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On the dominated coloring in graphs

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Abstract. A dominated coloring of a graph G is a proper vertex coloring where each color class is dominated by at least one vertex of G. The dominated chromatic number of G, denoted $\chi_{dom}(G)$, is the minimum number of colors required for a dominated coloring of G. In this paper, we provide new bounds for $\chi_{dom}(G)$ and characterize all graphs that achieve some of these bounds. Also we investigate graphs G for which $\chi_{dom}(G) = \chi(G)$ where $\chi(G)$ is the chromatic number of G, in particular we give a characterization of cubic graphs G such that $\chi_{dom}(G) = \chi(G)$.

1. Introduction

Throughout this paper, all graphs are assumed to be finite, undirected and without loops or multiple edges. For terminology and notation not presented here, we follow [2]. Consider a graph G with vertex set V and edge set E. The *complement graph* of G is denoted by $\overline{G}$. For a nonempty set $A \subseteq V$, we denote by $G[A]$ the subgraph of G induced by A. Let v be any vertex in G. The *open neighborhood* of v is defined as the set $N_G(v) = \{u : uv \in E\}$ and the *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of v in G, denoted $d_G(v)$, is the cardinality of $N_G(v)$. The *maximum degree* among all vertices in G is denoted by $\Delta(G)$. The *distance* between two vertices u and v in G, denoted by $d(u, v)$, is the length of a shortest path between u and v in G. The *diameter* of G, denoted $\mathrm{diam}(G)$, is $\max\{d(u, v) : u, v \in V(G)\}$. As usual, the *path*, *cycle*, *complete graph* of order n is denoted by P_n, C_n, K_n, respectively. The *complete bipartite graph* with parts of orders r and s is denoted by $K_{r,s}$. The *star* is the complete bipartite graph $K_{1,k}$. A *bistar* $B_{p,q}$ is a graph formed by two stars $K_{1,p}$ and $K_{1,q}$ by adding an edge between their center vertices. Given any graph F, a graph G is F-free if it does not have any induced subgraph isomorphic to F. A *tree* is any connected graph that contains no cycle.

A set $S \subseteq V$ is called an *independent set* in G if no two vertices of S are adjacent to each other. A *clique* in G is a set of pairwise adjacent vertices. The *independence number* $\alpha(G)$ (respectively, the *clique number* $\omega(G)$) is the largest cardinality among all independent (respectively, clique) sets of G. A *split graph* is a graph in which its vertices can be partitioned into a clique and an independent set.

A *proper vertex coloring* of a graph $G = (V, E)$ is a mapping $c : V \to \{1, 2, \ldots\}$ such that if $uv \in E$, then $c(u) \neq c(v)$. The *chromatic number* $\chi(G)$ of a graph G is the smallest integer k such that G admits a vertex

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proper coloring using k colors. It is an easy and well known observation that for every graph G with maximum degree $\Delta(G)$,

$$\chi(G) \leq \Delta(G) + 1. \tag{1}$$

A *dominated coloring* of a graph G is a proper vertex coloring of G such that every color class is dominated with at least one vertex. The *dominated chromatic number* of G, denoted $\chi_{dom}(G)$, is the minimum number of colors required for a dominated coloring of G. This parameter was introduced in 2015 by Boumediene Merouane et al. [10]. They showed that dominated coloring problem is $\mathcal{NP}$-complete for arbitrary graphs having $\chi_{dom}(G) \geq 4$, and they gave a polynomial time algorithm for recognizing graphs having $\chi_{dom}(G) \leq 3$. They also provided bounds for planar and star-free graphs and exact values for split graphs. In [6], Choopani et al. proved that the Vizing-type conjecture holds for dominated colorings of the direct product of two graphs and they gave Nordhaus-Gaddum type results for $\chi_{dom}(G)$. In [1], the authors investigated the impact on $\chi_{dom}(G)$ when G is modified by operations on vertex and edge of G. For more works, see for instance [5, 7–9, 11]. It is worth mentioning that this kind of coloring is defined only for graphs without isolated vertices.

The remainder of this paper is organized as follows: In Section 2, we provide some known results about $\chi(G)$ and $\chi_{dom}(G)$. In Section 3, we present new bounds for χ_{dom} and characterize all graphs that achieve some of these bounds. In Section 4, we investigate graphs G for which $\chi_{dom}(G) = \chi(G)$, in particular we give a characterization of cubic graphs G such that $\chi_{dom}(G) = \chi(G)$. We conclude the paper with a list of open problems.

2. Known Results

In this section, we recall some important results that will be useful in our investigations.

Observation 2.1 ([6]). *Let G be a graph of order n. Then $\chi_{dom}(G) \leq n$. Furthermore, equality is achieved if and only if each component of G is a complete graph.*

Observation 2.2 ([10]). *Let G be a graph of order n. Then $\chi_{dom}(G) \geq 2$. Moreover, $\chi_{dom}(G) = 2$ if and only if G is a bistar to which we can add some edges not inducing any triangle.*

Observation 2.3 ([10]). *If G is disconnected, then $\chi_{dom}(G) = \sum_i \chi_{dom}(G_i)$, where G_i is the i-th connected component of G.*

Observation 2.4 ([3]). *If G is disconnected, then $\chi(G) = \max_i\{\chi(G_i)\}$, where G_i is the i-th connected component of G.*

Theorem 2.5 ([6]). *For $n \geq 4$, we have,*

$$\chi_{dom}(P_n) = \chi_{dom}(C_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 (\text{mod } 4) \\ \left\lfloor \frac{n}{2} \right\rfloor + 1 & \text{otherwise} \end{cases}.$$

Theorem 2.6 ([10]). *Let G be a split graph. Then*

$$\chi_{dom}(G) = \chi(G) = \omega(G).$$

Theorem 2.7 ([4]). *Equality holds in (1) if and only if either $\Delta(G) = 2$ and G contains an odd cycle, or $\Delta(G) \neq 2$ and G contains a clique $K_{\Delta(G)+1}$.*

3. Bounds

In this section, we provide new upper and lower bounds on the dominated chromatic number. In [3], it was proved that the chromatic number of any graph G of order $n \geq 2$ is at most $n - \alpha(G) + 1$, where $\alpha(G)$ is the independence number of G. However, the same is not necessarily true for the dominated chromatic number of G, as we will establish in the following proposition.

Proposition 3.1. *For any positive integer k, there exists a graph G_k of order n_k such that*

$$\chi_{dom}(G_k) = n_k - \alpha(G_k) + k.$$

Proof. Let H_i be a path of order 5 with vertices x_i, y_i, z_i, u_i, v_i in this order. Let G_k be a tree obtained from $H_1, H_2, \ldots, H_{2k}$ by adding $2k - 1$ edges connecting z_i's so that they induce a path $P_{2k} : z_1\text{-}z_2\text{-}\cdots\text{-}z_{2k}$. Observe that $n_k = |V(G_k)| = 10k$. For example, the graph G_2 is illustrated in Figure 1. First, we will show that $\chi_{dom}(G_k) = 6k$. Notice that in every dominated coloring of G_k, no color can appear twice in $X = \cup_{i=1}^{2k}\{v_i, y_i, x_i\}$, since the distance between any two vertices in X is different from two. Thus $\chi_{dom}(G_k) \geq 6k$. To show equality, it suffices to exhibit a dominated coloring of G_k with $6k$ colors. We do this as follows. For each H_i, assign color $3i - 2$ to v_i and z_i, assign color $3i - 1$ to u_i and y_i, and assign color $3i$ to x_i. It is easy to check that this yields a dominated coloring of G with $6k$ colors, implying that $\chi_{dom}(G_k) \leq 6k$. Hence $\chi_{dom}(G_k) = 6k$. Let us now show that $\alpha(G_k) = 5k$. To this end, let S_0 be a maximum independent set of the path induced by $\{z_1, z_2, \ldots, z_{2k}\}$. It is well known that $|S_0| = \left\lceil \frac{2k}{2} \right\rceil = k$. Then any maximum independent set of G_k can contain at most k vertices among $z_1, z_2, \ldots, z_{2k}$ and 2 vertices from each set $\{x_i, y_i, u_i, v_i\}$, and thus $\alpha(G_k) \leq 5k$. On the other hand, since $S_0 \cup \left(\cup_{i=1}^{2k}\{x_i, v_i\}\right)$ is an independent set of size $5k$, $\alpha(G_k) \geq 5k$. Hence $\alpha(G_k) = 5k$. Finally, since $n_k = 10k$, we get $n_k - \alpha(G_k) + k = 6k$, and the required is done. □

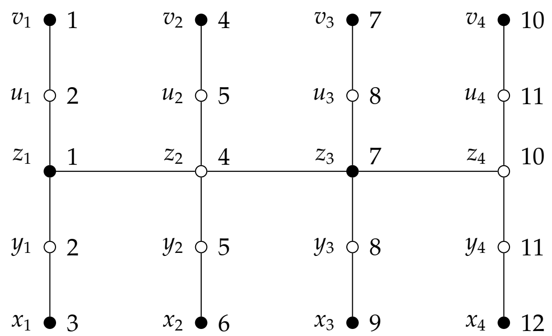

Figure 1: The graph G_2 of order $n_2 = 20$ with $\alpha(G_2) = 10$ and $\chi_{dom}(G_2) = 12$ satisfing $\chi_{dom}(G_2) = n_2 - \alpha(G_2) + 2$. The vertices in bold represent an $\alpha(G_2)$-set and the values assigned to the vertices represent a dominated coloring.

For every vertex v in a graph G, let $S(v)$ be a largest independent set of $G[N_G(v)]$ and $\alpha_0(G) = \max\{|S(v)| : v \in V(G)\}$. Clearly $\alpha_0(G) \leq \alpha(G)$ and $1 \leq \alpha_0(G) \leq \Delta(G)$, and $\alpha_0(G) = 1$ holds if and only if G is the union of complete graphs.

We next prove that $n - \alpha_0 + 1$ serves as an upper bound for $\chi_{dom}(G)$, and we characterize all graphs achieving equality for this bound.

We define families $\mathcal{H}$ and $\mathcal{F}$ as follows: The graphs $H \in \mathcal{H}$ are split graphs, with vertex set is partitioned into a clique Q and an independent set I such that H has a universal vertex and I has a vertex that is adjacent to all vertices of Q. A graph G belongs to $\mathcal{F}$ if G has exactly one component in $\mathcal{H}$ and all other components, if any, are complete graphs.

Remark that both K_n and $K_{1,n-1}$ are in $\mathcal{H}$, and further $\alpha_0(G) = |I|$ and $\omega(G) = n - \alpha_0 + 1$.

Theorem 3.2. *Let G be a graph of order n. Then*

$$\chi_{dom}(G) \leq n - \alpha_0 + 1,$$

with equality if and only if $G \in \mathcal{F}$.

Proof. Let $G_1, G_2, \ldots, G_p$ be the components of G, and let n_i be the order of G_i for $i \in \{1, 2, \ldots, p\}$. To prove the inequality, let x be a vertex of some component of G, say G_1, such that $|S(x)| = \alpha_0$. Define a coloring c of G as follows. Color every vertex in $S(x)$ with color 1, and color all the remaining vertices differently. Clearly c is a dominated coloring of G using $n - \alpha_0 + 1$ colors. Hence $\chi_{dom}(G) \leq n - \alpha_0 + 1$.

We now suppose that the equality in inequality holds. We assert that x is a universal vertex in G_1, for otherwise, there is at least one vertex in G_1 that is at distance exactly two from x. In such case, by coloring this vertex with the same color as x and maintaining the same coloring of c for the remaining vertices, we get a dominated coloring of G with $n - \alpha_0$ colors, a contradiction. Now we assert that $V(G_1) - S(x)$ is a clique. Suppose otherwise and let u and v be nonadjacent vertices in $V(G_1) - S(x)$. Note that $x \notin \{u, v\}$. Define a new coloring from c as follows. Color u and v by the same color among $c(u)$ and $c(v)$ and the remaining vertices keep their colors initially given by c. This gives a dominated coloring of G using less colors than c, a contradiction. Assert that $S(x)$ has at least one vertex that is adjacent to all vertices in $V(G_1) - S(x)$. Suppose otherwise and define a coloring from c as follows. Recolor each vertex in $S(x)$ with a color used by one of its non-neighbor in $V(G_1) - S(x)$. The remaining vertices keep their colors initially given by c. This gives a dominated coloring of G with $n - \alpha_0$ colors, a contradiction. Hence $G_1 \in \mathcal{H}$ with $I = S(x)$ and $Q = V(G_1) - S(x)$. We finally show that G_i is a complete graph for all $i \geq 2$. Suppose to the contrary that this is not true for some component, say G_2. Then by Observation 2.1, we have that $\chi_{dom}(G_2) \leq n_2 - 1$. Thus, using Observation 2.3, we get

$$\begin{aligned}\chi_{dom}(G) &= \chi_{dom}(G_1) + \chi_{dom}(G_2) + \textstyle\sum_{i=3}^{p} \chi_{dom}(G_i) \\ &\leq (n_1 - \alpha_0 + 1) + (n_2 - 1) + \textstyle\sum_{i=3}^{p} n_i \\ &= n - \alpha_0,\end{aligned}$$

a contradiction. From our previous discussions, we conclude that $G \in \mathcal{F}$.

Conversely, let G be a graph in $\mathcal{F}$, and let $G_1, G_2, \ldots, G_p$ be the components of G, where $G_1 \in \mathcal{H}$. Since G_1 is a split graph, by Theorem 2.6 and the remark before Theorem 3.2, we have $\chi_{dom}(G_1) = n_1 - \alpha_0 + 1$, and by Observation 2.1, we get $\chi_{dom}(G_i) = n_i$ for $i \geq 2$. Hence, Observation 2.3 leads to

$$\chi_{dom}(G) = \chi_{dom}(G_1) + \textstyle\sum_{i=2}^{p} \chi_{dom}(G_i) = (n_1 - \alpha_0 + 1) + \sum_{i=2}^{p} n_i = n - \alpha_0 + 1,$$

and this completes the proof. □

For the particular case $\alpha_0(G) = \Delta(G)$, we have the following immediate corollary from Theorem 3.2.

Corollary 3.3. *Let G be a graph of order n and maximum degree* $\Delta(G)$ *with* $\alpha_0(G) = \Delta(G)$. *Then* $\chi_{dom}(G) \leq n - \Delta(G) + 1$ *with equality if and only if each component of G is a complete graph except exactly one which is a star.*

We next give a lower bound for $\chi_{dom}(G)$ in terms of n and α_0.

Theorem 3.4. *If G is a graph of order n, then*

$$\chi_{dom}(G) \geq \frac{n}{\alpha_0}.$$

Proof. Let $k = \chi_{dom}(G)$. Let c be a dominated coloring of G with k colors, and let $C_1, C_2, \ldots, C_k$ be the color classes of c. Since $\alpha_0 \geq |C_i|$ for each $i \in \{1, 2, \ldots, k\}$, it follows that $n = |C_1| + |C_2| + \cdots + |C_k| \geq k\alpha_0$. □

This lower bound is sharp for complete graphs, and for complete bipartite graph $K_{p,p}$.

It is worth emphasizing that the above bound coincide with the lower bound $\chi_{dom}(G) \geq \frac{n}{k-1}$ ($k \geq 2$) for $K_{1,k}$-free graphs given by Boumediene-Merouane et al. [10], where $k = \alpha_0 + 1$, since every graph is K_{1,α_0+1}-free.

As $\alpha_0 \leq \Delta(G)$, the next corollary given in [10] follows immediately from Theorem 3.4.

Corollary 3.5 ([10]). *If G is a graph of order n and maximum degree $\Delta(G)$, then*

$$\chi_{dom}(G) \geq \frac{n}{\Delta(G)}.$$

The same authors of [10] characterized K_3-free graphs which attain the above bound.

Next, we give bounds, in terms of order and diameter of G for the dominated chromatic number of a connected graph.

Proposition 3.6. *Let G be a connected graph of order n and with diameter* diam(G)*. Then*

$$\frac{1}{2}(\text{diam}(G) + 1) \leq \chi_{dom}(G) \leq n - \frac{1}{2}(\text{diam}(G) - 1).$$

Moreover, lower (respectively, upper) bound is sharp for path P_{4k} (respectively, path P_{4k+2}), where k is a positive integer.

Proof. Let P be a diametral path in G of order $t = \text{diam}(G) + 1$. From Theorem 2.5, we can write

$$\frac{t}{2} \leq \chi_{dom}(P) \leq \frac{t}{2} + 1. \tag{2}$$

Therefore, to prove the statement it suffices to show the following

$$\chi_{dom}(P) \leq \chi_{dom}(G) \leq n + \chi_{dom}(P) - t. \tag{3}$$

Indeed, the right inequality in (3) follows since any dominated coloring of P with $\chi_{dom}(P)$ colors can be extended to a dominated coloring of G with $n + \chi_{dom}(P) - t$ colors by coloring the remaining vertices of G differently using $n - t$ colors. Now, let us prove the left inequality. To this end, let $P = v_1\text{-}v_2\text{-}\cdots\text{-}v_t$ and consider a dominated coloring c of G with $\chi_{dom}(G)$ colors. We assert that

$$\text{if } c(v_s) = c(v_r) \text{ for some } s \text{ and } r\ (1 \leq s < r \leq t), \text{ then } r = s + 2. \tag{4}$$

Suppose that $r \geq s + 3$ and let C be the class of color $c(v_s)$. This color class is dominated by some vertex $u \in V(G) - V(P)$ since v_s and v_r are at distance at least three. But in this case, $v_1\text{-}v_2\text{-}\cdots\text{-}v_s\text{-}u\text{-}v_r\text{-}\cdots\text{-}v_t$ would be a shorter path than P from v_1 to v_t, a contradiction. Thus (4) holds. From this, we conclude that each color of c is repeated at most twice in P. Taking this in conjunction with (4) we see that the restriction of c to P is a dominated coloring of P. Thus the left inequality in (3) holds, and so the required is obtained by combining (2) and (3). □

According to Proposition 3.6, one can easily see that if $\chi_{dom}(G) \leq 3$, then diam$(G) \leq 5$. The next result will improve this upper bound for the case $\chi_{dom}(G) = 3$.

Proposition 3.7. *Let G be a graph with $\chi_{dom}(G) \leq 3$. Then G is connected and further* diam$(G) \leq 4$*.*

Proof. Suppose that G is not connected. Then by Observation 2.3 and the fact that $\chi_{dom}(G) \leq 3$, there is a component of G with one vertex, which is impossible since G is without isolated vertices. Now, let us prove the second part. If $\chi_{dom}(G) = 2$, then Proposition 3.6 shows that $\text{diam}(G) \leq 3$. Assume now that $\chi_{dom}(G) = 3$ and let $\pi = \{X_1, X_2, X_3\}$ be a dominated coloring of G, and for each i in $\{1, 2, 3\}$, let a_i be a vertex that dominates X_i. Let x_1 and x_2 be vertices of G such that, $d(x_1, x_2) = \text{diam}(G)$ and assume without loss of generality that $x_1 \in X_1$. If $a_1 = a_2$ or $x_2 \in X_1$, then $d(x_1, x_2) \leq 2$. So, assume next that $a_1 \neq a_2$ and without loss of generality $x_2 \in X_2$. We assert that $d(a_1, a_2) \leq 2$. Suppose not. Then clearly a_1 and a_2 both must be in X_3. In this case, a_3 is adjacent to a_1 and a_2, and it is either in X_1 or in X_2, say X_1. If $a_3 = x_1$, then x_1, a_2, x_2 form a path of length 2, meaning that $d(x_1, x_2) \leq 2$. Otherwise, if $a_3 \neq x_1$, then x_1, a_1, a_3, a_2, x_2 form a path of length 4, giving that $d(x_1, x_2) \leq 4$. □

4. Graphs G with $\chi_{dom}(G) = \chi(G)$

We begin this section by improving the bound $\chi_{dom}(G) \geq \chi(G)$ for disconnected graphs.

Proposition 4.1. *Let G be a graph of order n with p components. Then $\chi_{dom}(G) \geq \chi(G) + 2(p-1)$.*

Proof. Let $G_1, G_2, \ldots, G_p$ be the components of G and without loss of generality, assume that $\chi(G_1) = \max\{\chi(G_i) : 1 \leq i \leq p\}$. Since $\chi_{dom}(G_1) \geq \chi(G_1)$ and $\chi(G) = \max\{\chi(G_i) : 1 \leq i \leq p\}$, we get $\chi_{dom}(G_1) \geq \chi(G)$. From this and Observation 2.3, we can write $\chi_{dom}(G) \geq \chi(G) + \sum_{i=2}^{p} \chi_{dom}(G_i)$. Since $\chi_{dom}(G_i) \geq 2$ for all $i \geq 2$, we get $\chi_{dom}(G) \geq \chi(G) + 2(p-1)$. □

Corollary 4.2. *If G is a graph of order n with $\chi(G) \leq \chi_{dom}(G) \leq \chi(G) + 1$, then G is connected.*

Observation 4.3. *Let G be a graph with maximum degree $\Delta(G) \leq 2$. Then $\chi_{dom}(G) = \chi(G)$ if and only if $G \in \{P_2, P_3, P_4, C_3, C_4, C_5\}$.*

Proof. Let G be a graph with $\Delta(G) \leq 2$ such that $\chi_{dom}(G) = \chi(G)$. Therefore since G is connected (by Corollary 4.2), it follows that G is either a path or a cycle. Since $\chi(G) \leq \Delta(G) + 1$, it follows that $\chi_{dom}(G) \leq 3$, implying that $\text{diam}(G) \leq 4$ according to Proposition 3.7. Taking this fact into consideration, by inspection we see that $G \in \{P_2, P_3, P_4, C_3, C_4, C_5\}$. □

The next observation follows by combining Brooks's Theorem together with Corollary 4.2 and Observation 4.3.

Observation 4.4. *Let G be a graph of order $n \geq 2$ with maximum degree $\Delta(G)$. Then $\chi_{dom}(G) = \chi(G) = \Delta(G) + 1$ if and only if $G \cong K_n$ or C_5.*

Next, we give a necessary conditions for which $\chi_{dom}(G) = \chi(G)$.

Theorem 4.5. *Let G be a graph of order n, maximum degree Δ such that $\chi_{dom}(G) = \chi(G)$. Then*

(i) *G is connected and further* $\text{diam}(G) \leq 5$.

(ii) *If $n \geq 3$ and $G \neq C_5$, then $n \leq \Delta^2(G)$.*

Proof. Set $k = \chi_{dom}(G) = \chi(G)$. Consider a dominated coloring c of G with k colors, and let $\pi = \{X_1, X_2, \ldots, X_k\}$ be the set of color classes of c.

(i) The connectedness of G follows from Corollary 4.2. To show that $\text{diam}(G) \leq 5$, let x_1 and x_2 be vertices of G such that $d(x_1, x_2) = \text{diam}(G)$ and assume without loss of generality that $x_1 \in X_1$, and let a_1 and a_2 be vertices that dominate X_1 and X_2, respectively. If $x_2 \in X_1$, then $\{x_1, a_1, x_2\}$ forms a path P_3. Now assume, without loss of generality, that $x_2 \in X_2$. If $a_1 = a_2$ or $a_1a_2 \in E$, then $d(x_1, x_2) \leq 3$. So, assume next that $a_1 \neq a_2$ and $a_1a_2 \notin E$. Then clearly $a_1 \notin X_2$ and $a_2 \notin X_1$. Assume first that a_1 and a_2 are in the same color class, say X_3.

Since X_3 is dominated by some vertex in G, $d(a_1, a_2) = 2$. Now, assume that a_1 and a_2 are in different classes. As c is a χ-coloring of G, we know that there is an edge between X_1 and X_2, say b_1b_2 such that $b_1 \in X_1$ and $b_2 \in X_2$. In this case a_1, b_1, b_2, a_2 form a path of length 3, and thus $d(a_1, a_2) \leq 3$. Hence in all cases we have that $d(x_1, x_2) \leq 5$. (ii) Remember that by (1), we have $k \leq \Delta(G) + 1$. If $k = \Delta(G) + 1$, then by Observation 4.4, we have $G \cong K_n$ or C_5. Since $n \geq 3$ and $G \neq C_5$, $n \leq (n-1)^2 = \Delta^2(G)$. If $k \leq \Delta(G)$, then by Corollary 3.5, we get $n \leq \Delta(G) k \leq \Delta^2(G)$. □

We close this section by characterizing cubic graphs G satisfying $\chi_{dom}(G) = \chi(G)$.

Theorem 4.6. *If G is a cubic graph, then $\chi_{dom}(G) = \chi(G)$ if and only if G is an element of the family of graphs described in Figure 2.*

Proof. Let G be a cubic graph with $k = \chi_{dom}(G) = \chi(G) \geq 2$. By Theorem 4.5-(i), G is connected and by (1), $k \in \{2, 3, 4\}$. If $k = 4$, then by Theorem 2.7 and the fact that G is cubic, we conclude that $G = K_4$. So from now on, assume that $k \in \{2, 3\}$. By Theorem 4.5-(ii), we get $n \leq 9$. We know that since G is cubic, n must be even implying that $n \in \{6, 8\}$. For $n = 6$, we see that $G \cong K_{3,3}$ or $\overline{C}_6$. We now look at cubic graphs of order 8. By Corollary 3.5, $k \geq \frac{n}{\Delta} = \frac{8}{3}$, implying that $k = 3$. Let c be a dominated coloring of G using 3 colors and let X, Y, Z be the color classes of c. Since $|X|, |Y|, |Z| \leq 3$ and $n = 8$, it follows that one among X, Y, Z has size 2, while the other two each have size 3. So, without loss of generality, we can let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$ and $Z = \{z_1, z_2\}$. We assert that

$$\text{for each } i \in \{1, 2\},\ z_i \text{ does not dominate neither } X \text{ nor } Y \tag{5}$$

To the contrary and by symmetry, suppose that z_1 dominates X. In such a case, no vertex in X can dominate Y as otherwise the vertex dominating Y would have degree equal to 4, which is impossible. Therefore, Y must be dominated by z_2. But then, no vertex from $X \cup Y$ can dominate Z (since, at this step, z_1 and z_2 each has degree 3), a contradiction. Thus (5) holds.

By (5), we can assume, without loss of generality, that x_1 dominates Y and y_1 dominates X. Since, at this step, x_1 and y_1 have degree 3, Z must be dominated by one among x_2, x_3, y_2, y_3, say y_2 (by symmetry). Now y_3 must have exactly two neighbors in $\{z_1, z_2, x_2, x_3\}$ to have degree equal to 3. However, y_3 cannot be adjacent to both z_1 and z_2 (respectively, both x_2 and x_3), for otherwise, one of x_2 or x_3 (respectively, z_1 or z_2) will not have a degree of 3, a contradiction. Thus, by symmetry, we can let $y_3z_2, y_3x_3 \in E(G)$ and $y_3z_1, y_3x_2 \notin E(G)$. To reach the degree 3, z_1 must be adjacent to x_2 and x_3 and likewise z_2 must be adjacent to x_2. Thus G is isomorphic to H (see Figure 2).

The converse is easy to check. An example of dominated coloring of each graph G belonging to $\{K_4, K_{3,3}, \overline{C}_6, H\}$ with $\chi_{dom}(G) = \chi(G)$ is given in Figure 2. □

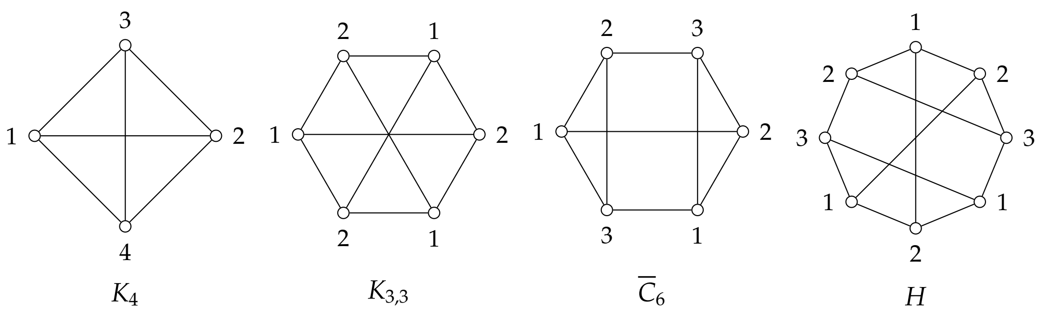

Figure 2: Family of all cubic graphs satisfying $\chi_{dom}(G) = \chi(G)$. For each graph, the values assigned to the vertices represent a dominated coloring.

5. Open questions

We conclude our paper with few open questions that might be interesting to study.

Question 5.1. *For what connected graphs of order $n \geq 4$ does $\chi_{dom}(G) = n - 2$?*

Question 5.2. *Characterize r-regular graphs ($r \geq 4$) for which $\chi_{dom}(G) = \chi(G)$.*

Question 5.3. *For what connected graphs does $\chi_{dom}(G) = \chi(G) + 1$?*

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