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A new class of Korovkin-type approximation theorems based on equi-statistical convergence of double sequences

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Abstract. In this article, we explore the concepts of point-wise statistical convergence, equi-statistical convergence, and uniform statistical convergence for sequences of functions of two variables, employing the deferred power-series method. We then demonstrate the inclusion relationships among these concepts, supplemented by several illustrative numerical examples. Furthermore, from an application perspective, we introduce a Korovkin-type theorem that utilizes our proposed method to examine the equi-statistical convergence of sequences of positive linear operators. Additionally, we consider an example involving the Meyer-König and Zeller operator and use MATLAB software to illustrate the convergence behavior of the operator. Finally, we provide an estimation of the equi-statistical convergence rates to assess the effectiveness of the findings in our research.

1. Introduction, Preliminaries and Motivation

The gradual development in the study of sequence spaces has significantly advanced the concept of statistical convergence, which is broader and more comprehensive than traditional convergence. This progress is largely attributed to the pioneering work of mathematicians Fast [12] and Steinhaus [46], who expanded the scope of statistical convergence analysis. Today, this influential concept finds applications across various fields of mathematics, as well as in analytical statistics. It is particularly valuable in areas such as operator theory, finance mathematics, industrial mathematics and probability theory. For more recent studies, interested readers may refer to [13] and [14].

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Statistical convergence is a generalization of the classical notion of convergence for real sequences. For a real sequence (a_k) statistical convergence is defined based on the concept of density of the set of indices. Specifically, a sequence (a_k) is said to be statistically convergent to a real number L if, for every $\epsilon > 0$, the set of indices k for which $|a_k - L| \geqq \epsilon$ has natural density zero. In other words, the proportion of terms in the sequence that deviate from L by at least ϵ becomes arbitrarily small as the sequence progresses. Unlike ordinary convergence, which requires the terms to approach L for all sufficiently large indices, statistical convergence allows for exceptions as long as they occur with vanishing frequency. This concept is particularly useful in dealing with sequences that exhibit irregular behavior or outliers, and it has important applications in various areas of mathematics, including number theory, probability, and functional analysis.
Let we write it as

$$\text{stat} \lim a_k = L \quad (k \to \infty).$$

Pringsheim [29], in the year 1900 studied the classical convergence of a double real sequence refers to the convergence of a sequence indexed by two integers, typically denoted as $(a_{\alpha,\beta})$ where α and β are positive integers. A double sequence $(a_{\alpha,\beta})$ is said to converge to a real number L if, for every $\epsilon > 0$, there exists positive integers M and N such that for all $\alpha \geqq M$ and $\beta \geqq N$, the inequality $|a_{\alpha,\beta} - L| < \epsilon$ holds. This definition implies that as both indices α and β tend to infinity, the terms of the sequence get arbitrarily close to the limit L. The notion of convergence here is essentially an extension of the concept of convergence of single-indexed sequences to a double-indexed scenario, requiring that the sequence stabilize around L in every direction of the (α, β) plane.

Statistical convergence of a double sequence $(a_{\alpha,\beta})$ is a generalization of convergence that involves a probabilistic approach rather than the traditional point-wise criterion. A double sequence $(a_{\alpha,\beta})$ is said to be statistically convergent to a real number L if, for every $\epsilon > 0$, the proportion of indices (α, β) for which $|a_{\alpha,\beta} - L| \geqq \epsilon$ becomes negligible as α and β increase. Formally, this can be expressed using the concept of the density of a set: $(a_{\alpha,\beta})$ is statistically convergent to L if

$$\delta(\mathcal{H}_\epsilon(i, j)) = 0$$

where

$$\delta(\mathcal{H}_\epsilon(i, j)) = \{(\alpha, \beta) : \alpha \leqq i, \beta \leqq j \ \text{ and } \ |a_{\alpha,\beta} - L| \geqq \epsilon\}.$$

Here, we write

$$\text{stat}^2 \lim_{\alpha,\beta} a_{\alpha,\beta} = L.$$

It is important to note that every double sequence that converges in the probabilistic sense (often denoted as P- convergence) will also converge statistically in the sense of stat^2 to the same limit. However, the converse is not necessarily true. This means that a double sequence which is stat^2 convergent may not always be P-convergent to the same limit.

Example 1.1. *Suppose we consider a double sequence $a = (a_{\alpha,\beta})$ as*

$$a_{\alpha,\beta} = \begin{cases} \sqrt{\alpha\beta} & (\alpha = k^2, \ \beta = l^2; \ \forall \, k, l \in \mathbb{N}) \\ 0 & \textit{otherwise}. \end{cases}$$

It is apparent that the sequence $(a_{\alpha,\beta})$ does not exhibit P-convergent in the usual sense. Nonetheless, 0 *serves as its statistical limit.*

Building on the concept of statistical convergence, Karakaya and Chishti [24] introduced the notion of weighted statistical convergence for single sequences. This concept has since been expanded by several researchers (see [6], [7] and [23]). Additionally, Srivastava *et al.* [44] introduced and examined deferred weighted statistical convergence, which has been explored further in subsequent works (see [16], [25], [28], [42], and [43]).

Recalling Agnew [1], let (ζ_β) and (η_β) be the sequences in $\mathbb{Z}^{0+}$ such that

$$\zeta_\beta < \eta_\beta \quad \text{and} \quad \lim_{\beta\to\infty} \eta_\beta = \infty,$$

are the regularity conditions for deferred summability technique.

We now introduce the deferred power series convergence technique for double sequence under a specific summability mean.

Let $(q_{\alpha,\beta})$ be a double sequence of non-negative numbers with $q_{00} > 0$. The associated deferred power series method is defined as

$$q(\chi,\psi) = \sum_{\substack{i=\zeta_\alpha+1\\ j=\zeta_\beta+1}}^{\eta_\alpha,\eta_\beta} q_{i,j}\,\chi^i\psi^j.$$

The radius of convergence R^- satisfies the condition $0 < R^- \leqq \infty$.

A double sequence $(a_{\alpha,\beta})$ is convergent to a finite real number a under the deferred power series (or DP-method) method, if

$$\lim_{0<\chi,\psi\to R^-} \frac{1}{q(\chi,\psi)} \sum_{\substack{i=\zeta_\alpha+1\\ j=\zeta_\beta+1}}^{\eta_\alpha,\eta_\beta} q_{i,j}\,\chi^i\psi^j a_{i,j} = a \quad (\because \chi,\psi \in I = [0,1/2]\times[0,1/2]).$$

Note that the DP-method is considered to be regular (refer to [8]) if and only if the following condition holds:

$$\lim_{0<\chi,\psi\to R^-} \frac{q_{i,j}\,\chi^i\psi^j}{q(\chi,\psi)} = 0 \qquad (\forall\ i,j \in \mathbb{N}).$$

We next introduce the statistical versions of convergence under the DP- method.

Let $\mathcal{G} \subset \mathbb{N}\times\mathbb{N}$ and let

$$\mathcal{G}_\epsilon = \left\{\alpha \leqq \eta_\alpha, \beta \leqq \eta_\beta \quad \text{and} \quad \alpha,\beta \in \mathcal{G}\right\}. \tag{1}$$

The DP-density of $\mathcal{G}$, denoted as $\delta_{\mathrm{DP}}(\mathcal{G}_\epsilon)$ is given by

$$\delta_{\mathrm{DP}}(\mathcal{G}_\epsilon) = \lim_{0<\chi,\psi\to R^-} \frac{1}{q(\chi,\psi)} \sum_{\alpha,\beta\in\mathcal{G}_\epsilon} q_{\alpha,\beta}\chi^\alpha\psi^\beta$$

provided that the limit exists.

Definition 1.2. *A double sequence $(a_{\alpha,\beta})$ is statistically convergent to L under the DP- method if, for every $\epsilon > 0$ the following condition is satisfied:*

$$\lim_{0<\chi,\psi\to R^-} \frac{1}{q\,(\chi,\psi)} \sum_{\alpha,\beta\in\mathcal{G}_\epsilon} q_{\alpha,\beta}\,\chi^\alpha\psi^\beta = 0,$$

where

$$\mathcal{G}_\epsilon = \left\{\alpha \leqq \eta_\alpha, \beta \leqq \eta_\beta \quad \text{and} \quad |a_{\alpha,\beta} - L| \geqq \epsilon\right\},$$

which implies

$$\delta_{\mathrm{DP}}(\mathcal{G}_\epsilon) = 0 \qquad (\forall\, \epsilon > 0).$$

We denote this by

$$\mathrm{stat}_{\mathrm{DP}} \lim a_{\alpha,\beta} = L.$$

The example below, demonstrates that statistical convergence and $\mathrm{stat}_{\mathrm{DP}}$ convergence are not fairly analogous.

Example 1.3. *Let*

$$q_{\alpha,\beta} = \begin{cases} 1 & (\alpha = k^2,\ \beta = \ell^2;\ k, \ell \in \mathbb{N}) \\ 0 & (\text{otherwise}) \end{cases}$$

and

$$a_{\alpha,\beta} = \begin{cases} 0, & (\alpha = k^2,\ \beta = \ell^2;\ k, \ell \in \mathbb{N}) \\ \alpha\beta, & (\text{otherwise}). \end{cases}$$

It is evident that $(a_{\alpha,\beta})$ *does not converge statistically to* 0*. However, according to Definition 1.2, we fairly have*

$$\lim_{0<s,t\to R^-} \frac{1}{q(\chi,\psi)} \sum_{\alpha,\beta \in \left\{\alpha \leqq \eta_\alpha, \beta \leqq \eta_\beta : \left|a_{\alpha,\beta}\right| \geqq \epsilon\right\}} q_{\alpha,\beta} \chi^\alpha \psi^\beta = 0,$$

where $\zeta_\beta = 2\beta$ *and* $\eta_\beta = 4\beta$*.*

Consequently, $(a_{\alpha,\beta})$ *converges statistically to* 0 *under the DP-method.*

Again, let

$$a_{\alpha,\beta} = \begin{cases} \frac{1}{\alpha\beta} & (\alpha = k^2,\ \beta = \ell^2;\ \alpha, \beta \in \mathbb{N}) \\ 0 & (\text{otherwise}), \end{cases}$$

where $\zeta_\beta = 2\beta$ *and* $\eta_\beta = 4\beta$*.*

Indeed, $(a_{\alpha,\beta})$ *statistically converges to* 0*.*

However,

$$\lim_{0<\chi,\psi\to R^-} \frac{1}{q(\chi,\psi)} \sum_{\alpha,\beta \in \left\{\alpha \leqq \eta_\alpha, \beta \leqq \eta_\beta : |a_{\alpha,\beta}| \geq \epsilon\right\}} q_{\alpha,\beta}\, \chi^\alpha \psi^\beta \neq 0.$$

Therefore, the sequence $(a_{\alpha,\beta})$ *is not* $\mathrm{stat}_{\mathrm{DP}}$ *convergent.*

The concept of fundamental statistical limits, along with suitable theorems and illustrative examples was first established by the distinguished scientist Móricz [21]. Subsequently, Mohiuddine *et al.* [20] provided significant insights into statistical Cesàro summability means, including illustrative examples and proofs of related Korovkin's results. Later, Mursaleen *et al.* [22] refined the elementary limit concepts and established inclusion relations among them. Recently, Saini *et al.* [31] proved Korovkin-type theorems using a certain class of equi-statistical convergence. In 2018, Srivastava *et al.* [42] investigated equi-statistical convergence using the deferred Nörlund mean and established related approximation theorems. Subsequently, Parida *et al.* [27] presented results concerning sequences converging equi-statistically through the deferred Cesàro mean, proving corresponding Korovkin-type theorems. More recently, Srivastava *et al.* [36] and Demirci *et al.* [9] examined equi-statistical convergence of sequences via the power-series method, deriving various approximation theorems. Interested readers can explore more recent studies in [5], [17], [32], [33], and [45] for further reference.

Inspired by the aforementioned results and advancements, our goal is to introduce the notions of point-wise statistical convergence, equi-statistical convergence, and uniform statistical convergence for sequences of functions of two variables, utilizing the deferred power series method (DP-method). We then explore inclusion relations among these convergence types and offer several illustrative numerical examples. Additionally, we present a Korovkin-type approximation theorem, which is derived from our approach for the equi-statistical convergence of sequences of linear operators. Lastly, we assess the rates of equi-statistical convergence to highlight the effectiveness of our findings.

Let $I \subseteqq \mathbb{R}^2$ and consider $g(\chi, \psi) \in C(I)$, where $C(I)$ denotes the class of real-valued continuous functions defined on $I = \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right]$. Additionally, let $g_{\alpha,\beta}(\chi, \psi) \in C(I)$ for each α, β. Define the supremum norm on $C(I)$ as $\| g(\chi, \psi) \|_{C(I)}$.

We now explore the concepts of deferred point-wise statistical ($\text{stat} - \text{pointwise}_\text{D}$) convergence, deferred equi-statistical ($\text{equi} - \text{stat}_\text{D}$) convergence, and deferred uniform statistical ($\text{stat} - \text{uniformly}_\text{D}$) convergence for sequences of functions in two variables as follows.

(a) If, for every $\epsilon > 0$ and for all $\chi, \psi \in I$, the following condition holds:

$$\lim \frac{\mathcal{K}_{\alpha,\beta}(\chi, \psi; \epsilon)}{\alpha\beta} = 0 \quad (\alpha, \beta \to \infty)$$

where

$$\mathcal{K}_{\alpha,\beta}(\chi, \psi; \epsilon) := |\{\alpha \leqq \eta_\alpha, \beta \leqq \eta_\beta \quad \text{and} \quad |g_{\alpha,\beta}(\chi, \psi) - g(\chi, \psi)| \geqq \epsilon\}|,$$

then the sequence $(g_{\alpha,\beta})$ is deferred statistically point-wise convergent to g on I. We denote this by

$$g_{\alpha,\beta} \to g \ \ (\text{stat} - \text{pointwise}_\text{D}).$$

(b) If, for every $\epsilon > 0$, the following holds uniformly with respect to $\chi, \psi \in \mathcal{I}$:

$$\lim_{\alpha,\beta\to\infty} \frac{\mathcal{K}_{\alpha,\beta}(\chi, \psi; \epsilon)}{\alpha\beta} = 0 \text{ uniformly with regards to } \chi, \psi \in I,$$

then the sequence $(g_{\alpha,\beta})$ is deferred equi-statistically convergent to g on I. We denote this by

$$g_{\alpha,\beta} \Longrightarrow g \ (\text{equi} - \text{stat}_\text{D}).$$

(c) If, for every $\epsilon > 0$, the following condition is satisfied:

$$\lim_{\alpha,\beta\to\infty} \frac{\mathcal{D}_{\alpha,\beta}(\chi, \psi; \epsilon)}{\alpha\beta} = 0,$$

where

$$D_k(\chi, \psi; \epsilon) = |\{\alpha \leqq \eta_\alpha, \beta \leqq \eta_\beta \quad \text{and} \quad \| g_{\alpha,\beta} - g \|_{C(I)} \geqq \epsilon\}|,$$

then the sequence $(g_{\alpha,\beta})$ is deferred statistically uniformly convergent to g on I. We denote this by

$$g_{\alpha,\beta} \rightrightarrows g \ (\text{stat} - \text{unifomly}_\text{D}).$$

For the purposes of this study, we now introduce the following definitions in accordance with the DP-method.

Definition 1.4. *For every $\epsilon > 0$ and for all $\chi, \psi \in I$, if*

$$\delta_{\mathrm{DP}}\left(\mathcal{K}_{\alpha,\beta}(\chi,\psi;\epsilon)\right) = \lim_{0<\chi,\psi\to R^-} \frac{1}{q(\chi,\psi)} \sum_{\alpha,\beta\in\mathcal{K}_{\alpha,\beta}(\chi,\psi;\epsilon)} q_{\alpha,\beta}\chi^\alpha\psi^\beta = 0,$$

then the sequence $(g_{\alpha,\beta})$ is said to be point-wise statistically convergent to g on I under the DP- method. We denote this by

$$g_{\alpha,\beta} \to g \quad (\mathrm{stat}-\mathrm{point}_{\mathrm{DP}}).$$

Definition 1.5. *For every $\epsilon > 0$, if*

$$\delta_{\mathrm{DP}}\left(\mathcal{K}_k(\chi,\psi;\epsilon)\right) = \lim_{0<\chi,\psi\to R^-} \frac{1}{q(\chi,\psi)} \sum_{\chi,\psi\in\mathcal{K}_{\alpha,\beta}(\chi,\psi;\epsilon)} q_{\alpha,\beta}\,\chi^\alpha\psi^\beta = 0 \text{ (uniformly in } \chi,\psi),$$

then the sequence $(g_{\alpha,\beta})$ is equi-statistically convergent to g on I under the DP- method. This is denoted by

$$g_{\alpha,\beta} \to g \quad (\mathrm{equi}-\mathrm{stat}_{\mathrm{DP}}).$$

Definition 1.6. *For every $\epsilon > 0$, if*

$$\delta_{\mathrm{DP}}\left(\mathcal{D}_{\alpha,\beta}(\chi,\psi;\epsilon)\right) = \lim_{0<\chi,\psi\to R^-} \frac{1}{q(\chi,\psi)} \sum_{\alpha,\beta\in D_{\alpha,\beta}(\chi,\psi;\epsilon)} q_{\alpha,\beta}\chi^\alpha\psi^\beta = 0,$$

then the sequence $(g_{\alpha,\beta})$ is uniformly statistically convergent to g on I under the DP- method. This is denoted by

$$g_{\alpha,\beta} \rightrightarrows g \quad (\mathrm{stat}-\mathrm{uni}_{\mathrm{DP}}).$$

Considering Definitions 1.4, 1.5, and 1.6, we now introduce an inclusion relation, which is further illustrated with various examples as detailed below.

Lemma 1.7. *For a double sequence $(g_{\alpha,\beta})$, the following implications are valid:*

$$\begin{aligned} g_{\alpha,\beta} \rightrightarrows g \quad (\mathrm{stat}-\mathrm{uni}_{\mathrm{DP}}) &\Longrightarrow g_{\alpha,\beta} \to g \quad (\mathrm{equi}-\mathrm{stat}_{\mathrm{DP}}) \\ &\Longrightarrow g_{\alpha,\beta} \to g \quad (\mathrm{stat}-\mathrm{point}_{\mathrm{DP}}). \end{aligned} \tag{2}$$

Also, the implications are strict, meaning that the reverse directions of the implications under (2) *generally do not hold.*

To illustrate that the implications are indeed strict, as stated in Lemma 1.7, we provide the following numerical examples.

Example 1.8. *Let $\zeta_\beta = 2\beta$ and $\eta_\beta = 4\beta$, and let*

$$g_{\alpha,\beta}(\chi,\psi) = \begin{cases} -2^{\alpha\beta}(\chi\psi - \frac{1}{2^{\alpha-1}}) & (\alpha = k^2, \beta = \ell^2, k,\ell \in \mathbb{N};\ \chi,\psi \in \mathcal{A}) \\ 2^{\alpha\beta}(\chi\psi - \frac{1}{2^{\alpha\beta}}) & (\alpha = k^2, \beta = \ell^2, k,\ell \in \mathbb{N};\ \chi,\psi \in \mathcal{B}) \\ 0 & (\alpha = k^2, \beta = \ell^2, k,\ell \in \mathbb{N}; \chi,\psi \notin \mathcal{A}\cup\mathcal{B}) \\ \alpha\beta & (\text{elsewhere}), \end{cases} \tag{3}$$

where

$$\mathcal{A} = \left[2^{-(\alpha\beta-1)} - 2^{-\alpha\beta}, 2^{-(\alpha\beta-1)}\right] \quad \text{and} \quad \mathcal{B} = \left[2^{-\alpha\beta}, 2^{-(\alpha\beta-1)} - 2^{-\alpha\beta}\right].$$

Let us also assume that

$$q_{\alpha,\beta} = \begin{cases} 1 & (\alpha = k^2, \beta = k^2, k \in \mathbb{N}) \\ 0 & (\text{otherwise}). \end{cases}$$

Clearly, from Definition 1.2, we have

$$\delta_{\text{DP}}\left(\left\{\alpha \leqq \eta_\alpha, \beta \leqq \eta_\beta \quad \text{and} \quad \left|g_{\alpha,\beta} - g\right| \geqq \epsilon\right\}\right) = 0.$$

Thus, for any $\chi, \psi \in I$,

$$\lim_{0<\chi,\psi\to R^-} \frac{1}{q(\chi,\psi)} \sum_{\left\{\alpha\leqq\eta_\alpha,\beta\leqq\eta_\beta \text{ and } |g_{\alpha,\beta}-g|\geqq\epsilon\right\}} q_{\alpha,\beta}\chi^\alpha\psi^\beta \leqq \lim_{0<\chi,\psi\to R^-} \frac{1}{q(\chi,\psi)} q_{\alpha_0\beta_0}\chi^{\alpha_0}\psi^{\beta_0} = 0.$$

We thus obtain

$$g_{\alpha,\beta} \to g \quad (\text{equi} - \text{stat}_{\text{DP}}) \quad \text{on } I.$$

However, since

$$\| g_{\alpha,\beta} - g \|_{C(I)} \neq 0,$$

the sequence $(g_{\alpha,\beta})$ *fails to converge to* 0 *statistically or uniformly statistically under the DP- method.*

Example 1.9. *Let* $I = \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right]$, *and let*

$$g_{\alpha,\beta}(\chi,\psi) = \begin{cases} 0 & (\alpha = k^2,\ \beta = \ell^2;\ k, \ell \in \mathbb{N}) \\ \chi^{\alpha\beta}\psi^{\alpha\beta} & (\text{otherwise}), \end{cases}$$

and

$$\lim_{\alpha,\beta\to\infty} g_{\alpha,\beta}(\chi,\psi) = g(\chi,\psi) \quad (\chi,\psi \in I),$$

where

$$g(\chi,\psi) = \begin{cases} 0 & \chi,\psi \in [0,1) \\ 1 & (\chi = 1 \ \&\ \psi = 1). \end{cases}$$

Also let

$$q_{\alpha,\beta} = \begin{cases} 0 & (\alpha = k^2,\ \beta = \ell^2;\ k, \ell \in \mathbb{N}) \\ 1 & (\text{otherwise}). \end{cases}$$

Then

$$g_{\alpha,\beta} \to g \quad (\text{stat} - \text{point}_{\text{DP}}).$$

Furthermore, if we set $\epsilon = \frac{1}{2}$, *then for every* χ, ψ *with* $\psi \in \left(\sqrt[\alpha\beta]{\frac{1}{2}}, 1 \right)$ *and* $\psi \in \left(\sqrt[\alpha\beta]{\frac{1}{2}}, 1 \right)$,

$$|g_{\alpha,\beta}(\chi, \psi)| = |\chi^{\alpha\beta}\psi^{\alpha\beta}| > \left| \left(\sqrt[\alpha\beta]{\frac{1}{2}} \right)^{\alpha\beta} \left(\sqrt[\alpha\beta]{\frac{1}{2}} \right)^{\alpha\beta} \right| = \frac{1}{4}.$$

Therefore, we conclude that the statement

$$g_{\alpha,\beta} \to g \quad (\mathrm{equi} - \mathrm{stat}_{\mathrm{DP}})$$

is ultimately not valid.

2. A New Korovkin-Type Theorem

Korovkin-type approximation theorems provide fundamental tools in approximation theory, particularly for studying the convergence of sequences of linear positive operators. These theorems are named after Korovkin, who introduced the concept in the 1950s. The classical Korovkin theorem gives simple conditions under which a sequence of linear operators converges to the identity operator for continuous functions. Specifically, the theorem states that if a sequence of positive linear operators converges on a set of test functions-typically polynomials like 1, x and x^2 then it converges uniformly for all continuous functions on a given interval. These results have broad applications, ranging from numerical analysis to functional analysis, and have been extended to various settings, such as multivariate functions, abstract spaces, and quantum spaces. Korovkin-type theorems offer a powerful framework for ensuring convergence while only verifying the behavior of the operators on a small set of simple functions.

Recent studies have investigated Korovkin-type results through different approaches to statistical convergence techniques, as documented in sources such as [3], [10], [13], [15], [20], [37], [38], [39], [40], and [41]. Additionally, Balcerzak *et al.* [4] presented a powerful result using equi-statistical convergence in place of uniform statistical convergence. Numerous researchers have established different results based on equi-statistical convergence in various contexts (see, for instance, [11], [19], [26], [27], and [42]). Building on these advanced studies, we apply the DP-method to develop a Korovkin-type theorem that utilizes the notion of equi-statistical convergence for sequences of functions.

Let $\mathfrak{L}$ be a linear operator acting on $C(I)$. The operator $\mathfrak{L}$ is called a positive linear operator if, for $g(\chi, \psi) \geqq 0$ it follows that

$$\mathfrak{L}(g(s,t); \chi, \psi) \geqq 0.$$

Building upon certain approximation theorems, we aim to establish a new Korovkin-type theorem in this section by utilizing our proposed DP-method under the equi-statistical convergence of sequences of positive linear operators. To prove the desired theorem, we consider the following test functions:

$$g_0(\chi, \psi) = 1,\ g_1(\chi, \psi) = \frac{\chi}{1-\chi},\ g_2(\chi, \psi) = \frac{\psi}{1-\psi} \text{ and } g_3(\chi, \psi) = \left(\frac{\chi}{1-\chi}\right)^2 + \left(\frac{\psi}{1-\psi}\right)^2.$$

Before presenting the main results, we begin by revisiting the classical Korovkin-type theorem (see [18]) as well as several statistical Korovkin-type theorems developed within the context of power series methods (see [48] and [9]) as follows.

Theorem 2.1. (see [18]) *Let* $(\mathfrak{L}_\alpha)$ *be a sequences of positive linear operators on* $C(I)$. *For each* $g \in C(I)$, *the following is true:*

$$\lim_{\alpha\to\infty} \| \mathfrak{L}_\alpha(g) - g \|_{C(I)} = 0$$

if and only if

$$\lim_{\alpha\to\infty} \parallel \mathfrak{L}_\alpha(g_i;\chi) - g_i \parallel_{C(I)} = 0 \qquad (i = 0,1,2).$$

Theorem 2.2. (see [48]) *Let* $(\mathfrak{L}_\alpha)$ *be a sequences of positive linear operators on* $C(I)$. *For each* $g \in C(I)$, *the following is true:*

$$\mathrm{stat_P}\lim_{\alpha\to\infty} \parallel \mathfrak{L}_\alpha(g;\chi) - g \parallel_{C(I)} = 0$$

if and only if

$$\mathrm{stat_P}\lim_{\alpha\to\infty} \parallel \mathfrak{L}_\alpha(g_i;\chi) - g_i \parallel_{C(I)} = 0 \qquad (i = 0,1,2).$$

Theorem 2.3. (see [9]) *Let* $(\mathfrak{L}_\alpha)$ *be a sequence of (positive) linear operators on* $C(I)$. *For each* $g \in C(I)$, *the following is true:*

$$\mathfrak{L}_\alpha(g;\chi) \longrightarrow g \quad (\mathrm{equi-stat_P}) \text{ on } I \tag{4}$$

if and only if

$$\mathfrak{L}_\alpha(g_i;\chi) \longrightarrow g_i \quad (\mathrm{equi-stat_P})\ (i = 0,1,2). \tag{5}$$

As a key result of this study, we now present below a new Korovkin-type theorem.

Theorem 2.4. *Let* (ζ_β) and (η_β) *be sequences in* $\mathbb{Z}^{0+}$, *and let* $(\mathfrak{L}_{\alpha,\beta})$ *denote a sequence of linear operators on* $C(I)$. *For every* $g \in C(I)$, *the following is true:*

$$\mathfrak{L}_{\alpha,\beta}(g(s,t);\chi,\psi) \longrightarrow g(\chi,\psi) \quad (\mathrm{equi-stat_{DP}}) \text{ on } I \tag{6}$$

if and only if

$$\mathfrak{L}_{\alpha,\beta}(g_i(s,t);\chi,\psi) \longrightarrow g_i(\chi,\psi) \quad (\mathrm{equi-stat_{DP}})\ (i = 0,1,2,3). \tag{7}$$

Proof. Since $g_i(\chi,\psi) \in C(I)$ for $i = 0,1,2,3$ is continuous, the implication:

$$(6) \Longrightarrow (7)$$

is clearly trivial.

To finish the proof of Theorem 2.4, we start by assuming that (7) holds. Let $g \in C(I)$, and for all $\chi,\psi \in I$, there exists a constant $\mathcal{E}$ such that

$$\left|g(\chi,\psi)\right| \leqq \mathcal{E} \quad (\chi,\psi \in I), \tag{8}$$

we have

$$\left|g(s,t) - g(\chi,\psi)\right| \leqq 2\mathcal{E} \quad (s,t,\chi,\psi \in I).$$

Therefore, for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$|g(s,t) - g(\chi,\psi)| < \epsilon \quad (\forall\ \chi,\psi,s,t \in I), \tag{9}$$

whenever

$$\left|\frac{s}{1-s} - \frac{\chi}{1-\chi}\right| < \delta \quad \text{and} \quad \left|\frac{t}{1-t} - \frac{\psi}{1-\psi}\right| < \delta.$$

Using (8) and (9), we have

$$|g(s,t) - g(\chi,\psi)| < \epsilon + \frac{2\mathcal{E}}{\delta^2}([\vartheta(s,\chi)]^2 + \vartheta(t,\psi)]^2) \quad (\forall\ \chi,\psi,s,t \in I),$$

where

$$\vartheta(s,\chi) = \frac{s}{1-s} - \frac{\chi}{1-\chi} \quad \text{and} \quad \vartheta(t,\psi) = \frac{t}{1-t} - \frac{\psi}{1-\psi}.$$

As a consequence of the linearity and positivity of $\left(\mathfrak{L}_m(1,\chi,\psi)\right)$, we obtain

$$\begin{aligned}
|\mathfrak{L}_{\alpha,\beta}(g(s,t);\chi,\psi) - g(\chi,\psi)| &= |\mathfrak{L}_{\alpha,\beta}(g(s,t) - g(\chi,\psi);\chi,\psi) + g(\chi,\psi)[\mathfrak{L}_{\alpha,\beta}(g_0;\chi,\psi) - g_0]| \\
&\leqq \mathfrak{L}_{\alpha,\beta}(|g(s,t) - g(\chi,\psi)|;\chi,\psi) + \mathcal{E}|\mathfrak{L}_{\alpha,\beta}(g_0;\chi,\psi) - g_0| \\
&\leqq \left|\mathfrak{L}_{\alpha,\beta}\left(\epsilon + \frac{2\mathcal{E}}{\delta^2}([\vartheta(s,\chi)]^2 + \vartheta(t,\psi)]^2);\chi,\psi\right)\right| + \mathcal{E}|\mathfrak{L}_{\alpha,\beta}(g_0;\chi,\psi) - g_0| \\
&= \epsilon + (\epsilon + \mathcal{E})\left|\mathfrak{L}_{\alpha,\beta}(g_0;\chi,\psi) - g_0(\chi,\psi)\right| \\
&\quad + \frac{2\mathcal{E}}{\delta^2}\left|\mathfrak{L}_{\alpha,\beta}(g_3;\chi,\psi) - g_3(\chi,\psi)\right| - \frac{4\mathcal{E}}{\delta^2}\left(\frac{\chi}{1-\chi}\right)\left|\mathfrak{L}_{\alpha,\beta}(g_1;\chi,\psi) - g_1(\chi,\psi)\right| \\
&\quad - \frac{4\mathcal{E}}{\delta^2}\left(\frac{\chi}{1-\chi}\right)\left|\mathfrak{L}_{\alpha,\beta}(g_2;\chi,\psi) - g_2(\chi,\psi)\right| \\
&\quad + \frac{2\mathcal{E}}{\delta^2}\left(\left(\frac{\chi}{1-\chi}\right)^2 + \left(\frac{\psi}{1-\psi}\right)^2\right)\left|\mathfrak{L}_{\alpha,\beta}(g_0;\chi,\psi) - g_0(\chi,\psi)\right| \\
&= \epsilon + \left(\epsilon + \mathcal{E} + \frac{4\mathcal{E}}{\delta^2}\right)\left|\mathfrak{L}_{\alpha,\beta}(g_0;\chi,\psi) - g_0(\chi,\psi)\right| \\
&\quad + \frac{4\mathcal{E}}{\delta^2}\left|\mathfrak{L}_{\alpha,\beta}(g_1;\chi,\psi) - g_1(\chi,\psi)\right| + \frac{4\mathcal{E}}{\delta^2}\left|\mathfrak{L}_{\alpha,\beta}(g_2;\chi,\psi) - g_2(\chi,\psi)\right| \\
&\quad + \frac{2\mathcal{E}}{\delta^2}\left|\mathfrak{L}_{\alpha,\beta}(g_3;\chi,\psi) - g_3(\chi,\psi)\right|.
\end{aligned}$$

By considering supremum norm, we consequently get

$$\left\|\mathfrak{L}_{\alpha,\beta}(g(s,t);\chi,\psi) - g(\chi,\psi)\right\|_{C(I)} \leqq \epsilon + M\sum_{j=0}^{3}\|\mathfrak{L}_{\alpha,\beta}(g_i(s,t);\chi,\psi) - g_i(\chi,\psi)\|_{C(I)}, \tag{10}$$

where

$$M = \left\{\epsilon + \mathcal{E} + \frac{4\mathcal{E}}{\delta^2}\right\}.$$

Next, for $\lambda > 0$, we select $\epsilon > 0$ with $0 < \epsilon < \lambda$. Then, by setting

$$\mathcal{H}_{\alpha,\beta} = \left\{\alpha \leqq \eta_\alpha, \beta \leqq \eta_\beta \quad \text{and} \quad |\mathfrak{L}_{\alpha,\beta}(g(s,t);\chi,\psi) - g(\chi,\psi)| \geqq \lambda\right\}$$

and

$$\mathcal{H}_{i;\alpha,\beta} = \left\{\alpha \leqq \eta_\alpha, \beta \leqq \eta_\beta \quad \text{and} \quad |\mathfrak{L}_{\alpha,\beta}(g_i(s,t);\chi,\psi) - g_i(\chi,\psi)| \geqq \frac{\lambda-\epsilon}{3M}\right\}, \textit{and}$$

using (10), we thus obtain

$$\lim_{0<\chi,\psi\to R^-}\frac{1}{q(\chi,\psi)}\sum_{(\alpha,\beta)\in\mathcal{H}_{\alpha,\beta}} q_{\alpha\beta}\chi^\alpha\psi^\beta \leqq \sum_{i=0}^{3}\lim_{0<\chi,\psi\to R^-}\frac{1}{q(\chi,\psi)}\sum_{(\alpha,\beta)\in\mathcal{H}_{i;\alpha,\beta}} q_{\alpha\beta}\chi^\alpha\psi^\beta. \tag{11}$$

Finally, under the given assumption related to the implication in (7) and by applying Definition 1.5, the right-hand side of (11) approaches zero. Consequently, we obtain

$$\lim_{0<\chi,\psi\to R^-} \frac{1}{q(\chi,\psi)} \sum_{(\alpha,\beta)\in\mathcal{H}_{\alpha,\beta}} q_{\alpha\beta}\chi^{\alpha}\psi^{\beta} = 0.$$

Hence, the implication in (6) is indeed valid, thereby proving Theorem 2.4. □

3. Computational and Geometrical Approaches of Theorem 2.4

In light of Theorem 2.4, we provide a numerical example below using certain specific positive linear polynomials known as the Meyer-König and Zeller (or MKZ-operator) operator for two variables (see [2], [47]).

Example 3.1. *Let $I = \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right]$, and consider the MKZ operator $\mathfrak{M}_{\alpha,\beta}(g(s,t);\chi,\psi)$ of two variables defined as*

$$\mathfrak{M}_{\alpha,\beta}(g(s,t);\chi,\psi) = \frac{1}{\Psi_{\alpha,\beta}(\chi,\psi,s,t)} \sum_{\substack{i=0\\ j=0}}^{\infty} f\left(\frac{a_{i,\alpha}}{a_{i,\alpha}+b_{\alpha}}, \frac{c_{j,\beta}}{c_{j,\beta}+d_{\beta}}\right) \Gamma_{i,j}^{\alpha,\beta}(s,t)\chi^{i}\psi^{j} \tag{12}$$

for

$$0 \leqq \frac{a_{i,\alpha}}{a_{i,\alpha}+b_{\alpha}} \leqq A \quad \text{and} \quad 0 \leqq \frac{c_{j,\beta}}{c_{j,\beta}+d_{\beta}} \leqq B, \quad A, B \in (0,1)$$

where $\{\Psi_{\alpha,\beta}(\chi,\psi,s,t)\}_{\alpha,\beta\in\mathbb{N}}$ represents the multiple generating functions for the sequence of functions $\{\Gamma_{i,j}^{\alpha,\beta}(s,t)\chi^{i}\psi^{j}\}_{\mathbb{N}\cup\{0\}}$ such that

$$\Psi_{\alpha,\beta}(\chi,\psi,s,t) = \sum_{\substack{i=0\\ j=0}}^{\infty} \Gamma_{i,j}^{\alpha,\beta}(s,t)\chi^{i}\psi^{j} \quad \text{with} \quad \Gamma_{i,j}^{\alpha,\beta}(s,t) \geqq 0;\ \chi,\psi \in I.$$

Let us assume that the following conditions hold:

(i) $\Psi_{\alpha,\beta}(\chi,\psi,s,t) = (1-\chi)\Psi_{\alpha+1,\beta}(\chi,\psi,s,t)$

(ii) $a_{i+1,\alpha}\Gamma_{i+1,j}^{\alpha,\beta}(\chi,\psi) = b_{\alpha}\Gamma_{i,j}^{\alpha+1,\beta}(\chi,\psi)$

(iii) $a_{i+1,\alpha} = a_{i,\alpha+1} + \varphi_{\alpha},\ |\varphi_{\alpha}| \leqq \alpha_1 < \infty$ *and* $a_{0,\alpha} = 0$

(iv) $b_{\alpha} \to \infty,\ \frac{b_{\alpha+1}}{b_{\alpha}} \to \infty$ *and* $b_{\beta} \neq 0,\ \forall\ \beta \in \mathbb{N}$

(v) $\Psi_{\alpha,\beta}(\chi,\psi,s,t) = (1-\psi)\Psi_{\alpha,\beta+1}(\chi,\psi,s,t)$

(vi) $c_{j+1,\beta}\Gamma_{i,j+1}^{\alpha,\beta}(\chi,\psi) = d_{\beta}\Gamma_{i,j}^{\alpha,\beta+1}(\chi,\psi)$

(vii) $c_{j+1,\beta} = c_{i,\beta+1} + \varphi_{\beta},\ |\varphi_{\beta}| \leqq \beta_1 < \infty$ *and* $c_{0,\beta} = 0$

(viii) $d_{\beta} \to \infty,\ \frac{d_{\beta+1}}{d_{\beta}} \to \infty$ *and* $d_{\beta} \neq 0,\ \forall\ \beta \in \mathbb{N}.$

We now define $\mathfrak{L}_{\alpha,\beta}(g(s,t);\chi,\psi)$ as the sequence of linear operators obtained from the composition of the MKZ operators for two variables and sequences of functions, as follows:

$$\mathfrak{L}_{\alpha,\beta}(g(s,t);\chi,\psi) = \Big(1 + g_{\alpha,\beta}(\chi,\psi)\Big)\mathfrak{M}_{\alpha,\beta}(g(s,t);\chi,\psi) \qquad \Big(\chi,\psi \in I;\ g \in C(I)\Big), \tag{13}$$

where the sequence $(g_{\alpha,\beta})$ is specified by (3) with

$$q_{\alpha,\beta} = \begin{cases} 1 & (\alpha = k^2,\ \beta = \ell^2;\ k,\ell \in \mathbb{N}) \\ 0 & (\text{elsewhere}). \end{cases}$$

We then evaluate the positive linear operators $\mathfrak{L}_{\alpha,\beta}(g_i(s,t);\chi,\psi)$ for each values of $i = 0,1,2,3$, which are given by:

$$\begin{aligned}\mathfrak{L}_{\alpha,\beta}(g_0;\chi,\psi) &= \Big(1 + g_{\chi,\psi}(\chi,\psi)\Big)\mathfrak{M}_{\alpha,\beta}(1;\chi,\psi)\\ &= \Big(1 + g_{\alpha,\beta}(\chi,\psi)\Big)\cdot 1,\end{aligned}$$

$$\begin{aligned}\mathfrak{L}_{\alpha,\beta}(g_1;\chi,\psi) &= \Big(1 + g_{\alpha,\beta}(\chi,\psi)\Big)\mathfrak{M}_{\alpha,\beta}(g_1;\chi,\psi)\\ &= \Big(1 + g_{\alpha,\beta}(\chi,\psi)\Big)\frac{\chi}{1-\chi},\end{aligned}$$

$$\begin{aligned}\mathfrak{L}_{\alpha,\beta}(g_2;\chi,\psi) &= \Big(1 + g_{\alpha,\beta}(\chi,\psi)\Big)\mathfrak{M}_{\alpha,\beta}(g_2;\chi,\psi)\\ &= \Big(1 + g_{\alpha,\beta}(\chi,\psi)\Big)\frac{\psi}{1-\psi}\end{aligned}$$

and

$$\begin{aligned}\mathfrak{L}_{\alpha,\beta}(g_3;\chi,\psi) &= \Big(1 + g_{\alpha,\beta}(\chi,\psi)\Big)\mathfrak{M}_{\alpha,\beta}(g_3;\chi,\psi)\\ &= \Big(1 + g_{\alpha,\beta}(\chi,\psi)\Big)\frac{\chi^2}{1-\chi^2}\frac{b_{\alpha+1}}{b_\alpha} + \frac{\psi^2}{1-\psi^2}\frac{d_{\beta+1}}{d_\beta} + \frac{\chi}{1-\chi}\frac{\varphi_\alpha}{b_\alpha} + \frac{\psi}{1-\psi}\frac{\xi_\beta}{d_\beta},\end{aligned}$$

where

$$\mathfrak{M}_{\alpha,\beta}(1;\chi,\psi) = 1,\ \ \mathfrak{M}_{\alpha,\beta}\left(\frac{s}{1-s};\chi,\psi\right) = \frac{\chi}{1-\chi},\ \ \mathfrak{M}_{\alpha,\beta}\left(\frac{t}{1-t};\chi,\psi\right) = \frac{\psi}{1-\psi},$$

and

$$\mathfrak{M}_{\alpha,\beta}\left\{\left(\frac{s}{1-s}\right)^2 + \left(\frac{t}{1-t}\right)^2;\chi,\psi\right\} = \frac{\chi^2}{1-\chi^2}\frac{b_{\alpha+1}}{b_\alpha} + \frac{\psi^2}{1-\psi^2}\frac{d_{\beta+1}}{d_\beta} + \frac{\chi}{1-\chi}\frac{\varphi_\alpha}{b_\alpha} + \frac{\psi}{1-\psi}\frac{\xi_\beta}{d_\beta}.$$

Since

$$g_{\alpha,\beta} \to g = 0 \quad (\text{equi} - \text{stat}_{\text{DP}}) \text{ on } I,$$

for the sequence $(g_{\alpha,\beta})$, as specified in Example 1.8, it follows that

$$\mathfrak{L}_{\alpha,\beta}(g_i(s,t);\chi,\psi) \to g_i(\chi,\psi) \quad (\text{equi} - \text{stat}_{\text{DP}}) \text{ on } I,$$

for each value of $i = 0,1,2,3$. Thus, according to Theorem 2.4, it can be observed that

$$\mathfrak{L}_{\alpha,\beta}(g(s,t);\chi,\psi) \to g(\chi,\psi) \quad (\text{equi} - \text{stat}_{\text{DP}}) \text{ on } I$$

for any $f \in C(I)$.

It is observed that the sequence $(g_{\alpha,\beta})$ defined in (3) does not converge statistically and uniformly to $g = 0$ over I using the DP-method. As a result, the findings of Srivastava *et al.* [36], Demirci *et al.* [9], and Ünver and Orhan [48] are not applicable to the operators $\left(\mathfrak{L}_{\alpha,\beta}(g; \chi, \psi)\right)$ introduced in (13). Moreover, since $(g_{\alpha,\beta})$ fails to converge uniformly to $g = 0$ in the conventional sense on I, the classical Korovkin's theorem [18] is also inapplicable. Therefore, the operators in (13) satisfy Theorem 2.4. We use MATLAB to illustrate the behavior of $\mathfrak{L}_{\alpha,\beta}(g_i; \chi, \psi)$, for $i = 0, 1, 2, 3$ in Figures 1 to 4.

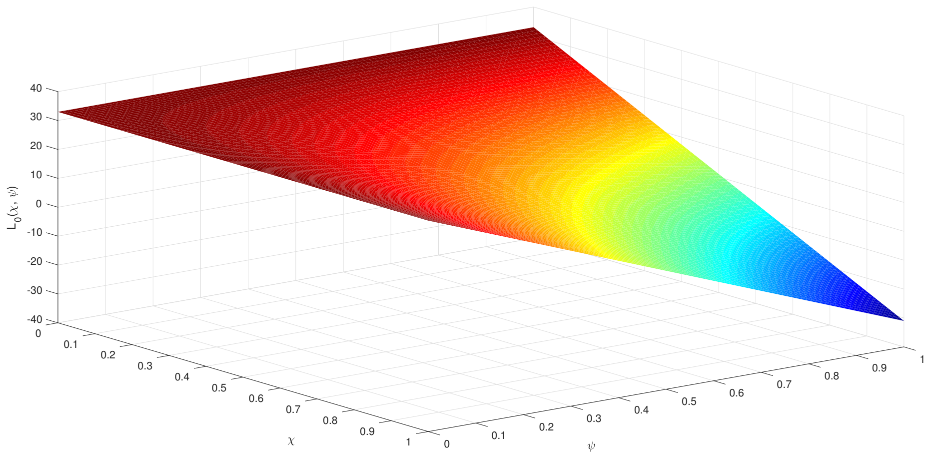

Figure 1: Behavior of the operator $\mathfrak{L}_{\alpha,\beta}(g_0; \chi, \psi)$

In Figure 1, the operator represents the simplest form, where the output is proportional to $\left(1 + g_{\alpha,\beta}(\chi, \psi)\right)$. The plot will show the effect of $g_{\alpha,\beta}(\chi, \psi)$ on the constant function. The shape of the surface indicates how the perturbation caused by $g_{\alpha,\beta}$ affects the identity operator.

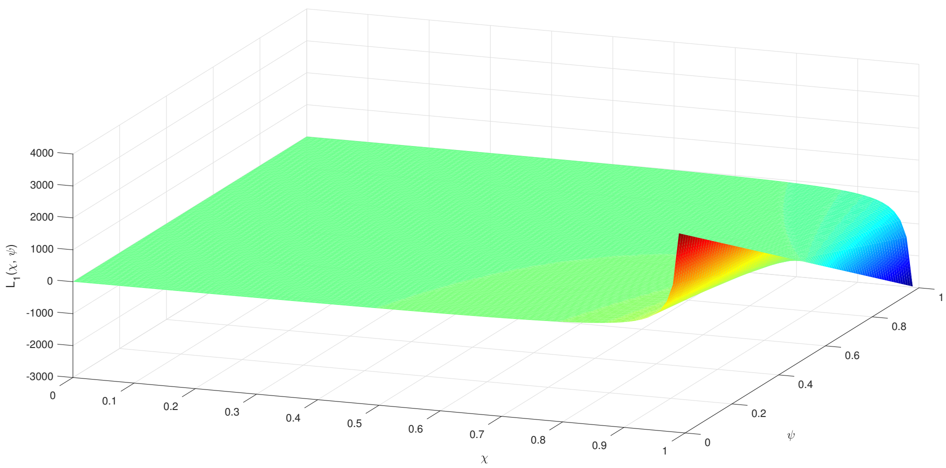

Figure 2: Behavior of the operator $\mathfrak{L}_{\alpha,\beta}(g_1; \chi, \psi)$

In Figure 2, the operator involves the term $\frac{\chi}{1-\chi}$, which increases rapidly as χ approaches 1. The plot will exhibit rapid growth near $\chi = 1$, showcasing how the function blows up near this boundary. It provides insight into the operator's behavior near critical values of χ.

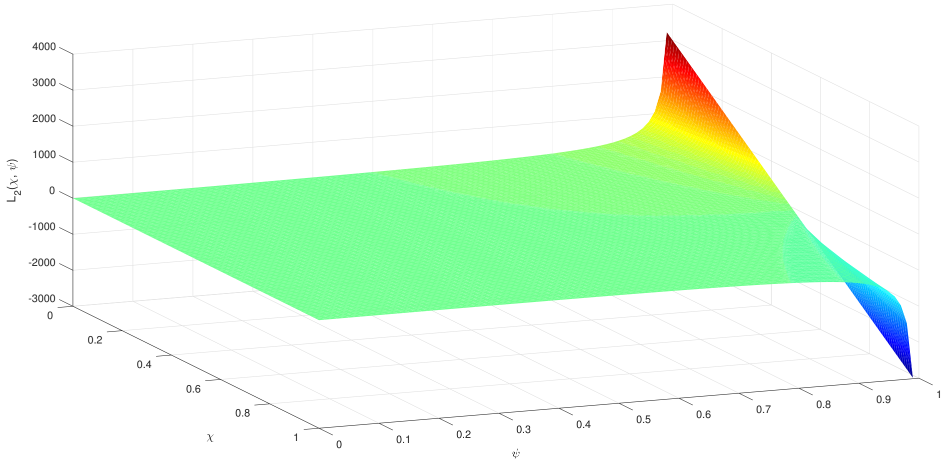

Figure 3: Behavior of the operator $\mathfrak{L}_{\alpha,\beta}(g_2; \chi, \psi)$

In Figure 3, the operator behaves similarly to Figure 2, but with respect to ψ. The term $\frac{\psi}{1-\psi}$ dominates as $\psi \to 1$. The plot displays a similar rapid increase near $\psi = 1$, showing how the operator behaves near the boundary.

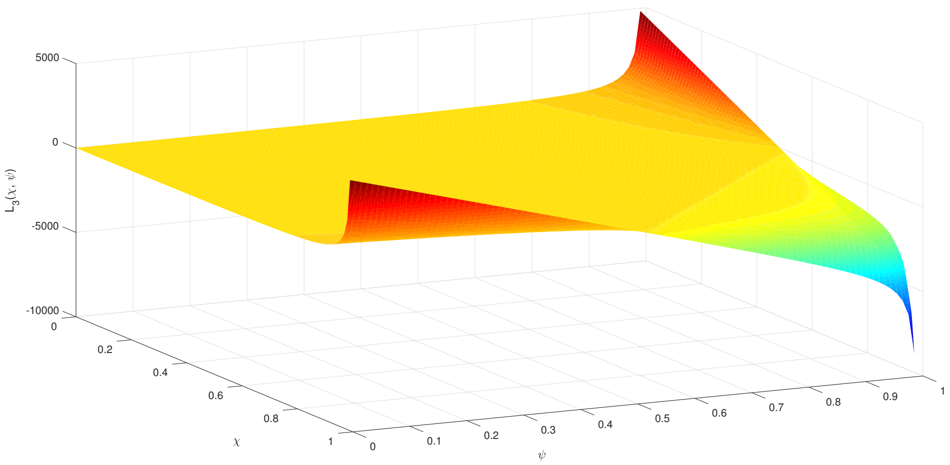

Figure 4: Behavior of the operator $\mathfrak{L}_{\alpha,\beta}(g_3; \chi, \psi)$

The Figure 4, is the most complex operator, involving both quadratic terms $\frac{\chi^2}{1-\chi^2}$ and $\frac{\psi^2}{1-\psi^2}$, along with linear terms. The plot has a more intricate shape, with noticeable growth near both $\chi = 1$ and $\psi = 1$. The combination of quadratic and linear terms will influence the curvature of the surface.

Overall, these plots demonstrate how the positive linear operators behave as a function of χ and ψ, especially in regions where χ and ψ approach critical values (close to 1). The rapid growth in these regions suggests that the operators exhibit significant sensitivity near the boundaries of the domain, which may be crucial for understanding their convergence properties as $g_{\alpha,\beta}$ tends to $g(\chi,\psi)$.

4. Rate of Equi-Statistical Convergence

We aim to explore the rates of ($\text{equi}-\text{stat}_{\text{DP}}$)-method for sequences of positive linear operators, with a focus on the modulus of continuity.

Definition 4.1. *Let $(a_{\alpha,\beta})$ be a positive, non-increasing sequence. If for each $\epsilon > 0$,*

$$\lim_{0<s,t\to R^-} \frac{1}{a_{\alpha,\beta}q(s,t)} \sum_{\alpha,\beta\in\mathcal{K}_{\alpha,\beta}(\chi,\psi;\epsilon)} q_{\alpha,\beta}s^\alpha t^\beta = 0 \text{ (uniformly) in } \chi,\psi \in I,$$

then the sequence $(g_{\alpha,\beta})$ converges equi-statistically to g using the DP- method, with a rate of convergence of $o(a_{\alpha,\beta})$. We denote this as

$$g_{\alpha,\beta} - g = o(a_{\alpha,\beta}) \quad (\text{equi} - \text{stat}_{\text{DP}}) \text{ on } I.$$

Prior to presenting the theorem on rates of equi-statistical convergence, we will first deduce Lemma 4.2.

Lemma 4.2. *Let $(g_{\alpha,\beta})$ and $(h_{\alpha,\beta})$ be sequences in $C(I)$ such that*

$$g_{\alpha,\beta}(\chi,\psi) - g(\chi,\psi) = o(u_{\alpha,\beta}) \quad (\text{equi} - \text{stat}_{\text{DP}}) \text{ on } I$$

and

$$h_{\alpha,\beta}(\chi,\psi) - h(\chi,\psi) = o(v_{\alpha,\beta}) \quad (\text{equi} - \text{stat}_{\text{DP}}) \text{ on } I.$$

As a result, each of the following statements is true:

(i) $[g_{\alpha,\beta}(\chi,\psi) + h_{\alpha,\beta}(\chi,\psi)] - [g(\chi,\psi) + h(\chi,\psi)] = o(w_{\alpha,\beta}) \quad (\text{equi} - \text{stat}_{\text{DP}}) \text{ on } I$

(ii) $[g_{\alpha,\beta}(\chi,\psi) - g(\chi,\psi)][h_{\alpha,\beta}(\chi,\psi) - h(\chi,\psi)] = o(u_{\alpha,\beta}v_{\alpha,\beta}) \quad (\text{equi} - \text{stat}_{\text{DP}}) \text{ on } I$

(iii) $\lambda[g_{\alpha,\beta}(\chi,\psi) - g(\chi,\psi)] = o(u_{\alpha,\beta}) \quad (\text{equi} - \text{stat}_{\text{DP}}) \text{ on } I$ for any scalar λ

(iv) $\sqrt{|g_{\alpha,\beta}(\chi,\psi) - g(\chi,\psi)|} = o(v_{\alpha,\beta}) \quad (\text{equi} - \text{stat}_{\text{DP}}) \text{ on } I,$

where

$$w_{\alpha,\beta} = \max\{u_{\alpha,\beta}, v_{\alpha,\beta}\}. \tag{14}$$

Proof. To prove assertion (i), let $\chi,\psi \in I$ and $\epsilon > 0$. The following sets are defined in the following way:

$$\mathfrak{A}_{\alpha,\beta}(\chi,\psi;\epsilon) = \left|\left\{\alpha \leqq \eta_\alpha, \beta \leqq \eta_\beta \quad \text{and} \quad |(g_{\alpha,\beta} + h_{\alpha,\beta})(\chi,\psi) - (g+h)(\chi,\psi)| \geqq \epsilon\right\}\right|,$$

$$\mathfrak{A}_{0;\alpha,\beta}(\chi,\psi;\epsilon) = \left|\left\{\alpha \leqq \eta_\alpha, \beta \leqq \eta_\beta \quad \text{and} \quad |g_{\alpha,\beta}(\chi,\psi) - g(\chi,\psi)| \geqq \frac{\epsilon}{2}\right\}\right|$$

and

$$\mathfrak{A}_{1;\alpha,\beta}(\chi,\psi;\epsilon) = \left|\left\{\alpha \leqq \eta_\alpha, \beta \leqq \eta_\beta \quad \text{and} \quad |h_{\alpha,\beta}(\chi,\psi) - h(\chi,\psi)| \geqq \frac{\epsilon}{2}\right\}\right|.$$

Clearly, this implies that

$$\mathfrak{A}_{\alpha,\beta}(\chi,\psi;\epsilon) \leqq \mathfrak{A}_{0;\alpha,\beta}(\chi,\psi;\epsilon) + \mathfrak{A}_{1;\alpha,\beta}(\chi,\psi;\epsilon).$$

Furthermore, given that

$$w_{\alpha,\beta} = \max\{u_{\alpha,\beta}, v_{\alpha,\beta}\},$$

and under the condition (6) of Theorem 2.4, we consequently obtain

$$\lim_{0<\chi,\psi\to R^-} \frac{1}{q(\chi,\psi)} \sum_{\alpha,\beta\in\mathfrak{A}_{\alpha,\beta}(\chi,\psi;\epsilon)} q_{\alpha\beta}\chi^{\alpha}\psi^{\beta} \leqq \lim_{0<\chi,\psi\to R^-} \frac{1}{q(\chi,\psi)} \sum_{\alpha,\beta\in\mathfrak{A}_{0;\alpha,\beta}(\chi,\psi;\epsilon)} q_{\alpha\beta}\chi^{\alpha}\psi^{\beta}$$
$$+ \lim_{0<\chi,\psi\to R^-} \frac{1}{q(\chi,\psi)} \sum_{\alpha,\beta\in\mathfrak{A}_{1;\alpha,\beta}(\chi,\psi;\epsilon)} q_{\alpha\beta}\chi^{\alpha}\psi^{\beta}.$$

Moreover, considering condition (7) of Theorem 2.4, we have

$$\lim_{0<\chi,\psi\to R^-} \frac{1}{q(\chi,\psi)} \sum_{\alpha,\beta\in\mathfrak{A}_{\alpha,\beta}(\chi,\psi;\epsilon)} q_{\alpha\beta}\chi^{\alpha}\psi^{\beta} = 0.$$

This completes the proof for condition (i). Conditions (ii) to (iv) are analogous to condition (i), so their details are omitted. Hence, the proof of Lemma 4.2 is complete. □

We now revisit the modulus of continuity $\omega(g,\mu)$ for a function $g \in C(I)$, which is defined as follows:

$$\omega(g,\mu) = \sup_{\chi,\psi,s,t\in I} \{|g(s,t) - g(\chi,\psi)| : |(s,t) - (\chi,\psi)| \leqq \mu\}, \textit{and}$$

demonstrate the following theorem.

Theorem 4.3. *Let* $\left(\mathfrak{L}_{\alpha,\beta}(g(s,t);\chi,\psi)\right) : C(I) \to C(I)$ *denote a sequence of positive linear operators. Let us assume that the following conditions hold:*

(i) $\mathfrak{L}_{\alpha,\beta}(1;\chi,\psi) - 1 = o(u_{\alpha,\beta}) \quad (\text{equi}-\text{stat}_{\text{DP}})$ on I;

(ii) $\omega(g,\mu_{\alpha,\beta}) = o(v_{\alpha,\beta}) \quad (\text{equi}-\text{stat}_{\text{DP}})$ on I,

where

$$\mu_{\alpha,\beta}(\chi,\psi) = \sqrt{\mathfrak{L}_{\alpha,\beta}(\vartheta^2;\chi,\psi)} \quad \text{with} \quad \vartheta(\chi,\psi) = \left(\frac{s}{1-s} - \frac{\chi}{1-\psi}\right)^2 + \left(\frac{t}{1-t} - \frac{\psi}{1-\psi}\right)^2.$$

Then, for $g \in C(I)$*, the following assertion holds:*

$$\mathfrak{L}_{\alpha,\beta}(g(s,t);\chi,\psi) - g(\chi,\psi) = o(w_{\alpha,\beta}) \quad (\text{equi}-\text{stat}_{\text{DP}}) \text{ on } I, \tag{15}$$

where $w_{\alpha,\beta}$ *is defined in* (14).

Proof. : Let $g \in C(I)$ and $\chi,\psi \in I$. Thus, we have:

$$|\mathfrak{L}_{\alpha,\beta}(g(s,t);\chi,\psi) - g(\chi,\psi)| \leqq \mathcal{M}|\mathfrak{L}_{\alpha,\beta}(1;\chi,\psi) - 1| + \left(\mathfrak{L}_{\alpha,\beta}(1;\chi,\psi) + \sqrt{\mathfrak{L}_{\alpha,\beta}(1;\chi,\psi)}\right)\omega(g,\mu_{\alpha,\beta}),$$

where

$$\mathcal{M} = \|g\|_{C[I]}.$$

This clearly results in

$$|\mathfrak{L}_{\alpha,\beta}(g;\chi,\psi) - g(\chi,\psi)| \leqq \mathcal{M}|\mathfrak{L}_{\alpha,\beta}(1;\chi,\psi) - 1| + 2\omega(g,\mu_{\alpha,\beta})$$
$$+ \omega(g,\mu_{\alpha,\beta})|\mathfrak{L}_{\alpha,\beta}(1;\chi,\psi) - 1| + \omega(g,\mu_{\alpha,\beta})\sqrt{|\mathfrak{L}_{\alpha,\beta}(1;\chi,\psi) - 1|}. \tag{16}$$

Given the conditions (i) and (ii) of Theorem 4.3 and applying Lemma 4.2, the final inequality in (16) allows us to derive the conclusion in (15) of Theorem 4.3. Consequently, Theorem 4.3 is proved. □

5. Conclusion and Future Scope

In this section, we provide several additional remarks and observations related to the various results we have established.

Remark 5.1. *Let $(g_{\alpha,\beta})_{\alpha,\beta\in\mathbb{N}}$ be the sequence of functions provided in Example 1.8. Then, since*

$$g_{\alpha,\beta} \to g \quad (\text{equi} - \text{stat}_{\text{DP}}) \quad \text{on } I,$$

we immediately have

$$\mathfrak{L}_{\alpha,\beta}(g_i; \chi, \psi) \to g_i(\chi, \psi) \quad (\text{equi} - \text{stat}_{\text{DP}}) \text{ on } \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right] \quad (i = 0, 1, 2, 3). \tag{17}$$

Thus, by applying Theorem 2.4, we obtain

$$\mathfrak{L}_{\alpha,\beta}(g; \chi, \psi) \to g_i(\chi, \psi) \quad (\text{equi} - \text{stat}_{\text{DP}}) \text{ on } I \tag{18}$$

for all $g \in C(I)$. Additionally, since $(g_{\alpha,\beta})$ does not converge uniformly statistically to $g = 0$ on I using the DP-method, and is not simply uniformly convergent, the classical Korovkin-type theorem does not apply to the operator described in (13). Therefore, these observations indicate that Theorem 2.4 provides a significant generalization of several well-established results (see [18], [26], and [27]).

Remark 5.2. *By substituting $(\zeta_{\alpha,0}) = 0$ and $(\eta_{\alpha,0}) = \alpha$ into our main Theorem 2.4, we recover the previously published results by Srivastava et al. [36], Demirci et al. [9], and Ünver and Orhan [48]. In this regard, Theorem 2.4 can be considered a significant generalization of these earlier results (see [9] and [48]).*

Remark 5.3. *In this study, we introduced the concept of statistical convergence for functions of two variables using the DP-technique. We defined several new concepts and established new theorems based on these definitions. We also estimated the rates of equi-statistical convergence for functions of two variables under our proposed DP-method applied to sequences of positive linear operators.*

Inspired by the work of Srivastava et al. [36] and Demirci et al. [9], we highlight the potential for developing Korovkin-type approximation theorems in both sequence spaces and probability spaces. Moreover, in light of the findings by Paikray et al. [26] and Saini and Raj [30], we encourage further research into fuzzy approximation theorems. Additionally, the recent contributions of Srivastava et al. [34, 35] underscore the scope for advancing Korovkin-type theorems through the formulation of a new class in double sequence spaces and the introduction of the statistical derivative of deferred weighted summability means for positive linear operators.

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