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Rough approximation of subgroup and conjugacy relation

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Abstract. Let G be a group and H be a rough subgroup of G with respect to the conjugacy relation, which is considered as an equivalence relation. An internal edge of H is defined as the difference between H and its lower approximation. Let E be a non-empty subset of G. In this paper, we aim to answer the following questions: Can E represent an internal edge of some subgroup of G (in other words, what are the conditions that E must satisfy in order to be an internal edge of some subgroup of G)? If the answer to this question is yes, what is this subgroup, and is it unique or not?

1. Introduction

Since the renowned mathematician scientist Zdzislaw Pawlak introduced the definition of rough sets and approximation space [11, 12], rough set theory has garnered significant interest of many researchers. This is due to its numerous applications in various fields, including data mining, machine learning, pattern recognition, decision support systems and others.

The equivalence relation, which play key role in rough set is replaced by arbitrary relation to handle more uncertainty (cf., Liu and Zhu [9], Yao [14]), shows that upper and lower approximations of a set are noting but closure and interior of the set and proposed several models of rough sets. Many interesting and constructive extensions to binary relations and the subsets have been proposed. Several researchers studied rough sets from an algebraic perspective and including rough semigroups, rough groups, rough rings, rough modules, and rough vector spaces (cf., Biswas and Nanda [1], Bonikowski [2], Wang and Chen [3], Iwinski [6], Kuroki and Wang [7], Miao et.al[10]). There are mainly two approaches for the development of rough set theory, the constructive and axiomatic approaches. By taking advantage of these two approaches, rough set theory has been combined with other mathematical theories such as Boolean algebra [4, 5], semigroup [8]. Among these research aspects, many papers has been focused on the connection between rough sets and algebraic systems. Biswas and Nanda [1] defined the notion of rough subgroups. Kuroki [8] introduced the notion of a rough ideal in a semigroup, studied approximations of a subset in a semigroup and discussed some structures of a rough ideal. Conjugacy is a very significant equivalence relation in the theory of groups and it has several important applications as well. In [13], the authors

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considering the equivalence relation conjugacy on a group and obtained their properties.

Numerous studies have investigated the properties of rough subgroups with respect to a conjugacy relation on a group by analyzing their upper and lower approximation, while often neglecting a thorough examination of their internal and external edges. Most studies on rough subgroups have focused on deriving the structure of the upper and lower approximations of a rough subgroup with known properties. However, to the best of my knowledge, no studies have attempted to determine the structure and properties of a rough subgroup based on the form of its upper or lower approximations, internal or external edges or boundary region. This gap represents the primary motivation for presenting this research and subsequent studies. In this paper, we aim to answer the following questions: What conditions must a subset E of a group G satisfy to be an internal edge of some rough subgroup of a group G? What is the structure of that rough subgroup? Furthermore, is it possible for E to be an internal edge of more than one rough subgroup?

2. Preliminaries

Definition 2.1. *Let $R : U \longrightarrow U$ be an equivalence relation defined on a set U, and let the equivalence class of an element $x \in U$ be denoted by $[x]$. If $X \subseteq U$, we define the following:*

1. *The lower approximation of X with respect to the equivalence relation R, denoted by $\underline{Apr}(X)$, is the set*

$$\underline{Apr}(X) = \{x \in U : [x] \subseteq X\}$$

2. *The upper approximation of X with respect to the equivalence relation R, denoted by $\overline{Apr}(X)$, is the set*

$$\overline{Apr}(X) = \{x \in U : [x] \cap X \neq \phi\}$$

3. *The boundary region of X is $B(X) = \overline{Apr}(X) - \underline{Apr}(X)$.*
4. *The internal edge of X is $\underline{Edge}(X) = X - \underline{Apr}(X)$*
5. *The external edge of X is $\overline{Edge}(X) = \overline{Apr}(X) - X$*

Definition 2.2. *Let $R : U \longrightarrow U$ be an equivalence relation defined on a set U. A subset X of U is said to be a rough set with respect to R if the boundary region of X is a non-empty set, in symbols $B(X) \neq \phi$. If U is a group and X is a subgroup of U, then a rough set X is called a rough subgroup.*

Definition 2.3. *Let G be a group and $R : G \longrightarrow G$ be a relation such that*

$$(x, y) \in R \Longleftrightarrow y = g^{-1}xg \text{ for some } g \in G.$$

A relation R is called a conjugacy relation. It is well known that, R represents an equivalence relation with the equivalence classes given by:

$$[x] = \{g^{-1}xg : g \in G\}$$

Note that: From now on, we fix the following notations:

G	A group.
H	A rough subgroup of a group G.
$\underline{Apr}(H)$	the lower approximation of H with respect to the conjugacy relation.
$\underline{Edge}(H)$	the internal edge of H with respect to the conjugacy relation.
$[\underline{Edge}(H)]^2$	the set $\{xy : x, y \in \underline{Edge}(H)\}$.
$\lvert\underline{Edge}(H)\rvert$	the cardinality of $\underline{Edge}(H)$ (the number of elements in $\underline{Edge}(H)$).

Proposition 2.4. [13, Prposition 4.4] *$\underline{Apr}(H)$ is a subgroup of G.*

3. Main results

Proposition 3.1. *$\underline{Apr}(H)$ is a normal subgroup of G.*

Proof. By Proposition 2.4, $\underline{Apr}(H)$ is a subgroup of G. Let $a \in \underline{Apr}(H)$. Then $[a] \subseteq H$.

$$\text{Let } b \in [a] \Longrightarrow [a] = [b] \Longrightarrow [b] \subseteq H \Longrightarrow b \in \underline{Apr}(H) \Longrightarrow [a] \subseteq \underline{Apr}(H)$$

Therefore, $g^{-1}ag \in \underline{Apr}(H)\ \forall g \in G, a \in \underline{Apr}(H)$ and hence $\underline{Apr}(H) \lhd G$. □

Proposition 3.2. *$\underline{Apr}(H)$ is the maximum normal subgroup of G contained in H.*

Proof. Let for a contradiction that, K be a normal subgroup of G with $\underline{Apr}(H) \subsetneqq K \subsetneqq H$.

$$\text{Let } x \in K \Longrightarrow g^{-1}xg \in K\ \forall\ g \in G \Longrightarrow [x] \subseteq K \Longrightarrow [x] \subseteq H \Longrightarrow x \in \underline{Apr}(H)$$

Hence $K \subseteq \underline{Apr}(H)$ which contradicts with our assumption that $\underline{Apr}(H) \subsetneqq K \subsetneqq H$. Thus $\underline{Apr}(H)$ is the maximum normal subgroup of G contained in H. □

By the previous proposition, if $H \lhd G$, then $\underline{Apr}(H) = H$, which follows that $\underline{Edge}(H) = \phi$. Therefore, we obtain the following corollary.

Corollary 3.3. *If $H \lhd G$, then $\underline{Edge}(H) = \phi$.*

Proposition 3.4. *Let $x \in \underline{Edge}(H)$ and $a \in \underline{Apr}(H)$. Then x^{-1}, xa and $ax \in \underline{Edge}(H)$.*

Proof. Since $H \leq G$ and $x, a \in H$, we get x^{-1}, $xa \in H$.

$$\text{If } x^{-1} \in \underline{Apr}(H) \Longrightarrow \text{by Proposition 2.4, } (x^{-1})^{-1} = x \in \underline{Apr}(H)$$
$$\text{If } xa \in \underline{Apr}(H) \Longrightarrow \exists b \in \underline{Apr}(H) \text{ such that } xa = b \Longrightarrow x = ba^{-1} \in \underline{Apr}(H)$$

Therefore, if x^{-1} or $xa \in \underline{Apr}(H)$, then $x \in \underline{Apr}(H)$ which contradicts with $x \in \underline{Edge}(H)$. Thus x^{-1} and $xa \in \underline{Edge}(H)$. Similarly, $ax \in \underline{Edge}(H)$. □

Proposition 3.5. $\underline{Apr}(H) \subseteq [\underline{Edge}(H)]^2$

Proof. Let $a \in \underline{Apr}(H)$. Assume that $x \in \underline{Edg}(H)$. By Proposition 3.4, $ax \in \underline{Edge}(H)$ $\Longrightarrow \exists y \in \underline{Edg}(H)$ such that $ax = y \Longrightarrow a = yx^{-1}$. Since $y, x^{-1} \in \underline{Edg}(H)$, we get $a \in [\underline{Edg}(H)]^2$. Hence $\underline{Apr}(H) \subseteq [\underline{Edge}(H)]^2$. □

Note that the converse of the previous proposition is not necessary true in general. For example consider the following: Let $G \cong A_4$ (Alternating group of degree 4), $H \cong A_3$. Clearly, $\underline{Apr}(H) = \{e\}$ and $\underline{Edge}(H) = \{(123), (132)\}$. Therefore, $[\underline{Edge}(H)]^2 \cong A_3$ which follows $[\underline{Edge}(H)]^2 \nsubseteq \underline{Apr}(H)$. We now need to address the following question: under what conditions is the converse of the previous proposition true? The answer will be given in the following lemma.

Lemma 3.6. *$\underline{Apr}(H) = [\underline{Edge}(H)]^2$ if and only if $[H : \underline{Apr}(H)] = 2$.*

Proof. ($\Longrightarrow$) Let $\underline{Apr}(H) = [\underline{Edge}(H)]^2$. Assume, for contradiction, that $[H : \underline{Apr}(H)] > 2$. Then there are at least three distinct left cosets of $\underline{Apr}(H)$ in H, which we can denote as $\underline{Apr}(H)$, $h_1\underline{Apr}(H)$ and $h_2\underline{Apr}(H)$. Consequently, $h_1\underline{Apr}(H) \cup h_2\underline{Apr}(H) \subseteq \underline{Edge}(H)$, which implies that $h_1, h_2 \in \underline{Edge}(H)$. By Proposition 3.4, $h_1^{-1}, h_2a \in \underline{Edge}(H)$, where $a \in \underline{Apr}(H)$. Thus $h_1^{-1}h_2a \in [\underline{Edge}(H)]^2$. Given our assumption that $\underline{Apr}(H) = [\underline{Edge}(H)]^2$, it follows that $h_1^{-1}h_2a \in \underline{Apr}(H)$. Let $h_1^{-1}h_2a = b$, which implies $h_2a = h_1b$. However, $h_1b \in h_1\underline{Apr}(H)$ and $h_2a \in h_2\underline{Apr}(H)$, leading to the conclusion that $h_1\underline{Apr}(H) \cap h_1\underline{Apr}(H) \neq \phi$, a contradiction. Thus $[H : \underline{Apr}(H)] = 2$.

($\Longleftarrow$) Let $[H : \underline{Apr}(H)] = 2$ and $\underline{Apr}(H)$, $h\underline{Apr}(H)$ be the distinct left cosets of $\underline{Apr}(H)$ in H. Since $H = \underline{Apr}(H) \cup h\underline{Apr}(H) = \underline{Apr}(H) \cup \underline{Edge}(H)$ with $\underline{Apr}(H) \cap h\underline{Apr}(H) = \underline{Apr}(H) \cap \underline{Edge}(H) = \phi$, we get

$$\underline{Edge}(H) = h\underline{Apr}(H) \rightarrow (*)$$

By Proposition 3.5, $\underline{Apr}(H) \subseteq [\underline{Edge}(H)]^2$. We need only show that $[\underline{Edge}(H)]^2 \subseteq \underline{Apr}(H)$. Let $z \in [\underline{Edge}(H)]^2$. Then there exist $x, y \in \underline{Edge}(H)$ such that $z = xy$. By Proposition 3.4, $x, y^{-1} \in \underline{Edge}(H)$. Applying $(*)$, we can find $a_1, a_2 \in \underline{Apr}(H)$ such that $x = ha_1$, $y^{-1} = ha_2$. Thus $z = xy = ha_1a_2^{-1}h^{-1}$. Since $\underline{Apr}(H)$ is a normal subgroup of G (Proposition 3.1), $z \in \underline{Apr}(H)$. Therefore $[\underline{Edge}(H)]^2 \subseteq \underline{Apr}(H)$ and hence $\underline{Apr}(H) = [\underline{Edge}(H)]^2$. □

Proposition 3.7. *Assume that H and K are rough subgroups of a group G. Then $H = K$ if and only if $\underline{Edge}(H) = \underline{Edge}(K)$.*

Proof. It is clear that, if $H = K$ then $\underline{Edge}(H) = \underline{Edge}(K)$. For the converse, assume that $\underline{Edge}(H) = \underline{Edge}(K)$. By Proposition 3.5,

$$\underline{Apr}(H) \subseteq [\underline{Edge}(H)]^2 = [\underline{Edge}(K)]^2 \subseteq K \Longrightarrow \underline{Apr}(H) \cup \underline{Edge}(K) \subseteq K \Longrightarrow H \subseteq K$$

Similarly, $K \subseteq H$ and hence $H = K$. □

Proposition 3.8. $[\underline{Edge}(H)]^4 \subseteq [\underline{Edge}(H)]^2$.

Proof. Assume that $Z \in [\underline{Edge}(H)]^4$. Then z can be written as $z = x_1x_2x_3x_4$, where $x_i \in \underline{Edge}(H)$ for $i = 1, 2, 3, 4$. Since $x_1x_2 \in H$, either $x_1x_2 \in \underline{Apr}(H)$ or $x_1x_2 \in \underline{Edge}(H)$. If $x_1x_2 \in \underline{Apr}(H)$, then by Proposition 3.4, $(x_1x_2)x_3 \in \underline{Edge}(H)$ and hence $z = [(x_1x_2)x_3]x_4 \in [\underline{Edge}(H)]^2$. Similarly, if $x_3x_4 \in \underline{Apr}(H)$, then $z \in [\underline{Edge}(H)]^2$. Now assume both x_1x_2 and $x_3x_4 \in \underline{Edge}(H)$. Hence $z = (x_1x_2)(x_3x_4) \in [\underline{Edge}(H)]^2$ and we are done. □

Lemma 3.9. *$[\underline{Edge}(H)]^2$ is a subgroup of G normal in H.*

Proof. Let z_1, $z_2 \in [\underline{Edge}(H)]^2$. Then $z_1 = x_1x_2$, $z_2 = x_3x_4$ where $x_i \in \underline{Edge}(H)$ for $i = 1, 2, 3, 4$. Therefore, $z_1z_2^{-1} = x_1x_2x_4^{-1}x_3^{-1} \in [\underline{Edge}(H)]^4 \subseteq [\underline{Edge}(H)]^2$ (by Proposition 3.8) and hence $[\underline{Edge}(H)]^2 \leq G$. Now we need to show that $[\underline{Edge}(H)]^2 \lhd H$. Let $h \in H$. If $h \in \underline{Edge}(H)$, then $h^{-1}z_1h = h^{-1}x_1x_2h \in [\underline{Edge}(H)]^4 \subseteq [\underline{Edge}(H)]^2$. Also if $h \in \underline{Apr}(H)$, then by Proposition 3.4, $h^{-1}x_1$, $x_2h \in \underline{Edge}(H)$ which follows that $h^{-1}z_1h \in [\underline{Edge}(H)]^2$. Hence $[\underline{Edge}(H)]^2 \lhd H$. □

The following example demonstrates that $[\underline{Edge}(H)]^2$ does not need to be a normal subgroup of G. Let $G \cong A_4$ and $H = \langle(234)\rangle$. Clearly, $\underline{Edge}(H) = \{(234), (243)\}$. Hence $[\underline{Edge}(H)]^2 \cong H$ is not normal in G.

Lemma 3.10. *Either* $[\underline{Edge}(H)]^2 = \underline{Apr}(H)$ *or* $[\underline{Edge}(H)]^2 = H$.

Proof. By Proposition 3.5, we have $\underline{Apr}(H) \subseteq [\underline{Edge}(H)]^2 \subseteq H$. To reach a contradiction, suppose that $\underline{Apr}(H) \neq [\underline{Edge}(H)]^2 \neq H$. Let $h \in H - [\underline{Edge}(H)]^2$ and $x, y \in \underline{Edge}(H)$. Then $hxy \in h[\underline{Edge}(H)]^2$. If $hx \in \underline{Edge}(H)$, then $(hx)y \in [\underline{Edge}(H)]^2$ which contradicts with $[\underline{Edge}(H)]^2 \cap h[\underline{Edge}(H)]^2 = \phi$. Thus

$$hx \in \underline{Apr}(H)\ \forall x \in \underline{Edge}(H) \longrightarrow (**)$$

Let $z \in [\underline{Edge}(H)]^2 - \underline{Apr}(H)$. Since $[\underline{Edge}(H)]^2 - \underline{Apr}(H) = [\underline{Edge}(H)]^2 \cap \underline{Edge}(H)$, we have $z \in [\underline{Edge}(H)]^2 \cap \underline{Edge}(H)$. Then $z = u_1u_2$ where u_1, u_2 and z are elements of $\underline{Edge}(H)$. By applying $(**)$, hz and $hu_1 \in \underline{Apr}(H)$. By Proposition 2.4, $(hu_1)^{-1}(hz) \in \underline{Apr}(H)$. However,

$$(hu_1)^{-1}(hz) = u_1^{-1}h^{-1}hu_1u_2 = u_2 \in \underline{Edge}(H)$$

Then we get a contradiction which follows that, either $[\underline{Edge}(H)]^2 = \underline{Apr}(H)$ or $[\underline{Edge}(H)]^2 = H$. □

Corollary 3.11. *The following conditions are equivalent:*

(i) $[\underline{Edge}(H)]^2 = H$.

(ii) $[H : \underline{Apr}(H)] > 2$.

(iii) $|\underline{Edge}(H)| \neq |[\underline{Edge}(H)]^2|$.

Corollary 3.12. *The following conditions are equivalent:*

(i) $[\underline{Edge}(H)]^2 = \underline{Apr}(H)$.

(ii) $[H : \underline{Apr}(H)] = 2$.

(iii) $|\underline{Edge}(H)| = |[\underline{Edge}(H)]^2|$.

We are now ready to answer the questions raised at the beginning of this paper. The answer is provided in the following theorem, which represents our main theorem.

Theorem 3.13. *Let E be a non-empty subset of a group G. Then $E \cup E^2$ is a unique subgroup of G with internal edge E if and only if the following conditions hold:*

(i) $E^2 \leq G$.

(ii) *Either $|E| = |E^2|$ and $E^2 \lhd G$ or $E \subseteq E^2$ and $E^2 - E$ is the maximum normal subgroup of G contained in E^2.*

(iii) *$E \cup E^2$ is not normal in G.*

Proof. Let $E \cup E^2$ be a subgroup of G with internal edge E. By Lemma 3.9, E^2 is a subgroup of G and condition (i) holds. Since E is a non-empty subset, Corollary 3.3 implies that $E \cup E^2$ is not normal in G, so condition (iii) holds. By Lemma 3.10, there are two possibilities for E^2, either $E^2 = \underline{Apr}(E \cup E^2)$ or $E^2 = E \cup E^2$.

- If $E^2 = E \cup E^2$, then $E \subseteq E^2$ and $E^2 - E = \underline{Apr}(E \cup E^2)$ is the maximum normal subgroup of G contained in $E \cup E^2 = E^2$ (by Proposition 3.2).
- If $E^2 = \underline{Apr}(E \cup E^2)$, then $E^2 \lhd G$ and by Lemma 3.6, $[E \cup E^2 : \underline{Apr}(E \cup E^2)] = 2$. Consequently, E is a left coset of $\underline{Apr}(E \cup E^2)$ within $E \cup E^2$. This implies that $|E| = |\underline{Apr}(E \cup E^2)| = |E^2|$.

Thus condition (iii) is satisfied, and hence the necessary direction is proven. For the converse, suppose E is a subset of G that satisfies conditions (i)-(iii). We will demonstrate in the following steps that $E \cup E^2$ is a unique subgroup of G with internal edge E.

1. If $|E| = |E^2|$, then each element in $E \cup E^2$ has its inverse also within $E \cup E^2$.
   Let $x \in E \cup E^2$. By condition (i), if $x \in E^2$, then $x^{-1} \in E^2 \subseteq E \cup E^2$. Now consider $x \in E$. Assume $E = \{g_1, g_2, ..., g_n\}$. Then $xE = \{xg_1, xg_2, ..., xg_n\}$. Clearly, if $xg_i = xg_j$ for some $g_i, g_j \in E$ with $i \neq j$, then $g_i = g_j$, a contradiction. Thus $|xE| = |E| = |E^2|$. However $xE \subseteq E^2$ which follows that $xE = E^2$. Similarly, $Ex = E^2$. Therefore, $xE = Ex = E^2 \ \forall x \in E$. By condition (i), $e \in E^2$ (where e is the identity element). Then there exist $g \in E$ such that $xg = gx = e$. This implies that $x^{-1} = g \in E$. Thus, $x^{-1} \in E^2 \cup E \ \forall \ x \in E \cup E^2$.
2. $E \cup E^2$ is a subgroup of G.
   By condition (ii), either $E \subseteq E^2$ or $|E| = |E^2|$. If $E \subseteq E^2$, then $E^2 \cup E = E^2$ and by condition (i), $E \cup E^2 \leq G$. Therefore, we can assume that $|E| = |E^2|$. Let $x, y \in E \cup E^2$. Clearly, if both x and $y \in E$, then $xy \in E^2 \subseteq E \cup E^2$. Similarly, if both x and $y \in E^2$, then by condition (i), $xy \in E^2 \subseteq E \cup E^2$. Consider the case where $x \in E$ and $y \in E^2$. As shown in (1), $x^{-1}E = E^2$. Then E has an element u, say, such that $y = x^{-1}u$. Then $xy = xx^{-1}u = u \in E \subseteq E \cup E^2$. Therefore $xy \in E \cup E^2 \ \forall x, y \in E \cup E^2$ and by (1), each element in $E \cup E^2$ has its inverse also within $E \cup E^2$. Hence $E \cup E^2$ is a subgroup of G.
3. $\underline{Edge}(E \cup E^2) = E$.
   By condition (ii), either $|E| = |E^2|$ and $E^2 \lhd G$ or $E \subseteq E^2$ and $E^2 - E$ is the maximum normal subgroup of G contained in E^2. If $E \subseteq E^2$, then according to Proposition 3.2, $E^2 - E = \underline{Apr}(E \cup E^2)$. Therefore, $E^2 = E \cup (E^2 - E) = \underline{Edge}(E \cup E^2) \cup \underline{Apr}(E \cup E^2)$ and $E \cap (E^2 - E) = \underline{Edge}(E \cup E^2) \cap \underline{Apr}(E \cup E^2) = \phi$, which implies that $\underline{Edge}(E \cup E^2) = E$. Now Let $|E| = |E^2|$ and $E^2 \lhd G$. Then

   $$|E \cup E^2| = |E| + |E^2| - |E \cap E^2| = 2|E^2| - |E \cap E^2|$$

   $$\Longrightarrow |E \cap E^2| = 2|E^2| - |E \cup E^2|$$

   Since $E^2 \leq E \cup E^2$, we have $|E \cup E^2| = \alpha|E^2|$ where α is a positive integer number. Consequently, $|E \cap E^2| = (2 - \alpha)|E^2|$. Clearly. if $\alpha = 1$, then $E \cup E^2 = E^2$ which implies $E \subseteq E^2$, leading to a contradiction. The only possible value for α is 2, which follows that $E \cap E^2 = \phi$ and $|E \cup E^2| = 2|E^2|$. Therefore E^2 is a maximal subgroup of $E \cup E^2$ and by Preposition 3.2, we get $E^2 = \underline{Apr}(E \cup E^2)$. Hence $\underline{Edge}(E \cup E^2) = E$.

By Preposition 3.7, $E \cup E^2$ is the unique subgroup of G with internal edge E, and the proof is complete. □

Corollary 3.14. *Let E be a non-empty subset of a group G. If E is an internal edge of a subgroup H of G, then $H = E \cup E^2$.*

We provide some examples to demonstrate that we cannot omit any hypothesis of Theorem 3.13. In these examples, we choose $E \subseteq G$ so that it does not represent an internal edge of any subgroup of G and attempt to determine which condition of the theorem is not satisfied.

1. $G = S_3$ and $E = \{(12), (13), (23)\}$. Then $E^2 = \langle(123)\rangle \leq G$, $|E| = |E^2|$ and $E^2 \lhd G$ but $E \cup E^2 = S_3 \lhd G$.
2. $G = A_5$ and $E = \{(12)(45), (13)(45), (23)(45)\}$. Then $E^2 = \langle(123)\rangle \leq G$, $E \cup E^2$ is not normal in G and $|E| = |E^2|$ but E^2 is not normal in G.
3. $G = A_5$ and $E = \{(123), (12)(45), (13)(45), (23)(45)\}$. Then $E^2 = \langle(123), (12)(45)\rangle \leq G$, $E \cup E^2$ is not normal in G and $E \subseteq E^2$ but $E^2 - E = \{e, (132)\}$ is not normal in G.
4. $G = D_6 = \{e, a^2, a^3, a^4, a^5, b, ba, ba^2, ba^3, ba^4, ba^5\}$ (Dihedral group of order 12) and $E = \{a^3, b, a^3b\}$. Then $E^2 = \{e, a^3, b, a^3b\} \leq G$, $E \cup E^2$ is not normal in G, $E \subseteq E^2$ and $E^2 - E = \{e\}$ is normal in G but $E^2 - E$ is not the maximum normal subgroup of G contained in E^2.

Proposition 3.15. *Let $f : G \longrightarrow G^{\backprime}$ be an isomorphism from a group G to a group $G^{\backprime}$ and E be a non-empty subset of G. If E represents an internal edge of a rough subgroup of G, then the following holds:*

(i) $f(E^2) = [f(E)]^2$.

(ii) $f(E \cup E^2) = f(E) \cup [f(E)]^2$.

(iii) $[f(E)]^2$ *and* $f(E) \cup [f(E)]^2$ *are subgroups of* $G^{\backprime}$.

(iv) $f(E) \cup [f(E)]^2$ *is not normal in* $G^{\backprime}$.

(v) *Either* $|f(E)| = |[f(E)]^2|$ *and* $[f(E)]^2 \lhd G^{\backprime}$ *or* $f(E) \subseteq [f(E)]^2$ *and* $[f(E)]^2 - f(E)$ *is the maximum normal subgroup of* $G^{\backprime}$ *contained in* $[f(E)]^2$.

Proof. **(i)** Let $f(z) \in f(E^2) \Longleftrightarrow z \in E^2 \Longleftrightarrow \exists\, x, y \in E$ such that $z = xy \Longleftrightarrow \exists\, f(x), f(y) \in f(E)$ such that $f(z) = f(xy) = f(x)f(y) \Longleftrightarrow f(z) \in [f(E)]^2$.

(ii) Let $f(z) \in f(E \cup E^2) \Longleftrightarrow z \in E \cup E^2 \Longleftrightarrow z \in E \vee z \in E^2 \Longleftrightarrow f(z) \in f(E) \vee f(z) \in f(E^2) \Longleftrightarrow f(z) \in f(E) \vee f(z) \in [f(E)]^2 \Longleftrightarrow f(z) \in f(E) \cup [f(E)]^2$.

(iii) Since E^2 and $E \cup E^2$ are subgroups of G, we get $f(E^2) = [f(E)]^2$ and $f(E \cup E^2) = f(E) \cup [f(E)]^2$ are subgroups of $G^{\backprime}$.

(iv) Since $E \cup E^2$ is not normal in $G \Longrightarrow \exists\, g \in G$ and $x \in E \cup E^2$ such that $g^{-1}xg \notin E \cup E^2 \Longrightarrow f(g^{-1}xg) = [f(g)]^{-1}f(x)f(g) \notin f(E \cup E^2) = f(E) \cup [f(E)]^2$. Thus $f(E) \cup [f(E)]^2$ is not normal in $G^{\backprime}$.

(v) As f is isomorphism and $[f(E^2)] = [f(E)]^2$, we have

$$\begin{gathered} |E| = |E^2| \Longleftrightarrow |f(E)| = |[f(E)]^2| \\ E^2 \lhd G \Longleftrightarrow [f(E)]^2 \lhd G^{\backprime} \\ E \subseteq E^2 \Longleftrightarrow f(E) \subseteq [f(E)]^2 \\ E^2 - E \lhd G \Longleftrightarrow [f(E)]^2 - f(E) \lhd G^{\backprime} \end{gathered}$$

Since $E^2 - E$ is the maximum normal subgroup of G contained in E^2 if and only if $[f(E)]^2 - f(E)$ is the maximum normal subgroup of $G^{\backprime}$ contained in $[f(E)]^2$, it follows that, by Theorem 3.13, the proof is complete.

□

Corollary 3.16. *Let $f : G \longrightarrow G^{\backprime}$ be an isomorphism from a group G to a group $G^{\backprime}$ and E be a non-empty subset of G. Then E represents an internal edge of a rough subgroup of G if and only if $f(E)$ represents an internal edge of a rough subgroup of $G^{\backprime}$.*

Conclusion: This work focuses on rough subgroups by considering the conjugacy relation as an equivalence relation on a group. Several interesting properties of internal edges and lower approximations of rough subgroups are investigated. However, some limitations of the study should be acknowledged. The results obtained hold specifically for the conjugacy relation, but they do not necessarily generalize to arbitrary equivalence relations. An open question remains as to whether these results can be extended to a broader class of equivalence relations. Moreover, this paper represents a first step in a series of studies. Future research could explore similar properties in the context of other algebraic structures related to rough subgroups. This naturally leads to the following question: What conditions must a subset E of a group G satisfy in order to be an external edge, boundary region, lower approximation, or upper approximation of some rough subgroup of G?

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