Filomat 39:18 (2025), 6161–6172
https://doi.org/10.2298/FIL2518161L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Relative entropy in unital JB-algebras and positive linear maps on unital JC-algebras

F. Lotfi[a], A. G. Ghazanfari[a,*]

[a]*Department of Mathematics, Lorestan University, Khoramabad, Iran*

Abstract. In this paper, we study the relative entropy in the setting of unital JB-algebras and give some refined inequalities involving it in such algebras. We also investigate some property of positive linear maps on unital JC-algebras.

1. Introduction and preliminary

Jordan algebras provide a foundation, for explaining the concept of an algebra of observables in quantum mechanics. In the principles of quantum physics it is accepted that observables form a Jordan algebra. This is mainly due to the fact that observables are usually depicted as self adjoint operators on a Hilbert space and the collection of these operators adheres to the principles of a Jordan product, than any type. For information readers may refer to [4].

A Jordan algebra over $\mathbb{R}$ is a vector space $\mathcal{A}$ over $\mathbb{R}$ equipped with a commutative bilinear product $\circ$ that satisfies the identity

$$x \circ (y \circ x^2) = (x \circ y) \circ x^2,$$

for all $x, y \in \mathcal{A}$.

Let $\mathcal{A}$ be an algebra and $x, y \in \mathcal{A}$. Let

$$x \circ y = \frac{xy + yx}{2}. \tag{1.1}$$

Then $\circ$ defines a bilinear, commutative product on $\mathcal{A}$, which is called the Jordan product. If $\mathcal{A}$ is associative, then $\mathcal{A}$ becomes a Jordan algebra when equipped with the product (1.1), as does any subspace closed under $\circ$. Such Jordan algebras are called special Jordan algebras, all others are called exceptional. The following algebras are examples of special Jordan algebras with product (1.1).

Example 1.1.

- The Jordan algebra of $n \times n$ self-adjoint real matrices $H_n(\mathbb{R})$.

2020 *Mathematics Subject Classification*. Primary 46H70; Secondary 47A63, 47A64.
Keywords. JB-algebra, JC-algebra, operator mean, operator inequality.
Received: 20 October 2024; Revised: 31 March 2025; Accepted: 09 April 2025
Communicated by Dragan S. Djordjević
* Corresponding author: A. G. Ghazanfari
Email addresses: lotfi.fa@fs.lu.ac.ir (F. Lotfi), ghazanfari.a@lu.ac.ir (A. G. Ghazanfari)

- The Jordan algebra of $n \times n$ self-adjoint complex matrices $H_n(\mathbb{C})$.
- The Jordan algebra of $n \times n$ self-adjoint quaternionic matrices $H_n(\mathbb{H})$.
- The Jordan algebra of $n \times n$ self-adjoint octonionic matrices $H_n(\mathbb{O})$, where $n \leq 3$.

Definition 1.2. A Jordan Banach algebra is a real Jordan algebra $\mathcal{A}$ equipped with a complete norm satisfying

$$\|A \circ B\| \leq \|A\|\|B\|, \quad A, B \in \mathcal{A}.$$

Jordan operator algebras are norm-closed spaces of operators on a Hilbert space which are closed under the Jordan product.

Basic examples are real symmetric and complex hermitian matrices with the Jordan product. A *JB*-algebra is a Jordan Banach algebra $\mathcal{A}$ in which the norm satisfies the following two additional conditions for $A, B \in \mathcal{A}$:

(i) $\|A^2\| = \|A\|^2$
(ii) $\|A^2\| \leq \|A^2 + B^2\|$.

A *JC*-algebra is a *JB*-algebra $\mathcal{A}$ that is isomorphic to a norm closed Jordan subalgebra of $B(H)_{sa}$. We will make the convention that our *JC*-algebras have an identity.

Let $\mathcal{A}$ be a *JB*-algebra, we say $A \in \mathcal{A}$ is invertible if there exists $B \in \mathcal{A}$, which is called Jordan inverse of A, such that

$$A \circ B = I \quad \text{and} \quad A^2 \circ B = A.$$

The spectrum of A, denoted by $Sp(A)$, is the set of $\lambda \in \mathbb{R}$ such that $A - \lambda$ does not have an inverse in A. Furthermore, if $Sp(A) \subset [0, \infty)$, we say A is positive, denoted $A \geq 0$.
In a *JB*-algebra we define

$$U_A B = \{ABA\} := 2(A \circ B) \circ A - A^2 \circ B.$$

Note that ABA is meaningless unless $\mathcal{A}$ is special (for example a *JC*-algebra), in which case $\{ABA\} = ABA$. Moreover, if $B \geq 0$, then $U_A B = \{ABA\} \geq 0$.

We mention some of properties of U_A that we will use frequently in sequel: U_A is a linear mapping and

$$U_{\{ABA\}} = U_A U_B U_A. \tag{1.2}$$

It also satisfies the following two Lemmas:

Lemma 1.3. *[1, Lemma 1.23] Let $\mathcal{A}$ be a JB-Banach algebra and $A \in \mathcal{A}$. Then A is an invertible element iff U_A has a bounded inverse, and in this case the inverse map is $U_{a^{-1}}$ i.e., $U_A^{-1} = U_{A^{-1}}$.*

Lemma 1.4. *[1, Lemma 1.24] If A and B are invertible elements of a JB-algebra, Then $\{ABA\}$ is invertible with inverse $\{A^{-1}B^{-1}A^{-1}\}$.*

For more details, we refer the reader to [1, 10].

Definition 1.5. Let $\mathcal{A}$ be a *JB*-algebra or a C^*-algebra, and $f : I \to \mathbb{R}$ be a real-valued continuous function on a (non trivial) interval $I \subset \mathbb{R}$.

(i) The function f is called $\mathcal{A}$-monotone if for any $A, B \in \mathcal{A}$ with spectrum in I, we have

$$A \leq_{\mathcal{A}} B \Rightarrow f(A) \leq_{\mathcal{A}} f(B). \tag{1.3}$$

The function f is $\mathcal{A}$-convex if for any $\lambda \in [0, 1]$ and any $A, B \in \mathcal{A}$ with spectrum in I, we have

$$f((1 - \lambda)A + \lambda B) \leq_{\mathcal{A}} (1 - \lambda)f(A) + \lambda f(B). \tag{1.4}$$

(ii) If $\mathcal{A} = B(H)$, the standard C^*-algebra of all bounded linear operators on a Hilbert space H, then a $\mathcal{A}$-monotone (resp. $\mathcal{A}$-convex) function is called operator monotone (resp. operator convex) function.
(iii) If $\mathcal{B} \subseteq B(H)$ is a Jordan operator algebra or a C^*-subalgebra of $B(H)$, then a $\mathcal{B}$-monotone (resp. $\mathcal{B}$-convex) is called operator monotone (resp. operator convex) function on $\mathcal{B}$.
(iv) If $\mathcal{A} = M_n$, the standard C^*-algebra of all complex $n \times n$ matrices or equivalently of all bounded linear operators on an n-dimensional complex Hilbert space, then a $\mathcal{A}$-monotone (resp. $\mathcal{A}$-convex) function is called matrix monotone (resp. matrix convex) function of order n.

Wang et al. [16] introduced some means for two positive invertible elements A, B in a unital JB-algebra $\mathcal{A}$ and $\nu \in [0, 1]$, such as

- ν-weighted harmonic mean: $A!_\nu B = \left((1-\nu)A^{-1} + \nu B^{-1}\right)^{-1}$;
- ν-weighted geometric mean: $A\sharp_\nu B = \{A^{1/2}\{A^{-1/2}BA^{-1/2}\}^\nu A^{1/2}\}$;
- ν-weighted arithmetic mean: $A\nabla_\nu B = (1-\nu)A + \nu B$.

The following relations among them are also proved in [16].

$$A\sharp_\nu B = B\sharp_{1-\nu} A,$$
$$(A\sharp_\nu B)^{-1} = A^{-1}\sharp_\nu B^{-1},$$
$$A!_\nu B \le A\sharp_\nu B \le A\nabla_\nu B,$$
$$(\alpha A\sharp_\nu \beta B) = (\alpha\sharp_\nu\beta)(A\sharp_\nu B) \qquad (\alpha > 0, \beta > 0)$$
$$\{C(A\sharp_\nu B)C\} = \{CAC\}\sharp_\nu\{CBC\} \qquad \text{for any invertible } C \in \mathcal{A}.$$

Wang et al. [17] introduced the relative entropy for two positive invertible elements A, B in a unital JB-algebra $\mathcal{A}$, as follows

$$S(A|B) = \left\{A^{\frac{1}{2}} \log\left(\left\{A^{\frac{-1}{2}} B A^{\frac{-1}{2}}\right\}\right) A^{\frac{1}{2}}\right\}. \tag{1.5}$$

For any $\lambda \in (0, 1]$, the Tsallis relative entropy $T_\lambda(A|B)$ in a unital JB-algebra is also defined by

$$T_\lambda(A|B) = \frac{\left\{A^{\frac{1}{2}}\left\{A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right\}^\lambda A^{\frac{1}{2}}\right\} - A}{\lambda} = \frac{A\sharp_\lambda B - A}{\lambda}. \tag{1.6}$$

Some property of these two topic and relation between them are also proved in [17], such as

Proposition 1.6. *[17, Proposition 2.4] The relative operator entropy $S(A|B)$ defined in unital JB-algebra $\mathcal{A}$ has the following properties:*
(i) $S(\alpha A|\alpha B) = \alpha S(A|B)$ for any positive number α.
(ii) If $B \le C$, then $S(A|B) \le S(A|C)$.
(iii) $S(A|B)$ is operator concave with respect to A, B individually.
(iv) $S(\{(CAC\}|\{CBC\}) = \{CS(A|B)C\}$, for any invertible C in $\mathcal{A}$.

Proposition 1.7. *[17, Proposition 2.7] The Tsallis relative operator entropy $T_\lambda(A|B)$ defined in unital JB-algebra $\mathcal{A}$ has the following properties:*
(i) $T_\lambda(\alpha A|\alpha B) = \alpha T_\lambda(A|B)$ for any positive number α.
(ii) If $B \le D$, then $T_\lambda(A|B) \le T_\lambda(A|D)$.
(iii) $T_\lambda(\{CAC\}|\{CBC\}) = \{CT_\lambda(A|B)C\}$ for any invertible C in $\mathcal{A}$.
(iv) $T_\lambda(A|B)$ is concave with respect to A, B individually.
(v) $\lim_{\lambda\to 0} T_\lambda(A|B) = S(A|B)$.

Motivated by the above studies, in this paper we extend some inequalities related to relative operator entropy to unital JB-algebras and obtain some property of positive linear maps on unital JC-algebras.

2. Entropy in unital *JB*-algebras

In 1850 Clausius[3] was the first to introduce the concept of entropy, in the field of thermodynamics. Over time there have been modifications and updates explored in arena. The concept of relative operator entropy $S(A|B)$ for operators, in information theory was developed by Fujii and Kamei in their work referenced as [5, 6]. It is defined by

$$S(A|B) = A^{\frac{1}{2}} \log(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}}.$$

for invertible positive operators A and B on a Hilbert space H, as an extension of the entropy defined by Nakamura and Umegaki [12], and the relative entropy defined by Umegaki [15]. We remark that $S(A|I) = -AlogA$ is the usual well known operator entropy.
The Tsallis relative operator entropy is defined by

$$T_\lambda(A|B) = \frac{A\sharp_\lambda B - A}{\lambda}, \tag{2.1}$$

since

$$\lim_{\lambda\to 0} T_\lambda(A|B) = S(A|B), \tag{2.2}$$

the Tsallis relative operator entropy is a generalization of the relative operator entropy. For more information on the Tsallis relative entropy the reader is referred to [6–8, 18] and the references therein.

Furuta in [9] obtained the following inequality for $a > 0$

$$(1 - \log a)A - \frac{1}{a} AB^{-1}A \le S(A|B) \le (\log a - 1)A + \frac{1}{a} B, \tag{2.3}$$

as a generalization of the upper and lower bounds of

$$A - AB^{-1}A \le S(A|B) \le B - A, \tag{2.4}$$

which was given in [5].

Zou in [19] obtained a refinement of the inequality (1.5) in the following forms:

$$\begin{aligned} &-\left(\log a + \frac{1 - a^\lambda}{\lambda a^\lambda}\right) A + a^{-\lambda} T_{-\lambda}(A|B) \\ &\qquad\qquad \le S(A|B) \le T_\lambda(A|B) - \frac{1 - a^\lambda}{\lambda} A\sharp_\lambda B - (\log a)A. \end{aligned} \tag{2.5}$$

Soleimani and Ghazanfari in [14] give a refinment of (2.5), as follows: Let $0 < \lambda \le 1$ be a real number, $n \in \mathbb{N}$ and let A and B be strictly positive operators on a Hilbert space H. Then there is an $\varepsilon > 0$ such that

$$\begin{aligned} &\int_0^1 (A!_t B)dt - 2^{-n-1} A\left(A^{-1} + B^{-1} + 2\sum_{i=1}^{2^n-1} (2^{-n} iB + (1 - 2^{-n} i)A)^{-1}\right) A \\ &\le S(A|B) \le \frac{1}{\lambda(1+\varepsilon)} \left(a^\lambda A\sharp_\lambda B - \varepsilon a^{-\lambda} A\sharp_{-\lambda} B + (\varepsilon - 1)A\right) - (\log a)A \\ &\le T_\lambda(A|B) - \frac{1 - a^\lambda}{\lambda} A\sharp_\lambda B - (\log a)A. \end{aligned} \tag{2.6}$$

for all $a \ge \varepsilon^{\frac{1}{2\lambda}}$.

In the following, we obtain improved inequalities involving the relative entropy in *JB*-algebras. In order to do that, we prove two lemmas which will turn out to be useful in the proof of our results.

Lemma 2.1. *let $\mathcal{A}$ be a unital JB-algebra, Let $g : \mathbb{R} \times [0,1] \to \mathbb{R}$ be a continuous map and assume $f(x) = \int_0^1 g(x,t)\,d\mu(t)$ for every x on bounded and closed intervals, where μ is a bounded positive measure on* $[0,1]$. *Then $f(A) = \int_0^1 g(A,t)\,d\mu(t)$ for any $A \in \mathcal{A}$.*

Proof. Function g is uniformly continuous on every bounded and closed subset $[m,M] \subseteq \mathbb{R}$. Therefore for every given $\epsilon > 0$ there exist a $\delta > 0$ such that if $|x_1 - x_2| < \delta$, $|t_1 - t_2| < \delta$, then $|g(x_1,t_1) - g(x_2,t_2)| < \epsilon$. We define

$$f_n(x) := \sum_{i=1}^{n} g\left(x, \frac{i}{n}\right)\mu\left(\left[\frac{i-1}{n}, \frac{i}{n}\right]\right).$$

Then, for each $x \in [m,M]$ and sufficiently large n

$$\begin{aligned}|f_n(x) - f(x)| &= \left|\sum_{i=1}^{n} g\left(x, \frac{i}{n}\right)\mu\left(\left[\frac{i-1}{n}, \frac{i}{n}\right]\right) - \int_0^1 g(x,t)\,\mathrm{d}\mu(t)\right| \\ &\le \sum_{i=1}^{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left|g\left(x, \frac{i}{n}\right) - g(x,t)\right| \mathrm{d}\mu(t) \\ &\le \sum_{i=1}^{n} \epsilon\,\mu\left(\left[\frac{i-1}{n}, \frac{i}{n}\right]\right) \\ &\le \epsilon\,\mu([0,1]).\end{aligned}$$

So, f_n is uniformly convergent to f on $[m,M]$. Consequently, $f(A) = \int_0^1 g(A,t)\,d\mu(t)$. □

Lemma 2.2. *Let $\mathcal{A}$ be a unital JB-algebra and $A,B,C,D \in \mathcal{A}$. If $A \le C$, $B \le D$, then $A\,!_\lambda\,B \le C\,!_\lambda\,D$.*

Proof. We start with the following relation, obtained from inequalities (1.11) in [1]:

$$A \ge 0 \Leftrightarrow \|\|A\|1 - A\| \le \|A\|. \tag{2.7}$$

From (2.7), we get

$$A, B \ge 0 \text{ and } 0 \le \lambda \le 1 \Rightarrow (1-\lambda)A + \lambda B \ge 0. \tag{2.8}$$

Now, let $A \le C$, $B \le D$, then

$$\begin{aligned}&A^{-1} \ge C^{-1},\ B^{-1} \ge D^{-1} && \text{(by [10, Lemma (3.5.3)])}\\ &\Rightarrow (1-\lambda)A^{-1} + \lambda B^{-1} \ge (1-\lambda)C^{-1} + \lambda D^{-1} && \text{(by (2.8))}\\ &\Rightarrow \left((1-\lambda)A^{-1} + \lambda B^{-1}\right)^{-1} \le \left((1-\lambda)C^{-1} + \lambda D^{-1}\right)^{-1} && \text{(by [10, Lemma (3.5.3)])}\\ &\Rightarrow A!_\lambda B \le C!_\lambda D.\end{aligned}$$

□

In the following Theorem, we show that relation (2.6) holds for unital JB-algebras.

Theorem 2.3. *Let $\mathcal{A}$ be a unital JB-algebra and A,B be two invertible positive elements in $\mathcal{A}$. Let $0 < \lambda \le 1$ be a real number and $n \in \mathbb{N}$. Then there is an $\varepsilon > 0$ such that*

$$\begin{aligned}&\int_0^1 (B!_tA)dt - 2^{-n-1}\left(A + \{AB^{-1}A\} + 2\sum_{i=1}^{2^n-1}\left(A!_{2^{-n}i}\{AB^{-1}A\}\right)\right)\\ &\le S(A|B) \le \frac{1}{\lambda(1+\varepsilon)}\left(a^\lambda A\sharp_\lambda B - \varepsilon a^{-\lambda}A\sharp_{-\lambda}B + (\varepsilon-1)A\right) - (\log a)A \\ &\le T_\lambda(A|B) - \frac{1-a^\lambda}{\lambda}A\sharp_\lambda B - (\log a)A.\end{aligned} \tag{2.9}$$

for all $a \ge \varepsilon^{\frac{1}{2\lambda}}$.

Proof. Let $\mathcal{A}$ be a unital JB-algebra. From relation (2.13) in [14], we have

$$\int_0^1 ((1-t)a^{-1}+t1)^{-1}\,dt - 2^{-n-1}\left[1+a^{-1}+2\sum_{i=1}^{2^n-1}\left(2^{-n}ia+(1-2^{-n}i)\right)^{-1}\right] \le \log a. \tag{2.10}$$

Using the functional calculus $a=\{A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\}$ in (2.10), and that U_X is a positive linear map, we get the first inequality in (2.9).

If $0<\lambda,\varepsilon\le 1$. From relation (2.3) in [14], we have

$$\begin{aligned}\log x &\le \frac{1}{\lambda(1+\varepsilon)}\left(a^\lambda x^\lambda-\varepsilon a^{-\lambda}x^{-\lambda}-1+\varepsilon\right)-\log a\\ &\le \frac{1}{\lambda}\left(a^\lambda x^\lambda-1\right)-\log a,\end{aligned} \tag{2.11}$$

for all $a,x\ge \varepsilon^{\frac{1}{2\lambda}}$.

Utilizing the functional calculus $x=\{A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\}$ in (2.11), we obtain

$$\begin{aligned}\log\{A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\} &\le \frac{1}{\lambda(1+\varepsilon)}\left(a^\lambda\{A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\}^\lambda-\varepsilon a^{-\lambda}\{A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\}^{-\lambda}-1+\varepsilon\right)-\log a\\ &\le \frac{1}{\lambda}\left(a^\lambda\{A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\}^\lambda-1\right)-\log a\end{aligned}$$

Since U_X is a positive linear map, we get

$$\begin{aligned}&U_{A^{\frac{1}{2}}}(\log\{A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\})\\ &\le \frac{1}{\lambda(1+\varepsilon)}\Big(a^\lambda U_{A^{\frac{1}{2}}}(\{A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\}^\lambda)-\varepsilon a^{-\lambda}U_{A^{\frac{1}{2}}}(\{A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\}^{-\lambda})+(\varepsilon-1)A\Big)-(\log a)A\\ &\le \frac{1}{\lambda}\left(a^\lambda U_{A^{\frac{1}{2}}}(\{A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\}^\lambda)-A\right)-(\log a)A\end{aligned} \tag{2.12}$$

Relation (2.12) implies second and third inequalities in (2.9) for unital JB-algebra $\mathcal{A}$, since

$$\frac{1}{\lambda}\left(a^\lambda A\sharp_\lambda B-A\right)-(\log a)A = T_\lambda(A|B)-\frac{1-a^\lambda}{\lambda}A\sharp_\lambda B-(\log a)A.$$

□

3. Unital positive linear maps on unital JC-algebras

In this section, we will present some properties of unital positive linear map on unital JC-algebras. Furthermore, we will prove some results related to unital positive linear map on JB-algebras.

Theorem 3.1. *Let $\mathcal{A}$ be a unital JC-algebra and $\varphi:\mathcal{A}\to\mathcal{B}$ be an isomorphism onto norm closed Jordan subalgebra $\mathcal{B}$ of $B(H)_{sa}$. Let I be an interval in $\mathbb{R}$ and $f:I\to\mathbb{R}$ be a continuous function. Then*

(i) f is $\mathcal{A}$-monotone if and only if f is operator monotone on $\mathcal{B}$.

(ii) f is $\mathcal{A}$-convex if and only if f is operator convex on $\mathcal{B}$.

Proof. (i) Suppose that f is $\mathcal{A}$-monotone and $C,D\in\mathcal{B}$, $C\le D$. There exist elements $A,B\in\mathcal{A}$ with spectra in I such that $\varphi(A)=C$, $\varphi(B)=D$. Then $A\le B$, since φ holds positivity. This implies that $f(A)\le f(B)$. Consequently $\varphi(f(A))\le\varphi(f(B))$. It is obvious that $\varphi(P_n(A))=P_n(\varphi(A))$ for every polynomial P_n. By the Stone-Weierstrass theorem, we get

$$\varphi(f(A))=f(\varphi(A)). \tag{3.1}$$

Therefore

$$\begin{aligned}\varphi(f(A)) \le \varphi(f(B)) &\Rightarrow f(\varphi(A)) \le f(\varphi(B))\\ &\Rightarrow f(C) \le f(D).\end{aligned}$$

Now suppose that f is operator monotone on $\mathcal{B}$, and $A, B \in \mathcal{A}$. Then

$$\begin{aligned}A \le B &\Rightarrow \phi(A) \le \phi(B) && (\phi \text{ is isomorphism})\\ &\Rightarrow f(\phi(A)) \le f(\phi(B)) && (f \text{ is operator monotone on } \mathcal{B}))\\ &\Rightarrow \phi(f(A)) \le \phi(f(B)) && (\text{by}(3.1))\\ &\Rightarrow f(A) \le f(B) && (\phi^{-1} \text{ is isomorphism}).\end{aligned}$$

(ii) Suppose that f is $\mathcal{A}$-convex, $C, D \in \mathcal{B}$ and $0 \le t \le 1$. There exist elements $A, B \in \mathcal{A}$ with spectra in I such that $\varphi(A) = C$, $\varphi(B) = D$. Therefore

$$\begin{aligned}f(tC + (1-t)D) &= f(t\varphi(A) + (1-t)\varphi(B))\\ &= f(\varphi(tA + (1-t)B)\\ &= \varphi(f(tA + (1-t)B)) && (\text{by } (3.1))\\ &\le t\varphi f(A) + (1-t)\varphi f(B) && (f \text{ is } \mathcal{A}\text{-convex and } \varphi \text{ is positive})\\ &= tf\varphi(A) + (1-t)f\varphi(B) && (\text{by } (3.1))\\ &= tf(C) + (1-t)f(D).\end{aligned}$$

Consequently, f is operator convex on $\mathcal{B}$. Now, let f be operator convex on $\mathcal{B}$, $A, B \in \mathcal{A}$ and $0 \le t \le 1$. Then

$$\begin{aligned}\varphi f((1-t)A + tB) &= f((1-t)\varphi(A) + t\varphi(B)) && (\text{by } (3.1))\\ &\le (1-t)f(\varphi(A)) + tf(\varphi(B)) && (f \text{ is operator convex on } \mathcal{B})\\ &= (1-t)\varphi(f(A)) + t\varphi(f(B)) && (\text{by } (3.1))\\ &= \varphi((1-t)f(A) + tf(B)).\end{aligned}$$

This implies that $f((1-t)A + tB) \le (1-t)f(A) + tf(B)$. Hence f is $\mathcal{A}$-convex. □

Lemma 3.2. *Let $\mathcal{A}$ be a unital JC-algebra. If f is a positive $\mathcal{A}$-monotone function on $(0, \infty) \subseteq \mathbb{R}$. Then*

$$f(A) = \int_0^1 (1!_t A)\, d\nu_f(t) \qquad (0 < A \in \mathcal{A}), \tag{3.2}$$

where ν_f is a bounded positive measure on $[0, 1]$.

Proof. By theorem 4.9 in [11], for every positive operator monotone function f on $(0, \infty)$, there exist a bounded positive measure ν_f on $[0, 1]$ such that

$$f(x) = \int_0^1 (1!_t x)\, d\nu_f(t). \tag{3.3}$$

By Theorem 3.1(i), the function f is operator monotone on a norm closed Jordan subalgebra $\mathcal{B}$ of $B(H)_{sa}$. Therefore relation (3.3) holds and applying Lemma 2.1 for positive element $A \in \mathcal{A}$, we get

$$f(A) = \int_0^1 (1!_t A)\, d\nu_f(t).$$

□

Let $\mathcal{A}$ be a unital JC-algebra, f a positive $\mathcal{A}$-monotone function and $A, B \in \mathcal{A}$. We define a mean σ_f on $\mathcal{A}$, as follows

$$A\sigma_f B := \left\{A^{\frac{1}{2}} f\left(\left\{A^{\frac{-1}{2}} B A^{\frac{-1}{2}}\right\}\right) A^{\frac{1}{2}}\right\}. \tag{3.4}$$

Then

$$\begin{aligned} A\sigma_f B &= \left\{A^{\frac{1}{2}}\left(\int_0^1 \left(1!_t \left\{A^{\frac{-1}{2}} B A^{\frac{-1}{2}}\right\}\right) d\nu_f(t)\right) A^{\frac{1}{2}}\right\} \\ &= \int_0^1 \left\{A^{\frac{1}{2}}\left(1!_t \left\{A^{\frac{-1}{2}} B A^{\frac{-1}{2}}\right\}\right) A^{\frac{1}{2}}\right\} d\nu_f(t) \int_0^1 \left\{A^{\frac{1}{2}}\left((1-t) + t\left\{A^{\frac{-1}{2}} B A^{\frac{-1}{2}}\right\}^{-1}\right)^{-1} A^{\frac{1}{2}}\right\} d\nu_f(t) \\ &= \int_0^1 (A!_t B)\, d\nu_f(t). \end{aligned}$$

Therefore

$$A\sigma_f B = \int_0^1 (A!_t B)\, d\nu_f(t) \tag{3.5}$$

Wang and Wang [16, Proposition 6(iv)], proved the following inequality for $\sharp_\lambda$:

$$\{C(A\sharp_\lambda B)C\} = \{CAC\}\sharp_\lambda\{CBC\}.$$

Theorem 3.3. *Let $\mathcal{A}$ be a unital JC-algebra. If A, B, C, D are invertible elements in $\mathcal{A}$ and f is a positive $\mathcal{A}$-monotone function on $(0, \infty) \subseteq \mathbb{R}$. Then*

(i) $\left\{C\left(A\sigma_f B\right)C\right\} = \{CAC\}\,\sigma_f\,\{CBC\}$

(ii) If $A \leq C$, $B \leq D$, then $A\sigma_f B \leq C\sigma_f D$.

Proof. (i) First, we show that

$$\begin{aligned} \{CAC\}!_t\{CBC\} &= \left((1-t)\{CAC\}^{-1} + t\{CBC\}^{-1}\right)^{-1} \\ &= \left((1-t)\left\{C^{-1}A^{-1}C^{-1}\right\} + t\left\{C^{-1}B^{-1}C^{-1}\right\}\right)^{-1} \\ &= \left\{C^{-1}\left((1-t)A^{-1} + tB^{-1}\right)C^{-1}\right\}^{-1} \\ &= \left\{C\left((1-t)A^{-1} + tB^{-1}\right)^{-1} C\right\} \\ &= \{C(A!_t B)C\}. \end{aligned}$$

Then, we obtain

$$\begin{aligned} \left\{C\left(A\sigma_f B\right)C\right\} &= \left\{C\left(\int_0^1 (A!_t B)\, d\nu_f(t)\right)C\right\} \\ &= \int_0^1 \{C(A!_t B)C\}\, d\nu_f(t) \\ &= \int_0^1 \{CAC\}!_t\{CBC\}\, d\nu_f(t) \\ &= \{CAC\}\,\sigma_f\,\{CBC\} \end{aligned}$$

(ii) From Lemma (2.2), we have $A\,!_\lambda\, B \leq C\,!_\lambda\, D$. This and relation (3.5) imply that $A\sigma_f B \leq C\sigma_f D$. □

Proposition 3.4. *If $\mathcal{A},\mathcal{B}$ are two unital JC-algebras and $\Phi : \mathcal{A} \to \mathcal{B}$ is a unital positive linear map. Then $\Phi(B)\Phi(A)^{-1}\Phi(B) \leq \Phi\left(BA^{-1}B\right)$ for every two positive invertible elements $A, B \in \mathcal{A}$.*

Proof. There exist two isomorphisms $\varphi : \mathcal{A} \to \mathcal{C}$, $\psi : \mathcal{B} \to \mathcal{D}$ where $\mathcal{C}$, $\mathcal{D}$ respectively are two closed Jordan subalgebras of $B(H)_{sa}$, and $B(K)_{sa}$. Suppose that $A \in \mathcal{A}$, then $C = \varphi(A) \in \mathcal{C}$. We define $\Psi := \psi \circ \Phi \circ \varphi^{-1}$.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\Phi} & \mathcal{B} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \mathcal{C} & \xrightarrow{\Psi} & \mathcal{D} \end{array}$$

Clearly $\Psi \geq 0$, and by [13, Theorem 1.17](Kadison's Schwarz inequality), we have $\Psi(C^2) \geq \Psi(C)^2$ and $\Psi(C^{-1}) \geq \Psi(C)^{-1}$. Therefore

$$\begin{aligned} \psi\left(\Phi(\varphi^{-1}(C^2))\right) &= \Psi(C^2) \\ &\geq \Psi(C)^2 = (\psi \circ \Phi \circ \varphi^{-1}(C))^2 \\ &= \psi\left(\left(\Phi(\varphi^{-1}(C))\right)^2\right), \qquad (\psi \text{ is an isomorphism }) \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \psi\left(\Phi(\varphi^{-1}(C^{-1}))\right) &= \Psi(C^{-1}) \\ &\geq \Psi(C)^{-1} = (\psi \circ \Phi \circ \varphi^{-1}(C))^{-1} \\ &= \psi\left(\left(\Phi(\varphi^{-1}(C))\right)^{-1}\right). \qquad (\psi \text{ is an isomorphism }) \end{aligned} \tag{3.7}$$

From (3.6), (3.7) and positivity ψ^{-1} (ψ^{-1} is an isomorphism), we obtain

$$\begin{aligned} \Phi(A^2) &= \Phi \circ \varphi^{-1}(C^2) \geq (\Phi \circ \varphi^{-1}(C))^2 = (\Phi(A))^2 \\ \Phi(A^{-1}) &= \Phi \circ \varphi^{-1}(C^{-1}) \geq (\Phi \circ \varphi^{-1}(C))^{-1} = (\Phi(A))^{-1}. \end{aligned}$$

Define

$$\Psi(X) = \Phi(B)^{-\frac{1}{2}}\Phi\left(B^{\frac{1}{2}}XB^{\frac{1}{2}}\right)\Phi(B)^{-\frac{1}{2}}.$$

It is obvious that $\Psi(I) = \Phi(B)^{-\frac{1}{2}}\Phi(B)\Phi(B)^{-\frac{1}{2}} = I$, therefore Ψ is a unital positive linear map. Since $\Psi\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right) = \Phi(B)^{-\frac{1}{2}}\Phi(A)\Phi(B)^{-\frac{1}{2}}$, we obtain

$$\begin{aligned} \Phi(B)^{\frac{1}{2}}\Phi(A)^{-1}\Phi(B)^{\frac{1}{2}} &= \Psi\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)^{-1} \\ &\leq \Psi\left(\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)^{-1}\right) \\ &= \Psi\left(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\right) \\ &= \Phi(B)^{-\frac{1}{2}}\Phi(BA^{-1}B)\Phi(B)^{-\frac{1}{2}}. \end{aligned}$$

Therefore $\Phi(B)\Phi(A)^{-1}\Phi(B) \leq \Phi\left(BA^{-1}B\right)$. □

Theorem 3.5. *Let $\mathcal{A}$ be a unital JC-algebra. If f is an $\mathcal{A}$-convex function on $(0,\infty)$ and Ψ is a unital positive linear map on $\mathcal{A}$, then*

$$\Psi(f(A)) \geq f(\Psi(A)) \quad (A \in \mathcal{A}). \tag{3.8}$$

Proof. It is known that every operator convex function f on $(0, \infty)$ has a special integral representation as follows (see [2, Prob.V.5.5])

$$f(t) = \alpha + \beta t + \gamma t^2 + \int_0^\infty \frac{\lambda t^2}{\lambda + t} d\mu(\lambda), \tag{3.9}$$

where α, β are real numbers, $\gamma \geq 0$, and μ is a bounded positive measure. Since $\mathcal{A}$-convex functions are operator convex over a norm closed Jordan subalgebra $\mathcal{B}$ of $B(H)_{sa}$. Applying functional calculus at A to (3.9), [1, Proposition 1.21], and evaluating the value of function Ψ on it, we obtain

$$\Psi(f(A)) = \alpha 1_K + \beta\Psi(A) + \gamma\Psi(A^2) + \int_0^\infty \Psi(\lambda A^2(\lambda + A)^{-1}) d\mu(\lambda).$$

From the functional calculus at $\Psi(A)$ to (3.9), we obtain

$$f(\Psi(A)) = \alpha 1_K + \beta\Psi(A) + \gamma\Psi(A)^2 + \int_0^\infty \lambda\Psi(A)^2(\lambda + \Psi(A))^{-1} d\mu(\lambda).$$

By Proposition 3.4, we have

$$\begin{aligned}\Psi(\lambda A^2(\lambda + A)^{-1}) &= \lambda\Psi(A^2(\lambda + A)^{-1}) = \lambda\Psi(A(\lambda + A)^{-1}A)\\ &\geq \lambda\Psi(A)(\Phi(\lambda + A))^{-1}\Psi(A) = \lambda\Psi(A)(\lambda + \Psi(A))^{-1}\Psi(A)\\ &= \lambda\Psi(A)^2(\lambda + \Psi(A))^{-1}.\end{aligned}$$

Therefore $\Psi(f(A)) \geq f(\Psi(A))$, since $\Psi(A^2) \geq \Psi(A)^2$. □

Theorem 3.6. *Let $\mathcal{A}$, $\mathcal{B}$ be two unital JC-algebras. If Φ is a unital positive linear map, and σ is a mean on $\mathcal{A}$, then*

$$\Phi(A\,\sigma\,B) \leq \Phi(A)\,\sigma\,\Phi(B) \tag{3.10}$$

for every two positive invertible elements $A, B \in \mathcal{A}$.

Proof. $\Phi(A)$ is invertible since A is invertible. Define a map

$$\Psi(X) = \Phi(A)^{-\frac{1}{2}}\Phi(A^{\frac{1}{2}}XA^{\frac{1}{2}})\Phi(A)^{-\frac{1}{2}}.$$

Then Ψ is unital positive linear map. So we have by Theorem 3.5

$$\Psi\left(f(X)\right) \leq f\left(\Psi(X)\right)$$

for every operator concave function f on $[0, \infty)$. Let f be the representing function for σ, by Theorem 3.5 f is operator concave. Therefore it follows that

$$\begin{aligned}\Phi(A\,\sigma\,B) &= \Phi\left(A^{\frac{1}{2}}f(X)A^{\frac{1}{2}}\right)\\ &= \Phi(A)^{\frac{1}{2}}\Psi(f(X))\Phi(A)^{\frac{1}{2}}\\ &\leq \Phi(A)^{\frac{1}{2}}f(\Psi(X))\Phi(A)^{\frac{1}{2}}\\ &= \Phi(A)\,\sigma\,\Phi(B)\end{aligned}$$

□

Theorem 3.7. *Let $\mathcal{A}$, $\mathcal{B}$ be two unital JB-algebras, and $\Phi : \mathcal{A} \to \mathcal{B}$ be a unital positive linear map, then Φ is bounded.*

Proof. We know that positive elements in a JB-algebra $\mathcal{A}$ generate it, i.e. $\mathcal{A} = \mathcal{A}^+ - \mathcal{A}^+$, since $a = (a + \|a\|1) - \|a\|1$. Therfore, it is sufficient, we show that Φ is bounded on positive elements in $\mathcal{A}$. If S is a bounded subset of $\mathcal{A}^+$, then for $a \in S$ we have $\| a \| \leq M$. This show that $0 \leq a \leq \| a \| 1 \leq M$. Consequently, $0 \leq \Phi(a) \leq \| a \| \Phi(1) = \| a \|$ so $\| \Phi(a) \| \leq \| a \|$ for any $a \in S$. □

Theorem 3.8. *Let $\mathcal{A}$, $\mathcal{B}$ be two unital JC-algebras. If Φ is a unital positive linear map, then*

$$\Phi(S(A|B)) \leq S(\Phi(A)|\Phi(B)). \tag{3.11}$$

Proof. By Proposition 1.7 (v), we have

$$S(A|B) = \lim_{\lambda\to 0} T_\lambda(A|B) = \lim_{\lambda\to 0} \frac{A\sharp_\lambda B - A}{\lambda}$$

Applying Theorem 3.7 and evaluating the value of function Φ on the above equality, we obtain

$$\Phi(S(A|B)) = \Phi\left(\lim_{\lambda\to 0} \frac{A\sharp_\lambda B - A}{\lambda}\right) = \lim_{\lambda\to 0} \frac{\Phi(A\sharp_\lambda B) - \Phi(A)}{\lambda}.$$

Now from inequality (3.10), we deduce

$$\begin{aligned}\Phi(S(A|B)) &\leq \lim_{\lambda\to 0} \frac{\Phi(A)\sharp_\lambda \Phi(B) - \Phi(A)}{\lambda} \\ &= S(\Phi(A)|\Phi(B)).\end{aligned}$$

□

In the following, we give a complementary inequality to inequality (3.11).

Theorem 3.9. *Let $\mathcal{A}$, $\mathcal{B}$ be two unital JB-algebras and Φ be a unital positive linear map and A, B be two invertible positive elements in $\mathcal{A}$ such that $A \leq B$, then there is an $0 < \varepsilon < 1$ such that $B \leq \frac{1}{\varepsilon}A$ and*

$$\varepsilon S(\Phi(A)|\Phi(B)) \leq \varepsilon(\Phi(B) - \Phi(A)) = \Phi(\varepsilon(B - A)) \leq \Phi(S(A|B)). \tag{3.12}$$

Proof. Using the last inequality in (2.4) for $\Phi(A)$ and $\Phi(B)$, we get

$$S(\Phi(A)|\Phi(B)) \leq \Phi(B) - \Phi(A).$$

Now, let X be a invertible element in $\mathcal{A}$. Then there exists an $0 < \varepsilon < 1$ such that $X \leq \|X\|1_H \leq \frac{1}{\varepsilon}1_H$, by inequality (1.11) in [1]. For $X = \{A^{\frac{-1}{2}} B A^{\frac{-1}{2}}\}$, we have $B \leq \frac{1}{\varepsilon}A$.

Next, we show that for all $0 < \varepsilon < 1$ and $1 \leq t \leq \frac{1}{\varepsilon}$

$$\varepsilon(t - 1) \leq \log t. \tag{3.13}$$

Define

$$f(t) = \log t - \varepsilon(t - 1),$$

then $f'(x) = \frac{1}{t} - \varepsilon \geq 0$, for $0 < t \leq \frac{1}{\varepsilon}$. Therefore $f(t) \geq f(1) = 0$ for all $1 \leq t \leq \frac{1}{\varepsilon}$. Utilizing the functional calculus $t = \{A^{\frac{-1}{2}} B A^{\frac{-1}{2}}\}$ in (3.13) and then multiplying both sides by $A^{\frac{1}{2}}$, we obtain $\varepsilon(B - A) \leq S(A|B)$. Since Φ is a positive linear map, therefore

$$\varepsilon\Phi(B - A) = \Phi(\varepsilon(B - A)) \leq \Phi(S(A|B)). \tag{3.14}$$

This proves the last inequality in (3.12). □

Acknowledgments

The authors wish to express their hearty thanks to the referees for their valuable comments, detailed corrections, and suggestions for revising the manuscript.

References

[1] E. M. Alfsen and F. W. Shultz, Geometry of State Spaces of Operator Algebras, Mathematics: theory and Applications, Birkhauser, (2003).
[2] R. Bhatia, Matrix analysis, Springer-Verlag, New York, 1997.
[3] R. Clausius, Über die bewegende Kraft der Ẅarme, Part I, Part II. Ann Phys. 79(2)(1850), 368-397.
[4] G. G. Emch, Algebraic Methods in Statistical Mechanics and Quantum Field Theory, Wiely-interscience, New York, (1972).
[5] J. I. Fujii, E. Kamei, Uhlmann's interpolational method for operator means, Math. Japon., Vol. 34, 4 (1989), pp. 541–547.
[6] J. I. Fujii, E. Kamei, Relative operator entropy in noncommutative information theory, Math. Japon., Vol. 34, 3 (1989), pp. 341-348.
[7] S. Furuichi, Inequalities for Tsallis relative entropy and generalized skew information, Linear Multilinear Algebra, Vol. 59, 10 (2011), pp. 1143-1158.
[8] S. Furuichi, K. Yanagiy, K. Kuriyama, Fundamental properties of Tsallis relative entropy, J. Math. Phys., Vol. 45, 12 (2004), pp. 4868-4877.
[9] T. Furuta, Furuta's inequality and its application to the relative operator entropy, J. Operator Theory, Vol. 30, 1 (1993), pp. 21-30.
[10] H. Hanche-Olsen and E. Størmer, Jordan Operator Algebras, Pitman Publishing Inc, Massachusetss, (2008).
[11] F. Hansen, The fast track to Lowners theorem, Linear Alg. Appl., 438(2013), 4557-4571.
[12] M. Nakamura, H. Umegaki, A note on the entropy for operator algebra. Proc Jpn Acad. 37(1961), 149-154.
[13] J.E. Pečarić, T. Furuta, J. Mićić Hot, Y. Seo, Mond-Pečarić Method in operator inequalities, Element, Zagreb, 2005.
[14] S. Soleimani, A.G. Ghazanfari, New refinements of some inequalities involving relative operator entropy, Complex Anal. Oper. Theory (2019) 13:3337-3345.
[15] H . Umegaki, Conditional expectation in an operator algebra, IV, (entropy and information), Kodai Math. Sem. Rep., 14(1962), 59-85.
[16] S. Wang, Z. Wang, Operator means in JB-algebras, Reports on mathematical physics, 88(3)383-398 (2021).
[17] S. Wang, Z. Wang, Relative operator entropy, Rocky Mountain J Math (2022) 52(3) 1073-1087.
[18] K. Yanagiy, K. Kuriyama, S. Furuichi, Generalized Shannon inequalities based on Tsallis relative operator entropy, Linear Algebra Appl., Vol. 394, 1 (2005), pp. 109-118.
[19] L. Zou, Operator inequalities associated with Tsallis relative operator entropy, Math. Ineq. Appl. 18 (2015), 401-406.