Filomat 39:18 (2025), 6423–6436
https://doi.org/10.2298/FIL2518423J

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Note on the rigidity of graphs

Long Jin[a], Jianxi Li[a,*], Peng Huang[b]

[a] *School of Mathematics and Statistics, Minnan Normal University, Zhangzhou, Fujian, P.R. China*
[b] *School of Mathematics and Statistics, Nantong University, Nantong, Jiansu, P.R. China*

Abstract. It is of interest to look for the sufficient conditions for the rigidity of a graph. Fan, Huang and Lin (2023) recently studied the rigidity of a graph from the perspective of its spectral radius of the adjacency matrix and established a sufficient condition involving the spectral radius to ensure a 2-connected (or a 3-connected) graph G with a fixed minimum degree to be rigid (or globally rigid). In this note, we establish a similar condition which relates $\lambda_1^{\alpha}(G)$, the spectral radius of the matrix $A_\alpha(G) := \alpha D(G) + (1-\alpha)A(G)$, where $\alpha \in (0,1)$, $A(G)$ and $D(G)$ are the adjacency matrix and the diagonal degree matrix of G, respectively.

1. Introduction

For an undirected simple graph $G = (V(G), E(G))$, let $p : V(G) \to \mathbb{R}^d$ be a mapping that assigns a point in $\mathbb{R}^d$ to each vertex of G. The pair (G, p) is referred to as a *d-dimensional bar-and-joint framework*. Two frameworks (G, p) and (G, q) are said to be *equivalent* if $\|p(u) - p(v)\| = \|q(u) - q(v)\|$ for every $uv \in E(G)$ and are said to be *congruent* if $\|p(u) - p(v)\| = \|q(u) - q(v)\|$ for any $u, v \in V(G)$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^d$. A framework (G, p) is said to be *generic* if the coordinates of its points are algebraically independent over $\mathbb{Q}$. A framework (G, p) is *rigid* in $\mathbb{R}^d$ if there exists $\varepsilon > 0$ such that any framework (G, q) that is equivalent to (G, p) and satisfies $\|p(u) - q(u)\| < \varepsilon$ for $u \in V(G)$ must be congruent to (G, p). A generic framework (G, p) is rigid in $\mathbb{R}^d$ if and only if every generic framework of G is rigid in $\mathbb{R}^d$. A graph G is *rigid* in $\mathbb{R}^d$ if every/some generic framework of G is rigid in $\mathbb{R}^d$, and is *redundantly rigid* in $\mathbb{R}^d$ if $G - e$ is rigid in $\mathbb{R}^d$ for every $e \in E(G)$. Moreover, a graph G is *globally rigid* in $\mathbb{R}^d$ if there exists a globally rigid generic framework (G, p) in $\mathbb{R}^d$. For more information on rigid and generic framework can be found in [1]. The problem of determining whether a graph G is rigid (or globally rigid) in $\mathbb{R}^d$ is interesting and received a lot attentions [9–12]. Hendrickson [8] established that any globally rigid graph in $\mathbb{R}^d$ with a minimum of $d + 2$ vertices is $(d + 1)$-connected and redundantly rigid. Consequently, it becomes imperative to impose the condition that G is 3-connected when examining the global rigidity of G in $\mathbb{R}^2$.

Recently, Fan, Huang and Lin [4] studied the the rigidity of a graph in $\mathbb{R}^2$ from its eigenvalues of the adjacency matrix and proposed the following problem:

2020 *Mathematics Subject Classification*. Primary 05C50.
Keywords. Graph, rigidity, A_α-spectral radius.
Received: 09 October 2024; Revised: 19 March 2025; Accepted: 08 April 2025
Communicated by Paola Bonacini
The research is partially supported by NSFC (Nos. 12171089, 12271235), NSF of Fujian (No. 2021J02048).
* Corresponding author: Jianxi Li
Email addresses: jinlongmn@126.com (Long Jin), ptjxli@hotmail.com (Jianxi Li), phuang@ntu.edu.cn (Peng Huang)
ORCID iDs: https://orcid.org/0009-0000-4260-2307 (Long Jin), https://orcid.org/0000-0001-7034-9062 (Jianxi Li), https://orcid.org/0000-0002-4499-3847 (Peng Huang)

Problem 1.1. *Which spectral conditions can guarantee that a graph is rigid or globally rigid in* $\mathbb{R}^2$*?*

For $\alpha \in [0,1]$, the $A_\alpha(G)$-matrix of a graph G was defined in [13] as

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G),$$

where $A(G)$ and $D(G)$ are the adjacency matrix and the diagonal degree matrix of G, respectively. In particular, $A_0(G) = A(G)$, $A_{1/2}(G) = \frac{1}{2}Q(G)$ and $A_1(G) = D(G)$, where $Q(G) = D(G) + A(G)$ is the signless Laplacian matrix of G. Since $A_\alpha(G)$ is a real symmetric matrix, it follows that all of its eigenvalues are real. Moreover, the matrix $A_\alpha(G)$ is irreducible when G is connected. Consequently, the largest eigenvalue of $A_\alpha(G)$ is the spectral radius of $A_\alpha(G)$, also called the *A_α-spectral radius* of G, denoted by $\lambda_1^\alpha(G)$.

Let K_n be the complete graph of order n, and B^k_{n,n_1} be the graph obtained from $K_{n_1} \cup K_{n-n_1}$ by adding k independent edges (with no common endvertex) between K_{n_1} and K_{n-n_1}. Fan, Huang and Lin [4] provided the following conditions involving the spectral radius ($\lambda_1^0(G)$) for the rigidity (or the globally rigid) of a 2-connected graph (or a 3-connected graph):

Theorem 1.2 ([4]). *Let G be a 2-connected graph of order $n \geqslant 2\delta + 4$, where $\delta \geqslant 6$ is the minimum degree of G. If $\lambda_1^0(G) \geqslant \lambda_1^0\left(B^2_{n,\delta+1}\right)$, then G is rigid unless $G \cong B^2_{n,\delta+1}$.*

Theorem 1.3 ([4]). *Let G be a 3-connected graph of order $n \geqslant 2\delta + 4$, where $\delta \geqslant 6$ is the minimum degree of G. If $\lambda_1^0(G) \geqslant \lambda_1^0\left(B^3_{n,\delta+1}\right)$, then G is globally rigid unless $G \cong B^3_{n,\delta+1}$.*

It is natural and interesting to know whether the above mentioned results can be deduced from the conditions involving $\lambda_1^\alpha(G)$ for $\alpha \in [0,1]$. In this note, we extend their conditions to $\lambda_1^\alpha(G)$ for $\alpha \in (0,1)$. Our results can be read as follows:

Theorem 1.4. *Let G be a 2-connected graph of order n with the maximum degree Δ and the minimum degree $\delta \geqslant 6$. For $\alpha \in (0,1)$,*

$$\Delta < \min\left\{n^2 - 24n + 170 + \frac{3n-36}{\alpha},\ n^2 - 21n + 116 + \frac{13}{\alpha},\ n^2 - 21n + 130 + \frac{4}{\alpha}\right\}$$

and

$$n \geqslant \max\left\{2\delta + 4,\ \left\lceil \frac{-g + \sqrt{g^2 - 4(1-\alpha)h_1}}{2(1-\alpha)} \right\rceil + 1\right\},$$

where

$$g = (\alpha^2 + \alpha - 2)\delta + 2\alpha(\alpha - 1) \text{ and } h_1 = (1-\alpha^2)\delta^2 + 2\alpha(1-\alpha)\delta - 4\alpha^3 + 3\alpha^2 + 2\alpha - 1,$$

if $\lambda_1^\alpha(G) \geqslant \lambda_1^\alpha\left(B^2_{n,\delta+1}\right)$, then G is rigid unless $G \cong B^2_{n,\delta+1}$.

Theorem 1.5. *Let G be a 3-connected graph of order n with the maximum degree Δ and the minimum degree $\delta \geqslant 6$. For $\alpha \in (0,1)$,*

$$\Delta < \min\left\{n^2 - 24n + 170 + \frac{3n-36}{\alpha},\ n^2 - 21n + 116 + \frac{13}{\alpha},\ n^2 - 21n + 130 + \frac{4}{\alpha}\right\}$$

and

$$n \geqslant \max\left\{2\delta + 4,\ \left\lceil \frac{-g + \sqrt{g^2 - 4(1-\alpha)h_2}}{2(1-\alpha)} \right\rceil + 1\right\},$$

where

$$g = (\alpha^2 + \alpha - 2)\delta + 2\alpha(\alpha - 1) \text{ and } h_2 = (1-\alpha^2)\delta^2 + 2\alpha(1-\alpha)\delta - 6\alpha^3 + 5\alpha^2 + 2\alpha - 1,$$

if $\lambda_1^\alpha(G) \geqslant \lambda_1^\alpha\left(B^3_{n,\delta+1}\right)$, then G is globally rigid unless $G \cong B^3_{n,\delta+1}$.

The remainder of this note is organized as follows: Section 2 includes some necessary preliminaries. By adopting the somewhat similar strategy which was used in [4], we provide the proofs of Theorems 1.4 and 1.5 in Section 3. The last section includes some concluding remarks.

2. Preliminary

Given a partition $\pi = (X_1, X_2, \ldots, X_k)$ of the set $\{1, 2, \ldots, n\}$ and a matrix M whose rows and columns are labeled with elements in $\{1, 2, \ldots, n\}$, then M can be expressed as the following partitioned matrix $M = \begin{bmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,k} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ M_{k,1} & M_{k,2} & \cdots & M_{k,k} \end{bmatrix}$ with respect to π. The quotient matrix M_π of M with respect to π is the $k \times k$ matrix (m_{ij}) such that m_{ij} is the average value of all row sums of $M_{i,j}$. The partition π is *equitable* if each block $M_{i,j}$ of M has constant row sum m_{ij}. Also, we say that the quotient matrix M_π is equitable if π is an equitable partition of M.

Lemma 2.1 ([2, 5]). *Let M be a real symmetric matrix and $\lambda(M)$ be its largest eigenvalue. If M_π is an equitable quotient matrix of M, then the eigenvalues of M_π are also eigenvalues of M. Furthermore, if M is nonnegative and irreducible, then $\lambda(M) = \lambda(M_\pi)$.*

Lemma 2.2 ([13]). *If H is a proper subgraph of a connected graph G, then for $\alpha \in [0, 1]$, we have $\lambda_1^\alpha(G) > \lambda_1^\alpha(H)$.*

Recall that B^k_{n,n_1} is the graph obtained from $K_{n_1} \cup K_{n-n_1}$ by adding k independent edges between K_{n_1} and K_{n-n_1}.

Lemma 2.3. *For $\alpha \in (0, 1)$, let $k \geqslant 1, b \geqslant k + 1$ and $n > \max\left\{2b + 1, \frac{-g+\sqrt{g^2-4(1-\alpha)h}}{2(1-\alpha)}\right\}$, where $g = (\alpha^2 + \alpha - 2)b + 2\alpha(\alpha - 1)$ and $h = (1 - \alpha^2)b^2 + 2\alpha(1 - \alpha)b - 2k\alpha^3 + (2k - 1)\alpha^2 + 2\alpha - 1$. Then we have $\lambda_1^\alpha\left(B^k_{n,b+1}\right) < \lambda_1^\alpha\left(B^k_{n,b}\right)$.*

Proof. Since $B^k_{n,b}$ contains K_{n-b} as a proper subgraph, by Lemma 2.2, we have

$$\lambda_1^\alpha\left(B^k_{n,b}\right) > \lambda_1^\alpha\left(K_{n-b}\right) = n - b - 1.$$

Note that $A_\alpha\left(B^k_{n,b}\right)$ has an equitable quotient matrix as

$$M^b_\pi = \begin{bmatrix} \alpha b + (1-\alpha)(k-1) & (1-\alpha)(b-k) & 1-\alpha & 0 \\ (1-\alpha)k & \alpha k + b - k - 1 & 0 & 0 \\ 1-\alpha & 0 & \alpha(n-b) + (1-\alpha)(k-1) & (1-\alpha)(n-b-k) \\ 0 & 0 & (1-\alpha)k & \alpha k + n - b - k - 1 \end{bmatrix},$$

and its characteristic polynomial is

$$\begin{aligned} & f\left(M^b_\pi, x\right) \\ =& (1-\alpha)^2 k(b+k-n)\left\{x^2 + [2-\alpha-(1+\alpha)b]x + \alpha(\alpha k - k - 1) + \alpha b^2 - b + 1\right\} \\ & - (x + k - \alpha k + b - n + 1)\left\{(1-\alpha)^2(x + 1 - b + k(1-\alpha)) \right. \\ & \left. + [(1-\alpha)(k-1) + \alpha(n-b) - x]\left[x^2 + (2-\alpha-(1+\alpha)b)x + \alpha(\alpha k - k - 1) + \alpha b^2 - b + 1\right]\right\}. \end{aligned}$$

Similarly, $A_\alpha\left(B^k_{n,b+1}\right)$ has an equitable quotient matrix M^{b+1}_π, substituting b with $b + 1$ in M^b_π, we then have

$$\begin{aligned} & f\left(M^{b+1}_\pi, x\right) - f\left(M^b_\pi, x\right) \\ =& (n - 2b - 1) \times \\ & \left\{(1+\alpha^2)x^2 + [(2-n)\alpha^2 - (2+n)\alpha + 2]x - 2k\alpha^3 - [2b(b-n+1) - 2k - n]\alpha^2 - n\alpha\right\}. \end{aligned}$$

As $n > 2b+1$, so $n-2b-1>0$. Let

$$f(x) = (1+\alpha^2)x^2 + [(2-n)\alpha^2 - (2+n)\alpha + 2]x - 2k\alpha^3 - [2(b-n+1)b - 2k - n]\alpha^2 - n\alpha.$$

In order to derive $f\left(M_\pi^{b+1}, x\right) - f\left(M_\pi^{b}, x\right) > 0$ for all $x \geqslant n-b-1$, we need to ensure that $f(x) > 0$ for all $x \geqslant n-b-1$, that is the largest root of $f(x)=0$ is less than $n-b-1$, *i.e.*,

$$n-b-1 > \frac{-c + \sqrt{c^2 - 4(1+\alpha^2)d}}{2(1+\alpha^2)},$$

where $c = (2-n)\alpha^2 - (2+n)\alpha + 2$ and $d = -2k\alpha^3 - [2b(b-n+1) - 2k - n]\alpha^2 - n\alpha$. By calculations, we have

$$n^2 - 2bn + b^2 - 2\alpha n - \alpha n^2 + 2\alpha b - 2\alpha^2 b + \alpha bn + \alpha^2 bn + 2\alpha - 1 - \alpha^2 + 2\alpha^2 n - \alpha^2 b^2 + 2\alpha^2 k - 2\alpha^3 k > 0,$$

that is

$$(1-\alpha)n^2 + \left[(\alpha^2+\alpha-2)b + 2\alpha(\alpha-1)\right]n + (1-\alpha^2)b^2 + 2\alpha(1-\alpha)b - 2k\alpha^3 + (2k-1)\alpha^2 + 2\alpha - 1 > 0.$$

It follows that $n > \frac{-g+\sqrt{g^2-4(1-\alpha)h}}{2(1-\alpha)}$, where

$$g = (\alpha^2+\alpha-2)b + 2\alpha(\alpha-1) \text{ and } h = (1-\alpha^2)b^2 + 2\alpha(1-\alpha)b - 2k\alpha^3 + (2k-1)\alpha^2 + 2\alpha - 1.$$

Hence, when $n > \max\left\{2b+1, \frac{-g+\sqrt{g^2-4(1-\alpha)h}}{2(1-\alpha)}\right\}$, we have $f\left(M_\pi^{b+1}, x\right) - f\left(M_\pi^{b}, x\right) > 0$ for $x \geqslant n-b-1$. It follows that $\lambda_1^\alpha\left(M_\pi^{b+1}\right) < \lambda_1^\alpha\left(M_\pi^{b}\right)$. This together with Lemma 2.1 implies that $\lambda_1^\alpha\left(B_{n,a+1}^k\right) < \lambda_1^\alpha\left(B_{n,a}^k\right)$, as desired. □

In particular, for $k = 2, 3$, we then have the following corollaries.

Corollary 2.4. *For $\alpha \in (0,1)$, $b \geqslant 3$ and $n \geqslant \max\left\{2b+4, \left\lceil \frac{-g+\sqrt{g^2-4(1-\alpha)h_1}}{2(1-\alpha)} \right\rceil + 1\right\}$ where $g = (\alpha^2+\alpha-2)b+2\alpha(\alpha-1)$ and $h_1 = (1-\alpha^2)b^2 + 2\alpha(1-\alpha)b - 4\alpha^3 + 3\alpha^2 + 2\alpha - 1$. Then $\lambda_1^\alpha\left(B_{n,b+1}^2\right) < \lambda_1^\alpha\left(B_{n,b}^2\right)$.*

Corollary 2.5. *For $\alpha \in (0,1)$, $b \geqslant 4$ and $n \geqslant \max\left\{2b+4, \left\lceil \frac{-g+\sqrt{g^2-4(1-\alpha)h_2}}{2(1-\alpha)} \right\rceil + 1\right\}$ where $g = (\alpha^2+\alpha-2)b+2\alpha(\alpha-1)$ and $h_2 = (1-\alpha^2)b^2 + 2\alpha(1-\alpha)b - 6\alpha^3 + 5\alpha^2 + 2\alpha - 1$. Then $\lambda_1^\alpha\left(B_{n,b+1}^3\right) < \lambda_1^\alpha\left(B_{n,b}^3\right)$.*

Lemma 2.6 ([14]). *Let G be a graph of order n with $e(G)$ edges, the maximum degree Δ and the minimum degree δ. Then for $\alpha \in [0,1]$, we have*

$$\lambda_1^\alpha(G) \leqslant \frac{1}{2}\left[(\delta-1) + \sqrt{(\delta-1)^2 + 4\{\alpha\Delta - \alpha(\delta-1)\delta + (1-\alpha)[2e(G) - \delta(n-1)]\}}\right].$$

Moreover, the equality holds if and only if G is regular.

Lemma 2.7 ([4]). *Let a and b be two positive integers. If $a \geqslant b$, then*

$$\binom{a}{2} + \binom{b}{2} < \binom{a+1}{2} + \binom{b-1}{2}.$$

For a subset $X \subseteq V(G)$, let $G[X]$ be the subgraph induced by X in G, let $e_G(X)$ and $e_G(G)$ (or simply $e(G)$) be the number of edges of $G[X]$ and G, respectively. For two subsets $X, Y \subseteq V(G)$, let $E_G(X,Y)$ be the set of edges having one endpoint in X and the other in Y, and $e_G(X,Y) = |E_G(X,Y)|$. For simplicity, we use $\partial_G(X)$ to denote $E_G(X, V(G) - X)$.

Lemma 2.8 ([7]). *Let G be a graph with the minimum degree δ and $U(\neq \emptyset) \subset V(G)$. If $|\partial_G(U)| \leqslant \delta - 1$, then $|U| \geqslant \delta + 1$.*

A part is *trivial* if it contains a single vertex. For any set $Z \subset V(G)$, let π be a partition of $V(G-Z)$ with n_0 trivial parts $\{v_1, v_2, \ldots, v_{n_0}\}$. Let $n_Z(\pi) = \sum_{i=1}^{n_0} |Z_i|$, where Z_i is the set of vertices in Z which are adjacent to v_i for $1 \leqslant i \leqslant n_0$. For any partition π of $V(G)$, let $E_G(\pi)$ be the set of edges in G whose endpoints lie in different parts of π, and $e_G(\pi) = |E_G(\pi)|$.

Lemma 2.9 ([6]). *A graph G contains k edge-disjoint spanning rigid subgraphs if for every $Z \subset V(G)$ and every partition π of $V(G-Z)$ with n_0 trivial parts and n_0' nontrivial parts,*

$$e_{G-Z}(\pi) \geqslant k(3 - |Z|)n_0' + 2kn_0 - 3k - n_Z(\pi).$$

Lemma 2.10 ([3, 10]). *Let G be a graph. Then G is globally rigid if and only if either G is a complete graph on at most three vertices or G is 3-connected and redundantly rigid.*

3. Proofs of Theorems 1.4 and 1.5

We say a graph G is *minimally rigid* if G is rigid but $G-e$ is not rigid for any $e \in E(G)$. Note that if a graph G is rigid, then it must contain a spanning subgraph that is also rigid. For minimal rigidity, this subgraph must remain rigid while the removal of any edge results in a non-rigid structure. On the other hand, if G has a minimally rigid spanning subgraph, the rigidity of this subgraph is sufficient to ensure the rigidity of G, as rigidity is inherently determined by the structural properties of the framework. Therefore, a graph G is rigid if and only if G has a minimally rigid spanning subgraph.

In this section, we will provide two key lemmas (Lemma 3.1 and Lemma 3.2), as well as the proofs of Theorems 1.4 and 1.5.

Lemma 3.1. *Let G be a 2-connected graph of order n with the minimum degree $\delta \geqslant 6$. If G is not rigid, then for every $Z \subset V(G)$ and every partition π of $V(G-Z)$ with n_0 trivial parts and n_0' nontrivial parts, we have $0 \leqslant |Z| \leqslant 2$ and $n_0' \geqslant 2$.*

Proof. Note that G does not contain any spanning rigid subgraphs since G is not rigid. Then Lemma 2.9 implies that there exists a subset $Z \subset V(G)$ and a partition π of $V(G-Z)$ with n_0 trivial parts $\{v_1, v_2, \ldots, v_{n_0}\}$ and n_0' nontrivial parts $\{V_1, V_2, \ldots, V_{n_0'}\}$ such that

$$e_{G-Z}(\pi) \leqslant (3 - |Z|)n_0' + 2n_0 - 4 - n_Z(\pi), \tag{1}$$

where $n_Z(\pi) = \sum_{j=1}^{n_0} |Z_j|$ and Z_j is the set of vertices in Z that are adjacent to v_j.

Since $d_{G-Z}(v_j) \geqslant \delta - |Z_j|$, $\delta \geqslant 6$ and $2e_{G-Z}(\pi) = \sum_{i=1}^{n_0'} |\partial_{G-Z}(V_i)| + \sum_{j=1}^{n_0} d_{G-Z}(v_j)$, then we have

$$2e_{G-Z}(\pi) \geqslant \sum_{i=1}^{n_0'} |\partial_{G-Z}(V_i)| + \delta n_0 - \sum_{j=1}^{n_0} |Z_j| \geqslant \sum_{i=1}^{n_0'} |\partial_{G-Z}(V_i)| + 6n_0 - n_Z(\pi). \tag{2}$$

It follows that

$$e_{G-Z}(\pi) \geqslant 3n_0 - \frac{1}{2} n_Z(\pi). \tag{3}$$

We now establish the possible values for $|Z|$ and n_0'.

Fact 1: $0 \leqslant |Z| \leqslant 2$.

Assume that $|Z| \geqslant 3$. Then by (1), we have $e_{G-Z}(\pi) \leqslant 2n_0 - 4 - n_Z(\pi)$. This together with (3) implies that $3n_0 - \frac{1}{2}n_Z(\pi) \leqslant 2n_0 - 4 - n_Z(\pi)$. It follows that $n_0 + 4 + \frac{1}{2}n_Z(\pi) \leqslant 0$. This is impossible since n_0 and $n_Z(\pi)$ are both non-negative. Therefore, $0 \leqslant |Z| \leqslant 2$.

Fact 2: $n_0' \geqslant 2$.

Assume that $n_0' \leqslant 1$. Then by (1) and Fact 1, we have $e_{G-Z}(\pi) \leqslant 2n_0 - 1 - n_Z(\pi)$. This together with (3) implies that $3n_0 - \frac{1}{2}n_Z(\pi) \leqslant 2n_0 - 1 - n_Z(\pi)$. It follows that $n_0 + 1 + \frac{1}{2}n_Z(\pi) \leqslant 0$. This is impossible since n_0 and $n_Z(\pi)$ are both non-negative. Therefore, $n_0' \geqslant 2$.

The proof is completed. □

Proof of Theorem 1.4: We prove it by contradiction. Assume that G is not rigid. Then Lemma 3.1 implies that there exists a subset $Z \subset V(G)$ and a partition π of $V(G-Z)$ into n_0 trivial parts $\{v_1, v_2, \ldots, v_{n_0}\}$ and n_0' nontrivial parts $\{V_1, V_2, \ldots, V_{n_0'}\}$, where $0 \leqslant |Z| \leqslant 2$ and $n_0' \geqslant 2$.

Note that

$$\lambda_1^{\alpha}(G) \geqslant \lambda_1^{\alpha}(B^2_{n,\delta+1}) > \lambda_1^{\alpha}(K_{n-\delta-1}) = n - \delta - 2.$$

This together with Lemma 2.6 implies that

$$\frac{1}{2}\left[(\delta-1) + \sqrt{(\delta-1)^2 + 4\{\alpha\Delta - \alpha(\delta-1)\delta + (1-\alpha)[2e(G) - \delta(n-1)]\}}\right] > n - \delta - 2.$$

Solving for $e(G)$, we obtain

$$e(G) > \frac{(2n-3\delta-3)^2 - (\delta-1)^2 - 4\alpha\Delta + 4\alpha(\delta-1)\delta + 4(1-\alpha)\delta(n-1)}{8(1-\alpha)}. \tag{4}$$

Moreover, as G is 2-connected, we have

$$|\partial_{G-Z}(V_i)| \geqslant 2 - |Z|, \text{ for } 1 \leqslant i \leqslant n_0'. \tag{5}$$

We now consider the following two cases according to the values of $|Z|$.

Case 1: $|Z| = 2$.

Then inequality (1) becomes

$$e_{G-Z}(\pi) \leqslant n_0' + 2n_0 - 4 - n_Z(\pi), \tag{6}$$

where $n_Z(\pi) = \sum_{j=1}^{n_0} |Z_j|$ and Z_j is the set of vertices in Z that are adjacent to v_j.

We will prove that $n_0' \geqslant 4$. If $2 \leqslant n_0' \leqslant 3$, then using (2), (5) and (6), we have

$$0 \leqslant \sum_{i=1}^{n_0'} |\partial_{G-Z}(V_i)| \leqslant 2n_0' - 8 - 2n_0 - n_Z(\pi) \leqslant -2,$$

a contradiction. Hence $n_0' \geqslant 4$.

Let δ' be the minimum degree of $G-Z$, then $\delta' \geqslant \delta - 2$. If the partition π contains at most one nontrivial part, say V_j $(1 \leqslant j \leqslant n_0')$, such that $|\partial_{G-Z}(V_j)| \leqslant \delta' - 1$, then $|\partial_{G-Z}(V_i)| \geqslant \delta'$ for all $i \in \{1, \ldots, n_0'\} \setminus \{j\}$. Note that

$$\begin{aligned}
2e_{G-Z}(\pi) &= \sum_{i=1}^{n_0'} |\partial_{G-Z}(V_i)| + \sum_{j=1}^{n_0} d_{G-Z}(v_j) \\
&\geqslant (n_0'-1)\delta' + \delta n_0 - n_Z(\pi) \quad (\text{as } d_{G-Z}(v_j) \geqslant \delta - |Z_j| \text{ and } n_Z(\pi) = \textstyle\sum_{j=1}^{n_0} |Z_j|) \\
&\geqslant (n_0'-1)(\delta-2) + \delta n_0 - n_Z(\pi) \quad (\text{as } \delta' \geqslant \delta - 2) \\
&= 2n_0' + 4n_0 - 8 - 2n_Z(\pi) + (\delta-4)n_0' - \delta + (\delta-4)n_0 + n_Z(\pi) + 10 \\
&\geqslant 2n_0' + 4n_0 - 8 - 2n_Z(\pi) + 3\delta - 6 \quad (\text{as } n_0' \geqslant 4,\ n_0 \geqslant 0 \text{ and } n_Z(\pi) \geqslant 0) \\
&> 2n_0' + 4n_0 - 8 - 2n_Z(\pi) \quad (\text{as } \delta \geqslant 6).
\end{aligned}$$

It follows that $e_{G-Z}(\pi) > n_0' + 2n_0 - 4 - n_Z(\pi)$, which contradicts (6). Hence, the partition π must contain at least two nontrivial parts, say V_1 and V_2, such that $|\partial_{G-Z}(V_1)| \leqslant \delta' - 1$ and $|\partial_{G-Z}(V_2)| \leqslant \delta' - 1$. Then Lemma 2.8 implies that $|V_i| \geqslant \delta' + 1 \geqslant \delta - 1$ for $i = 1, 2$ (as $\delta' \geqslant \delta - 2$).

In what follows, we determine the maximum value of $\sum_{i=1}^{n_0'} e_G(V_i)$. We assert that $\sum_{i=1}^{n_0'} e_G(V_i)$ is maximized when n_0' is minimized (i.e., $n_0' = 4$). Otherwise, if $n_0' \geq 5$, then we may increase the value of $\sum_{i=1}^{n_0'} e_G(V_i)$ by adding edges between V_4 and $V_{n_0'}$, which contradicts the maximality of $\sum_{i=1}^{n_0'} e_G(V_i)$.

For $n_0' = 4$, let V_1, V_2, V_3, and V_4 be the nontrivial parts of $G - Z$. If $|V_1|$ or $|V_2| = \max\{|V_1|, |V_2|, |V_3|, |V_4|\}$, since $|V_1|, |V_2| \geqslant \delta - 1$ and $|V_3|, |V_4| \geqslant 2$, then we have

$$\begin{aligned}\sum_{i=1}^{n_0'} e_G(V_i) \leqslant \sum_{i=1}^{4} e_G(V_i) &= \binom{|V_1|}{2} + \binom{|V_2|}{2} + \binom{|V_3|}{2} + \binom{|V_4|}{2} \\ &\leqslant \binom{\delta - 1}{2} + \binom{n - |Z| - \delta - 3}{2} + \binom{2}{2} + \binom{2}{2} \quad \text{(by Lemma 2.7).}\end{aligned}$$

Similarly, for $|V_3|$ or $|V_4| = \max\{|V_1|, |V_2|, |V_3|, |V_4|\}$, we have

$$\sum_{i=1}^{n_0'} e_G(V_i) \leqslant \sum_{i=1}^{4} e_G(V_i) \leqslant \binom{\delta - 1}{2} + \binom{\delta - 1}{2} + \binom{n - |Z| - 2\delta}{2} + \binom{2}{2}.$$

Recall that $n = |Z| + n_0 + \sum_{i=1}^{n_0'} |(V_i)|$, where $V_1, V_2, \ldots, V_{n_0'}$ are nontrivial parts, and $|V_1|, |V_2| \geqslant \delta - 1$. Then by calculation, we have

$$n_0' \leqslant \frac{n - |Z| - 2(\delta - 1)}{2} + 2 = \frac{n}{2} - \delta + 2, \text{ as } |Z| = 2 \text{ and } n_0 \geqslant 0. \tag{7}$$

Moreover, note that for $|Z| = 2$, we have

$$|\partial_G(Z)| + e_G(Z) - n_Z(\pi) \leqslant 2(n - 2) + 1 - 2n_0 = 2(n - 2 - n_0) + 1.$$

This together with (6) implies that

$$e_{G-Z}(\pi) + |\partial_G(Z)| + e_G(Z) \leqslant n_0' + 2n - 7.$$

Then by (7), we have

$$e_{G-Z}(\pi) + |\partial_G(Z)| + e_G(Z) \leqslant \frac{5n}{2} - \delta - 5. \tag{8}$$

Moreover, as $\delta \geqslant 6$ and $n \geqslant 2\delta + 4$, we then have

$$\begin{aligned}&e(G) \\ &= \sum_{i=1}^{n_0'} e_G(V_i) + \sum_{j=1}^{n_0} e_G(v_j) + e_{G-Z}(\pi) + |\partial_G(Z)| + e_G(Z) \\ &\leqslant \max\left\{\binom{\delta - 1}{2} + \binom{n - |Z| - \delta - 3}{2} + 2\binom{2}{2}, 2\binom{\delta - 1}{2} + \binom{n - |Z| - 2\delta}{2} + \binom{2}{2}\right\} \\ &\quad + 0 + e_{G-Z}(\pi) + |\partial_G(Z)| + e_G(Z) \\ &\leqslant \binom{\delta - 1}{2} + \binom{n - |Z| - \delta - 3}{2} + 2\binom{2}{2} + \frac{5n}{2} - \delta - 5 \quad \text{by (8)} \\ &\leqslant \frac{n^2}{2} - \frac{(2\delta + 6)n}{2} + \delta^2 + 3\delta + 13.\end{aligned}$$

This together with (4) implies that

$$\frac{(2n-3\delta-3)^2-(\delta-1)^2-4\alpha\Delta+4\alpha(\delta-1)\delta+4(1-\alpha)\delta(n-1)}{8(1-\alpha)}<\frac{n^2}{2}-\frac{(2\delta+6)n}{2}+\delta^2+3\delta+13.$$

Solving for δ, we get

$$\delta<\sqrt{\frac{\alpha\Delta+6\alpha n+24-3n-\alpha n^2-26\alpha}{3\alpha}+\frac{(3\alpha n+2-6\alpha)^2}{36\alpha^2}}+\frac{3\alpha n+2-6\alpha}{6\alpha}.$$

On the other hand, the condition

$$\sqrt{\frac{\alpha\Delta+6\alpha n+24-3n-\alpha n^2-26\alpha}{3\alpha}+\frac{(3\alpha n+2-6\alpha)^2}{36\alpha^2}}+\frac{3\alpha n+2-6\alpha}{6\alpha}<6$$

is equivalent to $\Delta<n^2-24n+170+\frac{3n-36}{\alpha}$. Therefore, when $\Delta<n^2-24n+170+\frac{3n-36}{\alpha}$, we have $\delta<6$, which contradicts our initial assumption $\delta\geqslant 6$.

Case 2: $0\leqslant |Z|\leqslant 1$.

This case can be analyzed in the following two subcases.

(A) $n_0'=2$.

In this case, the partition π consists of two nontrivial parts, V_1 and V_2, together with n_0 trivial parts. Substituting (5) into (2), we obtain

$$2e_{G-Z}(\pi)\geqslant|\partial_{G-Z}(V_1)|+|\partial_{G-Z}(V_2)|+6n_0-n_Z(\pi)\geqslant 4-2|Z|+6n_0-n_Z(\pi).$$

Consequently,

$$e_{G-Z}(\pi)\geqslant 2-|Z|+3n_0-\frac{1}{2}n_Z(\pi).$$

Given that $n_0'=2$, combining this inequality with (1), we have

$$2(3-|Z|)+2n_0-4-n_Z(\pi)\geqslant 2-|Z|+3n_0-\frac{1}{2}n_Z(\pi),$$

which simplifies to

$$-n_0-\frac{1}{2}n_Z(\pi)-|Z|\geqslant 0.$$

Since $n_0\geqslant 0, n_Z(\pi)\geqslant 0$ and $|Z|\geqslant 0$, we conclude that $n_0=0, n_Z(\pi)=0$ and $|Z|=0$. We find that the partition π consists of two nontrivial parts V_1 and V_2, and as $G-Z=G$, then $V(G)=V_1\cup V_2$. Using (1), we have $e_G(V_1,V_2)=e_G(\pi)\leqslant 2$. By (5), $e_G(V_1,V_2)=\frac{1}{2}(|\partial_G(V_1)|+|\partial_G(V_2)|)\geqslant 2$, making $e_G(V_1,V_2)=2$. We denote the edge set connecting V_1 and V_2 by $E_G(V_1,V_2)=\{f_1,f_2\}$. We claim that f_1 and f_2 are two independent edges. If not, assume $f_1\cap f_2=\{u\}$, then vertex u is a cut vertex of G, which is impossible as G is 2-connected. It is evident that G is a spanning subgraph of $B^2_{n,|V_1|}$, leading to

$$\lambda_1^\alpha(G)\leqslant\lambda_1^\alpha\left(B^2_{n,|V_1|}\right),\tag{9}$$

with equality if and only if $G\cong B^2_{n,|V_1|}$. Given that $\delta\geqslant 6$, we have $|\partial_G(V_1)|=|\partial_G(V_2)|=2<\delta-1$. Then, by Lemma 2.8, we have $\min\{|V_1|,|V_2|\}\geqslant\delta+1$. Applying Lemma 2.3, Corollary 2.4, and equation (9), we obtain

$$\lambda_1^\alpha(G)\leqslant\lambda_1^\alpha\left(B^2_{n,\delta+1}\right),$$

with equality if and only if $G\cong B^2_{n,\delta+1}$. However, this is impossible since we already have $\lambda_1^\alpha(G)\geqslant\lambda_1^\alpha\left(B^2_{n,\delta+1}\right)$ and $G\ncong B^2_{n,\delta+1}$.

(B) $n_0' \geqslant 3$.
Let δ' be the minimum degree of $G-Z$, then $\delta' \geqslant \delta-|Z|$. If the partition π contains at most one nontrivial part, say V_k $\left(1 \leqslant k \leqslant n_0'\right)$, such that $|\partial_{G-Z}(V_k)| \leqslant \delta'-1$, then $|\partial_{G-Z}(V_i)| \geqslant \delta'$ for all $i \in \left\{1,\ldots,n_0'\right\}\backslash\{k\}$. Note that

$$\begin{aligned}
&2e_{G-Z}(\pi)\\
=&\sum_{i=1}^{n_0'}|\partial_{G-Z}(V_i)| + \sum_{j=1}^{n_0} d_{G-Z}\left(v_j\right)\\
=&\sum_{i\in\left\{1,\ldots,n_0'\right\}\backslash\{k\}}|\partial_{G-Z}(V_i)| + |\partial_{G-Z}(V_k)| + \sum_{j=1}^{n_0} d_{G-Z}\left(v_j\right)\\
\geqslant&\left(n_0'-1\right)\delta' + 2 - |Z| + \delta n_0 - n_Z(\pi) \quad \text{(by (5), then } |\partial_{G-Z}(V_k)| \geqslant 2-|Z|)\\
\geqslant&\left(n_0'-1\right)(\delta-|Z|) + 2 - |Z| + \delta n_0 - n_Z(\pi) \quad (\text{as } \delta' \geqslant \delta-|Z|)\\
=&2(3-|Z|)n_0' + 4n_0 - 8 - 2n_Z(\pi) + (\delta-6+|Z|)n_0' + (\delta-4)n_0 - \delta + 10 + n_Z(\pi)\\
\geqslant&2(3-|Z|)n_0' + 4n_0 - 8 - 2n_Z(\pi) + 2\delta - 8 + 3|Z| + n_Z(\pi) \quad (\text{as } n_0' \geqslant 3 \text{ and } n_0 \geqslant 0)\\
>&2(3-|Z|)n_0' + 4n_0 - 8 - 2n_Z(\pi) \quad (\text{as } \delta \geqslant 3,\ n_Z(\pi) \geqslant 0 \text{ and } 0 \leqslant |Z| \leqslant 1),
\end{aligned}$$

which simplifies to

$$e_{G-Z}(\pi) > (3-|Z|)n_0' + 2n_0 - 4 - n_Z(\pi),$$

contradicting (1). Consequently, the partition π must contain at least two nontrivial parts, say V_1 and V_2, such that $|\partial_{G-Z}(V_1)| \leqslant \delta'-1$ and $|\partial_{G-Z}(V_2)| \leqslant \delta'-1$. Then Lemma 2.8 implies that $|V_1| \geqslant \delta'+1$ and $|V_2| \geqslant \delta'+1$. We now consider the following two situations according to the values of $|Z|$.

- $|Z| = 0$.
  For $|Z| = 0$, we have $\delta' = \delta$ and $|V_i| \geqslant \delta+1$ for $i = 1,2$. If $|V_1|$ or $|V_2| = \max\{|V_1|,|V_2|,\ldots,|V_{n_0'}|\}$, since $|V_i| \geqslant \delta+1$ and $\left|V_j\right| \geqslant 2$ for $i = 1,2$ and $j \in \{3,\ldots,n_0'\}$, then by a similar argument as that in Case 1, we have
  $$\sum_{i=1}^{n_0'} e_G(V_i) \leqslant \binom{\delta+1}{2} + \binom{n-\delta-3}{2} + \binom{2}{2}.$$
  Similarly, if $|V_1|$ and $|V_2| \neq \max\{|V_1|,|V_2|,\ldots,|V_{n_0'}|\}$, then we have
  $$\sum_{i=1}^{n_0'} e_G(V_i) \leqslant \binom{\delta+1}{2} + \binom{\delta+1}{2} + \binom{n-2\delta-2}{2}.$$
  As $|V_i| \geqslant \delta+1$ for $i = 1,2$ and $|V_3| \geqslant 2$, then $n_0 \leqslant n - \sum_{i=1}^{3}|V_i| \leqslant n-2\delta-4$. Recall that $n = |Z| + n_0 + \sum_{i=1}^{n_0'}|(V_i)| = n_0 + \sum_{i=1}^{n_0'}|(V_i)|$. By calculation, we have
  $$n_0' \leqslant \frac{n-(2\delta+4)-n_0}{2} + 3. \tag{10}$$
  As $|Z| = 0$, $G-Z = G$ and $n_Z(\pi) = 0$, by (1) we have
  $$\begin{aligned}
  e_G(\pi) &\leqslant 3n_0' + 2n_0 - 4\\
  &\leqslant \frac{3n}{2} - 3\delta - 1 + \frac{n_0}{2} \quad (\text{ by (10)})\\
  &\leqslant 2n - 4\delta - 3 \quad (\text{ as } n_0 \leqslant n-2\delta-4).
  \end{aligned}$$

Since $\delta \geqslant 6$ and $n \geqslant 2\delta+4$, then we have

$$\begin{aligned}
& e(G) \\
= & \sum_{i=1}^{n_0'} e_G(V_i) + \sum_{i=1}^{n_0} e_G(v_i) + e_G(\pi) \\
\leqslant & \max\left\{\binom{\delta+1}{2} + \binom{n-\delta-3}{2} + \binom{2}{2}, 2\binom{\delta+1}{2} + \binom{n-2\delta-2}{2}\right\} \\
& + 0 + e_G(\pi) \\
\leqslant & \binom{\delta+1}{2} + \binom{n-\delta-3}{2} + \binom{2}{2} + e_G(\pi) \quad (\text{ as } \delta \geqslant 6 \text{ and } n \geqslant 2\delta+4) \\
\leqslant & \frac{n^2}{2} - \frac{(2\delta+3)n}{2} + \delta^2 + 4.
\end{aligned}$$

This together with (4) implies that

$$\frac{(2n-3\delta-3)^2 - (\delta-1)^2 - 4\alpha\Delta + 4\alpha(\delta-1)\delta + 4(1-\alpha)\delta(n-1)}{8(1-\alpha)} < \frac{n^2}{2} - \frac{(2\delta+3)n}{2} + \delta^2 + 4.$$

By calculations, we have

$$\delta < \sqrt{\frac{\alpha\Delta + 3\alpha n + 6 - \alpha n^2 - 8\alpha}{3\alpha} + \frac{(3\alpha n-4)^2}{36\alpha^2}} + \frac{3\alpha n - 4}{6\alpha}.$$

On the other hand, the condition

$$\sqrt{\frac{\alpha\Delta + 3\alpha n + 6 - \alpha n^2 - 8\alpha}{3\alpha} + \frac{(3\alpha n-4)^2}{36\alpha^2}} + \frac{3\alpha n - 4}{6\alpha} < 6$$

is equivalent to $\Delta < n^2 - 21n + 116 + \frac{18}{\alpha}$. That is when $\Delta < n^2 - 21n + 116 + \frac{18}{\alpha}$, we have $\delta < 6$, a contradiction.

- $|Z| = 1$.

  When $|Z| = 1$, note that $\delta' \geqslant \delta - 1$, then $|V_i| \geqslant \delta' + 1 \geqslant \delta$ for $i = 1, 2$ and $|V_3| \geqslant 2$. Similarly, by calculation, we have

$$n_0' \leqslant \frac{n - |Z| - n_0 - \sum_{i\in\{1,2,3\}} |V_i|}{2} + 3 \leqslant \frac{n - n_0 - 2\delta + 3}{2}. \tag{11}$$

Let $Z = \{w\}$, then $d_G(w) - n_Z(\pi) \leqslant n - 1 - n_0$. By (1), we have

$$\begin{aligned}
e_{G-Z}(\pi) + d_G(w) &\leqslant (3 - |Z|)n_0' + 2n_0 - 4 - n_Z(\pi) + d_G(w) \\
&\leqslant 2n_0' + 2n_0 - 4 - n_Z(\pi) + d_G(w) \quad (\text{ as } |Z| = 1) \\
&\leqslant 2n_0' + 2n_0 - 4 - n_0 + n - 1 \quad (\text{as } d_G(w) - n_Z(\pi) \leqslant n - 1 - n_0) \\
&= 2n_0' + n_0 + n - 5 \\
&\leqslant 2n - 2\delta - 2 \quad (\text{ by (11)}).
\end{aligned}$$

Given that $\delta \geqslant 6$ and $n \geqslant 2\delta + 4$, we obtain

$$\begin{aligned}
& e(G) \\
= & \sum_{i=1}^{n_0'} e_G(V_i) + \sum_{i=1}^{n_0} e_G(v_i) + e_{G-Z}(\pi) + d_G(w) \\
\leqslant & \max\left\{\binom{\delta}{2} + \binom{n-|Z|-\delta-2}{2} + \binom{2}{2}, 2\binom{\delta}{2} + \binom{n-|Z|-2\delta}{2}\right\} \\
& + 0 + e_{G-Z}(\pi) + d_G(w) \\
\leqslant & \binom{\delta}{2} + \binom{n-|Z|-\delta-2}{2} + \binom{2}{2} \\
& + e_{G-Z}(\pi) + d_G(w) \quad (\text{ as } \delta \geqslant 6 \text{ and } n \geqslant 2\delta + 4) \\
\leqslant & \frac{n^2}{2} - \frac{(2\delta+3)n}{2} + \delta^2 + \delta + 5.
\end{aligned}$$

Similarly, combining this with (4), we get that when $\Delta < n^2 - 21n + 130 + \frac{4}{\alpha}$, $\delta < 6$, which is also a contradiction. This completes the proof. □

The following lemma is very important for proving Theorem 1.5.

Lemma 3.2. *Let G be a 3-connected graph of order n with the minimum degree $\delta \geqslant 6$. If G is not globally rigid, then there exists an edge $f \in E(G)$, such that for every $Z \subset V(G)$ and every partition π of $V(G - f - Z)$ with n_0 trivial parts and n_0' nontrivial parts, we have $0 \leqslant |Z| \leqslant 2$ and $n_0' \geqslant 2$.*

Proof. Assume that G is not globally rigid. Then Lemma 2.10 implies that G is not redundantly rigid since G is 3-connected. It means that there exists an edge $f \in E(G)$ such that $G - f$ is not rigid. Furthermore, Lemma 2.9 implies the existence of a subset $Z \subset V(G - f)$ and a partition π of $V(G - f - Z)$ with n_0 trivial parts $\{v_1, v_2, \ldots, v_{n_0}\}$ and n_0' nontrivial parts $\{V_1, V_2, \ldots, V_{n_0'}\}$ satisfying

$$e_{G-f-Z}(\pi) \leqslant (3 - |Z|)n_0' + 2n_0 - 4 - n_Z(\pi), \tag{12}$$

where $n_Z(\pi) = \sum_{i=1}^{n_0} |Z_j|$ and Z_j is the set of vertices in Z adjacent to v_j.

We now consider the following two cases:

Case 1: $f \in E_{G-Z}(\pi)$.

For $f \in E_{G-Z}(\pi)$, we have $e_{G-f-Z}(\pi) = e_{G-Z}(\pi) - 1$. Then from (12), we have

$$e_{G-Z}(\pi) \leqslant (3 - |Z|)n_0' + 2n_0 - 3 - n_Z(\pi). \tag{13}$$

On the other hand, since $d_{G-Z}(v_j) \geqslant \delta - |Z_j|$ and $\delta \geqslant 6$, we obtain

$$2e_{G-Z}(\pi) = \sum_{i=1}^{n_0'} |\partial_{G-Z}(V_i)| + \sum_{j=1}^{n_0} d_{G-Z}(v_j) \geqslant \sum_{i=1}^{n_0'} |\partial_{G-Z}(V_i)| + 6n_0 - n_Z(\pi). \tag{14}$$

It follows that

$$e_{G-Z}(\pi) \geqslant 3n_0 - \frac{1}{2}n_Z(\pi). \tag{15}$$

We now establish the possible values for $|Z|$ and n_0'.

Fact1: $0 \leqslant |Z| \leqslant 2$.
Assume to the contrary that $|Z| \geqslant 3$. Then, from (13), we have

$$e_{G-Z}(\pi) \leqslant (3-|Z|)n_0' + 2n_0 - 3 - n_Z(\pi) \leqslant 2n_0 - 3 - n_Z(\pi).$$

This together with (15) implies that $n_0 + \frac{1}{2}n_Z(\pi) + 3 \leqslant 0$. This is impossible since $n_0 \geqslant 0$ and $n_Z(\pi) \geqslant 0$. Therefore, we have $0 \leqslant |Z| \leqslant 2$.

Fact2: $n_0' \geqslant 2$.
If $n_0' \leqslant 1$, then by Fact 1 ($0 \leqslant |Z| \leqslant 2$) and (13), we have

$$e_{G-Z}(\pi) \leqslant (3-|Z|)n_0' + 2n_0 - 3 - n_Z(\pi) \leqslant 2n_0 - n_Z(\pi), \text{ since } 0 \leqslant |Z| \leqslant 2. \tag{16}$$

This together with (15) implies that $n_0 + \frac{1}{2}n_Z(\pi) \leqslant 0$. This means that all equalities hold in (15) and (16). Therefore, we have $n_0' = 1$, $n_0 = 0$, $n_Z(\pi) = 0$, and $|Z| = 0$. By (12), we have $e_{G-f-Z}(\pi) \leqslant -1$, which is impossible. Hence, we have $n_0' \geqslant 2$.

Case 2: $f \notin E_{G-Z}(\pi)$.

For $f \notin E_{G-Z}(\pi)$, we have

$$e_{G-Z}(\pi) = e_{G-f-Z}(\pi) \leqslant (3-|Z|)n_0' + 2n_0 - 4 - n_Z(\pi).$$

Then using an analogous argument as that in the proof of Lemma 3.1, we also have $0 \leqslant |Z| \leqslant 2$ and $n_0' \geqslant 2$. This completes the proof. □

Recall that, for any partition π of $V(G)$, $E_G(\pi)$ is the set of edges in G whose ends lie in different parts of π, and $e_G(\pi) = |E_G(\pi)|$.

Proof of Theorem 1.5: We prove it by contradiction. Assume to the contrary that G is not globally rigid. Then Lemma 2.10 implies that G is not redundantly rigid since G is 3-connected. It means that there exists an edge $f \in E(G)$ such that $G - f$ is not rigid. We now consider the following two cases:

Case 1: $f \in E_{G-Z}(\pi)$.

For $f \in E_{G-Z}(\pi)$, Lemma 3.2 implies that there exists a subset $Z \subset V(G)$ and a partition π of $V(G-f-Z)$ with n_0 trivial parts $\{v_1, v_2, \ldots, v_{n_0}\}$ and n_0' nontrivial parts $\{V_1, V_2, \ldots, V_{n_0'}\}$, where $0 \leqslant |Z| \leqslant 2$ and $n_0' \geqslant 2$.

Furthermore, since $\lambda_1^\alpha(G) \geqslant \lambda_1^\alpha(B^3_{n,\delta+1}) > \lambda_1^\alpha(K_{n-\delta-1}) = n - \delta - 2$, then by Lemma 2.6, we have

$$e(G) > \frac{(2n-3\delta-3)^2 - (\delta-1)^2 - 4\alpha\Delta + 4\alpha(\delta-1)\delta + 4(1-\alpha)\delta(n-1)}{8(1-\alpha)}. \tag{17}$$

Moreover, as G is 3-connected, we have

$$|\partial_{G-Z}(V_i)| \geqslant 3 - |Z| \text{ for } 1 \leqslant i \leqslant n_0', \tag{18}$$

We have the following two subcases according to the values of $|Z|$.

Subcase 1.1: $|Z| = 2$.

In this subcase, a similar argument as that used in the proof of Theorem 1.4 can be applied to obtain a contradiction, and we omit the details here.

Subcase 1.2: $0 \leqslant |Z| \leqslant 1$.

We further divide this subcase into the following two situations.

(A) $n_0' = 2$.
The partition π consists of two nontrivial parts, V_1 and V_2, together with n_0 trivial parts. Substituting (18) into (14), we obtain

$$2e_{G-Z}(\pi) \geqslant |\partial_{G-Z}(V_1)| + |\partial_{G-Z}(V_2)| + 6n_0 - n_Z(\pi) \geqslant 6 - 2|Z| + 6n_0 - n_Z(\pi).$$

Consequently,

$$e_{G-Z}(\pi) \geqslant 3 - |Z| + 3n_0 - \frac{1}{2}n_Z(\pi).$$

Since $n_0' = 2$, from (13), we have

$$-n_0 - \frac{1}{2}n_Z(\pi) - |Z| \geqslant 0.$$

As $n_0 \geqslant 0$, $n_Z(\pi) \geqslant 0$ and $|Z| \geqslant 0$, we conclude that $n_0 = 0$, $n_Z(\pi) = 0$ and $|Z| = 0$. As a consequence, the partition π comprises two nontrivial parts V_1 and V_2, $G-Z = G$ and $V(G) = V_1 \cup V_2$. By (13), $e_G(V_1, V_2) = e_G(\pi) \leqslant 3$. And from (18), we have $e_G(V_1, V_2) = \frac{1}{2}(|\partial_G(V_1)| + |\partial_G(V_2)|) \geqslant 3$. Therefore, $e_G(V_1, V_2) = 3$. Let $E_G(V_1, V_2) = \{f_1, f_2, f\}$. We claim that f_1, f_2, f are three independent edges. Otherwise, G cannot be 3-connected, leading to a contradiction. Thus, G is a spanning subgraph of $B^3_{n,|V_1|}$, and

$$\lambda_1^{\alpha}(G) \leqslant \lambda_1^{\alpha}\left(B^3_{n,|V_1|}\right), \tag{19}$$

with equality if and only if $G \cong B^3_{n,|V_1|}$. Since $\delta \geqslant 6$ and $|\partial_G(V_1)| = |\partial_G(V_2)| = 3 < \delta - 1$, by Lemma 2.8, we have $\min\{|V_1|, |V_2|\} \geqslant \delta + 1$. Combining this with Lemma 2.3, Corollary 2.5 and (19), we have

$$\lambda_1^{\alpha}(G) \leqslant \lambda_1^{\alpha}\left(B^3_{n,\delta+1}\right),$$

with equality if and only if $G \cong B^3_{n,\delta+1}$. This contradicts our initial assumption that $\lambda_1^{\alpha}(G) \geqslant \lambda_1^{\alpha}\left(B^3_{n,\delta+1}\right)$ and $G \ncong B^3_{n,\delta+1}$.

(B) $n_0' \geqslant 3$.

By using (17) and employing a similar approach as in the proof of Theorem 1.4, we can derive a contradiction under this scenario. We omit the details for brevity.

Case 2: $f \notin E_{G-Z}(\pi)$.

For $f \notin E_{G-Z}(\pi)$, utilizing similar arguments as those presented above, we can obtain a contradiction. This completes the proof. □

4. Concluding remarks

In this paper, we establish a criterion based on the A_α-spectral radius for determining the rigidity (or global rigidity) of 2-connected (or 3-connected) graphs with a prescribed minimum degree in $\mathbb{R}^2$. Specifically, we resolve the A_α-spectral radius characterization for Problem 1.1 for the cases $k = 2$ and $k = 3$. Note that every 6-connected graph is inherently rigid (or globally rigid). Consequently, the complexity of the A_α-spectral radius characterization for Problem 1.1 escalates for $k = 4$ and $k = 5$. For these cases, employing a similar analytical approach as in Theorems 1.4 and 1.5, we ascertain that a k-connected graph G is rigid (or globally rigid) if $\lambda_1^{\alpha}(G) > \lambda_1^{\alpha}\left(B^k_{n,\delta+1}\right)$. Since $B^k_{n,\delta+1}$ is rigid and globally rigid for $k = 4$ and 5, we conclude the paper by posing the subsequent problem for further exploration.

Problem 4.1. *Let $k \in \{4, 5\}$ and G be a k-connected graph with the maximum degree Δ and the minimum degree $\delta \geqslant 6$. For $\alpha \in (0, 1)$,*

$$\Delta < \min\left\{n^2 - 24n + 170 + \frac{3n - 36}{\alpha},\ n^2 - 21n + 116 + \frac{13}{\alpha},\ n^2 - 21n + 130 + \frac{4}{\alpha}\right\}$$

and

$$n \geqslant \max\left\{2\delta + 4, \left\lceil \frac{-g + \sqrt{g^2 - 4(1-\alpha)h}}{2(1-\alpha)} \right\rceil + 1\right\},$$

where

$$g = (\alpha^2 + \alpha - 2)\delta + 2\alpha(\alpha - 1) \text{ and } h = (1 - \alpha^2)\delta^2 + 2\alpha(1 - \alpha)\delta - 2k\alpha^3 + (2k - 1)\alpha^2 + 2\alpha - 1,$$

is it true that G is rigid (or globally rigid) when $\lambda_1^{\alpha}(G) \geqslant \lambda_1^{\alpha}\left(B^k_{n,\delta+1}\right)$?

Acknowledgements The authors would like to thank the referees for their constructive corrections and valuable comments, which have considerably improved the presentation of this paper. This work was supported by the Fujian Key Laboratory of Granular Computing and Applications.

Data Availibility Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

References

[1] L. Asimow, B. Roth. The rigidity of graphs. *Trans. Amer. Math. Soc.*, **245**(1978), 279–289.
[2] A.E. Brouwer, W.H. Haemers. *Spectra of Graphs*. Springer, Berlin, 2011.
[3] R. Connelly. Generic global rigidity. *Discrete Comput. Geom.*, **33**(2005), 549–563.
[4] D. Fan, X. Huang, H. Lin. Spectral radius conditions for the rigidity of graphs. *Electron. J. Combin.*, **30**(2023), P2.14.
[5] C. Godsil, G. Royle. Algebraic Graph Theory. Graduate Texts in Mathematices, 207, Springer-Verlag, New York, 2001.
[6] X. Gu. Spanning rigid subgraph packing and sparse subgraph covering. *SIAM J. Discrete Math.*, **32**(2018), 1305–1313.
[7] X. Gu, H.-J. Lai, P. Li, S. Yao. Edge-disjoint spanning trees, edge connectivity, and eigenvalues in graphs. *J. Graph Theory*, **81**(2016), 16–29.
[8] B. Hendrickson. Conditions for unique graph realizations. *SIAM J. Comput.*, **21**(1992), 65–84.
[9] B. Jackson, B. Servatius, H. Servatius. The 2-dimensional rigidity of certain families of graphs. *J. Graph Theory*, **54**(2007), 154–166.
[10] B. Jackson, T. Jordán. A sufficient connectivity condition for generic rigidity in the plane. *Discrete Appl. Math.*, **157**(2009), 1965–1968.
[11] H. Lai, Y. Shao, H. Wu, J. Zhou. Every 3-connected, essentially 11-connected line graph is Hamiltonian. *J. Combin. Theory Ser. B*, **96**(2006), 571–576.
[12] G. Laman. On graphs and rigidity of plane skeletal structures. *J. Engrg. Math.*, **4**(1970), 331–340.
[13] V. Nikiforov. Merging the A- and Q- spectral theories. *Appl. Anal. Discr. Math.*, **11**(2017), 81–107.
[14] J. da Silva, C. Oliveira, L. Cda Costa. Bounds for A_α-eigenvalues. *RAIRO-Oper. Res.*, **57**(2023), 2783–2798.