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Some Fredholm left invertible completion problems for operator matrices

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Abstract. The structure of the weak approximate spectrum of two-by-two upper triangular operator matrix

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

acting on a Hilbert space $\mathcal{H} \oplus \mathcal{K}$ is studied. First, we characterize the relationship between the spectrum $\sigma(M_C)$ and the weak approximate spectrum $\sigma_{_{Fa}}(M_C)$ and give the equivalent conditions that make

$$\sigma_{_{Fa}}(M_C) = \sigma(M_C)$$

and

$$\sigma_{_{Fa}}(M_C) = \sigma_{_{Fa}}(A) \cup \sigma_{_{Fa}}(B)$$

according to the properties that the operators A and B satisfy. Then, we study the weak property (ω_1) of M_C and explore the relationship between $\sigma(M_C) = \sigma_{_{Fa}}(M_C)$ and the weak property (ω_1) of M_C.

1. Introduction

In the study of linear operators theory, it is a popular method and idea to analyze the properties of linear operators by studying operator matrices. In fact, for a linear operator T from an infinite complex separable Hilbert space $\mathcal{H}$ to another infinite complex separable Hilbert space $\mathcal{K}$, if $\mathcal{H}$ and $\mathcal{K}$ can be decomposed into the direct sum of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and the direct sum of Hilbert spaces $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively, then T can naturally be written as the following operator matrix:

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} : \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{pmatrix} \to \begin{pmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \end{pmatrix}.$$

For this T, if the range of the restriction of T to the subspace $\mathcal{H}_1$ is contained in $\mathcal{K}_1$, then $T_{21} = 0$. Now, T is an upper triangular operator matrix. In recent years, researchers have characterized operator matrices

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from various aspects, such as the closedness (see [12]), the Fredholmness (see [11]), the Weylness (see [18]), the invertibility (see [13]), the Weyl type theorems (see [4]), the spectral problems (see [2, 15]) and so on. For these investigations, the most common problem is the exploration of completion problems for operator matrices. The completion problems for operator matrices refer to studying the properties of the known operator entries in the operator matrix such that it has given properties when adding unknown the operator entries.

This paper focuses on the completion problem of two-by-two upper triangular operator matrices. Let $\mathcal{B}(\mathcal{H},\mathcal{K})$ be the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. It is abbreviated as $\mathcal{B}(\mathcal{H})$ or $\mathcal{B}(\mathcal{K})$ if $\mathcal{H}=\mathcal{K}$. For given $A\in\mathcal{B}(\mathcal{H})$ and $B\in\mathcal{B}(\mathcal{K})$, we denote by M_C a two-by-two upper triangular operator matrix of the following form:

$$M_C=\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}:\begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix}\to\begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix},$$

where $C\in\mathcal{B}(\mathcal{K},\mathcal{H})$. If $C=0$, then we write M_C as M_0. Recently, many published articles focus on the completion problems for M_C. In 2009, Dou et al. characterize the closedness of the range for M_C (see [8]). Since then, Hai and Chen explore the consistent invertibility (see [9]), Yang and Cao characterize the upper semi-Fredholmness (see [19]), Dong and Cao investigate the consistency in Fredholm and index (see [7]), and so on. In this paper, we will consider some problems of the left invertibility for M_C.

2. Preparations

Let $\mathbb{C}$ and $\mathbb{N}$ be the set of complex numbers and the set of natural numbers, respectively. Let $T\in\mathcal{B}(\mathcal{H},\mathcal{K})$. We use $N(T)$, $R(T)$ and T^* to denote the null space, the range and the adjoint of T, respectively. Let $n(T)\triangleq \dim N(T)$ and $d(T)\triangleq \operatorname{codim} R(T)$. If $R(T)$ is closed, then, according to the values of $n(T)$ and $d(T)$ of T, the following operators can be defined:

(1) upper semi-Fredholm operator: $n(T)<\infty$;
(2) lower semi-Fredholm operator: $d(T)<\infty$;
(3) semi-Fredholm operator: $n(T)<\infty$ or $d(T)<\infty$;
(4) Fredholm operator: $n(T)<\infty$ and $d(T)<\infty$.

If T is semi-Fredholm, then T has the index $\operatorname{ind}(T)=n(T)-d(T)$, and when $\operatorname{ind}(T)=0$, we call T Weyl. If a Weyl operator T with a finite ascent $asc(T)$ or a finite descent $des(T)$, then we call T Browder, where $asc(T)=\inf\{n\in\mathbb{N}:N(T^n)=N(T^{n+1})\}$, $des(T)=\inf\{n\in\mathbb{N}:R(T^n)=R(T^{n+1})\}$. If $n(T)=0$ and $R(T)$ is closed, then T is left invertible; if $d(T)=0$, then T is right invertible; if $n(T)=d(T)=0$, then the T is invertible. Obviously, $d(T)=0$ implies that $R(T)$ is closed. In fact, if $d(T)<\infty$, then $R(T)$ is closed (see [1, Corollary 1.15]). Let $\sigma(T)$ be the normal spectrum of T:

$$\sigma(T)=\{\lambda\in\mathbb{C}:T-\lambda I \text{ is not invertible}\}.$$

Similarly defined, many local spectra of T can be given as follows:

$$\sigma_x(T)=\{\lambda\in\mathbb{C}:T-\lambda I \text{ is not adj.}_{(\mathrm{x})}\},$$

where $x\in\{a,b,w,e,SF,SF_+,SF_-\}$ and adj.$_{(\mathrm{e})}\triangleq$ Fredholm, adj.$_{(w)}\triangleq$ Weyl, adj.$_{(\mathrm{b})}\triangleq$ Browder, adj.$_{(a)}\triangleq$ left invertible, adj.$_{(\mathrm{SF}_+)}\triangleq$ upper semi-Fredholm, adj.$_{(\mathrm{SF}_-)}\triangleq$ lower semi-Fredholm, adj.$_{(\mathrm{SF})}\triangleq$ semi-Fredholm. In addition,

$$\sigma_{ea}(T)=\sigma_{SF_+}(T)\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)>d(T-\lambda I)\}$$

and

$$\sigma_{Fea}(T)=\sigma_e(T)\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)>d(T-\lambda I)\}$$

are called the essential approximate spectrum of T and the weak essential approximate spectrum of T, respectively. We call

$$\sigma_{Fa}(T)=\{\lambda\in\mathbb{C}:T-\lambda I \text{ is not Fredholm left invertible}\},$$

the weak approximate spectrum of T. For $\sigma_{\overline{x}}(T)$, we denote by $\rho_{\overline{x}}(T)$ the complementary set of $\sigma_{\overline{x}}(T)$, where $\overline{x} \in \{a, e, Fa, SF, SF_+, SF_-\}$.

Let $\mathcal{M}$ be a subspace of $\mathcal{H}$. Then we denote by $\mathcal{M}^\perp$ and $T|_{\mathcal{M}}$ the orthogonal complement of $\mathcal{M}$ and the restriction of the operator T to the subspace $\mathcal{M}$, respectively. Let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of $\mathcal{H}$. Then we denote by $\mathcal{M} \oplus \mathcal{N}$ and $\mathcal{M} \ominus \mathcal{N}$ the orthogonal sum and the orthogonal difference of $\mathcal{M}$ and $\mathcal{N}$, where $\mathcal{M} \ominus \mathcal{N} = \mathcal{M} \cap \mathcal{N}^\perp$. Let Ω be a subset of $\mathbb{C}$. Then we denote by isoΩ the set of isolated points of Ω.

For the convenience of expression, we define the FLI operator.

Definition 2.1. *Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. If T is Fredholm with $n(T) = 0$, then we call T a Fredholm left invertible operator, abbreviated a* **FLI** *operator.*

Next, we list some results that are useful in subsequent contents. The following conclusion can be gotten by the Lemma 2.3 in [19].

Lemma 2.2. *Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then:*

(1) *if $M \in \mathcal{B}(\mathcal{H})$ is invertible, then $R(TM)$ is closed if and only if $R(T)$ is closed, moreover, $n(TM) = n(T)$ and $d(TM) = d(T)$;*

(2) *if $N \in \mathcal{B}(\mathcal{K})$ is invertible, then $R(NT)$ is closed if and only if $R(T)$ is closed, moreover, $n(NT) = n(T)$ and $d(NT) = d(T)$.*

For M_0, it is easy to prove that $R(M_0)$ is closed if and only if $R(A)$ and $R(B)$ are closed, moreover, $n(M_0) = n(A) + n(B)$ and $d(M_0) = d(A) + d(B)$. For $M_C \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$, it can be written in the following form:

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \begin{pmatrix} I_{\mathcal{H}} & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I_{\mathcal{H}} & C \\ 0 & I_{\mathcal{K}} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & I_{\mathcal{K}} \end{pmatrix},$$

where $I_{\mathcal{H}}$ and $I_{\mathcal{K}}$ are identity operators on $\mathcal{H}$ and $\mathcal{K}$, respectively, the operator matrix $\begin{pmatrix} I_{\mathcal{H}} & C \\ 0 & I_{\mathcal{K}} \end{pmatrix}$ is invertible on $\mathcal{H} \oplus \mathcal{K}$, some Fredholm properties of M_C can be derived from [1, Remark 1.54] and [17, Theorem 13.1].

Lemma 2.3. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then, for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$:*

(1) *if M_C is upper semi-Fredholm, then A is upper semi-Fredholm;*

(2) *if M_C is lower semi-Fredholm, then B is lower semi-Fredholm;*

(3) *if M_C is Fredholm, then A is upper semi-Fredholm and B is lower semi-Fredholm;*

(4) *if A and B are Fredholm, then M_C is Fredholm and*

$$\text{ind}(M_C) = \text{ind}(A) + \text{ind}(B);$$

(5) *if M_C is Fredholm, then A is Fredholm if and only if B is Fredholm.*

From [10, Proposition 2.1], we know $n(A) \le n(M_C) \le n(A) + n(B)$ and $d(B) \le d(M_C) \le d(A) + d(B)$ for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. So, we have:

Corollary 2.4. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then, for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, we have:*

(1) *if M_C is left invertible, then A is left invertible;*

(2) *if M_C is right invertible, then B is right invertible;*

(3) *if A and B are left invertible, then M_C is left invertible;*

(4) *if A and B are invertible, then M_C is invertible;*

(5) *if M_C is invertible, then A is invertible if and only if B is invertible.*

Lemma 2.5. *Let $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K})$ and $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. If $C|_{N(B)}$ is injective from $N(B)$ into $N(A^*)$, then $n(A) = 0$ if and only if $n(M_C) = 0$.*

Proof. From [10, Proposition 2.1], it is obvious that $n(M_C) = 0$ implies $n(A) = 0$. In addition, let $\begin{pmatrix} x \\ y \end{pmatrix} \in N(M_C)$. Then $Ax + Cy = 0$ and $By = 0$, which shows $y \in N(B)$. So $Ax = -Cy \in R(A) \cap N(A^*) = \{0\}$. Due to A and $C|_{N(B)}$ are injective, it follows $x = y = 0$. Thus $n(M_C) = 0$. □

In this paper, we will explore the three questions in turn.

Question 1. What properties do A and B satisfy for

$$\sigma(M_C) = \sigma_{Fa}(M_C)$$

to be true for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$?

Question 2. What properties do A and B satisfy for

$$\sigma_{Fa}(M_C) = \sigma_{Fa}(A) \cup \sigma_{Fa}(B)$$

to be true for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$?

Question 3. Is there a relationship between $\sigma(M_C) = \sigma_{Fa}(M_C)$ and the weak property (ω_1) for M_C?

3. The structure of weak approximate spectrum $\sigma_{Fa}(M_C)$

If A and B are unconditional, then, for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, neither $\sigma(M_C) = \sigma_{Fa}(M_C)$ nor $\sigma_{Fa}(M_C) = \sigma_{Fa}(A) \cup \sigma_{Fa}(B)$ is necessarily true. We illustrate this with the following two examples.

Remark 3.1. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then there exists some $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $\sigma(M_C) \neq \sigma_{Fa}(M_C)$.*

Example 3.2. *Let $A, B, C \in \mathcal{B}(\ell^2)$ be defined by*

$$A(x_1, x_2, x_3, \cdots) = (0, x_1, 0, x_2, 0, x_3, \cdots),$$

$$B(x_1, x_2, x_3, \cdots) = (x_2, x_4, x_6, \cdots),$$

$$C(x_1, x_2, x_3, \cdots) = (0, 0, x_1, 0, x_3, 0, \cdots).$$

Let $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. We can prove that M_C is Fredholm with $n(M_C) = 0$, but $d(M_C) > 1$.

By calculations, A is upper semi-Fredholm with $n(A) = 0$, B is lower semi-Fredholm with $d(B) = 0$ and $A = B^*$. If M_C is decomposed in the spaces from $R(A^*) \oplus N(A) \oplus R(B^*) \oplus N(B)$ to $R(A) \oplus N(A^*) \oplus R(B) \oplus N(B^*)$, we have the following matrix form:

$$M_C = \begin{pmatrix} A_1 & 0 & C_1 & C_2 \\ 0 & 0 & C_3 & C_4 \\ 0 & 0 & B_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} R(A^*) \\ N(A) \\ R(B^*) \\ N(B) \end{pmatrix} \to \begin{pmatrix} R(A) \\ N(A^*) \\ R(B) \\ N(B^*) \end{pmatrix}.$$

Obviously, A_1 and B_1 are invertible. Thus, according to elementary transformation of matrices and Lemma 2.2, $R(M_C)$ is closed if and only if $R(C_4) (C_4 = C|_{N(B)})$ is closed, moreover, $n(M_C) = n(C_4) + n(A)$ and $d(M_C) = d(C_4) + d(B)$. By calculations, $R(C_4)$ is closed, and $n(C_4) = 0$, $d(C_4) = 1$. Thus, for this M_C, $0 \in \sigma(M_C)$ but $0 \notin \sigma_{Fa}(M_C)$, which shows $\sigma(M_C) \neq \sigma_{Fa}(M_C)$.

Remark 3.3. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then there exists some $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that*

$$\sigma_{Fa}(M_C) \neq \sigma_{Fa}(A) \cup \sigma_{Fa}(B).$$

Example 3.4. *Let $A, B, C \in \mathcal{B}(\ell^2)$ be defined by*

$$A(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots),$$

$$B(x_1, x_2, x_3, \cdots) = (0, 0, x_2, x_3, x_4, \cdots),$$

$$C(x_1, x_2, x_3, \cdots) = (x_1, 0, 0, 0, \cdots).$$

Let $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. Then, by calculations, we have $0 \notin \sigma_{_{Fa}}(M_C)$, but $0 \in \sigma_{_{Fa}}(B)$. Thus, for this M_C,

$$\sigma_{_{Fa}}(M_C) \neq \sigma_{_{Fa}}(A) \cup \sigma_{_{Fa}}(B).$$

From Remarks 3.1 and 3.3, it is natural to ask when $\sigma(M_C) = \sigma_{_{Fa}}(M_C)$ and $\sigma_{_{Fa}}(M_C) = \sigma_{_{Fa}}(A) \cup \sigma_{_{Fa}}(B)$ can be presented for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. To answer the questions, we need to give some lemmas first.

Lemma 3.5. *Assume that $A \in \mathcal{B}(\mathcal{H})$ is left invertible and $B \in \mathcal{B}(\mathcal{K})$ is lower semi-Fredholm. If $d(A) = n(B) = \infty$, then there exists some $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is FLI but not invertible.*

Proof. We discuss it from two aspects: $d(B) = 0$ and $d(B) > 0$.

Case 1: $d(B) = 0$.

Now, the operator B is right invertible. Take $\mu_1 \in N(A^*)$ with $\|\mu_1\| = 1$. Let $\mathcal{E} = \text{span}\{\mu_1\}$ and let C_{11} be an isometric invertible operator from $N(B)$ to $N(A^*) \ominus \mathcal{E}$. We construct a $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ as follows:

$$C = \begin{pmatrix} C_{11} & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} N(B) \\ R(B^*) \end{pmatrix} \to \begin{pmatrix} N(A^*) \ominus \mathcal{E} \\ R(A) \oplus \mathcal{E} \end{pmatrix}.$$

It is obvious that $C|_{N(B)} = C_{11}$ and C_{11} is injective from $N(B)$ into $N(A^*) \ominus \mathcal{E}$. Thus, it follows from $n(A) = 0$ and Lemma 2.5 that $n(M_C) = 0$.

In addition, we can prove that $R(M_C)$ is closed. Suppose

$$M_C \begin{pmatrix} x_n \\ y_n \end{pmatrix} \to \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \text{ as } n \to \infty,$$

that is,

$$\begin{cases} Ax_n + Cy_n \to u_0 \\ By_n \to v_0 \end{cases} \quad (n \to \infty).$$

Let $y_n = \alpha_n + \beta_n (n = 1, 2, 3, \cdots)$, where $\{\alpha_n\}_{n=1}^{\infty} \subseteq N(B)$ and $\{\beta_n\}_{n=1}^{\infty} \subseteq R(B^*)$. Then, it follows from $R(C|_{N(B)}) \subseteq N(A^*)$ and $R(C|_{R(B^*)}) = \{0\}$ that $\{Ax_n\}_{n=1}^{\infty}$ and $\{Cy_n\}_{n=1}^{\infty}$ are Cauchy sequences, and then $\{\alpha_n\}_{n=1}^{\infty}$ is a Cauchy sequence because $Cy_n = C_{11}\alpha_n (n = 1, 2, 3, \cdots)$ and C_{11} is invertible. Moreover, since $By_n = B\beta_n (n = 1, 2, 3, \cdots)$ and $B|_{R(B^*)}$ is invertible, it follows that $\{y_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Let $y_n \to y_0$ as $n \to \infty$. Since $R(A)$ is closed, there exists $x_0 \in \mathcal{H}$ such that $Ax_n \to Ax_0$ as $n \to \infty$. Thus $Ax_0 + Cy_0 = u_0$ and $By_0 = v_0$, which shows that $R(M_C)$ is closed.

Next, we will prove $d(M_C) = n(M_C^*) = 1$. Take $\begin{pmatrix} x \\ y \end{pmatrix} \in N(M_C^*)$. Then $x \in N(A^*)$. Let $x = x_1 + x_2$, where $x_1 \in N(A^*) \ominus \mathcal{E}$ and $x_2 \in \mathcal{E}$. Now, $C_{11}^* x_1 = C^* x = -B^* y \in N(B) \cap R(B^*) = \{0\}$. Thus $C_{11}^* x_1 = 0 = B^* y$, which implies $x_1 = 0$ and $y = 0$ because B is right invertible. So $N(M_C^*) = \text{span}\left\{\begin{pmatrix} x_2 \\ 0 \end{pmatrix}\right\}$. Due to $\dim \mathcal{E} = 1$, it follows that $n(M_C^*) = \dim \mathcal{E} = 1$.

Thus, the M_C is FLI but not invertible..

Case 2: $d(B) > 0$.

In this case, let C'_{11} be an isometric invertible operator from $N(B)$ to $N(A^*)$ and let

$$C = \begin{pmatrix} C'_{11} & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} N(B) \\ R(B^*) \end{pmatrix} \to \begin{pmatrix} N(A^*) \\ R(A) \end{pmatrix}.$$

Using the same method as in Case 1 above, we can get that $R(M_C)$ is closed and $n(M_C) = 0$. Moreover, we can prove $0 < d(M_C) < \infty$. In fact, let $\begin{pmatrix} x \\ y \end{pmatrix} \in N(M_C^*)$. Then we have $x \in N(A^*)$ and $C^*x = -B^*y \in N(B) \cap R(B^*)$, which implies $x = 0$ and $y \in N(B^*)$. Thus $N(M_C^*) = \{0\} \oplus N(B^*)$. Due to $d(M_C) = n(M_C^*) = n(B^*) = d(B)$, we have $0 < d(M_C) < \infty$. So, M_C is FLI but not invertible. □

It can be seen from the above proof of Case 1 that if $d(B) = 0$, the following conclusion can be obtained.

Corollary 3.6. *Assume that $A \in \mathcal{B}(\mathcal{H})$ is left invertible and $B \in \mathcal{B}(\mathcal{K})$ is right invertible. If $d(A) = n(B) = \infty$, then, for any positive integer k, there exists some $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is FLI but not invertible and $d(M_C) = k$.*

Lemma 3.7. *Assume that $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ are Fredholm. If $n(A) = 0$ and $n(B) < d(A)$, then there exists some $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is FLI but not invertible.*

Proof. Due to $n(B) < d(A)$, it follows from [3, Corollary 2.5] that M_C is not invertible for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Moreover, since A and B are Fredholm, it follows that M_C is Fredholm for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Next, we only need to prove that $n(M_C) = 0$ for some $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. If $n(B) = 0$, then it is obvious that $n(M_C) = 0$ for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. If $n(B) > 0$, let $\mathcal{F}$ be a closed subspace of $N(A^*)$ with $\dim\mathcal{F} = n(B)$ and let $N(A^*) = \mathcal{F} \oplus \mathcal{F}^{\perp}$. Then there exists an isometric invertible operator T from $N(B)$ onto $\mathcal{F}$. Let

$$C = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} N(B) \\ R(B^*) \end{pmatrix} \to \begin{pmatrix} \mathcal{F} \\ R(A) \oplus N(A^*) \ominus \mathcal{F} \end{pmatrix}.$$

From Lemma 2.5, we have $n(M_C) = n(A) = 0$. Thus M_C is FLI but not invertible for this C. □

Lemma 3.8. *Assume that $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ are Fredholm. If $n(A) = 0$, $n(B) = d(A)$ and $d(B) > 0$, then there exists some $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is FLI but not invertible.*

Proof. Due to $d(B) > 0$, it follows from [3, Corollary 2.5] that M_C is not invertible for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. So, if $n(B) = 0$, then M_C is FLI but not invertible for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. If $n(B) > 0$, by the same method as the Case 2 of Lemma 3.5, there exists some $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is FLI but not invertible. □

Remark 3.9. *In Lemma 3.8, the condition "$d(B) > 0$" is essential. In fact, if $d(B) = 0$, then M_C is Weyl for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Now, if M_C is left invertible, then M_C is invertible.*

From Lemmas 3.5, 3.7 and 3.8 and together with the Corollary 2.7 in [3], we can obtain the equivalent conditions that make $\sigma(M_C) = \sigma_{_{Fa}}(M_C)$ for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$.

Theorem 3.10. *Assume that $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then, for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, $\sigma(M_C) = \sigma_{_{Fa}}(M_C)$ if and only if the three sets $\mathcal{S}_1(A, B)$, $\mathcal{S}_2(A, B)$ and $\mathcal{S}_3(A, B)$ are empty sets, where*
$\mathcal{S}_1(A, B) = \{\lambda \in \rho_a(A) \cap \rho_{_{SF_-}}(B) : d(A - \lambda I) = n(B - \lambda I) = \infty\}$;
$\mathcal{S}_2(A, B) = \{\lambda \in \rho_{_{Fa}}(A) \cap \rho_e(B) : n(B - \lambda I) < d(A - \lambda I)\}$;
$\mathcal{S}_3(A, B) = \{\lambda \in \rho_{_{Fa}}(A) \cap \rho_e(B) : n(B - \lambda I) = d(A - \lambda I),\ d(B - \lambda I) > 0\}$.

Proof. From Lemmas 3.5, 3.7 and 3.8, we only need to prove the sufficiency. In fact, we only need to prove $\sigma(M_C) \subseteq \sigma_{_{Fa}}(M_C)$ for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Let $\lambda_0 \notin \sigma_{_{Fa}}(M_C)$. Then $M_C - \lambda_0 I$ is Fredholm with $n(M - \lambda_0 I) = 0$, which implies that $A - \lambda_0 I$ is left invertible and $B - \lambda_0 I$ is lower semi-Fredholm. Moreover, it follows from the Corollary 2.7 in [3] that $n(B - \lambda_0 I) \leq d(A - \lambda_0 I)$. So, from $\mathcal{S}_1(A, B) = \mathcal{S}_2(A, B) = \mathcal{S}_3(A, B) = \emptyset$, we can get two cases.

Case 1: $n(B - \lambda_0 I) < d(A - \lambda_0 I) = \infty$. In fact, this case cannot occur. Since $B - \lambda_0 I$ is lower semi-Fredholm, it follows from $n(B - \lambda_0 I) < \infty$ that $B - \lambda_0 I$ is Fredholm, which, from Lemma 2.3, implies that $A - \lambda_0 I$ is Fredholm because $M_C - \lambda_0 I$ is Fredholm. It contradicts the fact that $d(A - \lambda_0 I) = \infty$.

Case 2: $n(B - \lambda_0 I) = d(A - \lambda_0 I) < \infty$ and $d(B - \lambda_0 I) = 0$. In this case, we have that $M_C - \lambda_0 I$ is Weyl. Thus $M_C - \lambda_0 I$ is invertible because $n(M_C - \lambda_0 I) = 0$, that is, $\lambda_0 \notin \sigma(M_C)$. Thus $\sigma(M_C) \subseteq \sigma_{_{Fa}}(M_C)$. □

Example 3.11. *Let $A \in \mathcal{B}(\ell^2)$ and $B \in \mathcal{B}(\ell^2)$ be the unilateral shift and the backward shift, respectively. That is,*

$$A(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots),$$

$$B(x_1, x_2, x_3, \cdots) = (x_2, x_3, x_4, \cdots).$$

Then: (1) *if $\lambda \in \rho_a(A) \cap \rho_{SF_-}(B)$, then $n(B - \lambda I) < \infty$ and $d(A - \lambda I) < \infty$. So $\mathcal{S}_1(A, B) = \emptyset$.* (2) *If $\lambda \in \rho_{Fa}(A) \cap \rho_e(B)$, then $n(B - \lambda I) = d(A - \lambda I)$ but $d(B - \lambda I) = 0$. Thus $\mathcal{S}_2(A, B) = \mathcal{S}_3(A, B) = \emptyset$. From Theorem 3.10, we have $\sigma(M_C) = \sigma_{Fa}(M_C)$ for any $C \in \mathcal{B}(\ell^2)$.*

Next, we discuss $\sigma_{Fa}(M_C) = \sigma_{Fa}(A) \cup \sigma_{Fa}(B)$ for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. On the basis of Lemma 3.8 and Lemma 3.7, if $n(B) > 0$,then the following conclusions hold.

Lemma 3.12. *Assume that $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ are Fredholm. If $n(A) = 0$ and $0 < n(B) \leq d(A)$, then there exists some $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is FLI. In particular, when $n(B) < d(A)$, it can make the M_C non-invertible.*

Proof. Suppose $n(B) < d(A)$. Let $\mathcal{M}_1$ with $\dim\mathcal{M}_1 = n(B)$ and $\mathcal{N}$ with $\dim\mathcal{N} = d(A) - n(B)$ be the closed subspaces of $R(A)^\perp$ such that $R(A)^\perp = \mathcal{M}_1 \oplus \mathcal{N}$. Suppose $k = \dim\mathcal{N}$. Take a set of linearly independent vectors $\{e_1, e_2, \cdots, e_k\}$ of $N(B)^\perp$ and let $\mathcal{M} = \mathrm{span}\{e_1, e_2, \cdots, e_k\}$. Take a set of basis $\{u_1, u_2, \cdots, u_k\}$ in $\mathcal{N}$ such that $\|e_i\| = \|u_i\| = 1 (i = 1, 2, \cdots, k)$. Define an isometric invertible operator C_{11} from $N(B)$ to $\mathcal{M}_1$ and define an operator C_{22} from $\mathcal{M}$ to $\mathcal{N}$ such that $C_{22}e_i = u_i (i = 1, 2, \cdots, k)$. Let $\mathcal{H} = \mathcal{M}_1 \oplus \mathcal{N} \oplus R(A)$ and $\mathcal{K} = N(B) \oplus \mathcal{M} \oplus (N(B)^\perp \ominus \mathcal{M})$. Define a C as follows:

$$C = \begin{pmatrix} C_{11} & 0 & 0 \\ 0 & C_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} N(B) \\ \mathcal{M} \\ N(B)^\perp \ominus \mathcal{M} \end{pmatrix} \to \begin{pmatrix} \mathcal{M}_1 \\ \mathcal{N} \\ R(A) \end{pmatrix}. \quad (1)$$

Since A and B are Fredholm, it follows that M_C is Fredholm. Moreover, since C_{11} is injective from $N(B)$ into $\mathcal{M}_1 (\subseteq N(A^*))$ and $C|_{N(B)} = C_{11}$, it follows from $n(A) = 0$ and Lemma 2.5 that $n(M_C) = 0$. In addition, due to $\{e_i\}_{i=1}^k \subseteq N(B)^\perp = R(B^*)$, for every e_i, there exists v_i such that $B^* v_i = e_i$. Thus, it is easy to verify that $\left\{ \begin{pmatrix} u_i \\ -v_i \end{pmatrix} \right\}_{i=1}^k \subseteq N(M_C^*)$ because $\{u_i\}_{i=1}^k \subseteq N(A^*)$. Since $R(M_C)$ is closed, we have $d(M_C) = n(M_C^*) > 0$. So, M_C is non-invertible.

Suppose $n(B) = d(A)$. It can be seen from the above proof process that, at this time, $\mathcal{M} = \mathcal{N} = \{0\}$. Thus, the operator C in Equation (1) can be written in the following form:

$$C = \begin{pmatrix} C_{11} & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} N(B) \\ N(B)^\perp \end{pmatrix} \to \begin{pmatrix} R(A)^\perp \\ R(A) \end{pmatrix}.$$

Using Lemma 2.5 again, we can know that M_C is FLI. □

From Lemma 3.12, we see that $0 \in \sigma_{Fa}(A) \cup \sigma_{Fa}(B)$ but there exists C such that $0 \notin \sigma_{Fa}(M_C)$, which shows that $\sigma_{Fa}(A) \cup \sigma_{Fa}(B) \subseteq \sigma_{Fa}(M_C)$ is not always true for any C. We continue to explore the conditions under which $\sigma_{Fa}(M_C) = \sigma_{Fa}(A) \cup \sigma_{Fa}(B)$ is true for any C.

Theorem 3.13. *Assume that $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then, for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, $\sigma_{Fa}(M_C) = \sigma_{Fa}(A) \cup \sigma_{Fa}(B)$ if and only if $\mathcal{S}_1(A, B) = \mathcal{S}_4(A, B) = \emptyset$, where*

$$\mathcal{S}_4(A, B) = \{\lambda \in \rho_{Fa}(A) \cap \rho_e(B) : 0 < n(B - \lambda I) \leq d(A - \lambda I)\}.$$

Proof. Necessity. It is obvious that $\mathcal{S}_4(A, B) = \emptyset$ from Lemma 3.12. Moreover, if there exists $\lambda_0 \in \mathcal{S}_1(A, B)$, then there exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $\lambda_0 \notin \sigma_{Fa}(M_C)$ from Lemma 3.5 but $\lambda_0 \in \sigma_{Fa}(A) \cup \sigma_{Fa}(B)$. It contradicts $\sigma_{Fa}(A) \cup \sigma_{Fa}(B) = \sigma_{Fa}(M_C)$.

Sufficiency. It is obvious that $\sigma_{Fa}(M_C) \subseteq \sigma_{Fa}(A) \cup \sigma_{Fa}(B)$ for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. We only need to prove $\sigma_{Fa}(A) \cup \sigma_{Fa}(B) \subseteq \sigma_{Fa}(M_C)$ for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Let $\lambda_0 \notin \sigma_{Fa}(M_C)$. Then $M_C - \lambda_0 I$ is Fredholm and

$n(M_C - \lambda_0 I) = 0$, which follows that $A - \lambda_0 I$ is left invertible and $B - \lambda_0 I$ is lower semi-Fredholm and $n(B - \lambda_0 I) \leq d(A - \lambda_0 I)$ by the Corollary 2.7 in [3]. Due to $\mathcal{S}_1(A, B) = \mathcal{S}_4(A, B) = \emptyset$, it shows that $A - \lambda_0 I$ and $B - \lambda_0 I$ are Fredholm and $n(B - \lambda_0 I) = 0$. So $\lambda_0 \notin \sigma_{_{Fa}}(A) \cup \sigma_{_{Fa}}(B)$. Thus $\sigma_{_{Fa}}(A) \cup \sigma_{_{Fa}}(B) = \sigma_{_{Fa}}(M_C)$ for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. □

Due to $\sigma_{_{Fa}}(M_0) = \sigma_{_{Fa}}(A) \cup \sigma_{_{Fa}}(B)$, from Theorem 3.13, we can get the following conclusion.

Corollary 3.14. *Assume that $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then, for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$,*

$$\sigma(M_C) = \sigma_{_{Fa}}(M_C) = \sigma_{_{Fa}}(A) \cup \sigma_{_{Fa}}(B)$$

if and only if $\sigma(M_0) = \sigma_{_{Fa}}(M_0)$ and $\mathcal{S}_1(A, B) = \mathcal{S}_4(A, B) = \emptyset$.

Proof. Necessity. If $C = 0$, then $\sigma(M_0) = \sigma_{_{Fa}}(M_0)$. From Theorem 3.13, we have $\mathcal{S}_1(A, B) = \mathcal{S}_4(A, B) = \emptyset$.

Sufficiency. We only need to prove $\sigma(M_C) \subseteq \sigma_{_{Fa}}(M_C)$. From Theorem 3.13, for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, we have

$$\sigma_{_{Fa}}(M_C) = \sigma_{_{Fa}}(A) \cup \sigma_{_{Fa}}(B) = \sigma_{_{Fa}}(M_0) = \sigma(M_0) \supseteq \sigma(M_C).$$

Thus $\sigma(M_C) = \sigma_{_{Fa}}(M_C)$ for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. □

From Theorem 3.13 and Corollary 3.14, the following conclusion can be obtained immediately. Let

$$\mathcal{S}_4^*(A, B) = \{\lambda \in \rho_{_{Fa}}(A) \cap \rho_e(B) : 0 < n(B - \lambda I) = d(A - \lambda I)\}.$$

Corollary 3.15. *Assume that $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then, for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, $\sigma(M_C) = \sigma_{_{Fa}}(M_C)$ and $\mathcal{S}_4^*(A, B) = \emptyset$ if and only if the following statements hold:*

(1) $\sigma(M_0) = \sigma_{_{Fa}}(M_0)$;

(2) $\sigma_{_{Fa}}(M_C) = \sigma_{_{Fa}}(A) \cup \sigma_{_{Fa}}(B)$ *for any* $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$.

4. The weak property (ω_1) of M_C

For the FLI operators, one of its most common manifestations is the weak property (ω_1), which is a kind of Weyl type theorem. Recently, various kinds of Weyl type theorems have been studied (see [6, 20, 22]) and the structural characteristics of various local spectra have been shown. When it comes to the weak property (ω_1), we have to mention the property (ω_1) (see [16]). For $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, we say that T has property (ω_1) , if T satisfies

$$\sigma_a(T) \backslash \sigma_{ea}(T) \subseteq \pi_{00}(T),$$

where $\pi_{00}(T) = \{\lambda \in \text{iso}\sigma(T) : 0 < n(T - \lambda I) < \infty\}$. If T satisfies

$$\sigma_{_{Fa}}(T) \backslash \sigma_{_{Fea}}(T) \subseteq \pi_{00}(T),$$

then we say that T has weak property (ω_1). It is easy to get the following relation:

$$\text{property } (\omega_1) \Rightarrow \text{weak property } (\omega_1).$$

However, "weak property (ω_1)" does not necessarily follow "property (ω_1)".

Example 4.1. *Let $A \in \mathcal{B}(\ell^2)$ be defined by $A(x_1, x_2, x_3, \cdots) = (0, x_1, 0, x_2, 0, \cdots)$ and let $B \in \mathcal{B}(\ell^2)$ be defined by $B(x_1, x_2, x_3, \cdots) = (x_2, x_3, x_4, \cdots)$. Let $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. By calculations, we have $\sigma_a(T) \backslash \sigma_{ea}(T) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$, $\sigma_{_{Fa}}(T) \backslash \sigma_{_{Fea}}(T) = \emptyset$ and $\pi_{00}(T) = \emptyset$. Thus T has weak property (ω_1), but T has not property (ω_1).*

In the following, we study the weak property (ω_1) of M_C. Let's start with the following fact.

Remark 4.2. *Even if M_0 has weak property (ω_1), there still exist some C such that M_C has not weak property (ω_1).*

Example 4.3. *. Let $A, B, C \in \mathcal{B}(\ell^2)$ be defined by*

$$A(x_1, x_2, x_3, \cdots) = (0, x_1, 0, x_2, 0, x_3, \cdots),$$

$$B(x_1, x_2, x_3, \cdots) = (x_2, x_4, x_6, \cdots),$$

$$C(x_1, x_2, x_3, \cdots) = (0, 0, 0, 0, x_3, 0, x_5, 0, \cdots).$$

Let $M_0 = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. It is easy to verify that M_0 has weak property (ω_1). In addition, by calculations, we have $n(M_C) = 1$ and $d(M_C) = 2$. Then $\mathrm{ind}(M_C) = -1$. It follows that $0 \in \sigma_{Fa}(M_C)\backslash\sigma_{Fea}(M_C)$ but $0 \notin \pi_{00}(M_C)$.

Next, we give some useful conclusions for further study for the weak property (ω_1) of M_C.

Lemma 4.4. *([5, Proposition 6.9] and [17, Chapter V, Theorem 10.2]) Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then we have*

$$\mathrm{iso}\sigma(T)\backslash\sigma_{SF}(T) = \mathrm{iso}\sigma(T)\backslash\sigma_w(T) = \sigma(T)\backslash\sigma_b(T).$$

Lemma 4.5. *Let $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K})$ and $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Then M_C has weak property (ω_1) if and only if*

$$\sigma_{Fa}(M_C)\backslash\sigma_{Fea}(M_C) = \sigma(M_C)\backslash\sigma_b(M_C).$$

Proof. Necessity. Let $\lambda_0 \in \sigma_{Fa}(M_C)\backslash\sigma_{Fea}(M_C)$. Obviously, $\lambda_0 \in \sigma(M_C)$. Since M_C has weak property (ω_1), we have $\lambda_0 \in \mathrm{iso}\sigma(M_C)$. From Lemma 4.4, we know that $M_C - \lambda_0 I$ is Browder. Thus $\lambda_0 \in \sigma(M_C)\backslash\sigma_b(M_C)$. So $\sigma_{Fa}(M_C)\backslash\sigma_{Fea}(M_C) \subseteq \sigma(M_C)\backslash\sigma_b(M_C)$. In addition, let $\mu_0 \in \sigma(M_C)\backslash\sigma_b(M_C)$. Then $n(M_C - \mu_0 I) > 0$, which shows $\mu_0 \in \sigma_{Fa}(M_C)$. Combining with $\sigma_{Fea}(M_C) \subseteq \sigma_b(M_C)$, we get $\mu_0 \in \sigma_{Fa}(M_C)\backslash\sigma_{Fea}(M_C)$. Thus, we have $\sigma(M_C)\backslash\sigma_b(M_C) \subseteq \sigma_{Fa}(M_C)\backslash\sigma_{Fea}(M_C)$.

Sufficiency. Let $\mu_1 \in \sigma_{Fa}(M_C)\backslash\sigma_{Fea}(M_C)$. Due to $\sigma_{Fa}(M_C)\backslash\sigma_{Fea}(M_C) = \sigma(M_C)\backslash\sigma_b(M_C)$, it follows from Lemma 4.4 that $\mu_1 \in \pi_{00}(M_C)$. Thus M_C has weak property (ω_1). □

It is easy to prove that $\sigma_w(M_C) = \sigma_b(M_C)$ if M_C has weak property (ω_1). From Lemma 4.5, we can get the following conclusion.

Corollary 4.6. *Let $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K})$ and $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Then M_C has weak property (ω_1) and $\sigma_{Fa}(M_C) = \sigma(M_C)$ if and only if*

$$\sigma_{Fea}(M_C) = \sigma_w(M_C) = \sigma_b(M_C).$$

Proof. Necessity. Due to $\sigma_{Fea}(M_C) \subseteq \sigma_{Fa}(M_C)$ and $\sigma_b(M_C) \subseteq \sigma(M_C)$, combining with $\sigma_{Fa}(M_C) = \sigma(M_C)$, it follows from Lemma 4.5 that $\sigma_{Fea}(M_C) = \sigma_b(M_C)$. Moreover, it is easy to get $\sigma_{Fea}(M_C) \subseteq \sigma_w(M_C) \subseteq \sigma_b(M_C)$. Thus $\sigma_{Fea}(M_C) = \sigma_w(M_C) = \sigma_b(M_C)$.

Sufficiency. For $\sigma_{Fa}(M_C) = \sigma(M_C)$, we only prove $\sigma(M_C) \subseteq \sigma_{Fa}(M_C)$. Let $\lambda_1 \notin \sigma_{Fa}(M_C)$. Then $\lambda_1 \notin \sigma_{Fea}(M_C)$ and $n(M_C - \lambda_1 I) = 0$. Due to $\sigma_{Fea}(M_C) = \sigma_w(M_C)$, it shows $d(M_C - \lambda_1 I) = 0$. Thus $M_C - \lambda_1 I$ is invertible. So $\sigma_{Fa}(M_C) = \sigma(M_C)$. Now, we have $\sigma_{Fa}(M_C)\backslash\sigma_{Fea}(M_C) = \sigma(M_C)\backslash\sigma_b(M_C)$. From Lemma 4.5, M_C has weak property (ω_1). □

From [21, Lemma 3], we can get the following result.

Lemma 4.7. *Assume that $A \in \mathcal{B}(\mathcal{H})$ is upper semi-Fredholm and $B \in \mathcal{B}(\mathcal{K})$ is lower semi-Fredholm. If $d(A) = n(B) = \infty$, then there exists some non-zero $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $0 \in \sigma_{Fa}(M_C)\backslash\sigma_{Fea}(M_C)$ and $\mathrm{ind}(M_C) < 0$.*

Theorem 4.8. *Assume that $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then, for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, M_C has weak property (ω_1) if and only if M_0 has weak property (ω_1) and $\mathcal{S}_5(A, B) = \emptyset$, where*

$$\mathcal{S}_5(A, B) = \{\lambda \in \rho_{SF_+}(A) \cap \rho_{SF_-}(B) : d(A - \lambda I) = n(B - \lambda I) = \infty\}.$$

Proof. Necessity. If $C = 0$, then M_0 has weak property (ω_1). For $\mathcal{S}_5(A, B)$, if there exists $\lambda_0 \in \mathcal{S}_5(A, B)$, then from Lemma 4.7, there exists some non-zero $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $\lambda_0 \in \sigma_{_{Fa}}(M_C)\backslash\sigma_{_{Fea}}(M_C)$ and $\mathrm{ind}(M_C - \lambda_0 I) < 0$. Since M_C has weak property (ω_1), it follows from Lemma 4.4 that $\lambda_0 \notin \sigma_w(M_C)$, which implies $\mathrm{ind}(M_C - \lambda_0 I) = 0$. It contradicts $\mathrm{ind}(M_C - \lambda_0 I) < 0$. So $\mathcal{S}_5(A, B) = \emptyset$.

Sufficiency. Take an arbitrary $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Let $\mu_0 \in \sigma_{_{Fa}}(M_C)\backslash\sigma_{_{Fea}}(M_C)$. Then $0 < n(M_C - \mu_0 I) < \infty$, and we have that $A - \mu_0 I$ is upper semi-Fredholm and $B - \mu_0 I$ is lower semi-Fredholm. Since $\mathcal{S}_5(A, B) = \emptyset$, according to Lemma 2.3, we can only conclude that $A - \mu_0 I$ and $B - \mu_0 I$ are Fredholm. From $\mu_0 \in \sigma_{_{Fa}}(M_C)\backslash\sigma_{_{Fea}}(M_C)$, we have that

$$\mathrm{ind}(M_C - \mu_0 I) = \mathrm{ind}(A - \mu_0 I) + \mathrm{ind}(B - \mu_0 I) = \mathrm{ind}(M_0 - \mu_0 I) \leq 0$$

and

$$0 < n(M_C - \mu_0 I) \leq n(A - \mu_0 I) + n(B - \mu_0 I) = n(M_0 - \mu_0 I).$$

So $\mu_0 \in \sigma_{_{Fa}}(M_0)\backslash\sigma_{_{Fea}}(M_0)$. Since M_0 has weak property (ω_1), it follows that $\mu_0 \in \pi_{00}(M_0)$. Then $\mu_0 \in \mathrm{iso}\sigma(M_0)$. At this point, we can prove $\mu_0 \in \mathrm{iso}\sigma(M_C)$. In fact, there exists ϵ_1 such that $A - \mu I$ and $B - \mu I$ are invertible for any $0 < |\mu - \mu_0| < \epsilon_1$, which shows $\mu_0 \in \mathrm{iso}\sigma(M_C) \cup \rho(M_C)$, where $\rho(M_C)$ is the complement of $\sigma(M_C)$. We can claim that $\mu_0 \in \sigma(M_C)$. If not, from the Corollary 2.5 in [3], $A - \mu_0 I$ is left invertible and $B - \mu_0 I$ is right invertible. It follows from [14, Theorem 5.31] that there exists $\epsilon_2 (< \epsilon_1)$ such that $\mathrm{ind}(A - \mu_0 I) = \mathrm{ind}(A - \mu I) = 0$ and $\mathrm{ind}(B - \mu_0 I) = \mathrm{ind}(B - \mu I) = 0$ for any $0 < |\mu - \mu_0| < \epsilon_2$. Thus $A - \mu_0 I$ and $B - \mu_0 I$ are invertible. It contradicts $\mu_0 \in \sigma(M_0)$. So $\mu_0 \in \mathrm{iso}\sigma(M_C)$. Thus, we have $\mu_0 \in \pi_{00}(M_C)$. □

Next, we explore the relationship between $\sigma(M_C) = \sigma_{_{Fa}}(M_C)$ and the weak property (ω_1) of M_C.

Remark 4.9. (1) *Assume that $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Even if, for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, M_C has weak property (ω_1), it does not follow $\sigma(M_C) = \sigma_{_{Fa}}(M_C)$ for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$.*

Example 4.10. *Let $A, B \in \mathcal{B}(\ell^2)$ be defined by*

$$A(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots),$$

$$B(x_1, x_2, x_3, \cdots) = (0, 0, x_1, x_2, x_3, \cdots).$$

By calculations, M_0 has weak property (ω_1) and $\mathcal{S}_5(A, B) = \emptyset$. Thus, from Theorem 4.8, it follows that M_C has weak property (ω_1) for $C \in \mathcal{B}(\ell^2)$. However, $\sigma(M_0) \neq \sigma_{_{Fa}}(M_0)$.

(2) *Assume that $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Even if $\sigma(M_C) = \sigma_{_{Fa}}(M_C)$ for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, there exists some $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C has not weak property (ω_1).*

Example 4.11. *Let $A, B \in \mathcal{B}(\ell^2)$ be defined by*

$$A(x_1, x_2, x_3, \cdots) = (0, 0, 0, x_2, 0, x_3, 0, \cdots),$$

$$B(x_1, x_2, x_3, \cdots) = (x_2, x_4, x_6, \cdots).$$

By calculations, $\sigma(M_0) = \sigma_{_{Fa}}(M_0)$ and $\mathcal{S}_1(A, B) = \mathcal{S}_4(A, B) = \emptyset$. Thus, from Corollary 3.14, we have $\sigma(M_C) = \sigma_{_{Fa}}(M_C)$ for $C \in \mathcal{B}(\ell^2)$. However, $\mathcal{S}_5(A, B)$ is not an empty set. Thus, from Theorem 4.8, there exists $C_0 \in \mathcal{B}(\ell^2)$ such that M_{C_0} has not weak property (ω_1).

Theorem 4.12. *Assume that $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then, for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, M_C has weak property (ω_1) and $\sigma(M_0) = \sigma_{_{Fa}}(M_0)$ if and only if*

(1) *$\sigma(M_C) = \sigma_{_{Fa}}(M_C)$ for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$;*

(2) *M_0 has weak property (ω_1);*

(3) *$\mathcal{S}_6(A, B) = \{\lambda \in \rho_{_{SF_+}}(A) \cap \rho_{_{SF_-}}(B) : n(A - \lambda I) > 0,\ d(A - \lambda I) = n(B - \lambda I) = \infty\} = \emptyset$.*

Proof. Necessity. From Theorem 4.8, we know that M_0 has weak property (ω_1) and $\mathcal{S}_6(A,B) \subseteq \mathcal{S}_5(A,B) = \emptyset$. Next, we prove that the conclusion (1) holds. First, due to $\mathcal{S}_1(A,B) \subseteq \mathcal{S}_5(A,B)$, it follows from Theorem 4.8 that $\mathcal{S}_1(A,B) = \emptyset$. Second, let $\lambda_0 \in \mathcal{S}_2(A,B)$. Then, if $n(B-\lambda_0 I) = 0$, we have $\lambda_0 \notin \sigma_{_{Fa}}(M_0)$; if $n(B-\lambda_0 I) > 0$, we have $\lambda_0 \in \sigma_{_{Fa}}(M_0)\backslash\sigma_{_{Fea}}(M_0)$. Since $\sigma(M_0) = \sigma_{_{Fa}}(M_0)$ and M_0 has weak property (ω_1), it follows that $M_0 - \lambda_0 I$ is Browder either in the case $n(B-\lambda_0 I) = 0$ or in the case $n(B-\lambda_0 I) > 0$. Thus $A - \lambda_0 I$ is Weyl, which implies $d(A-\lambda_0 I) = 0$ because $n(A-\lambda_0 I) = 0$. It contradicts $n(B-\lambda_0 I) < d(A-\lambda_0 I)$. So $\mathcal{S}_2(A,B) = \emptyset$. Let $\mu_0 \in \mathcal{S}_3(A,B)$. Then we have that $M_0 - \mu_0 I$ is Fredholm and $\operatorname{ind}(M_0 - \mu_0 I) < 0$. Since $\sigma(M_0) = \sigma_{_{Fa}}(M_0)$ and M_0 has weak property (ω_1), it follows that $M_0 - \mu_0 I$ is Weyl either in the case $n(M_0 - \mu_0 I) = 0$ or in the case $n(M_0 - \mu_0 I) > 0$, which contradicts $\operatorname{ind}(M_0 - \lambda_0 I) < 0$. So $\mathcal{S}_3(A,B) = \emptyset$. From Theorem 3.10, we know $\sigma(M_C) = \sigma_{_{Fa}}(M_C)$ for any $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$.

Sufficiency. In the condition (1), let $C = 0$, then $\sigma(M_0) = \sigma_{_{Fa}}(M_0)$. Moreover, from Theorem 4.8, we only need to verify $\mathcal{S}_5(A,B) = \emptyset$ to prove that M_C has weak property (ω_1) for any $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$. Let $v_0 \in \mathcal{S}_5(A,B)$. From the condition (3), it follows $n(A - v_0 I) = 0$, which shows $v_0 \in \mathcal{S}_1(A,B)$. It contradicts $\mathcal{S}_1(A,B) = \emptyset$. So $\mathcal{S}_5(A,B) = \emptyset$. This completes the proof. □

Lemma 4.13. *Assume that $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. If M_0 has weak property (ω_1) and $\sigma(M_0) = \sigma_{_{Fa}}(M_0)$, then $\mathcal{S}_2(A,B) = \mathcal{S}_3(A,B) = \emptyset$.*

Proof. Let $\lambda \in \mathcal{S}_2(A,B) \cup \mathcal{S}_3(A,B)$. Then $M_0 - \lambda I$ is Fredholm and $\operatorname{ind}(M_0 - \lambda I) < 0$, which shows $\lambda \notin \sigma_{_{Fa}}(M_0)$ or $\lambda \in \sigma_{_{Fa}}(M_0)\backslash\sigma_{_{Fea}}(M_0)$. Since M_0 has weak property (ω_1) and $\sigma(M_0) = \sigma_{_{Fa}}(M_0)$, it follows that $M_0 - \lambda I$ is Browder. So $A - \lambda I$ and $B - \lambda I$ are Browder. Due to $n(A - \lambda I) = 0$, it shows that $A - \lambda I$ is invertible. If $\lambda \in \mathcal{S}_2(A,B)$, then it contradicts $n(B - \lambda I) < d(A - \lambda I)$. If $\lambda \in \mathcal{S}_3(A,B)$, then it contradicts $d(B - \lambda I) > 0$ because $d(B - \lambda I) = n(B - \lambda I) = d(A - \lambda I)$. So $\mathcal{S}_2(A,B) \cup \mathcal{S}_3(A,B) = \emptyset$. □

Due to $\mathcal{S}_1(A,B) \subseteq \mathcal{S}_5(A,B)$, it follows from Theorems 3.10 and 4.8 and Lemma 4.13 that the following corollary holds.

Theorem 4.14. *Assume that $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then, for any $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$, M_C has weak property (ω_1) and $\sigma(M_C) = \sigma_{_{Fa}}(M_C)$ if and only if*

(1) *$\sigma(M_0) = \sigma_{_{Fa}}(M_0)$;*

(2) *M_0 has weak property (ω_1);*

(3) *$\mathcal{S}_5(A,B) = \emptyset$.*

From Corollary 4.6 and Theorem 4.14, we have the following conclusion.

Corollary 4.15. *Assume that $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then, for any $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$, M_C has weak property (ω_1) and $\sigma(M_C) = \sigma_{_{Fa}}(M_C)$ if and only if $\sigma_{_{Fea}}(M_0) = \sigma_b(M_0)$ and $\mathcal{S}_5(A,B) = \emptyset$.*

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