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Refined Hilbert-Schmidt numerical radius inequalities

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Abstract. In this paper, we present some bounds for Hilbert-Schmidt numerical radius of 2×2 operator matrices. Some of these bounds are refinements of the existing ones. And some examples will be given to illustrate that our results are better than the existing bound.

1. Introduction and preliminaries

Let $\mathbb{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex separable Hilbert space $\mathcal{H}$. For $T \in \mathbb{B}(\mathcal{H})$, the adjoint, the real and imaginary parts of T are defined by T^*, $\Re(T)$ and $\Im(T)$, respectively. And according to the Cartesian decomposition, $T \in \mathbb{B}(\mathcal{H})$ can be presented as $T = \Re(T) + i\Im(T)$, where $\Re(T) = \frac{1}{2}(T + T^*)$ and $\Im(T) = \frac{1}{2i}(T - T^*)$. A compact operator $T \in \mathbb{B}(\mathcal{H})$ belongs to the Hilbert-Schmidt class C_2 if $\|T\|_2 = (trT^*T)^{\frac{1}{2}} < \infty$, where $tr(.)$ is the usual trace functional. Note that $\|.\|_2$ is unitarily invariant, that is for all $T \in C_2$ and all unitary operators $U, V \in \mathbb{B}(\mathcal{H})$, we have $\|UTV\|_2 = \|T\|_2$.

In 2019, A. Abu-Omar, F. Kittaneh[1] defined the Hilbert-Schmidt numerical radius as follows:

$$w_2(T) = \sup_{\theta \in \mathbb{R}} \left\| \Re(e^{i\theta}T) \right\|_2 .$$

It is well known that $w_2(T)$ defines an operator norm on $\mathbb{B}(\mathcal{H})$, which is equivalent to the Hilbert-Schmidt operator norm. In fact, for every $T \in C_2$,

$$\frac{1}{\sqrt{2}} \|T\|_2 \le w_2(T) \le \|T\|_2 . \tag{1}$$

The inequalities (1) are sharp. The first inequality becomes an equality if $T^2 = 0$. The second inequality becomes an equality if $T = T^*$.

A. Aldalabih, F. Kittaneh[4] proved that for $B, C \in C_2$,

$$w_2\left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right] \le \frac{1}{\sqrt{2}} (\|B\|_2 + \|C\|_2) . \tag{2}$$

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In [2], S. Aici et.al gave the following results: for $A, B, C \in C_2$,

$$w_2^2\left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right] \geq \max\left(w_2(BC), w_2(CB)\right) \tag{3}$$

and

$$w_2\left[\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\right] \geq \frac{1}{\sqrt{2}} \max\left(w_2(A), w_2(B)\right). \tag{4}$$

M. Hajmohamadi, R. Lashkaripour[7] proved that

$$w_2\left[\begin{pmatrix} A & B \\ -A & -B \end{pmatrix}\right] \geq \frac{1}{\sqrt{2}} \max\left(w_2(A+B), w_2(A-B)\right). \tag{5}$$

In [3], A. Frakis et.al gave the following results: for $A, B, C, D \in C_2$,

$$w_2^2\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right] \geq \frac{1}{2} \max\left(M, N, w_2^2(B+C), w_2^2(B-C)\right) \tag{6}$$

and

$$w_2^2\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right] \geq \frac{1}{\sqrt{2}} \max\left(w_2(AB+DC), w_2(AB-DC)\right) + \frac{1}{2}\left|trA^2 + trD^2 + 2trBC\right|, \tag{7}$$

where $M = w_2^2(A+D) + \inf_{\theta\in\mathbb{R}} \left\|\Re\left(e^{i\theta}(A-D)\right)\right\|_2^2$, $N = w_2^2(A-D) + \inf_{\theta\in\mathbb{R}} \left\|\Re\left(e^{i\theta}(A+D)\right)\right\|_2^2$.

In [6], M. Guesba obtained that for $A, B, C, D \in C_2$,

$$w_2\left[\begin{pmatrix} 0 & ACB^* \\ BDA^* & 0 \end{pmatrix}\right] \leq 2\|A\|_2 \|B\|_2 \, w_2\left[\begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}\right]. \tag{8}$$

In [9], F. Kittaneh proved that if A, B are positive operators, then for $A, B \in C_2$,

$$\|A+B\|_2 \leq \sqrt{\|A\|_2^2 + \|B\|_2^2 + 2^{\frac{1}{2}} \left\||A|^{\frac{1}{2}} |B|^{\frac{1}{2}}\right\|_2}. \tag{9}$$

The purpose of this paper is to establish some new upper and lower bounds for Hilbert-Schmidt numerical radius of 2×2 operator matrices. We first give a different proof of (3), then we give refinements of (3)–(5), (8) and (9). We also investigate some upper bounds for the Hilbert-Schmidt radius of the off-diagonal parts of 2×2 operator matrices by refining (9). Moreover, some examples will be given to illustrate that our results are better than the inequality (2).

2. Main results

Lemma 2.1 ([4]). *Let $B, C \in C_2$, then*

(a) $w_2\left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right] = w_2\left[\begin{pmatrix} 0 & C \\ B & 0 \end{pmatrix}\right]$;

(b) $w_2\left[\begin{pmatrix} 0 & B \\ e^{i\theta}C & 0 \end{pmatrix}\right] = w_2\left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right], \forall\theta \in \mathbb{R}$;

(c) $w_2\left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right] \geq \frac{1}{\sqrt{2}} \max\left(w_2(B+C), w_2(B-C)\right)$.

Lemma 2.2 ([3]). *Let $A, B \in C_2$, then*

$$w_2(AB) + \frac{1}{2}\left|tr(A^2 + B^2)\right| \le \frac{1}{2}\left(w_2^2(A+B) + w_2^2(A-B)\right).$$

First we apply the above results to give a different proof of (3).

Theorem 2.3. *Let $B, C \in C_2$, then*

$$w_2^2\left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right] \ge \max\left(w_2(BC), w_2(CB)\right).$$

Proof. Applying Lemma 2.2 to the operator matrices $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$, we see that

$$\begin{aligned} w_2^2\left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right] &= \frac{1}{2}\left(w_2^2\left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right] + w_2^2\left[\begin{pmatrix} 0 & B \\ -C & 0 \end{pmatrix}\right]\right) \text{(by Lemma 2.1(b))} \\ &\ge \max\left(w_2\left[\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}\right], w_2\left[\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}\right]\right) + \frac{1}{2}\left|tr\left(\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}^2 + \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}^2\right)\right| \\ &= \max\left(w_2(BC), w_2(CB)\right). \end{aligned}$$

□

Lemma 2.4 ([7]). *Let $T = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$, where $B, C \in C_2$, then*

$$w_2(T) = \frac{1}{\sqrt{2}}\sup_{\theta\in\mathbb{R}}\left\|e^{i\theta}B + e^{-i\theta}C^*\right\|_2.$$

Lemma 2.5 ([12]). *Let $A, B \in C_2$ be such that A, B are positive, then*

$$\|A - B\|_2 \le \left(\|A\|_2^2 + \|B\|_2^2\right)^{\frac{1}{2}}.$$

Now we give refinement of (3).

Theorem 2.6. *Let $B, C \in C_2$, then*

$$w_2^2\left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right] \ge \sqrt{2}\max\left(w_2(BC), w_2(CB)\right).$$

Proof. From the identity $4\Re(B^*C) = |B+C|^2 - |B-C|^2$, it follows that

$$\begin{aligned} \left\|4\Re(B^*C)\right\|_2 &= \left\||B+C|^2 - |B-C|^2\right\|_2 \\ &\le \left(\left\||B+C|^2\right\|_2^2 + \left\||B-C|^2\right\|_2^2\right)^{\frac{1}{2}} \text{ (by Lemma 2.5)} \\ &\le \left(\|B+C\|_2^4 + \|B-C\|_2^4\right)^{\frac{1}{2}}. \end{aligned}$$

Thus,

$$\left\|4\Re(B^*C)\right\|_2 \le \left(\|B+C\|_2^4 + \|B-C\|_2^4\right)^{\frac{1}{2}}. \tag{10}$$

Now, replacing B by $e^{i\theta}B$, and taking the supremum of both sides in the inequality (10) over $\theta \in \mathbb{R}$, we obtain

$$\begin{aligned} w_2(B^*C) &\leq \frac{1}{4}\left(\sup_{\theta\in\mathbb{R}} \left\|e^{i\theta}B + C\right\|_2^4 + \sup_{\theta\in\mathbb{R}} \left\|e^{i\theta}B - C\right\|_2^4\right)^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{2}} w_2^2\left[\begin{pmatrix} 0 & B \\ C^* & 0 \end{pmatrix}\right] (\text{by Lemma 2.4}). \end{aligned}$$

Hence, we can obtain

$$w_2^2\left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right] \geq \sqrt{2} \max\left(w_2(BC), w_2(CB)\right) (\text{by Lemma 2.1(a)}).$$

□

Next, we show the following formula for the off-diagonal block operator matrix $\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$. This will be used as a key tool to obtain our results.

Lemma 2.7. *Let $B, C \in C_2$, then*

$$w_2\left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right] = \sqrt{2} \sup_{\alpha^2+\beta^2=1} \left\|\alpha\frac{B + C^*}{2} + \beta\frac{B - C^*}{2i}\right\|_2.$$

Proof. By simple calculation, we have

$$\begin{aligned} \sup_{\alpha^2+\beta^2=1} \left\|\alpha\frac{B + C^*}{2} + \beta\frac{B - C^*}{2i}\right\|_2 &= \frac{1}{2} \sup_{\alpha^2+\beta^2=1} \left\|(\alpha - i\beta)B + (\alpha + i\beta)C^*\right\|_2 \\ &= \frac{1}{2}\left|\alpha - i\beta\right| \sup_{\alpha^2+\beta^2=1} \left\|B + \frac{\alpha + i\beta}{\alpha - i\beta}C^*\right\|_2. \end{aligned}$$

Let $\alpha = cos\theta, \beta = sin\theta, \theta \in [0, 2\pi]$. Then, we can see that

$$\begin{aligned} \sup_{\alpha^2+\beta^2=1} \left\|\alpha\frac{B + C^*}{2} + \beta\frac{B - C^*}{2i}\right\|_2 &= \frac{1}{2} \sup_{\theta\in\mathbb{R}} \left\|B + e^{2i\theta}C^*\right\|_2 \\ &= \frac{1}{\sqrt{2}} w_2\left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right] (\text{by Lemma 2.4}). \end{aligned}$$

□

Remark 2.8. *When $B = C$ in* Lemma 2.7*, we get*

$$w_2(B) = \sup_{\alpha^2+\beta^2=1} \left\|\alpha\mathfrak{R}(B) + \beta\mathfrak{I}(B)\right\|_2,$$

which has been given in [7].

The following lemma is the famous Cauchy inequality.

Lemma 2.9. *Let $a, b, c, d \in \mathbb{R}$, then*

$$(ab + cd)^2 \leq \left(a^2 + c^2\right)\left(b^2 + d^2\right).$$

Theorem 2.10. *Let $B, C \in C_2$, then*

$$w_2\left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right] \le \frac{1}{\sqrt{2}}\sqrt{\|B+C^*\|_2^2 + \|B-C^*\|_2^2}.$$

Proof. By Lemma 2.7, the triangle inequality and Lemma 2.9, we have

$$\begin{aligned} w_2\left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right] &= \sqrt{2}\sup_{\alpha^2+\beta^2=1}\left\|\alpha\frac{B+C^*}{2} + \beta\frac{B-C^*}{2i}\right\|_2 \\ &\le \frac{1}{\sqrt{2}}\sup_{\alpha^2+\beta^2=1}\left(|\alpha|\,\|B+C^*\|_2 + |\beta|\,\|B-C^*\|_2\right) \\ &\le \frac{1}{\sqrt{2}}\sup_{\alpha^2+\beta^2=1}\sqrt{|\alpha|^2+|\beta|^2}\sqrt{\|B+C^*\|_2^2 + \|B-C^*\|_2^2} \\ &= \frac{1}{\sqrt{2}}\sqrt{\|B+C^*\|_2^2 + \|B-C^*\|_2^2}. \end{aligned}$$

□

In the following, we obtain a generalization of (1) in terms of the real and imaginary parts of T.

Corollary 2.11. *Let $T = \Re(T) + i\Im(T)$ be the Cartesian decomposition of $T \in C_2$, then*

$$\frac{1}{\sqrt{2}}\|T\|_2 \le w_2\left[\begin{pmatrix} 0 & \Re(T) \\ i\Im(T) & 0 \end{pmatrix}\right] \le \|T\|_2.$$

Proof. Let $B = \Re(T)$ and $C = i\Im(T)$ in Lemma 2.7, and set $\alpha = 1, 0$, then

$$\frac{1}{\sqrt{2}}\max\left(\|\Re(T) + i\Im(T)\|_2, \|\Re(T) - i\Im(T)\|_2\right) \le w_2\left[\begin{pmatrix} 0 & \Re(T) \\ i\Im(T) & 0 \end{pmatrix}\right].$$

Hence,

$$\begin{aligned} \frac{1}{\sqrt{2}}\|T\|_2 &= \frac{1}{\sqrt{2}}\max\left(\|T\|_2, \|T^*\|_2\right) \\ &= \frac{1}{\sqrt{2}}\max\left(\|\Re(T) + i\Im(T)\|_2, \|\Re(T) - i\Im(T)\|_2\right) \\ &\le w_2\left[\begin{pmatrix} 0 & \Re(T) \\ i\Im(T) & 0 \end{pmatrix}\right] \\ &\le \frac{1}{\sqrt{2}}\sqrt{\|\Re(T) + i\Im(T)\|_2^2 + \|\Re(T) - i\Im(T)\|_2^2}\ (\text{by Theorem 2.10}) \\ &= \|T\|_2. \end{aligned}$$

□

Lemma 2.12 ([5]). *Let $A, B, C, D \in C_2$, then*

$$\left\|\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right\|_2^2 = \|A\|_2^2 + \|B\|_2^2 + \|C\|_2^2 + \|D\|_2^2.$$

Then, we will give the lower and upper bound estimates for $w_2\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right]$, where $A, B, C, D \in C_2$.

Theorem 2.13. *Let* $T=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$*, where* $A,B,C,D\in C_2$*, then*

$$w_2^2(T)\geq \frac{1}{\sqrt{2}}\max\left(w_2(AB+DC),w_2(AB-DC),w_2(BD+CA),w_2(BD-CA)\right)+\frac{1}{2}\left|trA^2+trD^2+2trBC\right|.$$

Proof. By Lemma 2.2 to the operators matrices $\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$ and $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$, we have

$$\begin{aligned} w_2^2(T) &= \frac{1}{2}\left(w_2^2(T)+w_2^2\left[\begin{pmatrix} -A & B \\ C & -D \end{pmatrix}\right]\right) \text{(by Lemma 2.12)} \\ &\geq w_2\left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\right]+\frac{1}{2}\left|trA^2+trD^2+2trBC\right| \\ &= w_2\left[\begin{pmatrix} 0 & BD \\ CA & 0 \end{pmatrix}\right]+\frac{1}{2}\left|trA^2+trD^2+2trBC\right| \\ &\geq \frac{1}{\sqrt{2}}\max\left(w_2(BD+CA),w_2(BD-CA)\right)+\frac{1}{2}\left|trA^2+trD^2+2trBC\right| \text{(by Lemma 2.1(c))}. \end{aligned}$$

Combining the above inequality and the inequality (7), we get the desired result. □

Remark 2.14. *It is clear that* Theorem 2.13 *is better than* (7).

Lemma 2.15 ([10]). *Let* $A,B\in C_2$ *be such that* AB *is self-adjoint, then*

$$\|AB\|_2\leq \left\|\Re(BA)\right\|_2.$$

Theorem 2.16. *Let* $A,B,C,D\in C_2$*, then*

$$w_2\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right]\leq \min(\alpha,\beta),$$

where

$$\alpha=\frac{1}{\sqrt{2}}\left(\sqrt{w_2^2(A)+\left\|\frac{I}{4}+AA^*+BB^*\right\|_2^2}+\sqrt{w_2^2(D)+\left\|\frac{I}{4}+CC^*+DD^*\right\|_2^2}\right),$$

$$\beta=\frac{1}{\sqrt{2}}\left(\sqrt{w_2^2(A)+\left\|\frac{I}{4}+AA^*+CC^*\right\|_2^2}+\sqrt{w_2^2(D)+\left\|\frac{I}{4}+BB^*+DD^*\right\|_2^2}\right).$$

Proof. By simple calculations, we have

$$\begin{aligned} \left\|\Re\left(e^{i\theta}\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\right)\right\|_2 &= \left\|\begin{pmatrix} \frac{1}{2}(e^{i\theta}A+e^{-i\theta}A^*) & \frac{1}{2}e^{i\theta}B \\ \frac{1}{2}e^{-i\theta}B^* & 0 \end{pmatrix}\right\|_2 \\ &= \left\|\begin{pmatrix} A^* & \frac{1}{2}e^{i\theta} \\ B^* & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2}e^{-i\theta} & 0 \\ A & B \end{pmatrix}\right\|_2 \\ &\leq \left\|\Re\left(\begin{pmatrix} \frac{1}{2}e^{-i\theta} & 0 \\ A & B \end{pmatrix}\begin{pmatrix} A^* & \frac{1}{2}e^{i\theta} \\ B^* & 0 \end{pmatrix}\right)\right\|_2 \textit{(by Lemma 2.15)} \\ &= \frac{1}{2}\left\|\begin{pmatrix} \Re(e^{i\theta}A) & \frac{I}{4}+AA^*+BB^* \\ \frac{I}{4}+AA^*+BB^* & \Re(e^{i\theta}A) \end{pmatrix}\right\|_2 \\ &= \frac{1}{\sqrt{2}}\sqrt{\left\|\Re(e^{i\theta}A)\right\|_2^2+\left\|\frac{I}{4}+AA^*+BB^*\right\|_2^2}. \end{aligned}$$

Next, by taking the supremum in the above inequalities over $\theta \in \mathbb{R}$, we have

$$w_2\left[\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\right] \leq \frac{1}{\sqrt{2}}\sqrt{w_2^2(A) + \left\|\frac{I}{4} + AA^* + BB^*\right\|_2^2}.$$

Let $U = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, where U is a unitary operator, then

$$\begin{aligned} w_2\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right] &\leq w_2\left[\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\right] + w_2\left[\begin{pmatrix} 0 & 0 \\ C & D \end{pmatrix}\right] \\ &= w_2\left[\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\right] + w_2\left[U^*\begin{pmatrix} D & C \\ 0 & 0 \end{pmatrix}U\right] \\ &= w_2\left[\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\right] + w_2\left[\begin{pmatrix} D & C \\ 0 & 0 \end{pmatrix}\right] \\ &\leq \frac{1}{\sqrt{2}}\left(\sqrt{w_2^2(A) + \left\|\frac{I}{4} + AA^* + BB^*\right\|_2^2} + \sqrt{w_2^2(D) + \left\|\frac{I}{4} + CC^* + DD^*\right\|_2^2}\right) \\ &= \alpha. \end{aligned}$$

Similarly,

$$\begin{aligned} w_2\left[\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix}\right] &\leq \frac{1}{\sqrt{2}}\left(\sqrt{w_2^2(A) + \left\|\frac{I}{4} + A^*A + C^*C\right\|_2^2} + \sqrt{w_2^2(D) + \left\|\frac{I}{4} + B^*B + D^*D\right\|_2^2}\right) \\ &= \beta. \end{aligned}$$

Thus, we get the desired result. □

Theorem 2.17. *Let $A, B \in C_2$, then*

$$w_2\left[\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\right] \geq \frac{1}{\sqrt{2}}\max\left(\sqrt{2}w_2(A), \|B\|_2\right).$$

Proof. By applying Lemma 2.12, we have

$$\begin{aligned} w_2\left[\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\right] &\geq \left\|\Re\left(e^{i\theta}\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\right)\right\|_2 \\ &= \left\|\begin{pmatrix} \Re\left(e^{i\theta}A\right) & \frac{1}{2}e^{i\theta}B \\ \frac{1}{2}e^{-i\theta}B^* & 0 \end{pmatrix}\right\|_2 \\ &\geq \frac{1}{\sqrt{2}}\|B\|_2 \end{aligned}$$

and

$$\begin{aligned} w_2\left[\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\right] &= \sup_{\theta\in\mathbb{R}}\left\|\Re\left(e^{i\theta}\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\right)\right\|_2 \\ &= \sup_{\theta\in\mathbb{R}}\left\|\begin{pmatrix} \Re\left(e^{i\theta}A\right) & \frac{1}{2}e^{i\theta}B \\ \frac{1}{2}e^{-i\theta}B^* & 0 \end{pmatrix}\right\|_2 \\ &\geq \sup_{\theta\in\mathbb{R}}\left\|\Re\left(e^{i\theta}A\right)\right\|_2 \\ &= w_2(A). \end{aligned}$$

Thus

$$w_2\left[\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\right] \geq \frac{1}{\sqrt{2}} \max\left(\sqrt{2}w_2(A), \|B\|_2\right).$$

□

Remark 2.18. Theorem 2.17 *is a refinement of* (4).

Lemma 2.19 ([6]). *Let $A, B, C, D \in C_2$, then*

$$w_2\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right] \geq \max\left(w_2\left[\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\right], w_2\left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right]\right).$$

Lemma 2.20 ([3]). *Let $A, D \in C_2$, then*

$$w_2^2\left[\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\right] \geq \frac{1}{2} \max\left(w_2^2(A-D) + m_2^2(A+D), w_2^2(A+D) + m_2^2(A-D)\right),$$

where $m_2(A) = \inf_{\theta \in \mathbb{R}} \left\|\Re(e^{i\theta}A)\right\|_2$.

The following theorem is a generalization of the inequalities (6) .

Theorem 2.21. *Let $A, B, C, D \in C_2$, then*

$$w_2^2\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right] \geq \frac{1}{2} \max\left(w_2^2(A-D) + m_2^2(A+D), w_2^2(A+D) + m_2^2(A-D), \|B+C^*\|_2^2, \|B-C^*\|_2^2\right).$$

Proof. Let $\alpha = 1, 0$ in Lemma 2.7, we can see that

$$w_2\left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right] \geq \frac{1}{\sqrt{2}} \max\left(\|B+C^*\|_2, \|B-C^*\|_2\right). \tag{11}$$

Therefore, by Lemma 2.19, Lemma 2.20 and the inequality (11), we obtain

$$\begin{aligned} w_2^2\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right] &\geq \max\left(w_2^2\left[\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\right], w_2^2\left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right]\right) \\ &\geq \frac{1}{2} \max\left(w_2^2(A-D) + m_2^2(A+D), w_2^2(A+D) + m_2^2(A-D), \|B+C^*\|_2^2, \|B-C^*\|_2^2\right). \end{aligned}$$

□

Corollary 2.22. *Let $A, B \in C_2$, then*

$$w_2^2\left[\begin{pmatrix} A & B \\ -A & -B \end{pmatrix}\right] \geq \frac{1}{2} \max\left(w_2^2(A+B) + m_2^2(A-B), w_2^2(A-B) + m_2^2(A+B), \|B+A^*\|_2^2, \|B-A^*\|_2^2\right).$$

Remark 2.23. Corollary 2.22 *is a refinement of* (5).

Lemma 2.24 ([11]). *Let $A, X \in C_2$, then*

$$w_2(AXA^*) \leq \|A\|^2 w_2(X),$$

where $\|.\|$ denotes the usual operator norm.

Theorem 2.25. *Let $A, B, C, D \in C_2$, then*

$$w_2\left[\begin{pmatrix} 0 & ACB^* \\ BDA^* & 0 \end{pmatrix}\right] \leq 2\,\|A\|\,\|B\|\,w_2\left[\begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}\right].$$

Proof. Let $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, $S = \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}$, then $TST^* = \begin{pmatrix} 0 & ACB^* \\ BDA^* & 0 \end{pmatrix}$. Noting that

$$\|T\|^2 = \left\|\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right\|^2 = \max\left(\|A\|^2, \|B\|^2\right),$$

(see [9]).

And by using Lemma 2.24, we can get

$$\begin{aligned} w_2\left(TST^*\right) &\leq \|T\|^2\, w_2(S) \\ &= \max\left(\|A\|^2, \|B\|^2\right) w_2\left(\begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}\right) \\ &\leq \left(\|A\|^2 + \|B\|^2\right) w_2\left(\begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}\right). \end{aligned}$$

Hence,

$$w_2\left[\begin{pmatrix} 0 & ACB^* \\ BDA^* & 0 \end{pmatrix}\right] \leq \left(\|A\|^2 + \|B\|^2\right) w_2\left(\begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}\right). \tag{12}$$

If $A = 0$ or $B = 0$, the result is clear.

If $A \neq 0$ and $B \neq 0$, then by replacing A and B by $\sqrt{\frac{\|B\|}{\|A\|}}A$ and $\sqrt{\frac{\|A\|}{\|B\|}}B$, respectively in the inequality (12), we get the desired inequality. □

Remark 2.26. *Since $\|.\| \leq \|.\|_2$(see [5]),* Theorem 2.25 *is a refinement of* (8)*.*

Lemma 2.27 ([8]). *Let $A, B \in C_2$, then*

$$\|A + B\|_2 \leq \||A^*| + |B^*|\|_2^{\frac{1}{2}}\, \||A| + |B|\|_2^{\frac{1}{2}}.$$

In particular, if A, B are normal, then

$$\|A + B\|_2 \leq \||A| + |B|\|_2.$$

Theorem 2.28. *Let $A, B \in C_2$ be such that A, B are normal, then for $r, s \in [0, 1]$,*

$$\|A + B\|_2 \leq \sqrt{\|A\|_2^2 + \|B\|_2^2 + \frac{1}{4}\left(M^2 + N^2\right)},$$

where $M = \left\|\frac{1}{t}|A|^r|B|^s + t|A|^{1-r}|B|^{1-s}\right\|_2$, $N = \left\|t|B|^{1-s}|A|^{1-r} + \frac{1}{t}|B|^s|A|^r\right\|_2$ for all t > 0.

Proof. First, we prove the result holds for positive operators A, B. For $r, s \in [0,1]$ and for any $t > 0$, we have

$$\begin{aligned}
\|A+B\|_2 &= \left\| \begin{pmatrix} A+B & 0 \\ 0 & 0 \end{pmatrix} \right\|_2 \\
&= \left\| \begin{pmatrix} tA^{1-r} & B^s \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{t}A^r & 0 \\ B^{1-s} & 0 \end{pmatrix} \right\|_2 \\
&\leq \left\| \Re \left(\begin{pmatrix} \frac{1}{t}A^r & 0 \\ B^{1-s} & 0 \end{pmatrix} \begin{pmatrix} tA^{1-r} & B^s \\ 0 & 0 \end{pmatrix} \right) \right\|_2 \text{ (by Lemma 2.15)} \\
&= \left\| \begin{pmatrix} A & \frac{1}{2}\left(\frac{1}{t}A^rB^s + tA^{1-r}B^{1-s}\right) \\ \frac{1}{2}\left(tB^{1-s}A^{1-r} + \frac{1}{t}B^sA^r\right) & B \end{pmatrix} \right\|_2 \\
&= \sqrt{\|A\|_2^2 + \|B\|_2^2 + \frac{1}{4}\left(\left\| \frac{1}{t}A^rB^s + tA^{1-r}B^{1-s} \right\|_2^2 + \left\| tB^{1-s}A^{1-r} + \frac{1}{t}B^sA^r \right\|_2^2 \right)}.
\end{aligned}$$

Next, we will prove the result holds for normal operators A, B. By applying Lemma 2.27, $\||A|\|_2 = \|A\|_2$ and $\||B|\|_2 = \|B\|_2$, we have

$$\|A+B\|_2 \leq \||A| + |B|\|_2 \leq \sqrt{\|A\|_2^2 + \|B\|_2^2 + \frac{1}{4}(M^2 + N^2)}.$$

□

The following corollary improves the inequalities (9).

Set $t = 1$, $r = s = \frac{1}{2}$ in Theorem 2.28, we obtain the following corollary.

Corollary 2.29. *Let $A, B \in C_2$ be such that A, B are normal, then*

$$\|A+B\|_2 \leq \sqrt{\|A\|_2^2 + \|B\|_2^2 + 2\left\| |A|^{\frac{1}{2}}|B|^{\frac{1}{2}} \right\|_2^2}.$$

Theorem 2.30. *Let $A, B \in C_2$, then for $r, s \in [0,1]$,*

$$\|A+B\|_2 \leq \sqrt{\|A\|_2^2 + \|B\|_2^2 + \frac{1}{8}(R^2 + S^2)},$$

where

$$R = \sqrt{\left\| \frac{1}{t}|A^*|^r|B^*|^s + t|A^*|^{1-r}|B^*|^{1-s} \right\|_2^2 + \left\| \frac{1}{t}|A|^r|B|^s + t|A|^{1-r}|B|^{1-s} \right\|_2^2},$$

$$S = \sqrt{\left\| t|B^*|^{1-s}|A^*|^{1-r} + \frac{1}{t}|B^*|^s|A^*|^r \right\|_2^2 + \left\| t|B|^{1-s}|A|^{1-r} + \frac{1}{t}|B|^s|A|^r \right\|_2^2}$$

for all t > 0.

Proof. Denote $\varphi(A) := \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$. Obviously, $\varphi(A)$ is self-adjoint. Then,

$$\|A+B\|_2 = \frac{1}{\sqrt{2}} \left\| \varphi(A+B) \right\|_2 = \frac{1}{\sqrt{2}} \left\| \varphi(A) + \varphi(B) \right\|_2.$$

By Theorem 2.28, we have

$$\|A+B\|_2 \leq \frac{1}{\sqrt{2}} \sqrt{\left\| \varphi(A) \right\|_2^2 + \|\varphi(B)\|_2^2 + \frac{1}{4}\left(R_1^2 + S_1^2\right)},$$

where

$$R_1 = \left\| \frac{1}{t}|\varphi(A)|^r|\varphi(B)|^s + t|\varphi(A)|^{1-r}|\varphi(B)|^{1-s} \right\|_2, S_1 = \left\| t|\varphi(B)|^{1-s}|\varphi(A)|^{1-r} + \frac{1}{t}|\varphi(B)|^s|\varphi(A)|^r \right\|_2.$$

Note

$$\left\|\varphi(A)\right\|_2 = \sqrt{2}\,\|A\|_2, \left\|\varphi(B)\right\|_2 = \sqrt{2}\,\|B\|_2, |\varphi(A)| = \begin{pmatrix} |A^*| & 0 \\ 0 & |A| \end{pmatrix}, \ |\varphi(B)| = \begin{pmatrix} |B^*| & 0 \\ 0 & |B| \end{pmatrix}.$$

Therefore,

$$\begin{aligned}
&\left\| \frac{1}{t}|\varphi(A)|^r|\varphi(B)|^s + t|\varphi(A)|^{1-r}|\varphi(B)|^{1-s} \right\|_2 \\
&= \left\| \frac{1}{t}\begin{pmatrix} |A^*| & 0 \\ 0 & |A| \end{pmatrix}^r \begin{pmatrix} |B^*| & 0 \\ 0 & |B| \end{pmatrix}^s + t\begin{pmatrix} |A^*| & 0 \\ 0 & |A| \end{pmatrix}^{1-r} \begin{pmatrix} |B^*| & 0 \\ 0 & |B| \end{pmatrix}^{1-s} \right\|_2 \\
&= \left\| \begin{pmatrix} \frac{1}{t}|A^*|^r|B^*|^s + t|A^*|^{1-r}|B^*|^{1-s} & 0 \\ 0 & \frac{1}{t}|A|^r|B|^s + t|A|^{1-r}|B|^{1-s} \end{pmatrix} \right\|_2 \\
&= \sqrt{\left\| \frac{1}{t}|A^*|^r|B^*|^s + t|A^*|^{1-r}|B^*|^{1-s} \right\|_2^2 + \left\| \frac{1}{t}|A|^r|B|^s + t|A|^{1-r}|B|^{1-s} \right\|_2^2}.
\end{aligned}$$

Similarity,

$$\begin{aligned}
&\left\| t|\varphi(B)|^{1-s}|\varphi(A)|^{1-r} + \frac{1}{t}|\varphi(B)|^s|\varphi(A)|^r \right\|_2 \\
&= \sqrt{\left\| t|B^*|^{1-s}|A^*|^{1-r} + \frac{1}{t}|B^*|^s|A^*|^r \right\|_2^2 + \left\| t|B|^{1-s}|A|^{1-r} + \frac{1}{t}|B|^s|A|^r \right\|_2^2}.
\end{aligned}$$

This completes the proof. □

Remark 2.31. *Set* $s = r = \frac{1}{2}, t = 1$ *in* Theorem 2.30*, we obtain*

$$\|A + B\|_2 \leq \sqrt{\|A\|_2^2 + \|B\|_2^2 + \left\||A^*|^{\frac{1}{2}}|B^*|^{\frac{1}{2}}\right\|_2^2 + \left\||A|^{\frac{1}{2}}|B|^{\frac{1}{2}}\right\|_2^2}.$$

Theorem 2.32. *Let* $B, C \in C_2$ *, then*

$$w_2\left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right] \leq \frac{1}{\sqrt{2}}\sqrt{\|B\|_2^2 + \|C\|_2^2 + \left\||B^*|^{\frac{1}{2}}|C|^{\frac{1}{2}}\right\|_2^2 + \left\||B|^{\frac{1}{2}}|C^*|^{\frac{1}{2}}\right\|_2^2}.$$

Proof. From Lemma 2.7 and Remark 2.31, we have

$$\begin{aligned}
w_2\left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right] &= \sqrt{2}\sup_{\alpha^2+\beta^2=1}\left\|\alpha\frac{B + C^*}{2} + \beta\frac{B - C^*}{2i}\right\|_2 \\
&= \frac{1}{\sqrt{2}}\sup_{\alpha^2+\beta^2=1}\left\|B + \frac{\alpha + i\beta}{\alpha - i\beta}C^*\right\|_2 \\
&\leq \frac{1}{\sqrt{2}}\sqrt{\|B\|_2^2 + \|C\|_2^2 + \left\||B^*|^{\frac{1}{2}}|C|^{\frac{1}{2}}\right\|_2^2 + \left\||B|^{\frac{1}{2}}|C^*|^{\frac{1}{2}}\right\|_2^2}.
\end{aligned}$$

□

Remark 2.33. *Considering the operator* $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$*, it is easy to see that* Theorem 2.32 *gives* $w_2(T) \leq \sqrt{2.5} \approx 1.5811$*. The inequality of* (2) *gives* $w_2(T) \leq 1 + \frac{1}{\sqrt{2}} \approx 1.7071$*. This indicates that for such operators the bound obtained by our result is better than that of* (2).

Theorem 2.34. *Let* $B, C \in C_2$ *, then*

$$w_2\left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right] \leq \frac{1}{\sqrt{2}}\left(\|B\|_2^2 + \|C\|_2^2 + 2\left\||B^*|^{\frac{1}{2}}|C|^{\frac{1}{2}}\right\|_2^2\right)^{\frac{1}{4}}\left(\|B\|_2^2 + \|C\|_2^2 + 2\left\||B|^{\frac{1}{2}}|C^*|^{\frac{1}{2}}\right\|_2^2\right)^{\frac{1}{4}}.$$

Proof. By Lemma 2.7, Lemma 2.27 and Corollary 2.29, we have

$$\begin{aligned} w_2\left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right] &= \sqrt{2}\sup_{\alpha^2+\beta^2=1}\left\|\alpha\frac{B+C^*}{2} + \beta\frac{B-C^*}{2i}\right\|_2 \\ &= \frac{1}{\sqrt{2}}\sup_{\alpha^2+\beta^2=1}\left\|B + \frac{\alpha+i\beta}{\alpha-i\beta}C^*\right\|_2 \\ &\leq \frac{1}{\sqrt{2}}\||B^*| + |C|\|_2^{\frac{1}{2}}\||B| + |C^*|\|_2^{\frac{1}{2}} \\ &\leq \frac{1}{\sqrt{2}}\left(\|B\|_2^2 + \|C\|_2^2 + 2\left\||B^*|^{\frac{1}{2}}|C|^{\frac{1}{2}}\right\|_2^2\right)^{\frac{1}{4}}\left(\|B\|_2^2 + \|C\|_2^2 + 2\left\||B|^{\frac{1}{2}}|C^*|^{\frac{1}{2}}\right\|_2^2\right)^{\frac{1}{4}}. \end{aligned}$$

□

Remark 2.35. *Considering the operator* $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $C = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$*, it is easy to see that* Theorem 2.34 *gives* $w_2(T) \leq \sqrt[4]{2} \approx 1.1892$*. The inequality of* (2) *gives* $w_2(T) \leq \sqrt{2} \approx 1.4142$*. This indicates that for such operators the bound obtained by our result is better than that of of* (2).

References

[1] A. Abu-Omar, F. Kittaneh, *A generalization of the numerical radius*, Linear Algebra Appl. **569** (2019) 323–334.

[2] S. Aici, A. Frakis, F. Kittaneh, *Further Hilbert-Schmidt numerical radius inequalities for* 2×2 *operator matrices*, Numer. Funct. Anal. Optim. **44** (5) (2023) 382–393.

[3] S. Aici, A. Frakis, F. Kittaneh, *Hilbert-Schmidt numerical radius of a pair of operators*, Acta Appl. Math. **188**(14) (2023), DOI: 10.1007/s10440-023-00624-z.

[4] A. Aldalabih, F. Kittaneh, *Hilbert-Schmidt numerical radius inequalities for operator matrices*, Linear Algebra Appl. **581** (2019) 72–84.

[5] R. Bhatia, *Matrix analysis*, Springer, New York 1997.

[6] M .Guesba, *Further results on Hilbert-Schmidt numerical radius inequalities*, J. Math. Ext. **17** (10) (2023), 41–58.

[7] M. Hajmohamadi, R. Lashkaripour,*Some inequalities involving Hilbert-Schmidt numerical radius on* 2×2 *operator matrices*, Filomat **34** (14) (2020) 4649–4657.

[8] R. Horn, X. Zhan, *Inequalities for C-S seminorms and Lieb functions*, Linear Algebra Appl. **291** (1-3) (1999), 103–113.

[9] F. Kittaneh, *Norm inequalities for certain operator sums*, J. Funct. Anal. **143** (2) (1997), 337–348.

[10] F. Kittaneh, *A note on the arithmetic-geometric mean inequality for matrices*, Linear Algebra Appl. **171** (1992), 1–8.

[11] C. Yang, M. Xu, *Some new numerical radius and Hilbert-Schmidt numerical radius inequalities for Hilbert space operators*, J. Math. Inequal. **17** (1) (2023), 269–282.

[12] X. Zhan, *Singular values of differences of positive semidefinite matrices*, SIAM J. Matrix Anal. Appl. **22** (2000), 819–823.