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On deferred statistical summability in neutrosophic n-normed linear space

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Abstract. In this research article, we introduce and define strong deferred summability and deferred statistical summability within the framework of neutrosophic-n-normed linear spaces (abbreviated as N-n-NLS), and we thoroughly explore their properties. Furthermore, we introduce the concepts of deferred statistical Cauchy sequences and deferred statistical completeness and provide a characterization of deferred statistical summability in these spaces. As a final step, we conduct a comparative analysis of deferred statistical summability for various sequence pairs $\alpha(\ell)$, $\beta(\ell), m(\ell)$ and $n(\ell)$, under the conditions $\alpha(\ell) \leq m(\ell) < n(\ell) \leq \beta(\ell)$.

1. Introduction

The concept of fuzzy logic to address uncertainty in various mathematical problems was introduced by Zadeh [50], In 1965. This concept proved to be particularly useful in areas such as population dynamics [6], chaos control [14], computer programming [17], and non-linear dynamical systems [20], among others. The development of fuzzy sets has led to the growth of numerous ideas in mathematical analysis.

One such concept is the idea of the fuzzy norm, initially introduced by Katsaras [23] as a generalization of the crisp norm for problems where norms cannot be precisely predicted due to significant uncertainty. This concept was redefined by Felbin [13] in 1992, using the idea of fuzzy real numbers. He demonstrated that a unique fuzzy norm exists, up to fuzzy equivalence, within finite-dimensional fuzzy normed linear spaces. Building on the concept of n-norm introduced by Gunawan and Mashadi [38], Vijayabalaji and Narayanan [18] introduced the idea of a fuzzy n-norm.

In their work, they obtained several significant results related to best approximation sets within α-n-normed spaces. Readers who wish to delve deeper into this subject are encouraged to refer to [4],[7], [15],[16],[24],[28],[31] for additional studies and discussions.

Atanassov [2],[3] recognized that the concept of a fuzzy set was insufficient for addressing many issues, such as complex uncertainty, ambiguity, and hesitation in decision-making processes. To improve upon

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this, he introduced intuitionistic fuzzy sets (IFS) as a generalization of fuzzy sets. Saadati and Park [41] expanded this concept by using IFS to develop the intuitionistic fuzzy norm (IFN), which includes both the non-membership function ψ and the membership function φ. Several extensions of IFNLS have since been published in the literature.

In their research, Mursaleen and Lohani [36] defined IF-2-NLS, contributing to the study of their topological properties. Subsequently, Vijayabalaji et al. [49] expanded this framework by defining IF-n-NLS, concentrating on the properties of convergent and Cauchy sequences in these normed linear spaces. In the past few year, researchers have looked into several summability methods like statistical convergence [22], lacunary statistical convergence [37], ideal convergence [32], λ-statistical convergence [26], Generalized weighted statistical convergence [59], and deferred statistical convergence [19],[27],[29] within Intuitionistic Fuzzy Normed Linear Spaces (IFNLS). More recently, these summability concepts were further extended and generalized in the broader context of Intuitionistic Fuzzy n-Normed Linear Spaces (IF-n-NLS), as detailed in the literature (see [9],[45],[46]). For readers seeking a thorough exploration of deferred statistical convergence, references such as [1],[5],[11],[25],[27],[39],[40],[47] offer extensive insights.

Decision-makers often face uncertainty, complicating decision-making processes. Some scenarios, like sports events or voting, involve three potential outcomes, requiring a nuanced approach. In 2005, Smarandache [57] introduced the neutrosophic set, combining fuzzy and intuitionistic fuzzy sets. Each element in this set is defined by truth (T), indeterminacy (I), and falsity (F) membership functions, offering a more flexible framework for decision-making.

Recently, Kirisci and Simsek [54] made a significant contribution by introducing neutrosophic normed linear spaces and exploring the concept of statistical convergence in this new context. Their work has inspired further research, with various scholars investigating multiple notions of sequence convergence in neutrosophic normed spaces. Those interested in a deeper exploration of these developments can consult additional studies [52],[55],[56],[58],[62],[63],[64].

In the present research article, we aim to explore the concepts of stronge deferred summability and statistical summability within the franework of N-n-NLS. we will systematically examine the properties of these summability concepts, highlighting their unique characteristics and potential applications. Through this analysis,we aim to reveal insights that contribute to the understanding of summability in N-n-NLS and their relevance to broader mathematical theories. This exploration will not only clarify these concepts but also enhance their applicability in future research.

2. Preliminaries

In this section delves into several fundamental definitions and key results necessary for the subsequent analysis. Our starting point is the concept of statistical convergence, fist independently introduced by Fast [48] and Steinhaus [12] through the application of natural density Δ for subsets of the set of positive integer $\mathbb{N}$. For any subset $\mathbb{H} \subseteq \mathbb{N}$, the natural density $\Delta(\mathbb{H})$ is defined by

$$\Delta(\mathbb{H}) = \lim_{\ell\to\infty} \frac{1}{\ell} |k \leq \ell : k \in \mathbb{H}|$$

provided the limit exists, where vertical bars indicate the cardinality of the enclosed set. A sequence $p = (p_k)$ is said to be statistically convergent to p_0 if for each $\varepsilon > 0$,

$$\lim_{\ell\to\infty} \frac{1}{\ell} \left|\left\{k \leq \ell : \left|p_k - p_0\right| \geq \varepsilon\right\}\right| = 0$$

i.e., $\Delta\left(\left\{k \in \mathbb{N} : \left|p_k - p_0\right| \geq \varepsilon\right\}\right) = 0$, which is denoted as $S - \lim p_k = p_0$. Statistical convergence has been used in many areas, like measure theory [10],[60], approximation theory [30],[61], trigo-nometric series [51]. It has also been investigate from the standpoint of sequence spaces and it connected to summability theory by researcher such as (e.g., "Connor [8]", "Salat [42]", "Schoenberg [43]") and also see [21], [33]-[35].

Definition 2.1. *[53]. Let $\mathfrak{I} = [0,1]$. A binary operation $\star : \mathfrak{I} \times \mathfrak{I} \to \mathfrak{I}$ is j-norm if for all $\mathfrak{s}, \mathfrak{e}, \mathfrak{h}, \mathfrak{r} \in \mathfrak{I}$ we have*
(1) $\star$ *is continuous commutative and associative.*
(2) $\mathfrak{s} = \mathfrak{s} \star 1$.
(3) $\mathfrak{s} \star \mathfrak{e} \leq \mathfrak{h} \star \mathfrak{r}$ *whenever* $\mathfrak{s} \leq \mathfrak{h}$ *and* $\mathfrak{e} \leq \mathfrak{r}$.

Definition 2.2. *[53]. Let $\mathfrak{R} = [0,1]$. A binary operation $\diamond : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$ is j-norm if for all $\mathfrak{s}, \mathfrak{e}, \mathfrak{h}, \mathfrak{r} \in \mathfrak{R}$ we have*
(1) $\diamond$ *is continuous commutative and associative.*
(2) $\mathfrak{s} = \mathfrak{s} \diamond 0$.
(3) $\mathfrak{s} \diamond \mathfrak{e} \leq \mathfrak{h} \diamond \mathfrak{r}$ *whenever* $\mathfrak{s} \leq \mathfrak{h}$ *and* $\mathfrak{e} \leq \mathfrak{r}$.

Using j-norm and j-conorm, Shyamal Debnath, Santonu Debnath and Chiranjib Choudhury [53] On Deferred Statistical Convergence of Sequences in Neutrosophic Normed Spaces as follows:

Definition 2.3. . *The five-tuple structure $(\mathcal{P}, \sigma, \tau, \upsilon, \star, \diamond)$ be an NNLS, where $\mathcal{P}$ is a linear space over a field F. σ, τ, υ are called neutrosophic normed space (NNS) on $\mathcal{P} \times (0,\infty)$ and represent the degree of membership and non-membership on $\mathcal{P} \times (0,1)$ if the following conditions hold, for every $p, w \in \mathcal{P}$ and $\lambda_1, \lambda_2 > 0$:*
(1) $\sigma(p,\lambda) + \tau(p,\lambda) \leq 1$.
(2) $\sigma(p,\lambda) > 0$.
(3) $\sigma(p,\lambda) = 1 \Leftrightarrow p = 0$.
(4) $\sigma(cp,\lambda) = \sigma\left(p, \frac{\lambda}{|c|}\right)$, *if* $c \neq 0, c \in F$.
(5) $\sigma(p,\lambda_1) \star \sigma(w,\lambda_2) \leq \sigma(p+w, \lambda_1+\lambda_2)$.
(6) $\sigma(p,.) : (0,\infty) \to [0,1]$ *is continuous.*
(7) $\lim_{\lambda\to\infty} \sigma(p,\lambda) = 1, \lim_{\lambda\to 0} \sigma(p,\lambda) = 0$.
(8) $\tau(p,\lambda) < 1$.
(9) $\tau(p,\lambda) = 0 \Leftrightarrow p = 0$.
(10) $\tau(cp,\lambda) = \tau\left(p, \frac{\lambda}{|c|}\right)$ *if* $c \neq 0, c \in F$.
(11) $\tau(p,\lambda_1) \diamond \tau(w,\lambda_2) \geq \tau(p+w, \lambda_1+\lambda_2)$.
(12) $\tau(p,.) : (0,\infty) \to [0,1]$ *is continuous.*
(13) $\lim_{\lambda\to\infty} \tau(p,\lambda) = 0, \lim_{\lambda\to 0} \tau(p,\lambda) = 1$.
(14) $\upsilon(p,\lambda) = 0 \Leftrightarrow p = 0$.
(15) $\upsilon(cp,\lambda) = \tau\left(p, \frac{\lambda}{|c|}\right)$ *if* $c \neq 0, c \in F$.
(16) $\upsilon(p,\lambda_1) \diamond \tau(w,\lambda_2) \geq \tau(p+w, \lambda_1+\lambda_2)$.
(17) $\upsilon(p,.) : (0,\infty) \to [0,1]$ *is continuous.*
(18) $\lim_{\lambda\to\infty} \upsilon(p,\lambda) = 0, \lim_{\lambda\to 0} \tau(p,\lambda) = 1$.
Then σ, τ, υ are called neutrosophic normed (NN).

Example 2.4. *Let $(\mathcal{P}, \|\bullet\|)$ be an NLS. Define $\mathfrak{s} \star \mathfrak{e} = \mathfrak{s}\mathfrak{e}$, $\mathfrak{s} \diamond \mathfrak{e} = \min\{\mathfrak{s}+\mathfrak{e}, 1\} \forall\, \mathfrak{s}, \mathfrak{e} \in [0,1]$.Consider $\sigma(p,\lambda) = \frac{\lambda}{\lambda+\|p\|}$ and $\tau(p,\lambda) = \frac{\|p\|}{\lambda+\|p\|}$ for every $\lambda > 0$, and $\upsilon(p,\lambda) = \frac{\|p\|}{\lambda}$ for every $\lambda > 0$ and $p \in \mathcal{P}$. Then $(\mathcal{P}, \sigma, \tau, \upsilon.\star, \diamond)$ be an NNLS. By using the idea of n-norm [53] Shyamal Debnath, Santonu Debnath and Chiranjib Choudhury, on deferred statistical convergence of sequences in neutrosophic normed paces(NNS). He also studied statistical convergent and statistical Cauchy sequences in these spaces.*

Definition 2.5. *The five-tuple structure $(\mathcal{P}, \sigma, \tau, \upsilon, \star, \diamond)$ be an N-n-NLS, where $\mathcal{P}$ is a linear space over a field F. σ, τ, υ are called neutrosophic normed space(NNS) on $\mathcal{P}^n \times (0,\infty)$ and represent the degree of membership and non-membership of $(p_1, p_2, \ldots, p_{n-1}, \quad p_n, \lambda) \in \mathcal{P}^n \times (0,1)$ if the following conditions are hold, for every $(p_1, p_2, \ldots, p_{n-1}, \quad p_n) \in \mathcal{P}^n$ and $\lambda_1, \lambda_2 > 0$:*
(1) $\sigma(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda) + \tau(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda) \leq 1$.
(2) $\sigma(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda) > 0$.
(3) $\sigma(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda) = 1 \quad \Leftrightarrow \quad p_1, p_2, \ldots, p_{n-1}, p_n$ *are linearly dependent.*
(4) $\sigma(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda)$ *is invariant under any permutation of* $p_1, p_2, \ldots, p_{n-1}, p_n$.
(5) $\sigma(p_1, p_2, \ldots, p_{n-1}, cp_n, \lambda) = \sigma\left(p_1, p_2, \ldots, p_{n-1}, p_n, \frac{\lambda}{|c|}\right)$ *if* $c \neq 0$ $c \in F$.

(6) $\sigma(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda_1) \star \sigma(p_1, p_2, \ldots, p_{n-1}, p'_n, \lambda_2) \leq$
$\sigma(p_1, p_2, \ldots, p_{n-1}, p_n + p'_n, \lambda_1 + \lambda_2)$.
(7) $\sigma(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda) : (0, \infty) \to [0, 1]$ *is continuous.*
(8) $\lim_{\lambda\to\infty} \sigma(p_1, \ldots, p_n, \lambda) = 1, \lim_{\lambda\to 0} \sigma(p_1, \ldots, p_n, \lambda) = 0$.
(9) $\tau(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda) < 1$.
(10) $\tau(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda) = 0 \quad \Leftrightarrow \quad p_1, p_2, \ldots, p_{n-1}, p_n$ *are linearly dependent.*
(11) $\tau(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda)$ *is invariant under any permutation of* $p_1, p_2, \ldots, p_{n-1}, p_n$.
(12) $\tau(p_1, p_2, \ldots, p_{n-1}, cp_n, \lambda) = \tau\left(p_1, p_2, \ldots, p_{n-1}, p_n, \frac{\lambda}{|c|}\right)$ *if* $c \neq 0, c \in F$.
(13) $\tau(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda_1) \diamond \tau(p_1, p_2, \ldots, p_{n-1}, p'_n, \lambda_2) \geq$
$\tau(p_1, p_2, \ldots, p_{n-1}, p_n + p'_n, \lambda_1 + \lambda_2)$.
(14) $\tau(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda) : (0, \infty) \to [0, 1]$ *is continuous.*
(15) $\lim_{\lambda\to\infty} \tau(p_1, \ldots, p_n, \lambda) = 0, \lim_{\lambda\to 0} \tau(p_1, \ldots, p_n, \lambda) = 1$.
(16) $\upsilon(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda) < 1$.
(17) $\upsilon(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda) = 0 \quad \Leftrightarrow \quad p_1, p_2, \ldots, p_{n-1}, p_n$ *are linearly dependent.*
(18) $\upsilon(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda)$ *is invariant under any permutation of* $p_1, p_2, \ldots, p_{n-1}, p_n$.
(19) $\upsilon(p_1, p_2, \ldots, p_{n-1}, cp_n, \lambda) = \tau\left(p_1, p_2, \ldots, p_{n-1}, p_n, \frac{\lambda}{|c|}\right)$ *if* $c \neq 0, c \in F$.
(20) $\upsilon(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda_1) \diamond \tau(p_1, p_2, \ldots, p_{n-1}, p'_n, \lambda_2) \geq$
$\upsilon(p_1, p_2, \ldots, p_{n-1}, p_n + p'_n, \lambda_1 + \lambda_2)$.
(21) $\upsilon(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda) : (0, \infty) \to [0, 1]$ *is continuous.*
(22) $\lim_{\lambda\to\infty} \tau(p_1, \ldots, p_n, \lambda) = 0, \lim_{\lambda\to 0} \upsilon(p_1, \ldots, p_n, \lambda) = 1$.
Then σ, τ, υ *are called neutrosophic normed(NN).*

Let $(\mathcal{P}, \|\bullet, \bullet, \ldots, \bullet\|)$ *be n-NLS. Define* $\mathfrak{s} \star \mathfrak{e} = \mathfrak{s}\mathfrak{e}, \mathfrak{s} \diamond \mathfrak{e} = \min\{\mathfrak{s} + \mathfrak{e}, 1\}\ \forall\ \mathfrak{s}, \mathfrak{e} \in [0, 1]$

$$\sigma(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda) = \frac{\lambda}{\lambda + \left\|p_1, p_2, \ldots, p_{n-1}, p_n\right\|}$$

and

$$\tau(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda) = \frac{\left\|p_1, p_2, \ldots, p_{n-1}, p_n\right\|}{\lambda + \left\|p_1, p_2, \ldots, p_{n-1}, p_n\right\|}$$

and

$$\upsilon(p_1, p_2, \ldots, p_{n-1}, p_n, \lambda) = \frac{\left\|p_1, p_2, \ldots, p_{n-1}, p_n\right\|}{\|\lambda\|}$$

for every $\lambda > 0$ *and* $p_1, p_2, \ldots, p_{n-1} \in \mathcal{P}$. *Then* $(\mathcal{P}, \sigma, \tau, \upsilon, \star, \diamond)$ *be an N-n-NLS. Let* $(\mathcal{P}, \sigma, \tau, \upsilon, \star, \diamond)$ *be an N-n-NLS. A sequence* $p = (p_k)$ *in* $\mathcal{P}$ *is said to be convergent sequence to* p_0 *w.r.t neutrosophic n-normed linear space* $(\sigma, \tau, \upsilon)^n$ *(briefly called N-n-NLS) if for every* $\varepsilon \in (0, 1), \lambda > 0$
and $p_1, p_2, \ldots, p_{n-1} \in \mathcal{P}, \exists\, k_0 \in \mathbb{N}$ *such that*

$$\sigma(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) > 1 - \varepsilon$$

and

$$\tau(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) < \varepsilon, \quad \forall k \geq k_0$$

and

$$\upsilon(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) < \varepsilon, \quad \forall k \geq k_0.$$

It is denoted by $(\sigma, \tau, \upsilon)^n - \lim p_k = p_0$. *A sequence* $p = (p_k)$ *in* $\mathcal{P}$ *is said to be Cauchy sequence w.r.t N-n-N* $(\sigma, \tau, \upsilon)^n$ *if for every* $\varepsilon \in (0, 1), \lambda > 0$ *and* $p_1, p_2, \ldots, p_{n-1} \in \mathcal{P}, \exists\, k_0 \in \mathbb{N}$ *such that* $\sigma(p_1, p_2, \ldots, p_{n-1}, p_k - p_r, \lambda) > 1 - \varepsilon$ *and* $\tau(p_1, p_2, \ldots, p_{n-1}, p_k - p_r, \lambda) < \varepsilon$*, and* $\upsilon(p_1, p_2, \ldots, p_{n-1}, p_k - p_r, \lambda) < \varepsilon$ *for all* $k, r \geq k_0$.

Definition 2.6. *Let $(\mathcal{P}, \sigma, \tau, \upsilon, \star, \diamond)$ be an N-n-NLS. A sequence $p = (p_k)$ in $\mathcal{P}$ is said to be statistically convergent to $p_0 \in \mathcal{P}$ w.r.t N-n-N $(\sigma, \tau, \upsilon)^n$ if for every $\varepsilon \in (0,1)$, $\lambda > 0$ and $p_1, p_2, \ldots, p_{n-1} \in \mathcal{P}$,*

$$\begin{aligned}\lim_{\ell\to\infty} \frac{1}{\ell}\Big|\{k \le \ell : &\sigma(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \le 1 - \varepsilon, \\ &\tau(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \ge \varepsilon, \\ &\upsilon(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \ge \varepsilon\}\Big| \\ &= 0.\end{aligned}$$

equivalently,

$$\begin{aligned}\Delta\{k \in \mathbb{N} : &\sigma(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \le 1 - \varepsilon, \\ &\tau(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \ge \varepsilon, \\ &\upsilon(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \ge \varepsilon\} \\ &= 0.\end{aligned}$$

It is denoted by $[S(\sigma, \tau, \upsilon)^n] - \lim p_k = p_0$.

Recently, Shyamal Debnath, Santonu Debnath and Chiranjib Choudhury [53] On Deferred Statistical Convergence of Sequences in Neutrosophic Normed Spaces and obtained some interesting results.

For any two sequences $\alpha = \{\alpha(\ell) : \ell \in \mathbb{N}\}$ and $\beta = \{\beta(\ell) : \ell \in \mathbb{N}\}$ of nonnegative integers satisfying

$$\alpha(\ell) < \beta(\ell) \quad \text{and} \quad \beta(\ell) \to \infty \quad \text{as} \quad \ell \to \infty \tag{2.1}$$

Agnew [1] generalized the Cesàro summability, called deferred Cesàro summability, and also Tauberian theorem for Cesaro summability in neutrosophic normed spaces [52] as follows: A sequence $p = (p_k)$ is called deferred Cesàro summable to p_0 if

$$\lim_{\ell\to\infty} \left(G_{\alpha,\beta}p\right)_\ell = p_0$$

where

$$\left(G_{\alpha,\beta}p\right)_\ell = \frac{1}{\beta(\ell) - \alpha(\ell)} \sum_{k=\alpha(\ell)+1}^{\beta(\ell)} p_k$$

kucukaslan and Yilmazturk [25] defined deferred density $\Delta_{\alpha,\beta}$ of subsets of $\mathbb{N}$ and studied deferred statistical summability for real number sequences. This method is called properly deferred if $\alpha(\ell)$ and $\beta(\ell)$ satisfy the condition $\frac{\alpha(\ell)}{\beta(\ell)-\alpha(\ell)}$, in addition to equation (2.1). For $\mathbb{H} \subseteq \mathbb{N}$, the deferred density $\Delta_{\alpha,\beta}$ of $\mathbb{H}$ is defined by

$$\Delta_{\alpha,\beta}(\mathbb{H}) = \lim_{\ell\to\infty} \frac{1}{\beta(\ell) - \alpha(\ell)} \left|\{\alpha(\ell) < k \le \beta(\ell) : k \in \mathbb{H}\}\right|$$

Definition 2.7. *Let $(\mathcal{P}, \sigma, \tau, \star, \upsilon, \diamond)$ be an NNLS. A sequence $p = (p_k)$ in $\mathcal{P}$ is said to be deferred statistical summable to p_0 w.r.t the NN (σ, τ, υ) if for every $\varepsilon \in (0,1)$, $\lambda > 0$, we have*

$$\begin{aligned}\lim_{\ell\to\infty} \frac{1}{\beta(\ell) - \alpha(\ell)}\Big|\{\alpha(\ell) < k \le \beta(\ell) : &\sigma(p_k - p_0, \lambda) \le 1 - \varepsilon, \\ &\tau(p_k - p_0, \lambda) \ge \varepsilon, \\ &\upsilon(p_k - p_0, \lambda) \ge \varepsilon\}\Big| \\ &= 0.\end{aligned}$$

equivalently,

$$\begin{aligned}\Delta_{\alpha,\beta}(\{\alpha(\ell) < k \le \beta(\ell) : &\sigma(p_k - p_0, \lambda) \le 1 - \varepsilon, \\ &\tau(p_k - p_0, \lambda) \ge \varepsilon, \\ &\upsilon(p_k - p_0, \lambda) \ge \varepsilon\}) \\ &= 0.\end{aligned}$$

It is denoted by $p_k \to p_0 \left(G_{\alpha,\beta}[S(\sigma,\tau,\upsilon)]\right)$.

Definition 2.8. *Let* $(\mathcal{P},\sigma,\tau,\upsilon,\star,\diamond)$ *be an NNLS. A sequence* $p=(p_k)$ *in* $\mathcal{P}$ *is said to be deferred statistical Cauchy w.r.t the NN* (σ,τ,υ) *if for every* $\varepsilon\in(0,1)$, $\lambda>0$ *and* $\exists\ r=r(\varepsilon)$ *such that*

$$\begin{aligned}\lim_{\ell\to\infty}\frac{1}{\beta(\ell)-\alpha(\ell)}\Big|\{\alpha(\ell)<k\le\beta(\ell):&\sigma(p_k-p_0,\lambda)\le 1-\varepsilon,\\ &\tau(p_k-p_0,\lambda)\ge\varepsilon,\\ &\upsilon(p_k-p_0,\lambda)\ge\varepsilon\}\Big|\\ &=0.\end{aligned}$$

equivalently,

$$\begin{aligned}\Delta_{\alpha,\beta}(\{\alpha(\ell)<k\le\beta(\ell):&\sigma(p_k-p_r,\lambda)\le 1-\varepsilon,\\ &\tau(p_k-p_r,\lambda)\ge\varepsilon,\\ &\upsilon(p_k-p_r,\lambda)\ge\varepsilon\})\\ &=0.\end{aligned}$$

3. $G_{\alpha,\beta}\,[S(\sigma,\tau,\upsilon,)^n]$-Summability of Sequence in N-n-NLS

In this section, we aim to introduce the concept of deferred statistical summability for sequences in an Neutrosophic n-dimensional linear space (N-n-NLS) and examine several of its key properties. Throughout this work by α and β we mean two sequences $\alpha=\{\alpha(\ell):\ell\in\mathbb{N}\}$ and $\beta=\{\beta(\ell):\ell\in\mathbb{N}\}$ of non negative integers satisfying $\alpha(\ell)<\beta(\ell)$ and $\beta(\ell)\to\infty$ as $\ell\to\infty$. Deferred summability extends the classical notion of Cesàro summability, addressing challenges related to ultimate bounds, oscillations, and the non-preservation of various convergence behaviors in function sequences. The framework established in this work provides a generalized approach to summability, specifically in the context of statistical summability.

Definition 2.9. *Let* $(\mathcal{P},\sigma,\tau,\upsilon,\star,\diamond)$ *be an N-n-NLS. A sequence* $p=(p_k)$ *in* $\mathcal{P}$ *is said to be strong deferred summable or* $G_{\alpha,\beta}$*-summable to* p_0 *w.r.t the N-n-norm* $(\sigma,\tau,\upsilon)^n$ *if for every* $\varepsilon\in(0,1)$, $\lambda>0$
and $p_1,p_2,\ldots,p_{n-1}\in\mathcal{P}$, $\exists\ k_0\in\mathbb{N}$ *such that*

$$\frac{1}{\beta(\ell)-\alpha(\ell)}\sum_{k=\alpha(\ell)+1}^{\beta(\ell)}\sigma(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda)>1-\varepsilon$$

and

$$\frac{1}{\beta(\ell)-\alpha(\ell)}\sum_{k=\alpha(\ell)+1}^{\beta(\ell)}\tau(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda)<\varepsilon$$

and

$$\frac{1}{\beta(\ell)-\alpha(\ell)}\sum_{k=\alpha(\ell)+1}^{\beta(\ell)}\upsilon(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda)<\varepsilon$$

holds for all $k\ge k_0$.

In this case, we write $G_{\alpha,\beta}\,[(\sigma,\tau,\upsilon)^n]-\lim p_k=p_0$, *or simply*
$p_k\to p_0\left(G_{\alpha,\beta}\,[(\sigma,\tau,\upsilon)^n]\right)$.

Let $G_{\alpha,\beta}\,[(\sigma,\tau,\upsilon)^n]=\left\{p=(p_k):p_k\to p_0\left(G_{\alpha,\beta}\,[(\sigma,\tau,\upsilon)^n]\right)\right\}$.

Definition 2.10. *Let $(\mathcal{P},\sigma,\tau,\upsilon,\star,\diamond)$ be an N-n-NLS. A sequence $p=(p_k)$ in $\mathcal{P}$ is said to be deferred statistical summable to p_0 w.r.t the N-n-norm $(\sigma,\tau,\upsilon)^n$ if for every $\varepsilon\in(0,1),\lambda>0$ and $p_1,p_2,\ldots,p_{n-1}\in\mathcal{P}$, we have*

$$\begin{aligned}\lim_{\ell\to\infty}\frac{1}{\beta(\ell)-\alpha(\ell)}\Big|\{\alpha(\ell)<k\leq\beta(\ell):&\sigma(p_k-p_0,\lambda)\leq 1-\varepsilon,\\ &\tau(p_k-p_0,\lambda)\geq\varepsilon,\\ &\upsilon(p_k-p_0,\lambda)\geq\varepsilon\}\Big|=0.\end{aligned}$$

or

$$\begin{aligned}\Delta_{\alpha,\beta}(\{\alpha(\ell)<k\leq\beta(\ell):&\sigma(p_k-p_0,\lambda)\leq 1-\varepsilon,\\ &\tau(p_k-p_0,\lambda)\geq\varepsilon,\\ &\upsilon(p_k-p_0,\lambda)\geq\varepsilon\})=0.\end{aligned}$$

In this case, we write $G_{\alpha,\beta}[S(\sigma,\tau,\upsilon)^n]-\lim p_k=p_0$, or simply $p_k\to p_0\left(G_{\alpha,\beta}[S(\sigma,\tau,\upsilon)^n]\right)$.

Let $G_{\alpha,\beta}[S(\sigma,\tau,\upsilon)^n]=\left\{p=(p_k):p_k\to p_0\left(G_{\alpha,\beta}[S(\sigma,\tau,\upsilon)^n]\right)\right\}$.

Remark: *(i) If $\beta(\ell)=\ell$ and $\alpha(\ell)=0$ then definition (9) coincides with the definition of statistical convergence in neutrosophic normed spaces given in [54].*
(ii) If $\beta(\ell)=\lambda_\ell$ and $\alpha(\ell)=0$ where (λ_ℓ) is a strictly increasing sequence of positive numbers tending to ∞ then definition (9) turns to the definition of λ-statistical convergence sequence in NNS [55].
(iii) If $\beta(\ell)=\theta_\ell$ and $\alpha(\ell)=\theta_{\ell-1}$ where (θ_ℓ) is a sequence of non-negative integers with $\theta_\ell-\theta_{\ell-1}\to\infty$ as $\ell\to\infty$, then definition (9)coincides with the definition of lacunary statisti- cal convergence in NNS [56].
definition (9) Immediately gives the following lemma.

Lemma 2.11. *Let $(\mathcal{P},\sigma,\tau,\upsilon,\star,\diamond)$ be an N-n-NLS. Then for every $\varepsilon\in(0,1),\lambda>0$ and $p_1,p_2,\ldots,p_{n-1}\in\mathcal{P}$, the following are equivalent:*

(*a*)

$$G_{\alpha,\beta}[S(\sigma,\tau,\upsilon)^n]-\lim p_k=p_0.$$

(*b*)

$$\begin{aligned}\Delta_{\alpha,\beta}(\{\alpha(\ell)<k\leq\beta(\ell):&\ \sigma(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda)\leq 1-\varepsilon,\\ &\ \tau(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda)\geq\varepsilon,\\ &\ \upsilon(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda)\geq\varepsilon\})=0.\end{aligned}$$

(*c*)

$$\begin{aligned}\Delta_{\alpha,\beta}(\{\alpha(\ell)<k\leq\beta(\ell):&\ \sigma(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda)\leq 1-\varepsilon,\\ &\ \tau(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda)\geq\varepsilon, \qquad .\\ &\ \upsilon(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda)\geq\varepsilon\})=1.\end{aligned}$$

(*d*)

$$\begin{aligned}\Delta_{\alpha,\beta}\{\alpha(\ell)<k\leq\beta(\ell):&\ \sigma(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda)\leq 1-\varepsilon\\ &=\tau(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda)\geq\varepsilon\\ &=\upsilon(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda)\geq\varepsilon\}=1.\end{aligned}$$

(*e*)

$$\begin{aligned}&G_{\alpha,\beta}S-\lim\sigma(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda)=1,\\ &G_{\alpha,\beta}S-\lim\tau(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda)=0,\\ &G_{\alpha,\beta}S-\lim\upsilon(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda)=0\end{aligned}$$

Theorem 2.12. *Let* $(\mathcal{P},\sigma,\tau,\upsilon,\star,\diamond)$ *be an N-n-NLS and* $p=(p_k)$ *be any sequence in* $\mathcal{P}$ *with* $(\sigma,\tau,\upsilon)^n - \lim p_k = p_0$ *then*
(*i*) $G_{\alpha,\beta}\left[(\sigma,\tau,\upsilon)^n\right] - \lim p_k = p_0$.
(*ii*) $G_{\alpha,\beta}\left[S(\sigma,\tau,\upsilon)^n\right] - \lim p_k = p_0$.

Proof. (*i*) For every $\varepsilon \in (0,1), \lambda > 0$ and $p_1,p_2,\ldots,p_{n-1} \in \mathcal{P}, \exists\, k_0 \in \mathbb{N}$ such that $\sigma(p_1,p_2,\ldots,p_{n-1},p_k - p_0,\lambda) > 1-\varepsilon$ and
$\tau(p_1,p_2,\ldots,p_{n-1},p_k - p_0,\lambda) < \varepsilon\ \forall\, k \geq k_0$ and
$\upsilon(p_1,p_2,\ldots,p_{n-1},p_k - p_0,\lambda) < \varepsilon\ \forall\, k \geq k_0$.
Assuming the above inequalities hold for values between $\alpha(\ell)+1$ and $\beta(\ell)$, the subsequent inequality is derived

$$\sum_{k=\alpha(\ell)+1}^{\beta(\ell)} \{\sigma(p_1,p_2,\ldots,p_{n-1},p_k - p_0,\lambda) > (1-\varepsilon)(\beta(\ell)-\alpha(\ell)),$$
$$\tau(p_1,p_2,\ldots,p_{n-1},p_k - p_0,\lambda) < \varepsilon(\beta(\ell)-\alpha(\ell)),$$
$$\upsilon(p_1,p_2,\ldots,p_{n-1},p_k - p_0,\lambda) < \varepsilon(\beta(\ell)-\alpha(\ell))\}$$

i.e.,

$$\frac{1}{\beta(\ell)-\alpha(\ell)} \sum_{k=\alpha(\ell)+1}^{\beta(\ell)} \{\sigma(p_1,p_2,\ldots,p_{n-1},p_k - p_0,\lambda) > (1-\varepsilon),$$
$$\tau(p_1,p_2,\ldots,p_{n-1},p_k - p_0,\lambda) < \varepsilon,$$
$$\upsilon(p_1,p_2,\ldots,p_{n-1},p_k - p_0,\lambda) < \varepsilon\}$$

and therefore we have $G_{\alpha,\beta}\left[(\sigma,\tau,\upsilon)^n\right] - \lim p_k = p_0$.

(*ii*) Since $(\sigma,\tau,\upsilon)^n - \lim p_k = p_0$. So clearly, we have the containment

$$\{\alpha(\ell) < k \leq \beta(\ell) : \sigma(p_1,p_2,\ldots,p_{n-1},p_k - p_0,\lambda) \leq (1-\varepsilon),$$
$$\tau(p_1,p_2,\ldots,p_{n-1},p_k - p_0,\lambda) \geq \varepsilon,$$
$$\upsilon(p_1,p_2,\ldots,p_{n-1},p_k - p_0,\lambda) \geq \varepsilon\}$$
$$\subseteq \{1,2,3,\ldots,k_0-1\},$$

gives immediately,

$$\Delta_{\alpha,\beta}(\{\alpha(\ell) < k \leq \beta(\ell) : \sigma(p_1,p_2,\ldots,p_{n-1},p_k - p_0,\lambda) \leq (1-\varepsilon),$$
$$\tau(p_1,p_2,\ldots,p_{n-1},p_k - p_0,\lambda) \geq \varepsilon,$$
$$\upsilon(p_1,p_2,\ldots,p_{n-1},p_k - p_0,\lambda) \geq \varepsilon\}$$
$$\subseteq \{1,2,3,\ldots,k_0-1\})$$
$$= 0$$

Hence, $G_{\alpha,\beta}\left[S(\sigma,\tau,\upsilon)^n\right] - \lim p_k = p_0$ □

In general, the converse of theorem (1) does not hold.

Example 2.13. *Let* $(\mathbb{R}^n, \|\bullet,\bullet,\ldots,\bullet\|)$ *be an n-NLS and* $\mathfrak{s} \star \mathfrak{e} = \mathfrak{s}\mathfrak{e}$, *and*
$\mathfrak{s}\diamond\mathfrak{e} = \min\{\mathfrak{s}+\mathfrak{e},1\}\ \forall\ \mathfrak{s},\mathfrak{e} \in [0,1]$. *For all* $\lambda > 0$, *and*
$p_1,p_2,\ldots,p_{n-1},p_n \in \mathbb{R}^n$, *define*

$$\sigma\left(p_1,p_2,\ldots,p_{n-1},p_n,\lambda\right)=\frac{\lambda}{\lambda+\left\|p_1,p_2,\ldots,p_{n-1},p_n\right\|}$$

and

$$\tau\left(p_1,p_2,\ldots,p_{n-1},p_n,\lambda\right)=\frac{\left\|p_1,p_2,\ldots,p_{n-1},p_n\right\|}{\lambda+\left\|p_1,p_2,\ldots,p_{n-1},p_n\right\|}$$

and

$$\upsilon\left(p_1,p_2,\ldots,p_{n-1},p_n,\lambda\right)=\frac{\left\|p_1,p_2,\ldots,p_{n-1},p_n\right\|}{\|\lambda\|}$$

then it is easy to check $(\mathbb{R}^n,\sigma,\tau,\upsilon,\star,\diamond)$ *is an N-n-NLS. Now consider the sequence* $p=(p_k)$ *where*

$$p_k=\begin{cases}(d^x,0,0,\cdots,0)\in{}^n; & [\sqrt[x]{\beta(\ell)}]-c_0<d<[\sqrt[x]{\beta(\ell)}]\\(0,0,0,\cdots,0)\in{}^n; & otherwise\end{cases}$$

where x *is positive integer and* $\beta(\ell)$ *is a monotone increasing sequence of positive integers and* $c_0(\neq 0)\in\mathbb{N}$ *is a fixed number. Now, for every* $\varepsilon\in(0,1)$, $\lambda>0$ *and* $p_1,p_2,\ldots,p_{n-1},p_n\in\mathbb{R}^n$, *consider the following set*

$$\begin{aligned}Q_n(\varepsilon,\lambda)=\{\alpha(\ell)<d\le\beta(\ell):\ &\sigma\left(p_1,p_2,\ldots,p_{n-1},p_k-0,\lambda\right)\le 1-\varepsilon,\\&\tau\left(p_1,p_2,\ldots,p_{n-1},p_k-0,\lambda\right)\ge\varepsilon,\\&\upsilon\left(p_1,p_2,\ldots,p_{n-1},p_k-0,\lambda\right)\ge\varepsilon\}\end{aligned}$$

$$\begin{aligned}&=\Big\{\alpha(\ell)<d\le\beta(\ell):\frac{\lambda}{\lambda+\left\|p_1,p_2,\ldots,p_{n-1},p_k\right\|}\le 1-\varepsilon,\\&\qquad\frac{\left\|p_1,p_2,\ldots,p_{n-1},p_k\right\|}{\lambda+\left\|p_1,p_2,\ldots,p_{n-1},p_k\right\|}\ge\varepsilon,\\&\qquad\frac{\left\|p_1,p_2,\ldots,p_{n-1},p_n\right\|}{\|\lambda\|}\Big\}\\&=\Big\{\alpha(\ell)<d\le\beta(\ell):\left\|p_1,p_2,\ldots,p_{n-1},p_k\right\|\ge\frac{\varepsilon\lambda}{1-\varepsilon}>0\Big\}\\&\subseteq\{\alpha(\ell)<d\le\beta(\ell):p_k=(d^x,0,0,\cdots,0)\in\mathbb{R}^n\}\\&=\Big\{\alpha(\ell)<d\le\beta(\ell):[|\sqrt[x]{\beta(\ell)}|]-c_0<d<[|\sqrt[x]{\beta(\ell)}|]\Big\}\\&\le\lim_{\ell\to\infty}\frac{1}{\beta(\ell)-\alpha(\ell)}c_0\to 0\end{aligned}$$

$\Delta_{\alpha,\beta}\left(Q_n(\varepsilon,\lambda)\right)=0$ *as density can't be negative.*
Thus, $G_{\alpha,\beta}\left[S(\sigma,\tau,\upsilon)^n\right]-\lim p_k=0$. *But it is clear that the sequence* $p=(p_k)$ *is not convergent in* $(\mathbb{R}^n,\sigma,\tau,\upsilon,\star,\diamond)$ *w.r.t the N-n-norm* $(\sigma,\tau,\upsilon)^n$.

Theorem 2.14. *Let* $(\mathcal{P},\sigma,\tau,\upsilon,\star,\diamond)$ *be an N-n-NLS. For any sequence* $p=(p_k)$ *in* $\mathcal{P}$, *if* $G_{\alpha,\beta}\left[S(\sigma,\tau,\upsilon)^n\right]-\lim p_k$ *exists, then it is unique.*

Proof. Let $G_{\alpha,\beta}\left[S(\sigma,\tau,\upsilon)^n\right]-\lim p_k=p_0$. Let $\varepsilon\in(0,1)$ be given. Choose $\zeta\in(0,1)$ such that

$$(1-\zeta)\star(1-\zeta)>1-\varepsilon,\quad \zeta\diamond\zeta<\varepsilon \tag{3.1}$$

For any $\lambda > 0$ and $p_1, p_2, \ldots, p_{n-1} \in \mathcal{P}$, if we define the following sets:

$$D_{\sigma,1}(\zeta,\lambda) = \left\{\alpha(\ell) < k \le \beta(\ell) : \sigma\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \frac{\lambda}{2}\right) \le 1-\zeta\right\} \tag{3.2}$$

$$D_{\sigma,2}(\zeta,\lambda) = \left\{\alpha(\ell) < k \le \beta(\ell) : \sigma\left(p_1, p_2, \ldots, p_{n-1}, p_k - q_0, \frac{\lambda}{2}\right) \le 1-\zeta\right\} \tag{3.3}$$

$$D_{\tau,1}(\zeta,\lambda) = \left\{\alpha(\ell) < k \le \beta(\ell) : \tau\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \frac{\lambda}{2}\right) \ge \zeta\right\} \tag{3.4}$$

$$D_{\tau,2}(\zeta,\lambda) = \left\{\alpha(\ell) < k \le \beta(\ell) : \tau\left(p_1, p_2, \ldots, p_{n-1}, p_k - q_0, \frac{\lambda}{2}\right) \ge \zeta\right\} \tag{3.5}$$

then from the lemma (1),

$$D_{\upsilon,1}(\zeta,\lambda) = \left\{\alpha(\ell) < k \le \beta(\ell) : \upsilon\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \frac{\lambda}{2}\right) \ge \zeta\right\} \tag{3.6}$$

$$D_{\upsilon,2}(\zeta,\lambda) = \left\{\alpha(\ell) < k \le \beta(\ell) : \upsilon\left(p_1, p_2, \ldots, p_{n-1}, p_k - q_0, \frac{\lambda}{2}\right) \ge \zeta\right\} \tag{3.7}$$

then from the lemma (1),

$$\Delta_{\alpha,\beta}\left(D_{\sigma,1}(\zeta,\lambda)\right) = \Delta_{\alpha,\beta}\left(D_{\tau,1}(\zeta,\lambda)\right) = \Delta_{\alpha,\beta}\left(D_{\upsilon,1}(\zeta,\lambda)\right) = 0$$

as $G_{\alpha,\beta}\left[S(\sigma,\tau,\upsilon)^n\right] - \lim p_k = p_0$. Similarly,

$$\Delta_{\alpha,\beta}\left(D_{\sigma,2}(\zeta,\lambda)\right) = \Delta_{\alpha,\beta}\left(D_{\tau,2}(\zeta,\lambda)\right) = \Delta_{\alpha,\beta}\left(D_{\upsilon,2}(\zeta,\lambda)\right) = 0$$

as $G_{\alpha,\beta}\left[S(\sigma,\tau,\upsilon)^n\right] - \lim p_k = q_0$.
Let

$$\begin{aligned} D_{\sigma,\tau,\upsilon}(\zeta,\lambda) = &\{D_{\sigma,1}(\zeta,\lambda) \cup D_{\sigma,2}(\zeta,\lambda)\} \\ &\cap \{D_{\tau,1}(\zeta,\lambda) \cup D_{\tau,2}(\zeta,\lambda)\} \\ &\cap \{D_{\upsilon,1}(\zeta,\lambda) \cup D_{\upsilon,2}(\zeta,\lambda)\} \end{aligned}$$

then $\Delta_{\alpha,\beta}\left(D_{\sigma,\tau,\upsilon}(\zeta,\lambda)\right) = 0$, and therefore $\Delta_{\alpha,\beta}\left(D_{\sigma,\tau,\upsilon}(\zeta,\lambda)\right)^{\complement} = 1$.
Let $k \in \left(D_{\sigma,\tau,\upsilon}(\zeta,\lambda)\right)^{\complement}$, then we have

$$\begin{aligned} k \in &\{(D_{\sigma,1}(\zeta,\lambda) \cup D_{\sigma,2}(\zeta,\lambda))^{\complement}, or \\ &(D_{\tau,1}(\zeta,\lambda) \cup D_{\tau,2}(\zeta,\lambda))^{\complement}, or \\ &(D_{\upsilon,1}(\zeta,\lambda) \cup D_{\upsilon,2}(\zeta,\lambda))^{\complement}\}. \end{aligned}$$

Case (i) If $k \in \left(\{D_{\sigma,1}(\zeta,\lambda) \cup D_{\sigma,2}(\zeta,\lambda)\}\right)^{\complement}$ then we have

$$\begin{aligned} \sigma\left(p_1, p_2, \ldots, p_{n-1}, p_0 - q_0, \lambda\right) &\ge \sigma\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \frac{\lambda}{2}\right) \\ &\quad \star\, \sigma\left(p_1, p_2, \ldots, p_{n-1}, p_k - q_0, \frac{\lambda}{2}\right) \\ &> (1-\zeta) \star (1-\zeta) \text{ by (3.2) and (3.3)} \\ &> (1-\varepsilon). \text{ by (3.1)} \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary so $\sigma\left(p_1, p_2, \ldots, p_{n-1}, p_0 - q_0, \lambda\right) = 1$. This implies that $p_0 - q_0 = 0$ i.e., $p_0 = q_0$.

Case (ii) If $k \in \left(\{D_{\tau,1}(\zeta,\lambda) \cup D_{\tau,2}(\zeta,\lambda)\}\right)^{\complement}$ then by similar argument as above

$$\begin{aligned}\tau(p_1,p_2,\ldots,p_{n-1},p_0-q_0,\lambda) &\le \tau\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\frac{\lambda}{2}\right)\\ &\diamond\,\tau\left(p_1,p_2,\ldots,p_{n-1},p_k-q_0,\frac{\lambda}{2}\right)\\ &< \zeta\diamond\zeta \text{ by (3.4) and (3.5)}\\ &< \varepsilon. \text{ by (3.1)}\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary so $\tau(p_1,p_2,\ldots,p_{n-1},p_0-q_0,\lambda) = 0$, and therefore, $p_0 = q_0$.
Hence, in all cases, $p_0 = q_0$.

Case (*iii*) If $k \in (\{D_{\upsilon,1}(\zeta,\lambda)\cup D_{\upsilon,2}(\zeta,\lambda)\})^{\complement}$ then by similar argument as above

$$\begin{aligned}\upsilon(p_1,p_2,\ldots,p_{n-1},p_0-q_0,\lambda) &\le \upsilon\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\frac{\lambda}{2}\right)\\ &\diamond\,\upsilon\left(p_1,p_2,\ldots,p_{n-1},p_k-q_0,\frac{\lambda}{2}\right)\\ &< \zeta\diamond\zeta \text{ by (3.6) and (3.7)}\\ &< \varepsilon. \text{ by (3.1)}\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary so $\upsilon(p_1,p_2,\ldots,p_{n-1},p_0-q_0,\lambda) = 0$, and therefore, $p_0 = q_0$.
Hence, in all cases, $p_0 = q_0$. □

Theorem 2.15. *Let $(\mathcal{P},\sigma,\tau,\upsilon,\star,\diamond)$ be an N-n-NLS. If $p=(p_k)$ in $\mathcal{P}$ then $G_{\alpha,\beta}\,[S(\sigma,\tau,\upsilon)^n]-\lim p_k = p_0$ if and only if $\exists$ a set $W=\{w_1<w_2<\cdots\}\subset\mathbb{N}$ such that $\Delta_{\alpha,\beta}(W)=1$ and $(\sigma,\tau,\upsilon)^n-\lim p_{w_j}=p_0$.*

Proof. Necessity: We first, assume that $G_{\alpha,\beta}\,[S(\sigma,\tau,\upsilon)^n]-\lim p_k = p_0$. For $\lambda>0$ and $\zeta=1,2,\ldots$, we denote,

$$\begin{aligned}Q_{\alpha,\beta}(\zeta,\lambda)=\{\alpha(\ell)<k\le\beta(\ell) : &\ \sigma(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda)>1-\frac{1}{\zeta},\\ &\ \tau(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda)<\frac{1}{\zeta},\\ &\ \upsilon(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda)<\frac{1}{\zeta}\}\end{aligned}$$

and

$$\begin{aligned}B_{\alpha,\beta}(\zeta,\lambda)=\{\alpha(\ell)<k\le\beta(\ell) : &\ \sigma(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda)\le 1-\frac{1}{\zeta},\\ &\ \tau(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda)\ge\frac{1}{\zeta},\\ &\ \upsilon(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda)\ge\frac{1}{\zeta}\}\end{aligned}$$

By assumption, we have

$$\Delta_{\alpha,\beta}\left(B_{\alpha,\beta}(\zeta,\lambda)\right)=0,\ \Delta_{\alpha,\beta}\left(Q_{\alpha,\beta}(\zeta,\lambda)\right)=1 \tag{3.8}$$

Further, we also have $Q_{\alpha,\beta}(\zeta+1,\lambda)\subset Q_{\alpha,\beta}(\zeta,\lambda)$. To complete the proof of the first part it is sufficient to show that for $k\in Q_{\alpha,\beta}(\zeta,\lambda)$ we have
$(\sigma,\tau,\upsilon)^n-\lim p_k=p_0$. Suppose it does not hold $(\sigma,\tau,\upsilon)^n-\lim p_k\ne p_0$ for some $k\in Q_{\alpha,\beta}(\zeta,\lambda)$. Then, there is

$\varepsilon \in (0,1), \lambda > 0$ and a positive
integer k_0 such that:

$$\begin{aligned}&\{\sigma(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \leq 1-\varepsilon, or\\ &\tau(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon, or\\ &\upsilon(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon\},\end{aligned}$$

holds for all $k \geq k_0$.

$$\begin{aligned}&\{\sigma(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) > 1-\varepsilon,\\ &\sigma(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) > \varepsilon,\\ &\upsilon(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) < \varepsilon\}\end{aligned}$$

holds for all $k < k_0$. Then

$$\begin{aligned}\Delta_{\alpha,\beta}(\{\alpha(\ell) < k \leq \beta(\ell) :&\{\sigma(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) > 1-\varepsilon,\\ &\tau(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) < \varepsilon,\\ &\upsilon(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) < \varepsilon\})\\ &= 0.\end{aligned}$$

Since $\varepsilon > \frac{1}{\zeta}$, then $\Delta_{\alpha,\beta}\left(A_{\alpha,\beta}(\zeta, \lambda)\right) = 0$. This contradicts equation (3.8). Hence we have $(\sigma, \tau, \upsilon)^n - \lim p_k = p_0$.

Sufficiency part: Suppose there exist a set $W = \{w_1 < w_2 < \ldots\} \subset \mathbb{N}$ such that $\Delta_{\alpha,\beta}(W) = 1$ and $(\sigma, \tau, \upsilon)^n - \lim_{j\to\infty} p_{w_j} = p_0$. By given hypothesis, $\exists\, k_0 \in \mathbb{N}$ such that for all $\varepsilon \in (0,1)$ and $\lambda > 0$,

$$\begin{aligned}&\{\sigma(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) > 1-\varepsilon,\\ &\tau(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) < \varepsilon,\\ &\upsilon(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) < \varepsilon\},\end{aligned}$$

holds for all $k \geq k_0, k \in W$. Further, the inclusion

$$\begin{aligned}&\{\sigma(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \leq 1-\varepsilon,\\ &\tau(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon,\\ &\upsilon(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon\}\\ &\subset \mathbb{N} - \{k_0, k_0+1, \ldots\}\end{aligned}$$

$$\begin{aligned}\Delta_{\alpha,\beta}(\{\alpha(\ell) < k \leq \beta(\ell) :&\ \sigma(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \leq 1-\varepsilon,\\ &\tau(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon,\\ &\upsilon(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon\})\\ &\leq \Delta_{\alpha,\beta}(\mathbb{N}) - \Delta_{\alpha,\beta}(\{k_0, k_0+1, \cdots\})\\ &= 0\end{aligned}$$

i.e.,

$$\begin{aligned}\Delta_{\alpha,\beta}\{\alpha(\ell) < k \leq \beta(\ell) :&\sigma(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \leq 1-\varepsilon,\\ &\tau(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon,\\ &\upsilon(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon\}\\ &= 0\end{aligned}$$

as it can't be negative. Thus, $G_{\alpha,\beta}\left[S(\sigma, \tau, \upsilon)^n\right] - \lim p_k = p_0$. □

Through the proof of theorem (4) and theorem (5),
we denote $1-\sigma(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) = \bar{\sigma}(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda)$ for simplicity. If $\sigma(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) > 1 - \varepsilon$ then we restate it as
$\bar{\sigma}(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) < \varepsilon$.

Theorem 2.16. Let*$(\mathcal{P}, \sigma, \tau, \upsilon, \star, \diamond)$ be an N-n-NLS and $p = (p_k)$ be in $\mathcal{P}$. Then $G_{\alpha,\beta}[(\sigma, \tau, \upsilon,)^n] - \lim p_k = x_0$ implies $G_{\alpha,\beta}[S(\sigma, \tau, \upsilon,)^n] - \lim p_k = p_0$.*

Proof. Let $G_{\alpha,\beta}[(\sigma, \tau, \upsilon,)^n] - \lim p_k = p_0$, then for every $\varepsilon \in (0, 1), \lambda > 0$ and $p_1, p_2, \ldots, p_{n-1} \in \mathcal{P}, \exists\, k_0 \in \mathbb{N}$ such that

$$\frac{1}{\beta(\ell) - \alpha(\ell)} \sum_{k=\alpha(\ell)+1}^{\beta(\ell)} \{\sigma(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) > (1 - \varepsilon),$$
$$\tau(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) < \varepsilon,$$
$$\upsilon(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) < \varepsilon\}$$

holds for all $k \geq k_0$. or equivalently,

$$\frac{1}{\beta(\ell) - \alpha(\ell)} \sum_{k=\alpha(\ell)+1}^{\beta(\ell)} \bar{\sigma}(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) < \varepsilon \tag{3.9}$$

and

$$\frac{1}{\beta(\ell) - \alpha(\ell)} \sum_{k=\alpha(\ell)+1}^{\beta(\ell)} \tau(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) < \varepsilon \tag{3.10}$$

and

$$\frac{1}{\beta(\ell) - \alpha(\ell)} \sum_{k=\alpha(\ell)+1}^{\beta(\ell)} \upsilon(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) < \varepsilon \tag{3.11}$$

holds for all $k \geq k_0$. Now

$$\frac{1}{\beta(\ell) - \alpha(\ell)} \sum_{k=\alpha(\ell)+1}^{\beta(\ell)} \bar{\sigma}(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) < \varepsilon \tag{3.12}$$

$$= \frac{1}{\beta(\ell) - \alpha(\ell)} \left(\sum_{\substack{k=\alpha(\ell)+1, \\ \bar{\sigma}(p_1, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon}}^{\beta(\ell)} + \sum_{\substack{k=\alpha(\ell)+1, \\ \bar{\sigma}(p_1, \ldots, p_{n-1}, p_k - p_0, \lambda) < \varepsilon}}^{\beta(\ell)} \right) \bar{\sigma}(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) < \varepsilon$$

$$\geq \frac{1}{\beta(\ell) - \alpha(\ell)} \sum_{\substack{k=\alpha(\ell)+1 \text{ and} \\ \bar{\sigma}(p_1, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon}}^{\beta(\ell)} \bar{\sigma}(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda)$$
$$\geq \frac{\varepsilon}{\beta(\ell) - \alpha(\ell)} \left| \{\alpha(\ell) < k \leq \beta(\ell) : \bar{\sigma}(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon\} \right|$$
$$\geq Q_\ell$$

where

$$Q_\ell = \frac{\varepsilon}{\beta(\ell)-\alpha(\ell)}\left|\{\alpha(\ell)<k\leq\beta(\ell):\bar{\sigma}\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\geq\varepsilon\}\right|$$

Now

$$\begin{aligned}\varepsilon &> \frac{1}{\beta(\ell)-\alpha(\ell)}\sum_{k=\alpha(\ell)+1}^{\beta(\ell)}\bar{\sigma}\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\\ &\geq Q_\ell \quad (\text{ by (3.9) and (3.12)})\end{aligned}$$

Since $\varepsilon>0$ is arbitrary thus

$$\lim_{\ell\to\infty} Q_\ell = 0.$$

This implies

$$\Delta_{\alpha,\beta}\left(Q_\ell\right)=0$$

as deferred density can't be negative.
By similar process we have

$$\frac{1}{\beta(\ell)-\alpha(\ell)}\sum_{k=\alpha(\ell)+1}^{\beta(\ell)}\tau\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\geq S_\ell \tag{3.13}$$

where

$$S_\ell = \frac{\varepsilon}{\beta(\ell)-\alpha(\ell)}\left|\{\alpha(\ell)<k\leq\beta(\ell):\tau\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\geq\varepsilon\}\right|$$

$$S_\ell = \frac{\varepsilon}{\beta(\ell)-\alpha(\ell)}\left|\{\alpha(\ell)<k\leq\beta(\ell):\upsilon\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\geq\varepsilon\}\right|$$

Also, by similar process

$$\Delta_{\alpha,\beta}\left(S_\ell\right)=0$$

As we have

$$\Delta_{\alpha,\beta}\left(Q_\ell\right)=0,\ \Delta_{\alpha,\beta}\left(S_\ell\right)=0.$$

Hence,

$$G_{\alpha,\beta}\left[S(\sigma,\tau,\upsilon)^n\right]-\lim p_k = p_0.$$

The converse of theorem (4) does not hold, in general. For this consider following example. □

Example 2.17. *Consider the N-n-NLS as in example* (2). *Define a sequence* $p=(p_k)$ *by*

$$p_k=\begin{cases}(d^x,0,0,\cdots,0)\in^n; & [\sqrt[x]{\beta(\ell)}]-c_0<d<[\sqrt[x]{\beta(\ell)}]\\ (0,0,0,\cdots,0)\in^n; & otherwise\end{cases}$$

where x is a positive integer and $\beta(\ell)$ is a monotone increasing sequence of positive integers and $c_0 \neq 0$ is a fixed natural number. Then we have shown in example (2), $G_{\alpha,\beta}\left[S(\sigma,\tau,\upsilon)^n\right] - \lim p_k = 0$. However, the sequence $p = (p_k)$ is not $G_{\alpha,\beta}\left[(\sigma,\tau,\upsilon)^n\right]$-convergent to zero as we have

$$\begin{aligned}
&\frac{1}{\beta(\ell)-\alpha(\ell)}\sum_{k=\alpha(\ell)+1}^{\beta(\ell)}\tau\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\\
&=\frac{1}{\beta(\ell)-\alpha(\ell)}\sum_{k=\alpha(\ell)+1}^{\beta(\ell)}\frac{\left\|p_1,p_2,\ldots,p_{n-1},p_k\right\|}{\lambda+\left\|p_1,p_2,\ldots,p_{n-1},p_k\right\|}\\
&=\frac{1}{\beta(\ell)-\alpha(\ell)}\sum_{k=\alpha(\ell)+1}^{\beta(\ell)}\frac{\left([|\sqrt[x]{\beta(\ell)}|]-c_0\right)^x}{\lambda+([|\sqrt{\beta(\ell)}|])^x}\\
&\geq\frac{\left([|\sqrt{\beta(\ell)}|]-c_0\right)^2}{([|\sqrt{\beta(\ell)}|])^2}\to 1 \quad as \quad \ell\to\infty
\end{aligned}$$

or

$$\begin{aligned}
&\frac{1}{\beta(\ell)-\alpha(\ell)}\sum_{k=\alpha(\ell)+1}^{\beta(\ell)}\upsilon\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\\
&=\frac{1}{\beta(\ell)-\alpha(\ell)}\sum_{k=\alpha(\ell)+1}^{\beta(\ell)}\frac{\left\|p_1,p_2,\ldots,p_{n-1},p_n\right\|}{\|\lambda\|}\\
&=\frac{1}{\beta(\ell)-\alpha(\ell)}\sum_{k=\alpha(\ell)+1}^{\beta(\ell)}\frac{\left([|\sqrt[x]{\beta(\ell)}|]-c_0\right)^x}{\lambda}\\
&\geq\frac{\left([|\sqrt{\beta(\ell)}|]-c_0\right)^x}{([|\sqrt{\beta(\ell)}|])^2}\to 1 \quad as \quad \ell\to\infty
\end{aligned}$$

The following result shows that the converse of theorem (4) specifically hold for bounded sequences. l_∞^n represents the collection of bounded sequences in N-n-NLS.

Theorem 2.18. *Let $(\mathcal{P},\sigma,\tau,\upsilon,\star,\diamond)$ be an N-n-NLS and sequence $p = (p_k)$ in l_∞^n then $G_{\alpha,\beta}\left[S(\sigma,\tau,\upsilon)^n\right] - \lim p_k = p_0$ implies $G_{\alpha,\beta}\left[(\sigma,\tau,\upsilon)^n\right] - \lim p_k = p_0$.*

Proof. Suppose $p = (p_k) \in l_\infty^n$ and $G_{\alpha,\beta}\left[S(\sigma,\tau,\upsilon)^n\right] - \lim p_k = p_0$. By the assumption on $p = (p_k)$ $\exists$ a positive real number M
such that $\sigma\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right) > 1-M$ and
$\tau\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right) < M$ and $\upsilon\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right) < M$or equivently,

$$\begin{aligned}
&\bar{\sigma}\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0\lambda\right) < M, and\\
&\{\tau(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda),\\
&\ \upsilon\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\} < M
\end{aligned}$$

holds for all k.
Now

$$\frac{1}{\beta(\ell)-\alpha(\ell)}\sum_{k=\alpha(\ell)+1}^{\beta(\ell)}\sigma\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right) \tag{3.14}$$

$$= \frac{1}{\beta(\ell)-\alpha(\ell)} \left(\sum_{\substack{k=\alpha(\ell)+1, \\ \sigma(p_1,\dots,p_{n-1},p_k-p_0,\lambda)\le 1-\varepsilon}}^{\beta(\ell)} + \sum_{\substack{k=\alpha(\ell)+1, \\ \sigma(p_1,\dots,p_{n-1},p_k-p_0,\lambda)> 1-\varepsilon}}^{\beta(\ell)} \right) \sigma\left(p_1,p_2,\dots,p_{n-1},p_k-p_0,\lambda\right) < \varepsilon$$

rewrite (3.14) as follows

$$\frac{1}{\beta(\ell)-\alpha(\ell)} \sum_{k=\alpha(\ell)+1}^{\beta(\ell)} \bar{\sigma}\left(p_1,p_2,\dots,p_{n-1},p_k-p_0,\lambda\right)$$

$$= \frac{1}{\beta(\ell)-\alpha(\ell)} \left(\sum_{\substack{k=\alpha(\ell)+1, \\ \sigma(p_1,\dots,p_{n-1},p_k-p_0,\lambda)\ge \varepsilon}}^{\beta(\ell)} + \sum_{\substack{k=\alpha(\ell)+1, \\ \bar{\sigma}(p_1,\dots,p_{n-1},p_k-p_0,\lambda)< \varepsilon}}^{\beta(\ell)} \right) \bar{\sigma}\left(p_1,p_2,\dots,p_{n-1},p_k-p_0,\lambda\right)$$

$$< \frac{1}{\beta(\ell)-\alpha(\ell)} \left(M \sum_{\substack{k=\alpha(\ell)+1, \\ \bar{\sigma}(p_1,\dots,p_{n-1},p_k-p_0,\lambda)\ge \varepsilon}}^{\beta(\ell)} 1 + \varepsilon \sum_{\substack{k=\alpha(\ell)+1, \\ \bar{\sigma}(p_1,\dots,p_{n-1},p_k-p_0,\lambda)< \varepsilon}}^{\beta(\ell)} 1 \right)$$

$$< \frac{M}{\beta(\ell)-\alpha(\ell)} \left|\left\{\alpha(\ell) < k \le \beta(\ell) : \bar{\sigma}\left(p_1,p_2,\dots,p_{n-1},p_k-p_0,\lambda\right) \ge \varepsilon\right\}\right|$$
$$+ \frac{\varepsilon}{\beta(\ell)-\alpha(\ell)} \left|\left\{\alpha(\ell) < k \le \beta(\ell) : \bar{\sigma}\left(p_1,p_2,\dots,p_{n-1},p_k-p_0,\lambda\right) < \varepsilon\right\}\right|.$$

Now, we take limit by considering $G_{\alpha,\beta}\left[S(\sigma,\tau,\upsilon)^n\right] - \lim p_k = p_0$, then we have

$$\frac{1}{\beta(\ell)-\alpha(\ell)} \sum_{k=\alpha(\ell)+1}^{\beta(\ell)} \bar{\sigma}\left(p_1,p_2,\dots,p_{n-1},p_k-p_0,\lambda\right) < M.(0) + \varepsilon.(1)$$
$$\frac{1}{\beta(\ell)-\alpha(\ell)} \sum_{k=\alpha(\ell)+1}^{\beta(\ell)} \bar{\sigma}\left(p_1,p_2,\dots,p_{n-1},p_k-p_0,\lambda\right) < \varepsilon$$

or equivently

$$\frac{1}{\beta(\ell)-\alpha(\ell)} \sum_{k=\alpha(\ell)+1}^{\beta(\ell)} \sigma\left(p_1,p_2,\dots,p_{n-1},p_k-p_0,\lambda\right) > 1-\varepsilon \tag{3.15}$$

Similarly, the following inequality

$$\frac{1}{\beta(\ell)-\alpha(\ell)} \sum_{k=\alpha(\ell)+1}^{\beta(\ell)} \tau\left(p_1,p_2,\dots,p_{n-1},p_k-p_0,\lambda\right)$$

$$= \frac{1}{\beta(\ell)-\alpha(\ell)}\left(\sum_{\substack{k=\alpha(\ell)+1,\\ \tau(p_1,\ldots,p_{n-1},p_k-p_0,\lambda)\geq\varepsilon}}^{\beta(\ell)} + \sum_{\substack{k=\alpha(\ell)+1,\\ \tau(p_1,\ldots,p_{n-1},p_k-p_0,\lambda)<\varepsilon}}^{\beta(\ell)}\right)\tau\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)$$

$$< \frac{1}{\beta(\ell)-\alpha(\ell)}\left(M\sum_{\substack{k=\alpha(\ell)+1,\\ \tau(p_1,\ldots,p_{n-1},p_k-p_0,\lambda)\geq\varepsilon}}^{\beta(\ell)} 1+\varepsilon\sum_{\substack{k=\alpha(\ell)+1,\\ \tau(p_1,\ldots,p_{n-1},p_k-p_0,\lambda)<\varepsilon}}^{\beta(\ell)} 1\right)$$

$$< \frac{M}{\beta(\ell)-\alpha(\ell)}\left|\{\alpha(\ell)<k\leq\beta(\ell):\tau\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\geq\varepsilon\}\right|$$
$$+\frac{\varepsilon}{\beta(\ell)-\alpha(\ell)}\left|\{\alpha(\ell)<k\leq\beta(\ell):\tau\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)<\varepsilon\}\right|$$

By similar process as above we have,

$$\frac{1}{\beta(\ell)-\alpha(\ell)}\sum_{k=\alpha(\ell)+1}^{\beta(\ell)}\tau\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)<\varepsilon \tag{3.16}$$

So, by considering equation (3.15) and (3.16) the proof is completed.
To prove the next theorem (5) we give the following lemma. □

Lemma 2.19. *If $a=(a_k)$ and $b=(b_k)$ be two sequences of positive integers such that $\lim_{k\to\infty}a_k=a_0$ and $\lim_{k\to\infty}b_k=\infty$, then $\lim_{k\to\infty}a_{b_k}=a_0$.*

Theorem 2.20. *Let $(\mathcal{P},\sigma,\tau,\star,\diamond)$ be an N-n-NLS and $p=(p_k)$ in $\mathcal{P}$ and sequence $\left(\frac{\alpha(\ell)}{\beta(\ell)-\alpha(\ell)}\right)$ is bounded then $[S(\sigma,\tau,\upsilon)^n]-\lim p_k=p_0$ implies*
$G_{\alpha,\beta}\left[S(\sigma,\tau,\upsilon)^n\right]-\lim p_k=p_0$.

Proof. Suppose $[S(\sigma,\tau,\upsilon)^n]-\lim p_k=p_0$ then every $\varepsilon\in(0,1),\lambda>0$ and $p_1,p_2,\ldots,p_{n-1}\in\mathcal{P}$,

$$\begin{aligned}\lim_{\ell\to\infty}\frac{1}{\ell}\Big|\{k\leq\ell:\ &\sigma\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\leq 1-\varepsilon,\\ &\tau\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\geq\varepsilon,\\ &\upsilon\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\geq\varepsilon\}\Big|=0.\end{aligned}$$

Let,

$$\begin{aligned}q_\ell=\lim_{\ell\to\infty}\frac{1}{\ell}\Big|\{k\leq\ell:\ &\sigma\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\leq 1-\varepsilon,\\ &\tau\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\geq\varepsilon,\\ &\upsilon\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\geq\varepsilon\}\Big|\end{aligned}$$

and $\beta(\ell)\to\infty$ as $\ell\to\infty$, so by lemma (2), we have

$$\begin{aligned}\lim_{\ell\to\infty}\frac{1}{\ell}\Big|\{k\leq\ell:\ &\sigma\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\leq 1-\varepsilon,\\ &\tau\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\geq\varepsilon,\\ &\upsilon\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right)\geq\varepsilon\}\Big|=0. \qquad (3.17)\end{aligned}$$

Now, from the inclusion

$$\begin{aligned}\{\alpha(\ell) < k \leq \beta(\ell) : &\sigma(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \leq 1 - \varepsilon,\\ &\tau(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon,\\ &\upsilon(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon\}\\ &\subset \{k \leq \beta(\ell) :\\ &\sigma(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \leq 1 - \varepsilon,\\ &\tau(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon,\\ &\upsilon(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon\},\end{aligned}$$

we have

$$\begin{aligned}\Big|\alpha(\ell) < k \leq \beta(\ell) : &\sigma(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \leq 1 - \varepsilon,\\ &\tau(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon,\\ &\upsilon(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon\Big|\\ &\leq \Big|k \leq \beta(\ell) :\\ &\sigma(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \leq 1 - \varepsilon,\\ &\tau(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon,\\ &\upsilon(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon\Big|,\end{aligned}$$

which immediately implies

$$\begin{aligned}&\frac{1}{\beta(\ell) - \alpha(\ell)}\Big|\{k \leq \beta(\ell) : \sigma(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \leq 1 - \varepsilon,\\ &\qquad\qquad \tau(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon,\\ &\qquad\qquad \upsilon(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon\}\Big|\\ &\leq \frac{1}{\beta(\ell) - \alpha(\ell)}\Big|\{k \leq \beta(\ell) : \sigma(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \leq 1 - \varepsilon,\\ &\qquad\qquad \tau(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon,\\ &\qquad\qquad \upsilon(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon\}\Big|\\ &= (1 + R(\ell)) \cdot \frac{1}{\beta(\ell)}\Big|\{k \leq \beta(\ell) : \sigma(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \leq 1 - \varepsilon,\\ &\qquad\qquad \tau(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon,\\ &\qquad\qquad \upsilon(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon\}\Big|\\ &\qquad\qquad \to 0\end{aligned}$$

As $R(\ell)$ is bounded and (3.14) holds where $R(\ell) = \frac{\alpha(\ell)}{\beta(\ell)-\alpha(\ell)}$. This shows that $G_{\alpha,\beta}\,[S(\sigma, \tau, \upsilon)^n]-\lim p_k = p_0$. □

4. Deferred Statistical Cauchy Sequence And Deferred Completeness IN N-*n*-NLS

In this work, we will develop and thoroughly examine the concepts of deferred statistical Cauchy sequences and deferred statistical completeness in N-*n*-NLS, contributing to a deeper understanding of sequence behavior in these spaces.

Definition 2.21. *Let $(\mathcal{P}, \sigma, \tau, \upsilon, \star, \diamond)$ be an N- n-NLS. A sequence $p = (p_k)$ in $\mathcal{P}$ is said to be deferred statistical Cauchy w.r.t the N-n-norm $(\sigma, \tau, \upsilon)^n$ if for every $\varepsilon \in (0,1), \lambda > 0$ and $p_1, p_2, \ldots, p_{n-1} \in \mathcal{P}, \exists\, r = r(\varepsilon)$ such that*

$$\lim_{\ell \to \infty} \frac{1}{\beta(\ell) - \alpha(\ell)} \Big| \{\alpha(\ell) < k \leq \beta(\ell) : \sigma(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \leq 1 - \varepsilon, \tau(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon, \upsilon(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon\} \Big| = 0.$$

or equivalently,

$$\Delta_{\alpha,\beta}(\{\alpha(\ell) < k \leq \beta(\ell) : \sigma(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \leq 1 - \varepsilon, \tau(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon, \upsilon(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon\}) = 0.$$

Definition 2.22. *Let $(\mathcal{P}, \sigma, \tau, \upsilon, \star, \diamond)$ be an N-n-NLS. A sequence $p = (p_k)$ in $\mathcal{P}$ is said to be deferred statistical complete if every deferred statistically Cauchy sequence w.r.t the N-n-norm $(\sigma, \tau, \upsilon)^n$ is deferred statistical convergent in $\mathcal{P}$.*

Theorem 2.23. *Let $(\mathcal{P}, \sigma, \tau, \upsilon, \star, \diamond)$ be an N-n-NLS. For any sequence $p = (p_k)$, if $G_{\alpha,\beta}[S(\sigma, \tau, \upsilon,)^n] - \lim p_k = p_0$, then $p = (p_k)$ is $G_{\alpha,\beta}[S(\sigma, \tau, \upsilon)^n]$ Cauchy.*

Proof. For $\varepsilon \in (0,1)$, choose $\zeta \in (0,1)$ such that (3.1) is satisfied. Since $G_{\alpha,\beta}[S(\sigma, \tau, \upsilon)^n] - \lim p_k = p_0$, so for every $\lambda > 0$ and $p_1, p_2, \ldots, p_{n-1} \in \mathcal{P}$ we have $\Delta_{\alpha,\beta}(D(\zeta, \lambda)) = 0$ and $\Delta_{\alpha,\beta}(D(\zeta, \lambda))^{\complement} = 1$, where

$$D(\zeta, \lambda) = \{\alpha(\ell) < k \leq \beta(\ell) : \sigma\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \frac{\lambda}{2}\right) \leq 1 - \zeta, \tau\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \frac{\lambda}{2}\right) \geq \zeta, \upsilon\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \frac{\lambda}{2}\right) \geq \zeta\}.$$

Let $r \in (D(\zeta, \lambda))^{\complement}$ then

$$\left\{\sigma\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \frac{\lambda}{2}\right) > 1 - \zeta, \tau\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \frac{\lambda}{2}\right) < \zeta, \upsilon\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \frac{\lambda}{2}\right) < \zeta\right\} \quad (4.1)$$

we define,

$$M(\varepsilon, \lambda) = \{\alpha(\ell) < k \leq \beta(\ell) : \sigma(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \leq 1 - \varepsilon, \tau(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon, \upsilon(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon\}$$

so to prove the result, it is sufficient to show that $M(\varepsilon, \lambda) \subseteq D(\zeta, \lambda)$.
Let $q \in M(\varepsilon, \lambda)$ then

$$\left\{\sigma(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \leq 1 - \varepsilon, \tau(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon, \upsilon(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda) \geq \varepsilon\right\} \quad (4.2)$$

Case 1: If $\sigma\left(p_1, p_2, \ldots, p_{n-1}, p_q - p_r, \lambda\right) \leq 1 - \varepsilon$, then we have

$$\sigma\left(p_1, p_2, \ldots, p_{n-1}, p_q - p_0, \frac{\lambda}{2}\right) \leq 1 - \zeta$$

therefore $q \in D(\zeta, \lambda)$. As otherwise,

$$\sigma\left(p_1, p_2, \ldots, p_{n-1}, p_q - p_0, \frac{\lambda}{2}\right) > 1 - \zeta$$

then by (3.1), (4.1) *and* (4.2) we have

$$\begin{aligned}
1 - \varepsilon &\geq \sigma\left(p_1, p_2, \ldots, p_{n-1}, p_q - p_r, \lambda\right) \\
&= \sigma\left(p_1, p_2, \ldots, p_{n-1}, p_q - p_0 + p_0 - p_r, \frac{\lambda}{2} + \frac{\lambda}{2}\right) \\
&\geq \sigma\left(p_1, \ldots, p_{n-1}, p_q - p_0, \frac{\lambda}{2}\right) \star \sigma\left(p_1, \ldots, p_{n-1}, p_0 - p_r, \frac{\lambda}{2}\right) \\
&> (1 - \zeta) \star (1 - \zeta) \\
&> 1 - \varepsilon. \quad (\text{ not possible })
\end{aligned}$$

Hence $M(\varepsilon, \lambda) \subseteq D(\zeta, \lambda)$.

Case 2: If $\tau\left(p_1, p_2, \ldots, p_{n-1}, p_q - p_r, \lambda\right) \geq \varepsilon$, then we have

$$\tau\left(p_1, p_2, \ldots, p_{n-1}, p_q - p_0, \frac{\lambda}{2}\right) \geq \zeta$$

therefore $q \in D(\zeta, \lambda)$. As otherwise

$$\tau\left(p_1, p_2, \ldots, p_{n-1}, p_q - p_0, \frac{\lambda}{2}\right) < \zeta$$

then by (3.1), (4.1) *and* (4.2) we have

$$\begin{aligned}
\varepsilon &\leq \tau\left(p_1, p_2, \ldots, p_{n-1}, p_q - p_r, \lambda\right) \\
&= \tau\left(p_1, p_2, \ldots, p_{n-1}, p_q - p_0 + p_0 - p_r, \frac{\lambda}{2} + \frac{\lambda}{2}\right) \\
&\leq \tau\left(p_1, \ldots, p_{n-1}, p_q - p_0, \frac{\lambda}{2}\right) \diamond \tau\left(p_1, \ldots, p_{n-1}, p_0 - p_r, \frac{\lambda}{2}\right) \\
&< \zeta \diamond \zeta \\
&< \varepsilon \quad (\text{not possible})
\end{aligned}$$

Case 3: If $\upsilon\left(p_1, p_2, \ldots, p_{n-1}, p_q - p_r, \lambda\right) \geq \varepsilon$, then we have

$$\upsilon\left(p_1, p_2, \ldots, p_{n-1}, p_q - p_0, \frac{\lambda}{2}\right) \geq \zeta$$

therefore $q \in D(\zeta, \lambda)$. As otherwise

$$\upsilon\left(p_1, p_2, \ldots, p_{n-1}, p_q - p_0, \frac{\lambda}{2}\right) < \zeta$$

then by (3.1), (4.1) *and* (4.2) we have

$$\begin{aligned}
\varepsilon &\leq \upsilon\left(p_1, p_2, \ldots, p_{n-1}, p_q - p_r, \lambda\right) \\
&= \upsilon\left(p_1, p_2, \ldots, p_{n-1}, p_q - p_0 + p_0 - p_r, \frac{\lambda}{2} + \frac{\lambda}{2}\right) \\
&\leq \upsilon\left(p_1, \ldots, p_{n-1}, p_q - p_0, \frac{\lambda}{2}\right) \diamond \upsilon\left(p_1, \ldots, p_{n-1}, p_0 - p_r, \frac{\lambda}{2}\right) \\
&< \zeta \diamond \zeta \\
&< \varepsilon \quad \text{(not possible)}
\end{aligned}$$

Hence $M(\varepsilon, \lambda) \subseteq D(\zeta, \lambda)$. Thus in all the cases, $M(\varepsilon, \lambda) \subseteq D(\zeta, \lambda)$. Since $\Delta_{\alpha,\beta}(D(\zeta, \lambda)) = 0$ so $\Delta_{\alpha,\beta}(M(\varepsilon, \lambda)) = 0$ and therefore $p = (p_k)$ is $G_{\alpha,\beta}\left[S(\sigma, \tau, \upsilon)^n\right]$-Cauchy. □

Theorem 2.24. *Let $(\mathcal{P}, \sigma, \tau, \upsilon, \star, \diamond)$ be an N-n-NLS. If every deferred statistical Cauchy sequence has deferred statistical convergent subsequence then $\mathcal{P}$ is deferred statistical complete.*

Proof. Given that $p = (p_k)$ be a $G_{\alpha,\beta}\left[S(\sigma, \tau, \upsilon)^n\right]$-Cauchy sequence in $\mathcal{P}$ and $\left(p_{k_j}\right)$ be a subsequence of (p_k) such that $p_{k_j} \to p_0\left(G_{\alpha,\beta}\left[S(\sigma, \tau, \upsilon)^n\right]\right)$. We have to prove that $p_k \to p_0\left(G_{\alpha,\beta}\left[S(\sigma, \tau, \upsilon)^n\right]\right)$. Let $\varepsilon \in (0, 1)$, choose $\zeta \in (0, 1)$ such that (3.1) holds. Since (p_k) is $G_{\alpha,\beta}\left[S(\sigma, \tau, \upsilon)^n\right]$-Cauchy so for any $\lambda > 0$ and $p_1, p_2, \ldots, p_{n-1} \in \mathcal{P}$ we have $\Delta_{\alpha,\beta}(E) = 0$, where

$$\begin{aligned}
E = \{\alpha(\ell) < k \leq \beta(\ell) : &\ \sigma\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \leq 1 - \zeta, \\
&\ \tau\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \geq \zeta, \\
&\ \upsilon\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \geq \zeta\}
\end{aligned}$$

Since $p_{k_j} \to p_0\left(G_{\alpha,\beta}\left[S(\sigma, \tau)^n\right]\right)$. So we have $\Delta_{\alpha,\beta}(F) = 0$, where

$$\begin{aligned}
F = \{\alpha(\ell) < k \leq \beta(\ell) : &\ \sigma\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \leq 1 - \zeta, \\
&\ \tau\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \geq \zeta, \\
&\ \upsilon\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \geq \zeta\}
\end{aligned}$$

Now define

$$\begin{aligned}
J = \{\alpha(\ell) < k \leq \beta(\ell) : &\ \sigma\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \leq 1 - \zeta, \\
&\ \tau\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \geq \zeta, \\
&\ \upsilon\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \geq \zeta\}
\end{aligned}$$

Now we claim that $E^{\complement} \cap F^{\complement} \subseteq J^{\complement}$. Let $b \in E^{\complement} \cap F^{\complement}$, then $b \in E^{\complement}$ and $b \in F^{\complement}$. Now first consider if $b \in E^{\complement}$, then

$$\begin{aligned}
\{&\sigma\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) > 1 - \zeta, \\
&\tau\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) < \zeta, \\
&\upsilon\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) < \zeta\} \qquad (4.3)
\end{aligned}$$

Now if $b \in F^{\complement}$, then $b = k_j$ for $j \in \mathbb{N}$ and therefore

$$\begin{aligned}
\{&\sigma\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) > 1 - \zeta, \\
&\tau\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) < \zeta, \\
&\upsilon\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) < \zeta\} \qquad (4.4)
\end{aligned}$$

Now,

$$\begin{aligned}
&\sigma\left(p_1, p_2, \ldots, p_{n-1}, p_m - p_0, \lambda\right)\\
&= \sigma\left(p_1, p_2, \ldots, p_{n-1}, p_m - p_{k_j} + p_{k_j} - p_0, \frac{\lambda}{2} + \frac{\lambda}{2}\right)\\
&\geq \sigma\left(p_1, p_2, \ldots, p_{n-1}, p_m - p_{k_j}, \frac{\lambda}{2}\right) \star \sigma\left(p_1, p_2, \ldots, p_{n-1}, p_{k_j} - p_0, \frac{\lambda}{2}\right)\\
&> (1-\zeta) \star (1-\zeta) \text{ for } r = k_j (\text{by } (4.3) \text{ and } (4.4))\\
&> 1 - \varepsilon \text{ by } (3.1)
\end{aligned}$$

and

$$\begin{aligned}
&\tau\left(p_1, p_2, \ldots, p_{n-1}, p_m - p_0, \lambda\right)\\
&= \tau\left(p_1, p_2, \ldots, p_{n-1}, p_m - p_{k_j} + p_{k_j} - p_0, \frac{\lambda}{2} + \frac{\lambda}{2}\right)\\
&\leq \tau\left(p_1, p_2, \ldots, p_{n-1}, p_m - p_{k_j}, \frac{\lambda}{2}\right) \diamond \tau\left(p_1, p_2, \ldots, p_{n-1}, p_{k_j} - p_0, \frac{\lambda}{2}\right)\\
&< \zeta \diamond \zeta \text{ for } r = k_j \quad (\text{by } (4.3) \text{ and } (4.4))\\
&< \varepsilon \quad (\text{by } (3.1))
\end{aligned}$$

and

$$\begin{aligned}
&\upsilon\left(p_1, p_2, \ldots, p_{n-1}, p_m - p_0, \lambda\right)\\
&= \upsilon\left(p_1, p_2, \ldots, p_{n-1}, p_m - p_{k_j} + p_{k_j} - p_0, \frac{\lambda}{2} + \frac{\lambda}{2}\right)\\
&\leq \upsilon\left(p_1, p_2, \ldots, p_{n-1}, p_m - p_{k_j}, \frac{\lambda}{2}\right) \diamond \upsilon\left(p_1, p_2, \ldots, p_{n-1}, p_{k_j} - p_0, \frac{\lambda}{2}\right)\\
&< \zeta \diamond \zeta \text{ for } r = k_j \quad (\text{by } (4.3) \text{ and } (4.4))\\
&< \varepsilon \quad (\text{by } (3.1))
\end{aligned}$$

which implies that $b \in J^{\complement}$, so $E^{\complement} \cap F^{\complement} \subseteq J^{\complement}$ or $J \subseteq E \cup F$. Therefore, $\Delta_{\alpha,\beta}(J) \leq \Delta_{\alpha,\beta}(E \cup F) = 0$. Thus, $p_k \to p_0 \left(G_{\alpha,\beta}\left[S(\sigma,\tau,\upsilon)^n\right]\right)$. Hence $\mathcal{P}$ is deferred statistically complete.

By similar process in the proof of theorem (3) and by theorem (7) we have the following theorem. □

Theorem 2.25. *Let $(\mathcal{P}, \sigma, \tau, \upsilon, \star, \diamond)$ be an N-n-NLS. If $p = (p_k)$ be any sequence in $\mathcal{P}$. Then the following conditions are equivalent:*
(i) $p = (p_k)$ is deferred statistical summable w.r.t the N-n-norm $(\sigma, \tau, \upsilon)^n$.
(ii) $p = (p_k)$ is deferred statistical Cauchy w.r.t the N-n-norm $(\sigma, \tau, \upsilon)^n$.
(iii) $\exists$ a set $T = \{t_1 < t_2 < \cdots\} \subset \mathbb{N}$ such that $\Delta_{\alpha,\beta}(T) = 1$ and the subsequence $(p_{t_j})_{t_j \in T}$ is Cauchy sequence w.r.t the N-n-norm $(\sigma, \tau, \upsilon)^n$.

5. Comparison of $G_{\alpha,\beta}\left[S(\sigma,\tau,\upsilon)^n\right]$ and $G_{m,n}\left[S(\sigma,\tau,\upsilon)^n\right]$

In this part, we consider two different pairs of integer sequences, $\alpha(\ell), \beta(\ell)$ and $m(\ell)), n(\ell)$, that satisfy the inequality

$$\alpha(\ell) \leq m(\ell)) < n(\ell) \leq \beta(\ell) \quad \text{for all } \ell \in \mathbb{N}. \tag{5.1}$$

Our goal to is study the relationship between the sets $G_{\alpha,\beta}\left[S(\sigma,\tau,\upsilon)^n\right]$ and $G_{m,n}\left[S(\sigma,\tau,\upsilon)^n\right]$.

The deferred limit of a sequence can be very different integer sequences $\alpha(\ell)$ and $\beta(\ell)$, leading to different sets $G_{\alpha,\beta}\left[S(\sigma,\tau,\upsilon)^n\right]$. Consequently, if we choosing different pairs of sequences $\alpha(\ell), \beta(\ell)$ and $m(\ell)), n(\ell)$, satisfying (5.1) *then we can lead to deferred limits, producing distinct sets.*

Theorem 2.26. *Let $m = m(\ell))$ and $n = n(\ell)$ be two positive integers sequences satisfying $\alpha(\ell) \leq m(\ell)) < n(\ell) \leq \beta(\ell)$ such that $\{k : \alpha(\ell) < k \leq m(\ell))\}$ and $\{k : n(\ell) < k \leq \beta(\ell)\}$ are finite sets for all $\ell \in \mathbb{N}$ then $G_{m,n}\left[S(\sigma,\tau,\upsilon)^n\right] - \lim p_k = p_0$ implies that $G_{\alpha,\beta}\left[S(\sigma,\tau,\upsilon)^n\right] - \lim p_k = p_0$.*

Proof. Suppose $G_{m,n}\left[S(\sigma,\tau,\upsilon)^n\right] - \lim p_k = p_0$ then for every $\varepsilon \in (0,1)$, $\lambda > 0$ and $p_1, p_2, \ldots, p_{n-1} \in \mathcal{P}$, we have

$$\lim_{\ell\to\infty} \frac{1}{n(\ell) - m(\ell))} \Big| \{m(\ell)) < k \leq n(\ell) : \sigma\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \leq 1 - \varepsilon, \\ \tau\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \geq \varepsilon, \\ \upsilon\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \geq \varepsilon\} \Big| = 0$$

Now define sets

$$D_{\alpha,\beta}(\sigma,\tau,\upsilon)^n = \{\alpha(\ell) < k \leq \beta(\ell) : \sigma\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \leq 1 - \varepsilon, \\ \tau\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \geq \varepsilon, \\ \upsilon\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \geq \varepsilon\}$$

$$D_{\alpha,m}(\sigma,\tau,\upsilon)^n = \{\alpha(\ell) < k \leq m(\ell)) : \sigma\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \leq 1 - \varepsilon, \\ \tau\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \geq \varepsilon, \\ \upsilon\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \geq \varepsilon\}$$

$$D_{m,n}(\sigma,\tau,\upsilon)^n = \{m(\ell)) < k \leq n(\ell) : \sigma\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \leq 1 - \varepsilon, \\ \tau\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \geq \varepsilon, \\ \upsilon\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \geq \varepsilon\}$$

$$D_{n,\beta}(\sigma,\tau,\upsilon)^n = \{n(\ell) < k \leq \beta(\ell) : \sigma\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \leq 1 - \varepsilon, \\ \tau\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \geq \varepsilon, \\ \upsilon\left(p_1, p_2, \ldots, p_{n-1}, p_k - p_0, \lambda\right) \geq \varepsilon\}$$

It is clear that,

$$D_{\alpha,\beta}(\sigma,\tau,\upsilon)^n = D_{\alpha,m}(\sigma,\tau,\upsilon)^n \cup D_{m,n}(\sigma,\tau,\upsilon)^n \cup D_{n,\beta}(\sigma,\tau,\upsilon)^n.$$

Now,

$$\frac{\beta(\ell) - \alpha(\ell)}{\beta(\ell) - \alpha(\ell)} \left|D_{\alpha,\beta}(\sigma,\tau,\upsilon)^n\right| \leq \Bigg(\frac{m(\ell)) - \alpha(\ell)}{m(\ell)) - \alpha(\ell)} \left|D_{\alpha,m}(\sigma,\tau,\upsilon)^n\right| \\ + \frac{n(\ell) - m(\ell))}{n(\ell) - m(\ell))} \left|D_{m,n}(\sigma,\tau,\upsilon)^n\right| \\ + \frac{\beta(\ell) - n(\ell)}{\beta(\ell) - n(\ell)} \left|D_{n,\beta}(\sigma,\tau,\upsilon)^n\right| \Bigg),$$

and therefore,

$$\frac{1}{\beta(\ell) - \alpha(\ell)} \left|D_{\alpha,\beta}(\sigma,\tau,\upsilon)^n\right| \leq \frac{1}{\beta(\ell) - \alpha(\ell)} \Bigg(\frac{m(\ell)) - \alpha(\ell)}{m(\ell)) - \alpha(\ell)} \left|D_{\alpha,m}(\sigma,\tau,\upsilon)^n\right| \\ + \frac{n(\ell) - m(\ell))}{n(\ell) - m(\ell))} \left|D_{m,n}(\sigma,\tau,\upsilon)^n\right| \\ + \frac{\beta(\ell) - n(\ell)}{\beta(\ell) - n(\ell)} \left|D_{n,\beta}(\sigma,\tau,\upsilon)^n\right| \Bigg).$$

Taking limit $\ell \to \infty$ and by using the fact that deferred density of a finite set is zero. We have

$$\lim_{\ell\to\infty} \frac{1}{\beta(\ell)-\alpha(\ell)} \left|D_{\alpha,\beta}(\sigma,\tau,\upsilon)^n\right| = 0$$

Hence,

$$G_{\alpha,\beta}\left[S(\sigma,\tau,\upsilon)^n\right] - \lim p_k = p_0.$$

We now give conditions on $\alpha(\ell), m(\ell)), n(\ell)$ and $\beta(\ell)$ such that the converse of the above theorem (10) holds. □

Theorem 2.27. *Let $\alpha(\ell), m(\ell)), n(\ell)$ and $\beta(\ell)$ be sequence of natural number satisfying $\alpha(\ell) \le m(\ell)) < n(\ell) \le \beta(\ell)$ such that $\lim_{\ell\to\infty} \frac{\beta(\ell)-\alpha(\ell)}{n(\ell)-m(\ell))} = r > 0$. Then $G_{\alpha,\beta}\left[S(\sigma,\tau,\upsilon)^n\right] - \lim p_k = p_0$ implies that $G_{m,n}\left[S(\sigma,\tau,\upsilon)^n\right] - \lim p_k = p_0$.*

Proof. Let us define set $D_{m,n}(\sigma,\tau,\upsilon)^n$ and $D_{\alpha,\beta}(\sigma,\tau,\upsilon)^n$ as follows

$$\begin{aligned} D_{m,n}(\sigma,\tau,\upsilon)^n = \{ & m(\ell)) < k \le n(\ell) : \\ & \sigma\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right) \le 1-\varepsilon, \\ & \tau\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right) \ge \varepsilon, \\ & \upsilon\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right) \ge \varepsilon\} \end{aligned}$$

and

$$\begin{aligned} D_{\alpha,\beta}(\sigma,\tau,\upsilon)^n = \{ & \alpha(\ell) < k \le \beta(\ell) : \\ & \sigma\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right) \le 1-\varepsilon, \\ & \tau\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right) \ge \varepsilon, \\ & \upsilon\left(p_1,p_2,\ldots,p_{n-1},p_k-p_0,\lambda\right) \ge \varepsilon\} \end{aligned}$$

It is clear to see

$$D_{m,n}(\sigma,\tau,\upsilon)^n \subset D_{\alpha,\beta}(\sigma,\tau,\upsilon)^n,$$

and therefore

$$\left|D_{m,n}(\sigma,\tau,\upsilon)^n\right| \le \left|D_{\alpha,\beta}(\sigma,\tau,\upsilon)^n\right|.$$

Thus, we have

$$\frac{1}{m(\ell))-n(\ell)} \left|D_{m,n}(\sigma,\tau,\upsilon)^n\right| \le \frac{\beta(\ell)-\alpha(\ell)}{m(\ell))-n(\ell)} \cdot \frac{1}{\beta(\ell)-\alpha(\ell)} \left|D_{\alpha,\beta}(\sigma,\tau,\upsilon)^n\right| \tag{5.2}$$

Since

$$G_{\alpha,\beta}\left[S(\sigma,\tau,\upsilon)^n\right] - \lim p_k = p_0,$$

so we have,

$$\lim_{\ell\to\infty} \frac{1}{m(\ell))-n(\ell)} \left|D_{m,n}(\sigma,\tau)^n\right| = 0.$$

Hence,

$$G_{m,n}\left[S(\sigma,\tau,\upsilon)^n\right] - \lim p_k = p_0.$$

□

6. Conclusion

The Neutrosophic norm, an evolution of fuzzy norms, which provides a more advanced versatile mathematical framework for handling uncertainty and vagueness. This norms ability to work in high dimensional spaces. Higher-dimensional spaces are crucial for capturing the complexity of real-world problems, modeling multi-dimensional data and providing a framework for advanced mathematical and computational techniques in various disciplines. In the present research, we studied deferred summability, deferred statistical summability in N-n-NLS. theorems [1] establish a link between usual convergence and these summability concepts in this space. The properties and relationships among these concepts are demonstrated in theorems [2], [3], [4] and [5]. The condition in theorem [6], explores cases where statistical summability implies deferred statistical summability. Additionally, we studied deferred statistical Cauchy sequences, deferred statistical completeness and their outcomes. Finally, theorems [10] and [11] present comparisons to understand the convergence behavior of deferred statistically summability with itself for different pairs of sequences in N- n-NLS.

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