate the uniqueness of solutions by the Schauder fixed point theorem. Lastly, a case study is given to exhibit the correctness of the acquired outcomes.

1. Introduction

Fractional Differential Equations may be observed of as an extension of Ordinary Differential Equations (ODE) to random fractional rank [20]. Agarwal et al. established the idea of Fuzzy Fractional Differential Equation (FFDE) in [21]. A number of articles have been published that handle FFDEs, for instance, [9, 12, 15, 38–41].

Recent advances in fuzzy fractional differential equations have demonstrated significant theoretical and computational progress. El Ghazouani and collaborators have established fundamental results including: existence and asymptotic behavior for nonlinear hybrid systems with fuzzy Caputo-Nabla differences [9]; semilinear elliptic equations [12]; mild solutions for fractional evolution equations [15]; and solvability of ABC-fractional coupled systems [22]. Their work on Volterra-Fredholm integro-differential equations [23] and boundary value problems via Hilfer derivatives [24] has extended stability analysis, while studies on neutral equations with Caputo generalized Hukuhara derivatives [25] and conformable derivatives [27] have enriched the operator theoretic framework. Novel solution techniques have been developed, including Chinchole-Bhadane interval methods [26], Laplace residual power series for wave equations [31, 35], and semi-analytical approaches for acoustic waves [30]. Important extensions cover: Langevin equations [29]; nonlocal conditions [32]; stochastic systems [33]; and intuitionistic fuzzy equations [34]. Recent breakthroughs include hybrid systems via Mönch's theorem [46] and ψ-Caputo nonlocal problems [37], with applications ranging from elliptic theory to fractional boundary value problems [28]. This collective

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work advances both the theoretical underpinnings and computational methods for fuzzy fractional models across mathematics and engineering applications.

In this article, we look at the FFDE using the F-derivative of Caputo of degree $\alpha \in (0,1]$, which means that the differentiation is supplied by an interactive derivation, as defined by Santo Pedro et al. [16].

$$\begin{aligned} &{}^{C_F}D^{\alpha}_{a^+}u(s) = Au(s) +_F f(s,u(s)),\\ &\quad u(a) = u_0 \in \mathcal{F}_{\mathbb{R}}, \quad J = [a,b] \end{aligned} \tag{1}$$

where $f : J \times \mathcal{F}_{\mathbb{R}} \to \mathcal{F}_{\mathbb{R}}$ is fuzzy continuous function and A is a linear operator.

The fundamental contribution of this study is the development of new theorems about the F-correlated fractional derivation of degree , such as:

- The relationships between the operators ${}^{RL_F}I^{\alpha}$ and ${}^{C_F}D^{\alpha}_{a^+}$
- The demonstration of: "If u is F-differentiable. Then u is continuous".
- We provide the Laplace transform of F-correlated fractional Caputo operator using the F-derivative.
- We present the derivation of the combination of two functions based on the F-derivative.

and use all of this to see if the problem (1) has a fuzzy mild solutions.

The generalized Hukuhara derivative (gH) is commonly utilized by scientists in fuzzy fractional mathematics. The outcomes that we achieve using the F-correlated derivative are comparable to those generated with (gH).

The afterwards is a breakdown of the work's sections: Section 2 covers the fundamental ideas of fuzzy set theory, along with the fuzzy derivative for self-correlated processes. Section 3 offers fuzzy interactive derivative and new theorems on this notion. Section 4 provide the existence and uniqueness conclusions for FFDE using the F-correlated derivative. Section 5 supplied an example of FFIVP under the F-correlated fractional Caputo derivative and Section 6 is the conclusion.

2. Preliminaries

Definition 1. *The fuzzy number is a fuzzy set $u : \mathbb{R} \to [0,1]$ that meets these conditions:*

1. *u is normal, i.e. there's a $t_0 \in \mathbb{R}$ such as $u(t_0) = 1$;*
2. *u is a fuzzy convex set;*
3. *u is upper semicontinuous;*
4. *The closure of $\{t \in \mathbb{R}, \quad u(t) > 0\}$ is compact.*

The set of all fuzzy elements on $\mathbb{R}$ is symbolized by $\mathcal{F}_{\mathbb{R}}$.

$$\mathcal{F}_{\mathbb{R}} = \{u : \mathbb{R} \to [0,1], \quad u \quad \texttt{satisfies} \ (1-4) \ \textit{below}\ \}.$$

The r-cut of a $\mathcal{F}_{\mathbb{R}}$ component is given by

$$[u]_r = \{s \in \mathbb{R}, u(s) \geq r\} \ \texttt{For all} \ r \in (0,1]$$

We may write using the previous items

$$[u(t)]_r = [u^+_r(t), u^-_r(t)]. \tag{2}$$

Allow $u \in \mathcal{F}_{\mathbb{R}}$. The size of the r-cut set of u is given by

$$\text{len}\,([u]_r) = u^+_r - u^-_r, \quad \forall r \in [0,1]. \tag{3}$$

If $r = 0$, then $\text{len}([u]_0) = \text{diam}(u)$.

The Hausdorff metric $d_\infty : \mathcal{F}_\mathbb{R} \times \mathcal{F}_\mathbb{R} \to \mathbb{R}_+ \cup \{0\}$, is represented by

$$d_\infty(u, v) = \sup_{0\le r\le 1} d_H([u]_r, [v]_r), \tag{4}$$

where d_H is the Hausdorff metric for compact elements of $\mathbb{R}$.

Let $u, v \in \mathcal{F}_\mathbb{R}$, then (4) turns to

$$d_\infty(u, v) = \sup_{0<r\le 1} \max\left\{\left|u_r^- - v_r^-\right|, \left|u_r^+ - v_r^+\right|\right\}. \tag{5}$$

Allow $u, v \in \mathcal{F}_\mathbb{R}$ and $J \in \mathcal{F}_J\left(\mathbb{R}^2\right)$. The relation J is a joint possibility distribution (*JPD* for short) of u and v if, [2]

$$\max_s \mu_J(t, s) = \mu_u(t) \ \ \texttt{and} \ \ \max_t \mu_J(t, s) = \mu_v(s), \forall t, s \in \mathbb{R}.$$

u and v are the marginal possibility distributions of J in this case.

The fuzzy numbers u and v are called non-interactive iff,

$$\mu_J(t, s) = \min\{\mu_u(t), \mu_v(s)\} \ \ \texttt{for all} \ \ t, s \in \mathbb{R}$$

If not, the fuzzy numbers are called interactive [2, 3].

Allow $u, v \in \mathcal{F}_\mathbb{R}$ with *JPD* J and $g : \mathbb{R}^2 \to \mathbb{R}$. The expansion of g in relation to J, applied to (u, v), is the fuzzy element $g_J(u, v)$ with the following membership function [4]

$$\mu_{g_J(u,v)}(x) = \begin{cases} \sup_{(z,y)\in g^{-1}(x)} \mu_J(z, y) & \texttt{if } g^{-1}(x) \neq \emptyset \\ 0 & \texttt{if } g^{-1}(x) = \emptyset \end{cases}, \tag{6}$$

where $g^{-1}(x) = \{(z, y) : g(z, y) = x\}$.

If J is supplied by the minimal t-norm, subsequently $g_J(u, v)$ is the Zadeh's extending concept g at u and v [4].

Theorem 1. *[4, 5]. Let $u, v \in \mathcal{F}_\mathbb{R}$, J be a JPD of u and v, and $g : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ a continuous function. At this scenario, $g_J : \mathcal{F}_\mathbb{R} \times \mathcal{F}_\mathbb{R} \longrightarrow \mathcal{F}_\mathbb{R}$ is clearly stated. and*

$$\left[g_J(u, v)\right]_r = g([J]_r) \quad \forall r \in [0, 1]. \tag{7}$$

Let concentrate on the unique relationship known as interaction. There are multiple kinds of *JPD* that provide various interactivities. This paper investigates the interaction known as linear correlation, which is achieved as proceeds.

Definition 2. *[2]. Allow $u, v \in \mathcal{F}_\mathbb{R}$ and a function $F : \mathbb{R} \to \mathbb{R}$. The fuzzy numbers u and v are said F-correlated if its JPD is expressed by*

$$\mu_J(w, z) = \chi_{\{(x,y=F(x))\}}(w, z)\mu_u(w) = \chi_{\{x,y=F(x)\}}(w, z)\mu_v(z). \tag{8}$$

It is worth noting that the fuzzy number v corresponds to the Zadeh's extended concept of the function F when performed at u. If F can be inverted, then $u = F - 1(v)$ and, in this instance,

$$[u]_r = \left\{(x, F(x)) \in \mathbb{R}^2 \mid x \in [u]_r\right\} = \left\{\left(F^{-1}(y), y\right) \in \mathbb{R}^2 \mid y \in [v]_r\right\}. \tag{9}$$

Furthermore, if F is a continuous function, the r-cuts of v are provided by [1]

$$[v]_r = F([u]_r). \tag{10}$$

If the function F is defined as $F(u) = qu + r$, then F is said to be linearly correlated (or linearly interactive).

Definition 3. *[4]. Allow u and v be F-correlated fuzzy numbers. The process $v \otimes_F u$ is expressed as follows:*

$$\mu_{v\otimes_F u}(w) = \begin{cases} \sup_{x\in\Phi_\otimes^{-1}(w)} \mu_u(x) & if\ \Phi_\otimes^{-1}(w) \neq \emptyset \\ 0 & if\ \Phi_\otimes^{-1}(w) = \emptyset \end{cases},$$

where $\Phi_\otimes^{-1}(w) = \{x \mid w = x \otimes y, y = F(x)\}$, and $\otimes \in \{+, -, \times, \div\}$.

According to the Theorem 1, For any $r \in [0, 1]$, the four mathematical computations of F-correlated fuzzy numbers are provided by

$$[v +_F u]_r = \{F(s) + s \in \mathbb{R} \mid s \in [u]_r\}; \tag{11}$$

$$[v -_F u]_r = \{F(s) - s \in \mathbb{R} \mid s \in [u]_r\}; \tag{12}$$

$$[v \cdot_F u]_r = \{sF(s) \in \mathbb{R} \mid s \in [u]_r\}; \tag{13}$$

$$[v \div_F u]_r = \{F(s) \div s \in \mathbb{R} \mid s \in [u]_r\}, \quad 0 \notin [u]_0. \tag{14}$$

Furthermore, scalar multiplying of λv, with $v = F(u)$, is stated by $[\lambda v]_r = \{\lambda F(s) \in \mathbb{R} \mid s \in [u]_r\}$.

Proposition 1. *[7]. Allow u and v be F-correlated fuzzy numbers, i.e., $[v]_r = F([u]_r)$, with F-differentiable, $[u]_r = [u_r^-, u_r^+]$ and $[v]_r = [v_r^-, v_r^+]$, then, $\forall r \in [0, 1]$,*

1)

$$[v -_F u]_r = \{F(s) - s \mid s \in [u]_r\} = \begin{cases} i. & [v_r^- - u_r^-, v_r^+ - u_r^+]\ if\ F'(s) > 1, \quad \forall s \in [u]_r \\ ii. & [v_r^+ - u_r^+, v_r^- - u_r^-]\ if\ 0 < F'(s) \leq 1, \forall s \in [u]_r; \\ iii. & [v_r^- - u_r^+, v_r^+ - u_r^-]\ if\ F'(s) \leq 0, \quad \forall s \in [u]_r \end{cases} \tag{15}$$

2)

$$[v +_F u]_r = \{F(s) + s \mid s \in [u]_r\} = \begin{cases} i. & [v_r^- + u_r^-, v_r^+ + u_r^+]\ if\ F'(s) > 0, \quad \forall s \in [u]_r \\ ii. & [v_r^+ + u_r^-, v_r^- + u_r^+]\ if\ -1 < F'(s) \leq 0, \forall s \in [u]_r. \\ iii. & [v_r^- + u_r^+, v_r^+ + u_r^-]\ if\ F'(s) \leq -1, \quad \forall s \in [u]_r \end{cases} \tag{16}$$

In the initial instance, (15)-i, we have len $([u]_r) <$ len $([v]_r)$, and $-_F$ matches with Hukuhara difference [6], but in (15)-ii, we have len $([u]_r) >$ len $([v]_r)$ and, $-_F$ is compatible with gH difference [8, 10]. In reality, the gH and Hukuhara differences are special examples of an interacting difference [8]. Furthermore, if $F'(s) \leq -1$, $-_F$ corresponds with standard difference, and $+_F$ matches with usual sum when $F'(s) > 0$.

Let $u, v \in \mathcal{F}_\mathbb{R}$ be linearly correlated, i.e. $F(y) = qy + r$, and $[v]_r = q[u]_r + r$, with $[u]_r = [u_r^-, u_r^+]$and $[v]_r = [v_r^-, v_r^+]$, (15) and (16) becomes

$$[v -_L u]_r = \begin{cases} \text{i.} & [v_r^- - u_r^-, v_r^+ - u_r^+] & \text{if } q \geq 1 \\ \text{ii.} & [v_r^+ - u_r^+, v_r^- - u_r^-] & \text{if } 0 < q < 1 \\ \text{iii.} & [v_r^- - u_r^+, v_r^+ - u_r^-] & \text{if } q < 0 \end{cases} \tag{17}$$

and

$$[v +_L u]_r = \begin{cases} \text{i.} & [v_r^- + u_r^-, v_r^+ + u_r^+] & \text{if } q > 0 \\ \text{ii.} & [v_r^+ + u_r^-, v_r^- + u_r^+] & \text{if } -1 \leq q < 0. \\ \text{iii.} & [v_r^- + u_r^+, v_r^+ + u_r^-] & \text{if } q < -1 \end{cases} \tag{18}$$

It is interesting that if q is positive, $+_L$ corresponds to traditional sum, while if q is negative, $-_L$ corresponds to normal difference [11]. Furthermore, when q is positive, $-_L$ corresponds with generalized Hukuhara difference [10], and for $q > 1$, it matches with Hukuhara difference. [6].

2.1. Auto-correlated Fuzzy Processes

Auto-correlated fuzzy processes are similar to auto-correlated statistical processes, such as [7, 11, 13]. Those fuzzy processes have been used in fields as diverse as epidemiology [14] and the evolution of populations [7].

Let $L(J, \mathcal{F}_{\mathbb{R}})$ be the field of all Lebesgue integrable functions, and $AC(J, \mathcal{F}_{\mathbb{R}})$ be the set of all absolutely continuous functions.

Definition 4. *[7]. A fuzzy function $u : J \longrightarrow \mathcal{F}_{\mathbb{R}}$ defines a fuzzy process u. Let $[u(t)]_r = [u_r^-(t), u_r^+(t)]$, for all $r \in [0,1]$, the process u is δ-locally F-auto-regressive at $t \in J$ if there's a family of real functions $F_{t,h}$ such as, for all $0 < |h| < \delta$*

$$[u(t+h)]_r = F_{t,h}([u(t)]_r), \forall r \in [0,1].$$

Definition 5. *[7]. If $u : J \to \mathcal{F}_{\mathbb{R}}$ is a F-auto-regressive fuzzy process, therefore u is F-correlated differentiable at $t_0 \in J$ if a fuzzy number $u'_F(t_0)$ exist in such a way that*

$$u'_F(t_0) = \lim_{h \to 0} \frac{u(t_0+h) -_F u(t_0)}{h}, \tag{19}$$

When the aforementioned limit exists and is equivalent to $u'_F(t_0)$ (by applying the metric d_∞). We state that u is F-differentiable if u'_F exists for every $t \in J$.

The following theorem characterizes the F derivative using r-cuts.

Theorem 2. *[7]. Allow $u : J \to \mathcal{F}_{\mathbb{R}}$ be F-differentiable at $t_0 \in J$, with $[u(t)]_r = [u_r^-(t), u_r^+(t)]$, where the associated function family $F_{t_0,h} : I \to \mathbb{R}$ is monotone continuous differentiable for every h, $r \in [0,1]$ and $F_{t,h}$, $\forall t \in J$. Then,*

$$\left[u'_F(t_0)\right]_r = \begin{cases} \left[(u_r^-)'(t_0), (u_r^+)'(t_0)\right] & \texttt{if} \quad F'_{t,h}(w) > 1 \\ \left[(u_r^+)'(t_0), (u_r^-)'(t_0)\right] & \texttt{if} \quad 0 < F'_{t,h}(w) \leq 1 \\ \left\{(u_r^-)'(t_0)\right\} = \left\{(u_r^+)'(t_0)\right\} & \texttt{if} \quad F'_{t,h}(w) \leq 0 \end{cases}.$$

for every $0 < |h| < \delta, \delta > 0$, and $w \in [u(t)]_r$.

The process u is referred to be expansive if the length of $u(t)$ is a non-decreasing function at t, and contractive if the length of $u(t)$ is a non-increasing function at t.

Theorem 3. *[16]. Consider $u \in AC(J, \mathcal{F}_{\mathbb{R}})$ be F-differentiable, and $[u(t)]_r = [u_r^-(t), u_r^+(t)]$.*

1. *Assume u is expansive, i.e., $\text{len}([u(t)]_r)$ is an increase function on J. If function u'_F is Riemann integrable then $(u_r^-)'(t)$ and $(u_r^+)'(t)$ are integrable on J, and*

$$\left[\int_a^t u'_F(s)ds\right]_r = \left[\int_a^t (u_r^-)'(s)ds, \int_a^t (u_r^+)'(s)ds\right].$$

2. *Assume u is contractive, i.e., $\text{len}([u(t)]_r)$ is a decrease function on J. If function u'_F is Riemann integrable then $(u_r^-)'(t)$ and $(u_r^+)'(t)$ are integrable on J, and*

$$\left[\int_a^t u'_F(s)ds\right]_r = \left[\int_a^t (u_r^+)'(s)ds, \int_a^t (u_r^-)'(s)ds\right].$$

2.2. Fractional Calculus

Definition 6. *The Gamma function is given by*

$$\forall x > 0, \quad \Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt. \tag{20}$$

Definition 7. *[15]. The Riemann Liouville fractional integral $I_{a^+}^{\alpha} u$ of $u \in L(J, \mathbb{R})$ of degree $\alpha \in (0, 1]$ is set up by*

$$\left(I_{a^+}^{\alpha} u\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s) ds, \quad \texttt{for } t > a$$

and $\Gamma(\alpha)$ is the gamma function. When $\alpha = 1$, we have $\left(I_{a^+}^{1} u\right)(t) = \int_a^t u(s) ds$.

Definition 8. *[15]. The Riemann Liouville derivative of degree $\alpha \in (0, 1]$, has been expressed like*

$$\left({}^{RL}D_{a^+}^{\alpha} u\right)(t) = \frac{d}{dt} I_{a^+}^{1-\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{-\alpha} u(s) ds, \quad \texttt{for } t > a.$$

Definition 9. *[15]. Allow $u \in L(J, \mathbb{R})$ and assume there's ${}^{RL}D_{a^+}^{\alpha} u$ on J. The Caputo fractional derivative ${}^{C}D_{a^+}^{\alpha} u$ is stated as*

$$\left({}^{C}D_{a^+}^{\alpha} u\right)(t) = \left({}^{RL}D_{a^+}^{\alpha} [u(\cdot) - u(a)]\right)(t), \quad \texttt{for } t \in J.$$

Besides that, if $u \in AC(J, \mathbb{R})$, then

$$\left({}^{C}D_{a^+}^{\alpha} u\right)(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} u'(s) ds, \forall t \in J,$$

and

$$\left({}^{\mathrm{RL}}D_{a^+}^{\alpha} u\right)(t) = \left({}^{\mathrm{C}}D_{a^+}^{\alpha} u\right)(t) + \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)} u(a), \quad \forall t \in J.$$

Definition 10. *[19] Allow u be a continuous function such that $e^{-st} \odot u(t)$ is integrable. Therefore the fuzzy Laplace transform of u, indicated by $\mathcal{L}[u(t)]$, is*

$$\mathcal{L}[u(t)] := U(s) = \int_0^{+\infty} e^{-st} \odot u(t) dt, s > 0. \tag{21}$$

A fuzzy function u is exponent bounded of degree α if there's $M > 0$ provided that

$$\exists t_0 > 0, d_{+\infty}(u(t), \tilde{0}) \leq M e^{\alpha t}, \forall t \geq t_0$$

Proposition 2. *[15]. If $u(t)$ is a fuzzy continuous function and of exponential degree α, thus*

$$\mathcal{L}((u \star v)(t)) = \mathcal{L}(u(t)) \odot \mathcal{L}(v(t)). \tag{22}$$

where $v(t)$ is a peace-wise continuous real function on $[0, \infty)$.

Proposition 3. *[15]. For all $\alpha > 0$, we get the following result*

$$\int_0^t E_{\alpha,1} \left(A s^{\alpha}\right) ds = t E_{\alpha,2} \left(A t^{\alpha}\right). \tag{23}$$

Lemma 1. *[15]. For all $\alpha > 0$ and $s > 0$,*

i. $s^{\alpha-1} \left(s^{\alpha} - A\right)^{-1} = \mathcal{L}\left(E_{\alpha,1}\left(A t^{\alpha}\right)\right)(s)$, *ii.* $s^{\alpha-2}(s^{\alpha}-A)^{-1} = \mathcal{L}(tE_{\alpha,2}(At^{\alpha}))(s)$,

iii. $(s^{\alpha}-A)^{-1} = \frac{1}{\Gamma(\alpha-1)}\mathcal{L}\left(\int_0^t (t-s)^{\alpha-2}E_{\alpha,1}(As^{\alpha})\,ds\right)$.

Now, we recall Schauder fixed point theorem and the Ascoli-Arzela theorem as follows.

Theorem 4. *(Schauder fixed point theorem) Allow $\mathcal{G}$ be a non-empty, closed, limited and convex subspace of a Banach space O, and assume that $Q : \mathcal{G} \to \mathcal{G}$ is a compact operator. Therefore Q has at least one fixed point in $\mathcal{G}$.*

Theorem 5. *(Ascoli-Arzela). Allow $\phi_n(t)$ be a series of functions which is uniformly limited and equi-continuous. Therefore, $\phi_n(t)$ has a uniformly convergent subsequence.*

The next parts considers u as a fuzzy process, instead of a deterministic function. The goal is to apply the ideas of fuzzy integral and fuzzy F-derivative.

3. Fuzzy Interactive Fractional Derivative

The fuzzy fractional Riemann Liouville integral of degree *alpha* > 0, of u can be specified by

$$\left[\left(I_{a^+}^{\alpha}u\right)(t)\right]_r = \frac{1}{\Gamma(\alpha)}\left[\int_a^t (t-s)^{\alpha-1}u_r^-(s)ds, \int_a^t (t-s)^{\alpha-1}u_r^+(s)ds\right], t > a.$$

For fuzzy fractional derivative of $u \in L(J, \mathbb{R}_{\mathcal{F}})$ we have

$$u_{1-\alpha}(t) = \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}u(s)ds, \ \texttt{for all } t \in J,$$

where $u_{1-\alpha}(a) = \lim_{t\to a^+} u_{1-\alpha}(t)$ according to Pompeiu Hausdorff distance.

Definition 11. *[16]. The fuzzy Riemann Liouville fractional derivative of u with regard to the F-derivative is characterized as*

$$\left({}^{RL_F}D_{a^+}^{\alpha}u\right)(t) = \frac{1}{\Gamma(1-\alpha)}\left(\int_a^t (t-s)^{-\alpha}u(s)ds\right)_F' = (u_{1-\alpha}(t))_F', \tag{24}$$

where $\int_a^t (t-s)^{-\alpha}u(s)ds$ is a F-correlated fuzzy process, F-differentiable for all $t \in J$.

It is critical to note that $\int_a^t (t-s)^{-\alpha}u(s)ds$ might be an expansive or contractive fuzzy process. nevertheless it is expansive if $u(\cdot)$ is expansive [17]. As a result, if $u_{1-\alpha}(\cdot)$ or $u(\cdot)$ is expansive, therefore

$$\left[{}^{RL_F}D_{a^+}^{\alpha}u(t)\right]_r = \frac{1}{\Gamma(1-\alpha)}\left[\frac{d}{dt}\int_a^t (t-s)^{-\alpha}u_r^-(s)ds, \frac{d}{dt}\int_a^t (t-s)^{-\alpha}u_r^+(s)ds\right].$$

Thus,

$$\left[{}^{RL_F}D_{a^+}^{\alpha}u(t)\right]_r = \begin{cases} \texttt{i.} \quad \left[D_{a^+}^{\alpha}u_r^-(t), D_{a^+}^{\alpha}u_r^+(t)\right] & \texttt{if } u_{1-\alpha}(\cdot) \texttt{ or } u(\cdot) \texttt{ is expansive} \\ \texttt{ii.} \quad \left[D_{a^+}^{\alpha}u_r^+(t), D_{a^+}^{\alpha}u_r^-(t)\right] & \texttt{if } u_{1-\alpha}(\cdot) \texttt{ is contractive} \end{cases}$$

Definition 12. *Allow u be a F-correlated fuzzy process. The fuzzy Caputo fractional derivative ${}^{C_F}D_{a^+}^{\alpha}u$ with regard to the F-derivative is stated as follows:*

$$\left({}^{C_F}D_{a+}^{\alpha}u\right)(t) = \left({}^{RL_F}D_{a+}^{\alpha}\left[u(\cdot) -_F u(a)\right]\right)(t), \quad \forall t \in J.$$

Then,

$$\left({}^{C_F}D_{a+}^{\alpha}u\right)(t) = \frac{1}{\Gamma(1-\alpha)}\left(\int_a^t (t-s)^{-\alpha}\left(u(s) -_F u(a)\right)ds\right)_F'.$$

According to (24) if $u_{1-\alpha}(\cdot)$ is contractive, therefore

$$\left[\left({}^{C_F}D^{\alpha}_{a^+}u\right)(t)\right]_r = \begin{cases} \left[\left({}^{C}D^{\alpha}_{a^+}u^-_r\right)(t), \left({}^{C}D^{\alpha}_{a^+}u^+_r\right)(t)\right] & \texttt{if } u(\cdot) \texttt{ is expansive} \\ \left[\left({}^{C}D^{\alpha}_{a^+}u^+_r\right)(t), \left({}^{C}D^{\alpha}_{a^+}u^-_r\right)(t)\right] & \texttt{if } u(\cdot) \texttt{ is contractive} \end{cases}$$

Theorem 6. *[16]. Let $u \in AC(J, \mathcal{F}_{\mathbb{R}})$ be a F-correlated fuzzy process, F-differentiable alongside $[u(t)]_r = [u^-_r(t), u^+_r(t)]$, for $0 < \alpha \leq 1$, and $r \in [0,1]$. we have,*

$$\left[\left({}^{C_F}D^{\alpha}_{a^+}u\right)(t)\right]_r = \begin{cases} \left[\int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\left(u^-_r\right)'(s)ds, \int_a^t \frac{(t-s)^{-a}}{\Gamma(1-\alpha)}\left(u^+_r\right)'(s)ds\right] & \textit{if } u \textit{ is expansive} \\ \left[\int_a^t \frac{(t-s)^{-a}}{\Gamma(1-\alpha)}\left(u^+_r\right)'(s)ds, \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\left(u^-_r\right)'(s)ds\right] & \textit{if } u \textit{ is contractive} \end{cases},$$

for $t \in J$.

Theorem 7. *If $u : J \to \mathcal{F}_{\mathbb{R}}$ is F-differentiable at $t_0 > 0$, Denote $[u(t)]_r = [u^-_r(t), u^+_r(t)], r \in [0,1]$. Then $u^-_r(t)$ and $u^+_r(t)$ are continuous at t_0 so u is continuous at t_0.*

Proof. Let $\epsilon > 0$ and $r \in [0,1]$, we have :

$$[u(t_0+h) -_F u(t_0)]_r =$$
$$\begin{cases} \texttt{i.} \quad [u^-_r(t_0+h) - u^-_r(t_0), u^+_r(t_0+h) - u^+_r(t_0)] & \texttt{if } F'(w) > 1, \quad \forall w \in [u]_r \\ \texttt{ii.} \quad [u^+_r(t_0+h) - u^+_r(t_0), u^-_r(t_0+h) - u^-_r(t_0)] & \texttt{if } 0 < F'(w) \leq 1, \forall w \in [u]_r; \\ \texttt{iii.} \quad [u^-_r(t_0+h) - u^+_r(t_0), u^+_r(t_0+h) - u^-_r(t_0)] & \texttt{if } F'(w) \leq 0, \quad \forall w \in [u]_r \end{cases}$$

Dividing and multiplying by h, we have :

$$[u(t_0+h) -_F u(t_0)]_r =$$
$$\begin{cases} \texttt{i.} \quad \left[\frac{u^-_r(t_0+h)-u^-_r(t_0)}{h}\cdot h, \frac{u^+_r(t_0+h)-u^+_r(t_0)}{h}\cdot h\right] & \texttt{if } F'(w) > 1, \quad \forall w \in [u]_r \\ \texttt{ii.} \quad \left[\frac{u^+_r(t_0+h)-u^+_r(t_0)}{h}\cdot h, \frac{u^-_r(t_0+h)-u^-_r(t_0)}{h}\cdot h\right] & \texttt{if } 0 < F'(w) \leq 1, \forall w \in [u]_r; \\ \texttt{iii.} \quad \left[\frac{u^-_r(t_0+h)-u^+_r(t_0)}{h}\cdot h, \frac{u^+_r(t_0+h)-u^-_r(t_0)}{h}\cdot h\right] & \texttt{if } F'(w) \leq 0, \quad \forall w \in [u]_r \end{cases}$$

Then

$$\lim_{h\to 0}[u(t_0+h) -_F u(t_0)]_r =$$
$$\begin{cases} \texttt{i.} \quad \left[\lim\limits_{h\to 0}\frac{u^-_r(t_0+h)-u^-_r(t_0)}{h}\cdot\lim\limits_{h\to 0}h, \lim\limits_{h\to 0}\frac{u^+_r(t_0+h)-u^+_r(t_0)}{h}\cdot\lim\limits_{h\to 0}h\right] & \texttt{if } F'(w) > 1, \quad \forall w \in [u]_r \\ \texttt{ii.} \quad \left[\lim\limits_{h\to 0}\frac{u^+_r(t_0+h)-u^+_r(t_0)}{h}\cdot\lim\limits_{h\to 0}h, \lim\limits_{h\to 0}\frac{u^-_r(t_0+h)-u^-_r(t_0)}{h}\cdot\lim\limits_{h\to 0}h\right] & \texttt{if } 0 < F'(w) \leq 1, \forall w \in [u]_r; \\ \texttt{iii.} \quad \left[\lim\limits_{h\to 0}\frac{u^-_r(t_0+h)-u^+_r(t_0)}{h}\cdot\lim\limits_{h\to 0}h, \lim\limits_{h\to 0}\frac{u^+_r(t_0+h)-u^-_r(t_0)}{h}\cdot\lim\limits_{h\to 0}h\right] & \texttt{if } F'(w) \leq 0, \quad \forall w \in [u]_r \end{cases}$$

Then

$$\lim_{h\to 0}[u(t_0+h) -_F u(t_0)]_r =$$
$$\begin{cases} \texttt{i.} \quad [u^{-\prime}_r\cdot 0, u^{+\prime}_r\cdot 0] & \texttt{if } F'(w) > 1, \quad \forall w \in [u]_r \\ \texttt{ii.} \quad [u^{+\prime}_r\cdot 0, u^{-\prime}_r\cdot 0] & \texttt{if } 0 < F'(w) \leq 1, \forall w \in [u]_r; \\ \texttt{iii.} \quad \left[\lim\limits_{h\to 0}\frac{u^-_r(t_0+h)-u^+_r(t_0)}{h}\cdot 0, \lim\limits_{h\to 0}\frac{u^+_r(t_0+h)-u^-_r(t_0)}{h}\cdot 0\right] & \texttt{if } F'(w) \leq 0, \quad \forall w \in [u]_r \end{cases}$$

which implies that

$$\lim_{h\to 0}[u(t_0+h)]_r = [u(t_0)]_r$$

Hence, u is continuous at t_0. □

Theorem 8. *Let $u \in AC(J, \mathcal{F}_{\mathbb{R}})$ be a F-correlated fuzzy process, F-correlated differentiable with $[u(t)]_r = [u^-_r(t), u^+_r(t)]$, for $0 < \alpha \leq 1$, and $r \in [0,1]$. Then*

1.

$$ {}^{RL_F}I^{\alpha}\left({}^{C_F}D^{\alpha}_{a^+}u(t)\right) = {}^{RL_F}I^{\alpha}\left({}^{RL_F}D^{\alpha}_{a+}\left[u(\cdot) -_F u(a)\right]\right)(t) = \begin{cases} u(t) -_F u(a), & \text{if } u(\cdot) \text{ is expansive} \\ -_F(-1)\left(u(t) -_F u(a)\right), & \text{if } u(\cdot) \text{ is contractive} \end{cases}. \tag{25} $$

2.

$$ {}^{C_F}D^{\alpha}_{a^+}\left({}^{RL_F}I^{\alpha}u(t)\right) = \begin{cases} u(t), & \text{if } u(\cdot) \text{ is expansive} \\ -_F(-1)u(t), & \text{if } u(\cdot) \text{ is contractive} \end{cases} \tag{26} $$

Proof. Using the r-cuts, we get

$$ \begin{aligned} \left[{}^{RL_F}I^{\alpha}\left({}^{C_F}D^{\alpha}_{a^+}u(t)\right)\right]_r &= {}^{RL}I^{\alpha}\left(\left[{}^{C_F}D^{\alpha}_{a^+}u(t)\right]_r\right) \\ &= \begin{cases} {}^{RL}I^{\alpha}\left(\left[\left({}^{C}D^{\alpha}_{a^+}u^-_r\right)(t), \left({}^{C}D^{\alpha}_{a^+}u^+_r\right)(t)\right]\right), & \text{if } u(\cdot) \text{ is expansive} \\ {}^{RL}I^{\alpha}\left(\left[\left({}^{C}D^{\alpha}_{a^+}u^+_r\right)(t), \left({}^{C}D^{\alpha}_{a^+}u^-_r\right)(t)\right]\right), & \text{if } u(\cdot) \text{ is contractive} \end{cases} \\ &= \begin{cases} \left[{}^{RL}I^{\alpha}\left({}^{C}D^{\alpha}_{a^+}u^-_r\right)(t), {}^{RL}I^{\alpha}\left({}^{C}D^{\alpha}_{a^+}u^+_r\right)(t)\right], & \text{if } u(\cdot) \text{ is expansive} \\ \left[{}^{RL}I^{\alpha}\left({}^{C}D^{\alpha}_{a^+}u^+_r\right)(t), {}^{RL}I^{\alpha}\left({}^{C}D^{\alpha}_{a^+}u^-_r\right)(t)\right], & \text{if } u(\cdot) \text{ is contractive} \end{cases} \end{aligned} $$

where

$$ \begin{aligned} {}^{RL}I^{\alpha}\left({}^{C}D^{\alpha}_{a^+}u^-_r\right)(t) &= {}^{RL}I^{\alpha}\left(\frac{1}{\Gamma(1-\alpha)}\int_a^t (t-s)^{-\alpha}u^{-\prime}_r(s)ds\right) \\ &= u^-_r(t) - u^-_r(a), \end{aligned} $$

and

$$ \begin{aligned} {}^{RL}I^{\alpha}\left({}^{C}D^{\alpha}_{a^+}u^+_r\right)(t) &= {}^{RL}I^{\alpha}\left(\frac{1}{\Gamma(1-\alpha)}\int_a^t (t-s)^{-\alpha}u^{+\prime}_r(s)ds\right) \\ &= u^+_r(t) - u^+_r(a), \end{aligned} $$

Thus,

$$ \left[{}^{RL_F}I^{\alpha}\left({}^{C_F}D^{\alpha}_{a^+}u(t)\right)\right]_r = \begin{cases} \left[u^-_r(t) - u^-_r(a), u^+_r(t) - u^+_r(a)\right], & \text{if } u(\cdot) \text{ is expansive} \\ \left[u^+_r(t) - u^+_r(a), u^-_r(t) - u^-_r(a)\right], & \text{if } u(\cdot) \text{ is contractive} \end{cases} $$

Therefore, using the F-difference (12),we obtain

$$ {}^{RL_F}I^{\alpha}\left({}^{C_F}D^{\alpha}_{a^+}u(t)\right) = \begin{cases} u(t) -_F u(a), & \text{if } u(\cdot) \text{ is expansive} \\ -_F(-1)\left(u(t) -_F u(a)\right), & \text{if } u(\cdot) \text{ is contractive} \end{cases} \tag{27} $$

For the second property, since u is continuous, then ${}^{RL_F}I^{\alpha}u(t)$ is clearly F-correlated differentiable. Hence, Using the r-cuts, we obtain

$$ \begin{aligned} \left[{}^{C_F}D^{\alpha}_{a^+}\left({}^{RL_F}I^{\alpha}u(t)\right)\right]_r &= \begin{cases} \left[{}^{C}D^{\alpha}_{a^+}\left({}^{RL}I^{\alpha}u^-_r\right)(t), {}^{C}D^{\alpha}_{a^+}\left({}^{RL}I^{\alpha}u^+_r\right)(t)\right], & \text{if } u(\cdot) \text{ is expansive} \\ \left[{}^{C}D^{\alpha}_{a^+}\left({}^{RL}I^{\alpha}u^+_r\right)(t), {}^{C}D^{\alpha}_{a^+}\left({}^{RL}I^{\alpha}u^-_r\right)(t)\right], & \text{if } u(\cdot) \text{ is contractive} \end{cases} \\ &= \begin{cases} \left[u^-_r(t), u^+_r(t)\right], & \text{if } u(\cdot) \text{ is expansive} \\ \left[u^+_r(t), u^-_r(t)\right], & \text{if } u(\cdot) \text{ is contractive} \end{cases} \\ &= \begin{cases} [u(t)]_r, & \text{if } u(\cdot) \text{ is expansive} \\ -_F(-1)\,[u(t)]_r, & \text{if } u(\cdot) \text{ is contractive} \end{cases} \end{aligned} $$

□

Theorem 9. *Let $u \in AC(J, \mathcal{F}_{\mathbb{R}})$ be a F-correlated fuzzy process, F-correlated differentiable with $[u(t)]_r = [u_r^-(t), u_r^+(t)]$, for $0 < \alpha \leq 1$, and $r \in [0,1]$. Then*

$$\mathcal{L}\left({}^{C_F}D_{a^+}^{\alpha} u(t)\right) = \begin{cases} s \odot \mathcal{L}(u(t)) -_F u(a), & \textit{if } u(\cdot) \textit{ is expansive} \\ -_F(-1)\left(s \odot \mathcal{L}(u(t)) -_F u(a)\right), & \textit{if } u(\cdot) \textit{ is contractive} \end{cases}. \tag{28}$$

Proof. Using the r-cuts, we get

$$\begin{aligned} \left[\mathcal{L}\left({}^{C_F}D_{a^+}^{\alpha} u(t)\right)\right]_r &= \begin{cases} \left[\mathcal{L}\left({}^{C}D_{a^+}^{\alpha} u_r^-\right)(t), \mathcal{L}\left({}^{C}D_{a^+}^{\alpha} u_r^+\right)(t)\right], & \text{if } u(\cdot) \text{ is expansive} \\ \left[\mathcal{L}\left({}^{C}D_{a^+}^{\alpha} u_r^+\right)(t), \mathcal{L}\left({}^{C}D_{a^+}^{\alpha} u_r^-\right)(t)\right], & \text{if } u(\cdot) \text{ is contractive} \end{cases} \\ &= \begin{cases} \left[s\mathcal{L}(u_r^-(t)) - u_r^-(a), s\mathcal{L}(u_r^+(t)) - u_r^+(a)\right], & \text{if } u(\cdot) \text{ is expansive} \\ \left[s\mathcal{L}(u_r^+(t)) - u_r^+(a), s\mathcal{L}(u_r^-(t)) - u_r^-(a)\right], & \text{if } u(\cdot) \text{ is contractive} \end{cases} \\ &= \begin{cases} s \odot \mathcal{L}(u(t)) -_F u(a), & \text{if } u(\cdot) \text{ is expansive} \\ -_F(-1)\left(s \odot \mathcal{L}(u(t)) -_F u(a)\right), & \text{if } u(\cdot) \text{ is contractive} \end{cases} \end{aligned}$$

The evidence is finished. □

Theorem 10. *Assume $0 < \alpha \leq 1$. If $\phi, \psi : J \to \mathcal{F}_{\mathbb{R}}$ are F-differentiable and $\lambda \in \mathbb{R}$ then*

1.

$$(\phi + \psi)'_F(s) = \phi'_F(s) + \psi'_F(s) \tag{29}$$

2.

$$(\lambda\phi)'_F(s) = \lambda\phi'_F(s) \tag{30}$$

Proof. We just offer the specifics for instance 1 because the second instance is analogous. Since ϕ is F-differentiable, as a result of this $\phi(s+h) -_F \phi(s)$ exists i.e. there is $u_1(s,h)$ such as

$$\phi(s+h) = \phi(s) + u_1(s,h)$$

Analogously since ψ is F-differentiable, then there is $v_1(s,h)$ such as

$$\psi(s+h) = \psi(s) + v_1(s,h)$$

and we get

$$\phi(s+h) + \psi(s+h) = \phi(t) + \psi(s) + u_1(s,h) + v_1(s,h)$$

that is the F-difference

$$\left(\phi(s+h) + \psi(s+h)\right) -_F (\phi(s) + \psi(s)) = u_1(s,h) + v_1(s,h) \tag{31}$$

We notice that

$$\lim_{h\to 0} \frac{u_1(s,h)}{h} = \phi'_F(s) \quad \text{and} \quad \lim_{h\to 0} \frac{v_1(s,h)}{h} = \psi'_F(s).$$

Finally, by multiplying (31) with $\frac{1}{h}$ and passing to limit with $\lim\limits_{h\to 0}$ we get the desired results. □

Theorem 11. *Let $u : J \to \mathcal{F}_{\mathbb{R}}$ and $\phi : J \to \mathbb{R}$. Assume that $\phi(t)$ is real differentiable function and the fuzzy function $u(t)$ is F-differentiable. Then*

$$(u \odot \phi)'_F(t) = u'_F(t) \odot \phi(t) +_F u(t) \odot \phi'_F(t). \tag{32}$$

Proof. First of all, we have

$$(u(t) \odot \phi(t))'_F = \lim_{h \to 0} \frac{u(t+h) \odot \phi(t+h) -_F u(t) \odot \phi(t)\big)}{h}$$

Evaluating the difference between the two edges of the equation, it is easy to see that the metric is zero.

$$\begin{aligned}
& d_\infty\left(\frac{u(t+h) \odot \phi(t+h) -_F u(t) \odot \phi(t)\big)}{h}, u'_F(t) \odot \phi(t) +_F u(t) \odot \phi'_F(t)\right) \\
= \; & d_\infty\left(\tfrac{u(t+h)\odot\phi(t+h) +_F u(t)\odot\phi(t+h) -_F u(t)\odot\phi(t+h) -_F u(t)\odot\phi(t)\big)}{h}, u'_F(t) \odot \phi(t) +_F u(t) \odot \phi'_F(t)\right) \\
= \; & d_\infty\left(\tfrac{(u(t+h) -_F u(t))\odot\phi(t+h) +_F u(t)\odot\big(\phi(t+h) -_F \odot\phi(t)\big)}{h}, u'_F(t) \odot \phi(t) +_F u(t) \odot \phi'_F(t)\right) \\
\leq & \; d_\infty\left(\frac{(u(t+h) -_F u(t)) \odot \phi(t+h)}{h}, u'_F(t) \odot \phi(t)\right) \\
+ & \; d_\infty\left(\frac{u(t) \odot \big(\phi(t+h) -_F \odot\phi(t)\big)}{h}, u(t) \odot \phi'_F(t)\right)
\end{aligned}$$

Now limit of two edges when $h \to 0$ are,

$$\begin{aligned}
& \lim_{h \to 0} d_\infty\left(\frac{u(t+h) \odot \phi(t+h) -_F u(t) \odot \phi(t)\big)}{h}, u'_F(t) \odot \phi(t) +_F u(t) \odot \phi'_F(t)\right) \\
& = \lim_{h \to 0} d_\infty\left(\frac{(u(t+h) -_F u(t)) \odot \phi(t+h)}{h}, u'_F(t) \odot \phi(t)\right) \\
& + \lim_{h \to 0} d_\infty\left(\frac{u(t) \odot \big(\phi(t+h) -_F \odot\phi(t)\big)}{h}, u(t) \odot \phi'_F(t)\right)
\end{aligned}$$

The proof is finished by considering the qualities of the limits and distances. □

Theorem 12. *Take $u(t)$ as a continuous and F-differentiable fuzzy correlated proses on $[a, b]$ respectively and $\phi(t)$ is a real continuous and differentiable function on $[a, b]$. Then there's $t_0 \in (a, b)$ such as*

$$[u(b) -_F u(a)] \odot \phi'(t_0) = \big[\phi(b) - \phi(a)\big] \odot u'_F(t_0) \tag{33}$$

Proof. Construct the following novel function:

$$\psi(t) = [u(b) -_F u(a)] \odot \phi(t) -_F \big[\phi(b) - \phi(a)\big] \odot u(t) \tag{34}$$

Since this function is continuous,

We can take $u(b) -_F u(a) = k$ and $\phi(b) - \phi(a) = l$ so we gain

$$\psi(t) = k \odot \phi(t) -_F l \odot u(t)$$

Afterwards, for every $\epsilon > 0$, $\exists\delta > 0$ if for each x in $|t - t_0| < \delta$ at an random point t we demonstrate,

$$d_\infty(\psi(t), \psi(t_0)) = d_\infty\big(k \odot \phi(x) -_F l \odot u(t), k \odot \phi(t_0) -_F l \odot u(t_0)\big)$$

In accordance with the characteristics of the Hausdorff metric,

$$\leq d_\infty(k \odot \phi(t), k \odot \phi(t_0)) + d_\infty(l \odot u(t), l \odot u(t_0))$$

We may produce, on the basis of the definitions of Hausdorff metric and the value of a fuzzy element,

$$\leq |k||\phi(t) - \phi(t_0)| + |l| d_\infty(u(t), u(t_0))$$

Furthermore, this total is smaller than ϵ.

On the opposite side, $\psi(a) = \psi(b)$ so it is F-differentiable at t_0 and equals to zero. Therefore

$$\psi'_F(t) = [u(b) -_F u(a)] \odot \phi'(t) -_F [\phi(b) - \phi(a)] \odot u'_F(t)$$

And $\psi'_F(t_0) = 0$ so,

$$[u(b) -_F u(a)] \odot \phi'(t_0) -_F [\phi(b) - \phi(a)] \odot u'_F(t_0) = 0$$

The proof is completed. □

Remark 1. *Our findings using F-correlated derivative are analogous with those found using* gH*. Yet, the levels of computation using F-correlated process and via* gH *have distinct characteristics as can be observed in* (11)-(14)*. Despite the difference* (12) *corresponds to the difference* (gH)*, the F-correlated multiplication and division operations are not compatible with normal operations employed with* (gH)*. These evidences suggest that the results of fuzzy differential equations obtained by* gH *and F may differ. For instance, [18] using computational simulations.*

4. Solutions via Fuzzy Interactive Fractional Derivative

Considering the next fuzzy fractional starting point issue, which is supplied by the F-correlated fractional Caputo derivative of degree $0 < \alpha \leq 1$

$$\begin{aligned} {}^{C_F}D^{\alpha}_{a^+} u(t) &= Au(t) +_F f(t, u(t)), \\ u(a) &= u_0 \in \mathbb{R}_{\mathcal{F}}, \quad J = [0, b] \end{aligned} \tag{35}$$

where $f : J \times \mathcal{F}_{\mathbb{R}} \to \mathcal{F}_{\mathbb{R}}$ is fuzzy continuous function and A is a linear operator.

Definition 13. *The F-correlated fuzzy process $u : J \to \mathcal{F}_{\mathbb{R}}$ is considered to be a solution of* (35) *if and only if*

1. *$u \in C(J, \mathcal{F}_{\mathbb{R}})$, $u(a) = u_0$ and*
2. *$\left({}^{C_F}D^{\alpha}_{a^+} u\right)(t) = Au(t) +_F f(t, u(t))$, for all $t \in J$.*

Lemma 2. *Let A be a linear operator, the solution $u(t)$ of* (35) *is provided by*

- *if $u(t)$ is expansive,*

$$u(t) = E_{\alpha,1}(At^{\alpha}) \odot u_0 +_F \int_0^t \int_s^t \frac{(t-\delta)^{\alpha-2}}{\Gamma(\alpha-1)} \odot E_{\alpha,1}(A(\delta - s)^{\alpha}) \odot f(s, u(s))\, d\delta ds. \tag{36}$$

- *if $u(t)$ is contractive,*

$$u(t) = E_{\alpha,1}(At^{\alpha}) \odot u_0 +_F -_F(-1) \int_0^t \int_s^t \frac{(t-\delta)^{\alpha-2}}{\Gamma(\alpha-1)} \odot E_{\alpha,1}(A(\delta - s)^{\alpha}) \odot f(s, u(s))\, d\delta ds. \tag{37}$$

Proof. By performing the Riemann-Liouville fractional integral operator with respect to F-derivative ${}^{RL}I^{\alpha}_{a^+}$ to both side of the Eq. (35) we get
if $u(t)$ is expansive

$$ {}^{RL_F}I^{\alpha}\left({}^{C_F}D^{\alpha}_{a^+}u(t)\right) = u(t) -_F u(a) = {}^{RL_F}I^{\alpha}_{a^+}\left(Au(t) +_F f\left(t, u(t)\right)\right), \tag{38} $$

And if $u(t)$ is contractive, we obtain

$$ {}^{RL_F}I^{\alpha}\left({}^{C_F}D^{\alpha}_{a^+}u(t)\right) = -_F(-1)\left(u(t) -_F u(a)\right) = {}^{RL_F}I^{\alpha}_{a^+}\left(Au(t) +_F f\left(t, u(t)\right)\right), \tag{39} $$

We may derive these claims from the concept of F-difference:

- if $u(t)$ is expansive

$$ u(t) = u(a) +_F {}^{RL_F}I^{\alpha}_{a^+}\left(Au(t) +_F f\left(t, u(t)\right)\right), \tag{40} $$

- if $u(t)$ is contractive,

$$ u(t) = u(a) +_F -_F(-1)^{RL_F}I^{\alpha}_{a^+}\left(Au(t) +_F f\left(t, u(t)\right)\right), \tag{41} $$

Now we use the fuzzy Laplace transform on (40) and (41)

- if $u(t)$ is expansive,

$$ U(s) = \frac{1}{s} \odot u_0 +_F \frac{1}{s^{\alpha}} \odot AU(s) +_F \frac{1}{s^{\alpha}} \odot F(s), $$

- if $u(t)$ is contractive,

$$ U(s) = \frac{1}{s} \odot u_0 +_F -_F(-1)\frac{1}{s^{\alpha}} \odot AU(s) +_F -_F(-1)\frac{1}{s^{\alpha}} \odot F(s), $$

which implies

- if $u(t)$ is expansive,

$$ s^{\alpha} \odot U(s) = s^{\alpha-1} \odot u_0 +_F AU(s) + F(s) $$
$$ (s^{\alpha} \odot Id - A) \odot U(s) = s^{\alpha-1} \odot u_0 +_F F(s) $$

  Then

$$ U(s) = (s^{\alpha} \odot Id - A)^{-1} \odot s^{\alpha-1} \odot u_0 +_F (s^{\alpha} \odot Id - A)^{-1} \odot F(s). \tag{42} $$

- if $u(t)$ is contractive,

$$ s^{\alpha} \odot U(s) = s^{\alpha-1} \odot u_0 +_F -_F(-1)AU(s) +_F -_F(-1)F(s) $$
$$ (s^{\alpha} \odot Id - A) \odot U(s) = s^{\alpha-1} \odot u_0 +_F -_F(-1)F(s) $$

  Then

$$ U(s) = (s^{\alpha} \odot Id - A)^{-1} \odot s^{\alpha-1} \odot u_0 +_F -_F(-1)(s^{\alpha} \odot Id - A)^{-1} \odot F(s). \tag{43} $$

By the Lemma 1 we gain

- if $u(t)$ is expansive,

$$ U(s) = \mathcal{L}\left(E_{\alpha,1}\left(At^{\alpha}\right)\right) \odot u_0 +_F \mathcal{L}\left(g \star f\right), \tag{44} $$

- if $u(t)$ is contractive,

$$U(s) = \mathcal{L}\left(E_{\alpha,1}\left(At^{\alpha}\right)\right) \odot u_0 +_F -_F(-1)\mathcal{L}\left(g \star f\right). \tag{45}$$

with $g(t) = \int_0^t \frac{(t-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \odot E_{\alpha,1}\left(A\tau^{\alpha}\right) d\tau$.

Finally by applying the inverse Laplace transformation on both sides of the equations (44) and (45), we get the mild solution of the Eq. (35). The evidence is complete. □

The next hypothesis will be utilized in the findings that follow:

(**Hyp1**) : For all $t \in J$, the function $f \in C(J \times \mathcal{T}, \mathcal{T})$ is continuous and for every $u \in C(J, \mathcal{T})$, $f(., u) : J \to \mathcal{T}$ is strongly measurable.

(**Hyp2**) : There exist $\alpha_2 \in [0, \alpha), B_\kappa := \left\{u \in \mathcal{T}, d_\infty(u, \tilde{0}) \le \kappa\right\} \subset \mathcal{T}, \kappa > 0$, and $\rho(.) \in L^{\frac{1}{\alpha_2}}(J, \mathbb{R}^+)$ such as for any $u, v \in C(J, B_\kappa)$ we obtain

$$d_\infty(f(t, u(t)), f(t, v(t)) \le \rho(t) d_\infty(u(t), v(t)), t \in J. \tag{46}$$

(**Hyp3**) : There is a constant $\alpha_1 \in [0, \alpha)$ and $m \in L^{\frac{1}{\alpha_1}}(J, \mathbb{R}^+)$ suchas

$$d_\infty(f(t, u(t)), \tilde{0}) \le \mu(t). \tag{47}$$

for all $u \in C(J, \mathcal{T})$ and for almost all $t \in J$.

(**Hyp4**) : $E_{\alpha,n}\left(At^{\alpha}\right)$ is a compact operator for any $t > 0$ and $n \in \mathbb{N}$.

Theorem 13. *Given assumptions* (***Hyp1***) − (***Hyp4***) *the Eq.* (35) *has a expansive mild fuzzy solution in the set* $C(J, \mathcal{T})$.

Proof. Assume $u \in C(J, \mathcal{T})$. as u is continuous according to t and hypothesis (**Hyp1**), $f(s, u(s))$ is a measurable function on J. Let

$$\sigma = \frac{\alpha - 1}{1 - \alpha_1}, \quad M_1 = \|\mu\|_{L^{\frac{1}{\alpha_1}}(J)}. \tag{48}$$

For $t \in J$, by using Holder's inequality and (**Hyp3**), we have

$$\begin{aligned} d_\infty\left(\int_0^t (t-s)^{\alpha-1} \odot f(s, u(s)) ds, \tilde{0}\right) &\le \int_0^t (t-s)^{\alpha-1} \odot d_\infty(f(s, u(s)), \tilde{0}) ds \\ &\le \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_1}} ds\right)^{1-\alpha_1} \|\mu\|_{L^{\frac{1}{q\alpha_1}}[0,t]} \\ &\le \frac{M_1 b^{(1+\sigma)(1-\alpha_1)}}{(1+\sigma)^{1-\alpha_1}}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} & d_\infty\left(\int_0^t \int_s^t \left(\frac{(t-\delta)^{\alpha-2}}{\Gamma(\alpha-1)} \odot E_{\alpha,1}\left(A(\delta-s)^{\alpha}\right) \odot f\left(s, u(s)\right) d\delta ds, \tilde{0}\right)\right. \\ &\le \int_0^t \int_s^t \frac{(t-\delta)^{\alpha-2}}{\Gamma(\alpha-1)} \odot d_\infty\left(E_{\alpha,1}\left(A(\delta-s)^{\alpha}\right) \odot f\left(s, u(s)\right), \tilde{0}\right) d\delta ds \\ &\le \frac{M}{\Gamma(\alpha)} \odot \int_0^t (t-s)^{\alpha-1} d_\infty(f\left(s, u(s)\right), \tilde{0}) ds \\ &\le \frac{M_1 M b^{(1+\sigma)(1-\alpha_1)}}{\Gamma(\alpha)(1+\sigma)^{1-\alpha_1}}. \end{aligned}$$

for all $t \in J$.

Therefore $\int_0^t \int_s^t \frac{(t-\delta)^{\alpha-2}}{\Gamma(\alpha-1)} \odot E_{\alpha,1}\left(A(\delta-s)^{\alpha}\right) \odot f\left(s,u(s)\right) d\delta ds$ is limited for any $t \in J$.

For $u \in C(J,\mathcal{T})$, we state

$$
\begin{aligned}
&(Q_1u)(t) = E_{\alpha,1}\left(At^{\alpha}\right) \odot u_0 \quad t \in J = [0,b] \\
&(Q_2u)(t) = \int_0^t \int_s^t \frac{(t-\delta)^{\alpha-2}}{\Gamma(\alpha-1)} \odot E_{\alpha,1}\left(A(\delta-s)^{\alpha}\right) \odot f\left(s,u(s)\right) d\delta ds,
\end{aligned}
$$

Set

$$
\lambda = M\left(\|u_0\| + \frac{M_1 b^{(1+\sigma)(1-q\alpha_1)}}{\Gamma(\alpha)(1+\sigma)^{1-\alpha_1}}\right).
$$

and $\mathcal{B}_\lambda := \left\{u(.) \in C(J,\mathcal{T}) : d_\infty(u(t),\tilde{0}) \leq \lambda \ \texttt{for all} \ t \in J\right\}$. We will prove that $Q_1u + Q_2u$ has a fixed point on $\mathcal{B}_\lambda$.

Step 1. we show for every $u \in \mathcal{B}_\lambda, Q_1u + Q_2u \in \mathcal{B}_\lambda$. Indeed, with $0 \leq t_1 \leq t_2 \leq b$ we have

$$
\begin{aligned}
d_\infty\left((Q_2u)(t_2)\ ,(Q_2u)(t_1)\right) &= d_\infty\Bigg(\int_0^{t_2}\int_s^{t_2} \frac{(t_2-\delta)^{\alpha-2}}{\Gamma(\alpha-1)} \odot E_{\alpha,1}\left(A(\delta-s)^{\alpha}\right) \odot f\left(s,u(s)\right) d\delta ds, \\
&\int_0^{t_1}\int_s^{t_1} \frac{(t_1-\delta)^{\alpha-2}}{\Gamma(\alpha-1)} \odot E_{\alpha,1}\left(A(\delta-s)^{\alpha}\right) \odot f\left(s,u(s)\right) d\delta ds\Bigg) \\
&= d_\infty\Bigg(\int_{t_1}^{t_2}\int_s^{t_2} \frac{(t_2-\delta)^{\alpha-2}}{\Gamma(\alpha-1)} \odot E_{\alpha,1}\left(A(\delta-s)^{\alpha}\right) \odot f\left(s,u(s)\right) d\delta ds \\
&+ \int_0^{t_1}\int_s^{t_2} \frac{(t_2-\delta)^{\alpha-2}}{\Gamma(\alpha-1)} \odot E_{\alpha,1}\left(A(\delta-s)^{\alpha}\right) \odot f\left(s,u(s)\right) d\delta ds \\
&- \int_0^{t_1}\int_s^{t_1} \frac{(t_1-\delta)^{\alpha-2}}{\Gamma(\alpha-1)} \odot E_{\alpha,1}\left(A(\delta-s)^{\alpha}\right) \odot f\left(s,u(s)\right) d\delta ds, \tilde{0}\Bigg) \\
&= d_\infty\Bigg(\int_{t_1}^{t_2}\int_s^{t_2} \frac{(t_2-\delta)^{\alpha-2}}{\Gamma(\alpha-1)} \odot E_{\alpha,1}\left(A(\delta-s)^{\alpha}\right) \odot f\left(s,u(s)\right) d\delta ds \\
&+ \int_0^{t_1}\int_s^{t_1} \frac{\left[(t_2-\delta)^{\alpha-2} - (t_1-\delta)^{\alpha-2}\right]}{\Gamma(\alpha-1)} \odot E_{\alpha,1}\left(A(\delta-s)^{\alpha}\right) \odot f\left(s,u(s)\right) d\delta ds \\
&+ \int_0^{t_1}\int_{t_1}^{t_2} \frac{(t_2-\delta)^{\alpha-2}}{\Gamma(\alpha-1)} \odot E_{\alpha,1}\left(A(\delta-s)^{\alpha}\right) \odot f\left(s,u(s)\right) d\delta ds, \tilde{0}\Bigg) \\
&\leq d_\infty\left(\int_{t_1}^{t_2}\int_s^{t_2} \frac{(t_2-\delta)^{\alpha-2}}{\Gamma(\alpha-1)} \odot E_{\alpha,1}\left(A(\delta-s)^{\alpha}\right) \odot f\left(s,u(s)\right) d\delta ds, \tilde{0}\right) \\
&+ d_\infty\left(\int_0^{t_1}\int_s^{t_1} \frac{\left[(t_2-\delta)^{\alpha-2} - (t_1-\delta)^{\alpha-2}\right]}{\Gamma(\alpha-1)} \odot E_{\alpha,1}\left(A(\delta-s)^{\alpha}\right) \odot f\left(s,u(s)\right) d\delta ds, \tilde{0}\right) \\
&+ d_\infty\left(\int_0^{t_1}\int_{t_1}^{t_2} \frac{(t_2-\delta)^{\alpha-2}}{\Gamma(\alpha-1)} \odot E_{\alpha,1}\left(A(\delta-s)^{\alpha}\right) \odot f\left(s,u(s)\right) d\delta ds, \tilde{0}\right) \\
&= I_1 + I_2 + I_3.
\end{aligned}
$$

where

$$I_1 = d_\infty\left(\int_{t_1}^{t_2}\int_{s}^{t_2} \frac{(t_2-\delta)^{\alpha-2}}{\Gamma(\alpha-1)} \odot E_{\alpha,1}\left(A(\delta-s)^{\alpha}\right)\odot f\left(s,u(s)\right)d\delta ds, \tilde{0}\right)$$
$$I_2 = d_\infty\left(\int_{0}^{t_1}\int_{s}^{t_1} \frac{\left[(t_2-\delta)^{\alpha-2}-(t_1-\delta)^{\alpha-2}\right]}{\Gamma(\alpha-1)} \odot E_{\alpha,1}\left(A(\delta-s)^{\alpha}\right)\odot f\left(s,u(s)\right)d\delta ds, \tilde{0}\right)$$
$$I_3 = d_\infty\left(\int_{0}^{t_1}\int_{t_1}^{t_2} \frac{(t_2-\delta)^{\alpha-2}}{\Gamma(\alpha-1)} \odot E_{\alpha,1}\left(A(\delta-s)^{\alpha}\right)\odot f\left(s,u(s)\right)d\delta ds, \tilde{0}\right).$$

We have :

$$\begin{aligned} I_1 &= d_\infty\left(\int_{t_1}^{t_2}\int_{s}^{t_2} \frac{(t_2-\delta)^{\alpha-2}}{\Gamma(\alpha-1)} \odot E_{\alpha,1}\left(A(\delta-s)^{\alpha}\right)\odot f\left(s,u(s)\right)d\delta ds, \tilde{0}\right)\\ &\leq \frac{MM_1}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2}(t_2-s)^{1-\alpha_1}\right)\\ &\leq \frac{MM_1}{(1+\sigma)^{1-\alpha_1}\Gamma(\alpha)}(t_2-t_1)^{(\sigma+1)(1-\alpha_1)}, \end{aligned}$$

also

$$\begin{aligned} I_2 &= d_\infty\left(\int_{0}^{t_1}\int_{s}^{t_1} \frac{\left[(t_2-\delta)^{\alpha-2}-(t_1-\delta)^{\alpha-2}\right]}{\Gamma(\alpha-1)} \odot E_{\alpha,1}\left(A(\delta-s)^{\alpha}\right)\odot f\left(s,u(s)\right)d\delta ds, \tilde{0}\right)\\ &\leq \frac{M}{\Gamma(\alpha)}\left(\int_{0}^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}-(t_2-t_1)^{\alpha-1}\right]\odot d_\infty\left(f\left(s,u(s)\right)ds,\tilde{0}\right)\right)\\ &\leq \frac{M}{\Gamma(\alpha)}\left(\int_{0}^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\odot d_\infty\left(f\left(s,u(s)\right)ds,\tilde{0}\right)\right.\\ &\left.-\int_{0}^{t_1}(t_2-t_1)^{\alpha-1}\odot d_\infty\left(f\left(s,u(s)\right)ds,\tilde{0}\right)\right)\\ &\leq \frac{M}{\Gamma(\alpha)}\left(\int_{0}^{t_1}\left[(t_2-s)^{b}-(t_1-s)^{\sigma}\right]^{1-\alpha_1}\odot M_1-(t_2-t_1)^{\alpha-1}\odot t_1^{1-\alpha_1}M_1\right)\\ &\leq \frac{MM_1}{(\sigma+1)^{1-\alpha_1}\Gamma(\alpha)}\left(-(t_2-t_1)^{\sigma+1}+t_2^{\sigma+1}-t_1^{\sigma+1}-(t_2-t_1)^{\alpha-1}(\sigma+1)^{1-\alpha_1}\right). \end{aligned}$$

likewise

$$\begin{aligned} I_3 &= d_\infty\left(\int_{0}^{t_1}\int_{t_1}^{t_2} \frac{(t_2-\delta)^{\alpha-2}}{\Gamma(\alpha-1)} \odot E_{\alpha,1}\left(A(\delta-s)^{\alpha}\right)\odot f\left(s,u(s)\right)d\delta ds, \tilde{0}\right)\\ &\leq \frac{M}{\Gamma(\alpha)}\left(\int_{0}^{t_1}(t_2-t_1)^{\alpha-1}\odot d_\infty\left(f\left(s,u(s)\right)ds,\tilde{0}\right)\right)\\ &\leq \frac{MM_1(t_2-t_1)^{\alpha-1}}{\Gamma(\alpha)}t_1^{1-\alpha_1}. \end{aligned}$$

Then it is straightforward that I_1, I_2, and I_3 tend to 0 as $t_2-t_1 \to 0$. So $(Q_2u)(t)$ is continuous in $t \in J$. It's easy to see that $(Q_1u)(t)$ is also continuous in $t \in J$.

Now, for any $u \in \mathcal{B}_\lambda$ and $t \in J$, we have

$$d_\infty\left((Q_1u)(t)+(Q_2u)(t),\tilde{0}\right) \leq M\left(\|u_0\|\right)+\frac{M_1Mb^{(1+\sigma)(1-\alpha_1)}}{\Gamma(\alpha)(1+\sigma)^{1-\alpha_1}} \leq \lambda. \tag{49}$$

Then $Q_1 + Q_2$ is an operator from $\mathcal{B}_\lambda$ into $\mathcal{B}_\lambda$.

Step 2 . We prove that Q_2 is a fully continuous operator that can be decomposed into several small steps.

First, we show that Q_2 is continuous in $\mathcal{B}_\lambda$.

Let $\{u_n\} \subseteq \mathcal{B}_\lambda$ with $u_n \to u$ on $\mathcal{B}_\lambda$. Applying hypothesis (**Hyp2**), we get

$$f(s, u_n(s)) \to f(s, u(s)) \quad \text{as} \quad n \to \infty, \tag{50}$$

almost everywhere $t \in J$.

From the hypothesis (**Hyp3**), $d_\infty(f(s, u_n(s)), f(s, u(s))) \leq 2\mu(s)$.

Therefore, by the domination convergence theorem, we get

$$\begin{aligned} d_\infty((Q_2 u_n)(t), (Q_2 u)(t)) &\leq \int_0^t \int_s^t \frac{M(t-\delta)^{\alpha-2}}{\Gamma(\alpha-1)} \odot d_\infty(f(s, u_n(s)), f(s, u(s)))\, d\delta ds \\ &\leq \int_0^t (t-s)^{\alpha-1} \frac{\alpha M}{\Gamma(1+\alpha)} \odot d_\infty(f(s, u_n(s)), f(s, u(s)))\, ds \to 0, \end{aligned}$$

when $n \to \infty$, This means Q_2 is continuous.

Next, we show that $Q_2(\mathcal{B}_\lambda)$ is relatively compact. This is the family of functions $\{Q_2 u : u \in \mathcal{B}_\lambda\}$ and $\{(Q_2 u)(t)\}$ relative compactness:$u \in \mathcal{B}_\lambda$, where $t \in J$.

We proved this for all $u \in \mathcal{B}_\lambda$ and $0 \leq t_1 \leq t_2 \leq b$

$$d_\infty((Q_2 u)(t_2), (Q_2 u)(t_1)) \leq I_1 + I_2 + I_3.$$

We now have

$$\begin{aligned} I_1 &\leq \frac{MM_1}{(1+\sigma)^{1-\alpha_1}\Gamma(\alpha)}(t_2 - t_1)^{(\sigma+1)(1-\alpha_1)} \\ I_2 &\leq \frac{MM_1}{(\sigma+1)^{1-\alpha_1}\Gamma(\alpha)}\left(-(t_2-t_1)^{\sigma+1} + t_2^{\sigma+1} - t_1^{\sigma+1} - (t_2-t_1)^{\alpha-1}(\sigma+1)^{1-\alpha_1}\right) \\ I_3 &\leq \frac{MM_1(t_2-t_1)^{\alpha-1}}{\Gamma(\alpha)} t_1^{1-\alpha_1}. \end{aligned}$$

From **step 1**, it is easy to see that $Q_2(\mathcal{B}_\lambda)$ is equi-continuous.

Proving this is enough for each $t \in J, V(t) = \{(Q_2 u)(t) : u \in \mathcal{B}_\lambda\}$ is relatively compact. For any fixed $0 < t \leq b, \forall \epsilon \in (0, t)$ and $\forall \delta > 0$, let the operator $Q_{\epsilon,\delta}$ be define as

$$\begin{aligned} (Q_{\epsilon,\delta} u)(t) &= \int_0^{t-\epsilon} \int_{s+\eta}^t \frac{(t-\delta)^{\alpha-2}}{\Gamma(\alpha-1)} \odot E_{\alpha,1}(A(\delta-s)^\alpha) \odot f(s, u(s))\, d\delta ds \\ &= \int_0^{t-\epsilon} \int_{s+\eta}^t \frac{(t-\delta)^{\alpha-2}}{\Gamma(\alpha-1)} \odot E_{\alpha,1}(A(\delta-s)^\alpha - A(\eta-\epsilon) + A(\eta-\epsilon)) \odot f(s, u(s))\, d\delta ds \\ &= E_{\alpha,1}(A(\eta-\epsilon)) \int_0^{t-\epsilon} \int_{s+\eta}^t \frac{(t-\delta)^{\alpha-2}}{\Gamma(\alpha-1)} \odot E_{\alpha,1}(A(\delta-s)^\alpha - A(\eta-\epsilon)) \odot f(s, u(s))\, d\delta ds, \end{aligned}$$

where $u \in \mathcal{B}_\lambda$. From hypothesis (**Hyp4**), $E_{\alpha,1}(A(\eta-\epsilon))$ is a compact operator, then $V_{\epsilon,\delta}(t) = \{(Q_{\epsilon,\delta} u)(t) : u \in \mathcal{B}_\lambda\}$

is relatively compact. Moreover, $\forall u \in \mathcal{B}_\lambda$, we have

$$\begin{aligned}
d_\infty\left((Q_2u)(t),(Q_{\epsilon,\delta}u)(t)\right) &= d_\infty\Bigg(\int_0^t\int_s^t \frac{(t-\delta)^{\alpha-2}}{\Gamma(\alpha-1)}\odot E_{\alpha,1}\left(A(\delta-s)^\alpha\right)\odot f\left(s,u(s)\right)d\delta ds \\
&- \int_0^{t-\epsilon}\int_{s+\eta}^t \frac{(t-\delta)^{\alpha-2}}{\Gamma(\alpha-1)}\odot E_{\alpha,1}\left(A(\delta-s)^\alpha\right)\odot f\left(s,u(s)\right)d\delta ds\Bigg) \\
&= d_\infty\Bigg(\int_0^t\int_s^{s+\eta} \frac{(t-\delta)^{\alpha-2}}{\Gamma(\alpha-1)}\odot E_{\alpha,1}\left(A(\delta-s)^\alpha\right)\odot f\left(s,u(s)\right)d\delta ds \\
&+ \int_0^t\int_{s+\eta}^t \frac{(t-\delta)^{\alpha-2}}{\Gamma(\alpha-1)}\odot E_{\alpha,1}\left(A(\delta-s)^\alpha\right)\odot f\left(s,u(s)\right)d\delta ds \\
&- \int_0^{t-\epsilon}\int_{s+\eta}^t \frac{(t-\delta)^{\alpha-2}}{\Gamma(\alpha-1)}\odot E_{\alpha,1}\left(A(\delta-s)^\alpha\right)\odot f\left(s,u(s)\right)d\delta ds, \tilde{0}\Bigg) \\
&\le d_\infty\left(\int_0^t\int_s^{s+\eta} \frac{(t-\delta)^{\alpha-2}}{\Gamma(\alpha-1)}\odot E_{\alpha,1}\left(A(\delta-s)^\alpha\right)\odot f\left(s,u(s)\right)d\delta ds, \tilde{0}\right) \\
&+ d_\infty\left(\int_{t-\epsilon}^t\int_{s+\eta}^t \frac{(t-\delta)^{\alpha-2}}{\Gamma(\alpha-1)}\odot E_{\alpha,1}\left(A(\delta-s)^\alpha\right)\odot f\left(s,u(s)\right)d\delta ds, \tilde{0}\right) \\
&\le \frac{M_1M}{(\sigma+1)\Gamma(\alpha)}\left[\left((-\eta)^{\sigma+1}-(a-\eta)^{\sigma+1}+b^{\sigma+1}\right)^{1-\alpha_1}\left(-(-\eta)^{\sigma+1}+(-\epsilon-\eta)^{\sigma+1}\right)^{1-\alpha_1}\right]\to 0,
\end{aligned}$$

when $\eta,\epsilon\to 0$.

Then we have a relatively compact set arbitrarily close to $V(t), t>0$, which means that $V(t), t>0$ is also relatively compact.

Applying the Ascoli-Arzela theorem 5 shows that $Q_2(\mathcal{B}_\lambda)$ is relatively compact. Since Q_2 is continuous and $Q_2(\mathcal{B}_\lambda)$ is relatively compact, Q_2 is a fully continuous operator.

According to Schauder's fixed point theorem 4, Q_1+Q_2 has a fixed point at $\mathcal{B}_\lambda$. So the Eq. (35) has a expansive mild solution. □

Set

$$\hat{Q}[u](t) = E_{\alpha,1}\left(At^\alpha\right)\odot u_0 -_F (-1)\int_0^t\int_s^t \frac{(t-\delta)^{\alpha-2}}{\Gamma(\alpha-1)}\odot E_{\alpha,1}\left(A(\delta-s)^\alpha\right)\odot f\left(s,u(s)\right)d\delta ds.$$

$$\hat{C}(J,\mathcal{T}) = \left\{u\in C(J,\mathcal{T}) : \hat{Q}[u](t) \texttt{ exists for all } t\in J\right\}. \tag{51}$$

The following results show that there's a contractive mild solution for the Eq. (35) in the space $C(J,\mathcal{T})$.

Theorem 14. *The hypothesis (**Hyp1**) – (**Hyp4**) is true and*

Hyp5 $\hat{C}(J,\mathcal{T})\neq\emptyset$.

Hyp6 *if $u\in\hat{C}(J,\mathcal{T})$, hence $\hat{Q}[u]\in\hat{C}(J,\mathcal{T})$.*

In this case the Eq. (35) *has a contractive mild solution $C(J,\mathcal{T})$.*

Proof. For $u\in\hat{C}(J,\mathcal{T})$, $\quad \hat{Q}[u](t) = (Q_1u)(t) -_F (-1)\odot(Q_2u)(t)$.
Set

$$\lambda = M\left(\|x_0\| + \frac{M_1b^{(1+\sigma)(1-\alpha_1)}}{\Gamma(\alpha)(1+\sigma)^{1-\alpha_1}}\right).$$

Using a similar method as before, we get : $Q_1u -_F (-1)\odot Q_2v\in\mathcal{B}_\lambda$ for any pair $u,v\in\mathcal{B}_\lambda\subset\hat{C}(J,\mathcal{T})$, where $(Q_1u)(t)$ and $(Q_2u)(t)$ are continuous in $t\in J$.

Now for any $u, v \in \mathcal{B}_\lambda$ we have,

$$d_\infty\left(\left(Q_1 u\right)(t) -_F (-1) \odot \left(Q_2 u\right)(t), \tilde{0}\right) \leq d_\infty\left(\left(Q_1 u\right)(t), \tilde{0}\right) + d_\infty\left(\left(Q_2 u\right)(t), \tilde{0}\right)$$
$$\leq M\left(\|u_0\| + \frac{M_1 b^{(1+\sigma)(1-\alpha_1)}}{\Gamma(\alpha)(1+\sigma)^{1-\alpha_1}}\right) = \lambda,$$

which means that $Q_1 -_F (-1) \odot Q_2$ is an operator from $\mathcal{B}_\lambda$ into $\mathcal{B}_\lambda$.

Since Q_2 is a fully continuous operator, according to the Schauder fixed point theorem 4 $Q_1 -_F (-1) \odot Q_2$ has a fixed point in $\mathcal{B}_\lambda$, This indicate that the Eq. (35) has a contractive mild solution. □

5. An example

Consider the following equations.

$$\begin{cases} {}^{C_F}D_{0^+}^{\frac{3}{2}} u(t,x) = \frac{\partial}{\partial t} u(t,x) + \frac{e^{-t}}{9+e^t}\left(\frac{|u(t,\,x)|}{1+|u(t,\,x)|}\right), \ (t,x) \in]0,1[\times]0,1[, \\ u(t,0) = u(t,1) = 0, \quad t \in]0,1[, \\ u(0,x) = \psi(x), \quad x \in]0,1[. \end{cases} \tag{52}$$

We choose $\mathbb{E} = C([0,1] \times \mathcal{T}, \mathcal{T})$ And the operator $A : D(A) \subset \mathbb{E} \to \mathbb{E}$ descripted by

$$D(A) = \left\{u \in \mathbb{E} : \tfrac{\partial}{\partial t} u \in \mathbb{E} \quad \text{and} u(0,0) = u(0,1) = 0\right\},$$

$$Au = \frac{\partial}{\partial t} u.$$

Then, we get

$$\overline{D(A)} = \left\{u \in \mathbb{E} : u(t,0) = u(t,1) = 0\right\}. \tag{53}$$

This implies that A satisfies (**Hyp4**).

Let's pose

$$V(t) = u(t,\cdot), \ \texttt{that is} \ V(t)(x) = u(t,x), \quad \forall (t,x) \in]0,1[\times]0,1[.$$

In this example, we have $f :]0,1[\times\mathcal{T} \to \mathcal{T}$ provided by

$$f(t, V(t)) = \frac{e^{-t}}{9+e^t}\left(\frac{|V(t)|}{1+|V(t)|}\right).$$

It is obvious that for each $V, W \in C([0,1], B_\lambda)$ we obtain

$$d_\infty\left(f(t,V(t)), f(t,W(t))\right) \leq \rho(t) d_\infty(V(t), W(t)), \ \texttt{with} \ \rho(t) = \frac{e^{-t}}{9+e^t} \in L^1, \forall t \in]0,1[,$$

And that

$$d_\infty\left(f(t,V(t)), \tilde{0}\right) \leq \mu(t), \ \texttt{with} \ \mu(t) = \frac{1}{9+e^t} \in L^1, \forall t \in]0,1[.$$

Moreover f is continuos, therefore it is strongly measurable.

Hence, according to the theorem 13 and the lemma 2, the problem (52) admits two types of solutions expressed as follow

- if $V(t)$ is expansive,

$$V(t) = E_{\alpha,1}\left(\frac{\partial}{\partial t}t^{\alpha}\right) \odot u_0 +_F \int_0^t \int_s^t \frac{(t-\delta)^{\alpha-2}}{\Gamma(\alpha-1)} \odot E_{\alpha,1}\left(\frac{\partial}{\partial s}(\delta-s)^{\alpha}\right) \odot \frac{e^{-s}}{9+e^{s}}\left(\frac{|V(s)|}{1+|V(s)|}\right) d\delta ds. \tag{54}$$

- if $V(t)$ is contractive,

$$V(t) = E_{\alpha,1}\left(\frac{\partial}{\partial t}t^{\alpha}\right) \odot u_0 -_F (-1) \int_0^t \int_s^t \frac{(t-\delta)^{\alpha-2}}{\Gamma(\alpha-1)} \odot E_{\alpha,1}\left(\frac{\partial}{\partial s}(\delta-s)^{\alpha}\right) \odot \frac{e^{-s}}{9+e^{s}}\left(\frac{|V(s)|}{1+|V(s)|}\right) d\delta ds. \tag{55}$$

6. Conclusion

An interactive fuzzy derivative is applied to a fuzzy fractional starting value problem in this study. The equation's starting point is represented by a fuzzy subset, and the differentiation is offered by the F-derivative. Initially we present fresh fundamental theorems about the F-correlated fractional derivative of degree $\alpha \in (0, 1]$. In addition, we employ those theorems to extract the mild solution of the basic problem. After that, we use the Schauder fixed point theorem to guarantee the existence and uniqueness of solutions. Finally, an example is provided to corroborate and confirm the viability of the acquired findings. We anticipate that the presented results will inspire scholars to pursue more study on the issue.

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