Filomat 39:18 (2025), 6413–6421
https://doi.org/10.2298/FIL2518413T

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On extinction and non-extinction of solutions for a p-Kirchhoff problem with logarithmic nonlinearity

Ihya Talibi[a], Brahim El Boukari[a], Jalila El Ghordaf[a], Abdellah Taqbibt[a,*]

[a] *Laboratory LMACS, Sultan Moulay Slimane University, Beni Mellal, BP 523, 23000, Morocco*

Abstract. In this work, we study a class of fast diffusion Kirchhoff-type p-Laplace equation with logarithmic nonlinearity. Under appropriate conditions, by applying energy estimates in combination with the Galerkin method and Sobolev inequality, we establish the global existence of solutions. Moreover, we analyze the criteria for the extinction and non-extinction of these solutions.

1. Introduction

In this paper, we consider the following initial boundary value problem:

$$\begin{cases} \frac{\partial u}{\partial t} - M(\|\nabla u\|_p^p)\Delta_p u = |u|^{q-2}u\log(|u|) & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0 & \text{in } \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded open domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $\Delta_p = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and $1 < p, q < 2$. $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function defined by:

$$(M) \qquad \text{There exist constant } m_1 > 0 \text{ such that } m_1 = \inf_{s\in\mathbb{R}^+} M(s).$$

Problem (1) belongs to the class of quasilinear diffusion problems which have garnered increasing interest in recent years due to their applications in various scientific fields. More precisely, when $M = 1$ in (1), it describes the motion of a compressible fluid moving through a porous medium, this movement is governed by the conservation of mass, expressed by the equation:

$$\theta(x)\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\vec{V}) - f(\rho) = 0 \tag{2}$$

2020 *Mathematics Subject Classification*. Primary 35A01; Secondary 35A02.
Keywords. extinction, non-extinction, weak solution, global existence, Kirchhoff equation, logarithmic nonlinearity, energy functional.
Received: 09 November 2024; Revised: 08 July 2025; Accepted: 12 July 2025
Communicated by Maria Alessandra Ragusa
* Corresponding author: Abdellah Taqbibt
Email addresses: talibiihya@gmail.com (Ihya Talibi), elboukaribrahim@yahoo.fr (Brahim El Boukari), elg_jalila@yahoo.fr (Jalila El Ghordaf), abdellah.taqbibt@usms.ma (Abdellah Taqbibt)

where $\theta(x)$ represents the volumetric moisture content, ρ is the fluid density, $\vec{V}$ is the fluid velocity and $f(u)$ is a source term (see [28, 29]). The velocity is influenced by Darcy's law for nonlinear diffusion is given by:

$$\rho\vec{V} = -\lambda|\nabla\rho|^{\alpha-2}\nabla\rho, \tag{3}$$

where α and λ are characteristics of the medium. By substituting this velocity relation into the conservation equation, and with specific values like $\theta(x) = |x|^{-r}$, $r = 0$, $\lambda = 1$ and $f(\rho) = |\rho|^{p-2}\rho\log(|\rho|)$, we arrive at the formulation of the problem (1). Moreover, as already mentioned in [9], the function $u(x,t)$ represents the population density at time t and spatial position x, $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ accounts for the diffusion of the population density and $|u|^{q-2}u\log(|u|)$ is the source.

There are also several studies concerning global solutions for problems similar to (1). Based on the potential well theory and variational methods, the authors in [11] obtained the global existence and finite time blow-up of solutions to the following initial boundary value problem of Kirchhoff type

$$\begin{cases} u_t - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(u), & (x,t) \in \Omega \times (0,T), \\ u = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{4}$$

where $M(r) = a + br$, a, b are postitive constants and $f(u) = |u|^{q-1}u$. Equation (4) was investigated in [6, 12, 17, 24, 25, 33], where authors have proved global existence, uniqueness and asymptotic behavior of a weak or strong solution.

The research with logarithmic nonlinearity can be found in many physical applications, including the theory of superfluidity, diffusion and transport phenomena, and nuclear physics. We refer the readers to [34] for more information. Within the framework of partial differential equations (see [15, 19, 27]), the authors [31] studied the existence of global solution to the following semilinear pseudo-parabolic problem

$$u_t - \Delta u_t - M(\|\nabla u\|_p^p)\Delta u = |u|^{q-2}\ln|u|, \tag{5}$$

with Dirichlet boundary condition by applying the logarithmic Sobolev inequality. By using the potential well method, the authors in [32] studied the global existence and finite-time blow-up for the weak solutions. See [20, 21] for Kirchhoff type problems with logarithmic nonlinearity. We also refer the reader to [1, 4, 5, 7, 8, 13, 14, 16, 23, 26, 30] where the theory of logarithmic nonlinearity find its applications for the same evolution equations.

In the present work, we aim to study the combined effects of the p-Laplacian and logarithmic nonlinearity to discuss the global existence and extinction properties of solutions. Our approach relies on energy estimates and Sobolev inequalities to establish the global existence of weak solutions for the problem (1). To the best of our knowledge, this is the first study to explore both the global existence and extinction properties of solutions for evolution equations involving the p-Laplacian and logarithmic nonlinearity.

Next, we will introduce the energy functional $E : W_0^{1,p}(\Omega) \to \mathbb{R}$ associated with problem (1) defined by

$$E(u) = \frac{1}{p}\int_0^{\|u(t)\|^p_{W_0^{1,p}(\Omega)}} M(s)ds - \frac{1}{q}\int_\Omega |u|^q \log(|u|)dx + \frac{1}{q^2}\int_\Omega |u|^q dx \tag{6}$$

For simplicity reasons, throughout this paper, we adopt the following abbreviations:

$$\|u\|_p = \|u\|_{L^p(\Omega)}, \quad \|u\|_2 = \|u\|_{L^2(\Omega)}.$$

2. Preliminaries

In this section, we give some lemmas and definitions which will be needed in our proofs of the main results.

In the first instance, for $1 \le p < \infty$, we introduce the Hilbert space

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{\partial u}{\partial x_i} \in L^p(\Omega), \quad i = 1, 2, \dots, n \right\}$$

endowed with the norm

$$\|u\|^p_{W^{1,p}(\Omega)} = \|u\|^p_p + \|\nabla u\|^p_p.$$

Denote

$$W_0 := W^{1,p}_0(\Omega) = \left\{ u \in W^{1,p}(\Omega) : \, u|_{\partial\Omega} = 0 \right\}.$$

Due to Poincaré's inequality, one can know that $\|\nabla u\|_p$ is an equivalent norm to $\|u\|_{W^{1,p}(\Omega)}$ in W_0.
Secondly, we denote the maximal existence time of a solution $u = u(t)$ to problem (1) by T_m, which is defined as follows:

Definition 2.1. ***(1)*** *If there exists a $\hat{t} \in (0, +\infty)$ such that u exists for $0 \le t < \hat{t}$, but u blows up in W_0 as $t \to \hat{t}^-$ i.e*

$$\lim_{t \to \hat{t}^-} \|u(t)\|_{W_0} = +\infty \tag{7}$$

then $T_m = \hat{t}$.

(2) *If* (7) *does not happen at any finite time, then $T_m = +\infty$ and we say u exists globally.*

Definition 2.2. *Let u be a solution to the problem* (1). *We say $u = u(t)$ vanishes in finite time if there exists a $T^* > 0$ such that*

$$u(x,t) \equiv 0 \quad on \ \Omega, \quad t \ge T^* \tag{8}$$

Lemma 2.3. *[3] Let $1 \le p \le \infty$, we have*

$$\|u\|_{L^{\frac{np}{n-p}}} \le C_p \|u\|_{W_0} \quad \forall u \in W_0. \tag{9}$$

Lemma 2.4. *[10] Suppose that l, m and s are the positive constants, and $\varphi(t)$ is absolutely continuous and nonnegative function such that $\varphi'(t) + m\varphi^l(t) \ge s$, $t > 0$. Then*

$$\varphi(t) \ge \min\left\{ \varphi(0), \left(\frac{s}{m}\right)^{\frac{1}{l}} \right\}.$$

Lemma 2.5. *[18] If $0 < r < s \le 1$ and $h(t)$ solve*

$$\begin{cases} \frac{dh}{dt} + \gamma_1 h^r \le \gamma_2 h^s, \quad t > 0 \\ h(0) = h_0 > 0 \end{cases} \tag{10}$$

with $\gamma_1 > 0, 0 < \gamma_2 < \frac{1}{2}\gamma_1 h_0^{r-s}$. Then, there exists $q_1, q_2 > 0$ such that

$$0 \le h(t) \le q_2 e^{-q_1 t} \text{ for all } t \ge 0.$$

Lemma 2.6. *[2] Let σ be a positive constant. Then for all $r \geqslant 1$, we have:*

$$|\log(r)| \le \frac{1}{\sigma} r^\sigma,$$

for all $r \in [1, +\infty)$.

3. Main results

In this section, we will define a weak solution to the problem (1) and prove the main results. We start with the following definition:

Definition 3.1. *Fix $T > 0$ and assume that $u_0 \in W_0$. A function $u \in L^\infty(0, T; W_0)$ is called a weak solution of* (1)*, if $\frac{\partial u}{\partial t} \in L^2(0, T, L^2(\Omega))$ and*

$$\int_\Omega \frac{\partial u}{\partial t}\phi dxdt + \int_\Omega M(\|u(t)\|_{W_0})|\nabla u|^{p-2}\nabla u \nabla \phi dx = \int_\Omega |u|^{q-2}u\log(|u|)\phi dx, \tag{11}$$

holds fo a.e $t \in (0, T)$ and for all $\phi \in W_0$.

The main results of our paper is the following:

Theorem 3.2. *Let $u_0 \in W_0$ and suppose that (M) hold. The problem* (1) *admits a global weak solution $u = u(t)$ if there exists a constant $\sigma > 0$ such that $1 < q + \sigma \leq 2$.*

Proof. Assume that u is a weak solution of (1). As in [14, Theorem 3.2] and [22, Theorem 1], the local existence of weak solutions to problem (1) is established using the Galerkin method. Moreover, we establish the global existence of the weak solution. For this reason, let assume that the weak solution of (1) blows up in finite time as in definition 2.1.
We let $\phi = u$ in (11), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 + M(\|u(t)\|_{W_0}^p)\|u(t)\|_{W_0}^p = \int_\Omega |u(t)|^q \log|u(t)|dx. \tag{12}$$

According to Lemma 2.6, we have

$$\int_\Omega |u(t)|^q \log|u(t)|dx \leq \frac{1}{\sigma}\int_\Omega |u(t)|^{q+\sigma}dx = \frac{1}{\sigma}\|u(t)\|_{q+\sigma}^{q+\sigma}. \tag{13}$$

Combining (12), (13), (M) and the Hölder inequality, we obtain

$$\|u(t)\|_2 \frac{d}{dt}\|u(t)\|_2 + m_1\|u(t)\|_{W_0}^p \leq \frac{1}{\sigma}\|u(t)\|_{L^{q+\sigma}(\Omega)}^{q+\sigma} \leq \frac{|\Omega|^{\frac{2-(q+\sigma)}{2}}}{\sigma}\|u(t)\|_2^{q+\sigma}.$$

Therefore

$$\frac{d}{dt}\|u(t)\|_2 \leq \frac{|\Omega|^{\frac{2-(q+\sigma)}{2}}}{\sigma}\|u(t)\|_2^{q+\sigma-1}. \tag{14}$$

Next, by choosing $\phi = \frac{du}{dt}$ in (11) and by using (M), the Hölder inequality and the Young inequality, we get

$$\begin{aligned}
\left\|\frac{du(t)}{dt}\right\|_2^2 + \frac{m_1}{p}\frac{d}{dt}\|u(t)\|_{W_0}^p &\leq \int_\Omega |u(t)|^{q-2}u(t)\log|u(t)|\frac{du(t)}{dt}dx \\
&\leq \int_\Omega \frac{1}{\sigma}|u(t)|^{q+\sigma-1}\left|\frac{du(t)}{dt}\right|dx \\
&\leq \frac{1}{2\sigma}\int_\Omega |u(t)|^{2(q+\sigma-1)}dx + \frac{1}{2}\int_\Omega \left|\frac{du(t)}{dt}\right|^2 dx \\
&\leq \frac{|\Omega|^{1-\frac{2(q+\sigma-1)}{p}}}{2\sigma}\|u(t)\|_p^{2(q+\sigma-1)} + \frac{1}{2}\left\|\frac{du(t)}{dt}\right\|_2^2.
\end{aligned} \tag{15}$$

Case 1: If $1 < q + \sigma < 2$, after a simple calculation on (14), we obtain

$$\|u(t)\|_2 \leq \left(\frac{(2-q-\sigma)\,|\Omega|^{1-\frac{q+\sigma}{2}}}{\sigma} t + \|u_0\|_2^{2-q-\sigma} \right)^{\frac{1}{2-q-\sigma}} \quad \forall t \in (0,T) \tag{16}$$

for sufficiently large T.
Since $p < 2$, combining (16) with Hölder inequality, we conclude that

$$\|u(t)\|_p \leq |\Omega|^{\frac{2-p}{2p}} \|u(t)\|_2 \leq \left(\frac{(2-q-\sigma)\,|\Omega|^{\frac{2-(q+\sigma)}{2}}}{\sigma} t + \|u_0\|_2^{2-q-\sigma} \right)^{\frac{1}{2-q-\sigma}} |\Omega|^{\frac{2-p}{2p}}. \tag{17}$$

Substituting the inequality (17) into (15), we get

$$\frac{d}{dt}\|u\|_{W_0}^p \leq \left(\frac{(2-q-\sigma)\,|\Omega|^{\frac{2-(q+\sigma)}{2}}}{\sigma} t + \|u_0\|_2^{2-q-\sigma} \right)^{\frac{2(q+\sigma-1)}{2-q-\sigma}} \frac{p|\Omega|^{2-q-\sigma}}{2m_1\sigma}. \tag{18}$$

Since the right-hand sides of (17) and (18) is defined on $t \in [0,+\infty)$ and

$$\left(\frac{(2-q-\sigma)\,|\Omega|^{\frac{2-(q+\sigma)}{2}}}{\sigma} t + \|u_0\|_2^{2-q-\sigma} \right)^{\frac{1}{2-q-\sigma}} |\Omega|^{\frac{2-p}{2p}} \in [0,+\infty) \quad \forall t \geq 0,$$

$$\left(\frac{(2-q-\sigma)\,|\Omega|^{\frac{2-(q+\sigma)}{2}}}{\sigma} t + \|u_0\|_2^{2-q-\sigma} \right)^{\frac{2(q+\sigma-1)}{2-q-\sigma}} \frac{p|\Omega|^{2-q-\sigma}}{2m_1\sigma} \in [0,+\infty) \quad \forall t \geq 0.$$

Moreover, for $\hat{t} \in (0,+\infty)$,

$$\lim_{t\to\hat{t}} \left(\frac{(2-q-\sigma)\,|\Omega|^{\frac{2-(q+\sigma)}{2}}}{\sigma} t + \|u_0\|_2^{2-q-\sigma} \right)^{\frac{1}{2-q-\sigma}} |\Omega|^{\frac{2-p}{2p}} < +\infty,$$

$$\lim_{t\to\hat{t}} \left(\frac{(2-q-\sigma)\,|\Omega|^{\frac{2-(q+\sigma)}{2}}}{\sigma} t + \|u_0\|_2^{2-q-\sigma} \right)^{\frac{2(q+\sigma-1)}{2-q-\sigma}} \frac{p|\Omega|^{2-q-\sigma}}{2m_1\sigma} < +\infty,$$

this contradicts (7). As a result, the problem (1) has a weak solution u that exists globally.
Case 2: If $q + \sigma = 2$, according to inequality (14) that

$$\frac{d}{dt}\|u(t)\|_2 \leq \frac{1}{\sigma}\|u(t)\|_2,$$

we obtain

$$\|u(t)\|_2 \leq \|u_0\|_2\, e^{\frac{t}{\sigma}}. \tag{19}$$

According to the Hölder inequality combined with (19) with $p < 2$, we have

$$\|u(t)\|_p \leq |\Omega|^{\frac{2-p}{2p}} \|u(t)\|_2 \leq |\Omega|^{\frac{2-p}{2p}} \|u_0\|_2\, e^{\frac{t}{\sigma}}. \tag{20}$$

Substituting (20) into (15) with $q + \sigma = 2$, we obtain that

$$\frac{d}{dt}\|u\|_{W_0}^p \leq \frac{p|\Omega|^{1-\frac{2}{p}}}{2m_1\sigma}\|u(t)\|_p^2 \leq \frac{p\,\|u_0\|_2^2}{2m_1\sigma^2} e^{\frac{2t}{\sigma}}. \tag{21}$$

Since the right hand sides in (20) and (21) is defined on $t \in [0, +\infty)$ and

$$|\Omega|^{\frac{2-p}{2p}} \|u_0\|_2 e^{\frac{t}{\sigma}} \in [0, +\infty) \quad t \geq 0,$$

$$\frac{p\|u_0\|_2^2}{2m_1\sigma^2} e^{\frac{2t}{\sigma}} \in [0, +\infty) \quad t \geq 0.$$

Moreover, for $\hat{t} \in (0, +\infty)$,

$$\lim_{t \to \hat{t}} |\Omega|^{\frac{2-p}{2p}} \|u_0\|_2 e^{\frac{t}{\sigma}} < +\infty,$$

$$\lim_{t \to \hat{t}} \frac{p\|u_0\|_2^2}{2m_1\sigma^2} e^{\frac{2t}{\sigma}} < +\infty$$

this contradicts (7). Consequently, the global solution is obtained. □

Theorem 3.3. *Let $u_0 \in W_0 \cap L^2(\Omega)$ and (M) hold. Assume that problem* (1) *admits a global weak solution. Then*

i) *if $\frac{2n}{n+2} < p < \min\{n, q+\sigma\}$ and*

$$\|u_0\|_2^{q+\sigma-p} < \frac{m_1\sigma}{2C_p} |\Omega|^{\frac{n(q+\sigma-p)-2p}{2n}} \tag{22}$$

with some $0 < \sigma \leq 2 - q$, then u vanishes in finite time.

ii) *Suppose that $\lambda \geq 1$ such that $\gamma(t)M(\gamma(t)) \leq \lambda \int_0^{\gamma(t)} M(s)ds$ for any $t \geq 0$. If*

$$\|u_0\|_2 \neq 0 \text{ and } E(u_0) \leq 0, \text{ when } q = p\lambda$$

or

$$\|u_0\|_2 \neq 0 \text{ and } E(u_0) < 0, \text{ when } q < p\lambda.$$

Then u cannot vanish in finite time.

Proof. Theorem 3.2 leads to the conclusion that the problem (1) has a global weak solution u. First, we prove conclusion i). From (11) and (M), we deduced that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 + m_1\|u(t)\|_{W_0}^p \leq \int_\Omega |u(t)|^q \log|u(t)| \mathrm{d}x \tag{23}$$

Plugging (13) into (23) and using Lemma 2.3, we get

$$\|u(t)\|_2 \frac{d}{dt}\|u(t)\|_2 + \frac{m_1}{C_p}\|u(t)\|^p_{L^{\frac{np}{n-p}}(\Omega)} \leq \frac{1}{\sigma}\|u(t)\|_{q+\sigma}^{q+\sigma}. \tag{24}$$

Using again the Hölder inequality in (24), it follows that

$$\frac{d}{dt}\|u(t)\|_2 + \frac{m_1|\Omega|^{1-\frac{p(n+2)}{2n}}}{C_p}\|u(t)\|_2^{p-1} \leq \frac{|\Omega|^{\frac{2-(q+\sigma)}{2}}}{\sigma}\|u(t)\|_2^{q+\sigma-1}$$

Let $z(t) = \|u(t)\|_2$, the inequality mentioned above can be reduced to

$$\frac{dz}{dt} + \frac{m_1|\Omega|^{1-\frac{p(n+2)}{2n}}}{C_p} z^{p-1}(t) \leq \frac{|\Omega|^{\frac{2-(q+\sigma)}{2}}}{\sigma} z^{q+\sigma-1}(t) \tag{25}$$

Furthermore, from inequality (22) and $1 \leq \frac{2n}{n+2} < p < \min\{n, q+\sigma\}$, we ensures that $p - 1 < q + \sigma - 1$ and

$$0 < \frac{|\Omega|^{\frac{2-(q+\sigma)}{2}}}{\sigma} < \frac{m_1|\Omega|^{\frac{2n-p(n+2)}{2n}}}{2C_p}\|u_0\|_2^{p-q-\sigma}$$

Thus, it follows from Lemma 2.5 that constants $q_1 > 0$ and $q_2 > 0$ exist such that

$$0 \le z(t) \le q_2 e^{-q_1 t}, \quad t \ge 0. \tag{26}$$

We take the constant $T^* > 0$ such that

$$\left(q_2 e^{-q_1 t}\right)^{q+\sigma-p} \le \frac{m_1 \sigma}{2C_p} |\Omega|^{\frac{n(q+\sigma-p)-2p}{2n}}, \quad t \ge T^* \tag{27}$$

Combining inequalities (26) and (27), it is concluded that

$$z^{q+\sigma-1}(t) = \left(z^{q+\sigma-p} z^{p-1}\right)(t) \le \frac{\sigma m_1}{2C_p} |\Omega|^{\frac{n(q+\sigma-p)-2p}{2n}} z^{p-1}(t), \quad t \ge T^* \tag{28}$$

Plugging (28) into (25), we get

$$\frac{dz}{dt} + \frac{m_1 |\Omega|^{1-\frac{p(n+2)}{2n}}}{2C_p} z^{p-1}(t) \le 0, \quad t \ge T^*$$

After a simple calculation, we obtain

$$\begin{cases} \|u(t)\|_2 \le \left(z^{2-p}\left(T^*\right) - \frac{m_1(2-p)|\Omega|^{1-\frac{p(n+2)}{2n}}}{2C_p}\left(t - T^*\right)\right)^{\frac{1}{2-p}}, & T^* \le t < T_v^* \\ \|u(t)\|_2 \equiv 0 & t \ge T_v^* \end{cases}$$

where $T_v^* = \frac{2C_p}{m_1(2-p)|\Omega|^{1-\frac{p(n+2)}{2n}}} z^{2-p}\left(T^*\right) + T^*$. Then, u vanishes in finite time.

Secondly, we prove conclusion ii). Choosing $\phi = \frac{du}{dt}$ in (11), we have

$$\int_\Omega \left(\frac{du(t)}{dt}\right)^2 dx + \frac{1}{p} M(\|u(t)\|_{W_0}^p) \frac{d}{dt} \|u(t)\|_{W_0}^p = \int_\Omega |u(t)|^{q-2} u(t) \log|u(t)| \frac{du(t)}{dt} dx. \tag{29}$$

We can see that (29) simplifies as follows:

$$\begin{aligned} \int_\Omega \left(\frac{du(t)}{dt}\right)^2 dx + \frac{1}{p} M(\|u(t)\|_{W_0}^p) \frac{d}{dt} \|u(t)\|_{W_0}^p &= \frac{1}{q} \frac{d}{dt} \int_\Omega |u(t)|^q \log|u(t)| dx \\ &\quad - \frac{1}{q^2} \frac{d}{dt} \int_\Omega |u(t)|^q dx. \end{aligned} \tag{30}$$

Differentiating (6) with respect to t, we get

$$\frac{d}{dt} E(u(t)) = \frac{1}{p} M(\|u(t)\|_{W_0}^p) \frac{d}{dt} \|u(t)\|_{W_0}^p - \frac{1}{q} \frac{d}{dt} \int_\Omega |u|^q \log|u| \mathrm{d}x + \frac{1}{q^2} \frac{d}{dt} \int_\Omega |u|^q dx. \tag{31}$$

According to (30) and (31), we can conclude that

$$\frac{d}{dt} E(u(t)) = -\int_\Omega \left(\frac{du(t)}{dt}\right)^2 dx \le 0. \tag{32}$$

Thus, $E(u)$ is non-increasing with respect to t. Consequently, by integrating (32) from 0 to t, we have

$$E(u(t)) = E\left(u_0\right) - \int_0^t \int_\Omega \left(\frac{du(s)}{ds}\right)^2 dx ds \tag{33}$$

Let $g(t) = \frac{1}{2}\|u(t)\|_2^2$. Then, by (6), (12) and (33), we can get

$$\begin{aligned} g'(t) &= -M(\|u(t)\|_{W_0}^p)\|u(t)\|_{W_0}^p + \int_\Omega |u(t)|^q \log|u(t)|\mathrm{d}x \\ &\geq -\int_0^{\|u(t)\|_{W_0}^p} M(s)ds + \frac{q-p\lambda}{q}\int_\Omega |u(x,t)|^q \log|u(t)|dx + \frac{p\lambda}{q^2}\int_\Omega |u(t)|^q dx \\ &= -p\lambda E(u_0) + p\lambda\int_0^t\int_\Omega\left(\frac{d}{ds}u(t)\right)^2 dxds + \frac{q-p\lambda}{q}\int_\Omega |u(t)|^q\log|u(t)|dx \\ &+ \frac{p\lambda}{q^2}\int_\Omega |u(t)|^q dx \\ &\geq -p\lambda E(u_0) + \frac{q-p\lambda}{q}\int_\Omega |u(t)|^q \log|u(t)|dx. \end{aligned} \tag{34}$$

Case 1: If $q = p\lambda$, hence, (34) can be reduced to $g'(t) \geq -p\lambda E(u_0)$, which gives

$$g(t) \geq g(0) - p\lambda E(u_0) > 0, \quad \forall t > 0$$

therefore, u cannot vanish in finite time.

Case 2: If $q < p\lambda$. Thanks to $1 < q < 2$, there exists a constant $\sigma > 0$ such that $q + \sigma < 2$. Consequently, inequality (34) can be simplified as follows

$$\begin{aligned} g'(t) &\geq -p\lambda E(u_0) + \frac{q-p\lambda}{q\sigma}\int_\Omega |u(t)|^{q+\sigma}dx \\ &\geq -p\lambda E(u_0) + \frac{|\Omega|^{\frac{2-q-\sigma}{2}}(q-p\lambda)}{q\sigma}\left(\int_\Omega |u(t)|^2\,\mathrm{d}x\right)^{\frac{q+\sigma}{2}} \\ &= -p\lambda E(u_0) - \frac{(p\lambda-q)(\sqrt{2})^{q+\sigma}|\Omega|^{\frac{2-q-\sigma}{2}}}{q\sigma} g^{\frac{q+\sigma}{2}}(t). \end{aligned}$$

Consequently, using Lemma 2.4 allow us to get

$$g(t) \geq \min\left\{g(0), \frac{-p\lambda q\sigma E(u_0)}{(p\lambda-q)(\sqrt{2})^{q+\sigma}|\Omega|^{\frac{2-q-\sigma}{2}}}\right\} > 0, \quad \forall t > 0$$

As a results, u cannot vanish in finite time. This completes the proof of Theorem 3.3. □

References

[1] A. Alharbi, A. Choucha, S. Boulaaras, *Blow-up of solutions for a viscoelastic Kirchhoff equation with a logarithmic nonlinearity, delay and Balakrishnan-Taylor damp- ing terms*, Filomat, **38**(26), 9237–9247, (2024)

[2] T. Boudjeriou, *Global Existence and Blow-Up for the Fractional p-Laplacian with Logarithmic Nonlinearity*, Mediterr. J. Math. **17**(2020), no. 5.

[3] H. Brézis, *Functional analysis. Theory and applications. (Analyse fonctionnelle. Théorie et applications.)*, Collection Mathématiques Appliquées pour la Maîtrise, 1994.

[4] N. Chems Eddine, P.D. Nguyen, M.A. Ragusa, *Existence and multiplicity of so- lutions for a class of critical anisotropic elliptic equations of Schrodinger-Kirchhoff-type*, Mathematical Methods in the Applied Sciences, (2023)

[5] H. Chen, P. Luo, G. Liu, *Global solution and blow-up of a semilinear heat equation with logarithmic nonlinearity*, J. Math. Anal. Appl. **422**(2015), no. 1, 84–98 .

[6] M. Chipot, V. Valente, G.V. Cafarelli, *Remarks on a nonlocal problems involving the Dirichlet energy*, Rend. Sem. Math. Univ. Padova **110**(2003), 199–220.

[7] P. Dai, C. Mu, G. Xu, *Blow-up phenomena for a pseudo-parabolic equation with p-Laplacian and logarithmic nonlinearity terms*, J. Math. Anal. Appl. **481**(2020), no. 1, 123439.

[8] H. Ding, J. Zhou, *Global existence and blow-up for a mixed pseudo-parabolic p-Laplacian type equation with logarithmic nonlinearity.*, J. Math. Anal. Appl. **478**(2019), no. 2, 393-420.

[9] P. Fife, *Some nonclassical trends in parabolic and parabolic-like evolutions*, Trends in nonlinear analysis (2003), 153-191.
[10] B. Guo, W. Gao, *Non-extinction of solutions to a fast diffusive p-Laplace equation with Neumann boundary conditions*, J. Math. Anal. Appl. **422**(2015) no. 2, 1527-1531.
[11] Y. Han, Q. Li, *Threshold results for the existence of global and blow-up solutions to Kirchhoff equations with arbitrary initial energy*, Comput. Math. Appl. **75**(2018), no. 9, 3283–3297.
[12] Y. Han, W. Gao, Z. Sun, H. Li, *Upper and lower bounds of blow-up time to a parabolic type Kirchhoff equation with arbitrary initial energy*, Comput. Math. Appl. **76**(2018), no. 10, 2477-2483.
[13] Y. He, H. Gao, H. Wang, *Blow-up and decay for a class of pseudo-parabolic p-Laplacian equation with logarithmic nonlinearity*, Comput. Math. Appl. **75**(2018), no. 2, 459-469.
[14] C.N. Le, X. T. Le, *Global solution and blow-up for a class of p-Laplacian evolution equations with logarithmic nonlinearity*, Acta Appl. Math. **151**(2017), no. 1, 149-169.
[15] J. Li, Y. Han, *Global existence and finite time blow-up of solutions to a nonlocal p-Laplace equation*, Math. Model. Anal. **24**(2019), no. 2, 195-217.
[16] M. Liao, Q. Li, *A class of fourth-order parabolic equations with logarithmic nonlinearity*, Taiwanese J. Math. **24**(2020), no. 4, 975-1003.
[17] Y. Li, D.B. Wang, J. Zhang, *Sign-changing solutions for a class of p-Laplacian Kirchhoff-type problem with logarithmic nonlinearity*, AIMS Math. **5**(2020), no. 3, 2100-2112.
[18] W. Liu, *Extinction properties of solutions for a class of fast diffusive p-Laplacian equations*, Nonlinear Anal. **74**(2011), no. 13, 4520-4532.
[19] N. Mezouar, S.M. Boulaaras, A. Allahem, *Global existence of solutions for the viscoelastic Kirchhoff equation with logarithmic source terms*, Complexity **2020**(2020), no. 1, 7105387.
[20] E. Piskin, N. Irkıl, *Mathematical behavior of solutions of the Kirchhoff type equation with logarithmic nonlinearity*, Bol. Soc. Paran. Mat. **40**(2022), 1–13.
[21] X. Shao, G.J. Tang, *Blow-up phenomena for a Kirchhoff-type parabolic equation with logarithmic nonlinearity*, Appl. Math. Lett. **116**(2021), 106969.
[22] X. Shao, *Global Existence and Blow-Up for a Kirchhoff-Type Hyperbolic Problem with Logarithmic Nonlinearity*, Appl Math Optim **84**(2021), 2061–2098.
[23] D.D. Sun, *Ground state solutions of Schrodinger-Kirchhoff equations with poten-tials vanishing at infinity, Journal of Function Spaces*, **2023**(2023)
[24] I. Talibi, B. El Boukari, J. El Ghordaf, *existence of solutions for the fractional kirchhoff parabolic equation with logarithmic nonlinearity*. Journal of Mathematical Sciences, 1-10. (2025)
[25] I. Talibi, B.E. Boukari, J. El Ghordaf, A. Taqbibt, F. Balaadich, *Kirchhoff-type problem involving the fractional p-Laplacian via Young measures*. Filomat, **38**(23), 8211-8223 (2024)
[26] A. Taqbibt, I. Talibi, B. E. Boukari, J.E. Ghordaf, *Global existence, extinction, and non-extinction of solutions to a fast diffusion p-Laplace evolution equation with logarithmic nonlinearity*. Filomat, **39**(11), 3719-3727 (2025)
[27] S. Toualbia, A. Zaraï, S. Boulaaras, *Decay estimate and non-extinction of solutions of p-Laplacian nonlocal heat equations*, AIMS Math. **5**(2020), no. 3, 1663-1680.
[28] J.L. Vázquez, *The porous medium equation: mathematical theory*, Oxford university press, 2007.
[29] Z. Wu, J. Zhao, J. Yin, H. Li, *Nonlinear diffusion equations*, 2001.
[30] H. Yang, , Y. Han, *Blow-up for a damped p-Laplacian type wave equation with logarithmic nonlinearity*, J. Differential Equations **306** (2022), 569-589.
[31] H. Yang, *Blow-up for a pseudo-parabolic p -Kirchhoff equation with logarithmic nonlinearity*, Evol. Equ. Control Theory **13**(2024), no. 2, 445-457.
[32] Q. Zhao, Y. Cao, *Initial boundary value problem of pseudo-parabolic Kirchhoff equations with logarithmic nonlinearity*, Math. Methods Appl. Sci. **47**(2024) no. 2, 799-816.
[33] S. Zheng, M. Chipot, *Asymptotic behavior of solutions to nonlinear parabolic equations with nonlocal terms*, Asymptot. Anal. **45**(2005) no. 3-4, 301–312.
[34] K.G. Zloshchastiev, *Logarithmic nonlinearity in theories of quantum gravity: Origin of time and observational consequences*, Gravit. Cosmol. **16**(2010) no. 4, 288-297.