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Improved geometric properties of unified Struve function

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Abstract. In this article, we prove certain improved geometric properties like univalence, starlike, convex, close-to-convex, and strongly starlike of order α for the unified Struve function in the open unit disk. We present new procedures to prove these results, which rely on differential inequalities for normalized analytic functions. Additionally, we have established inclusion relations for the unified Struve function, and results on univalency of integral operators containing unified Struve function. Finally, we obtain certain special cases to demonstrate the improvement in the existing results.

1. Introduction

It is well known that special functions, such as Bessel, Struve, Lommel, Mittag-Leffler, and Wright functions, have many beautiful geometric and monotonicity properties. Among these, the Struve function, which is associated with the Bessel functions, appears in various applications. While it is not feasible to mention all existing applications of Struve function, some include beamforming, Wave problem, describing the effect of confining interface on Brownian motion of colloidal particles at low Reynolds numbers, leakage inductance in transformer winding, *etc.*; see the papers [4, 8, 27] and the references therein. In recent years, the geometric and monotonicity properties of the Struve functions were investigated in a series of papers [6, 17].

Now we present some basic concepts of geometric function theory. Let $\mathcal{H}$ denote the class of analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\mathcal{H}_0 \subset \mathcal{H}$ contains functions f such that $f(0) = 0$, and $\mathcal{A} \subset \mathcal{H}_0$ contains functions f such that $f'(0) = 1$. Let $f, g \in \mathcal{H}$, then the function f is subordinate to g, written as $f(z) \prec g(z)$ $(z \in \mathbb{D})$, if there exist a Schwartz function w, with $w(0) = 0$ and $|w(z)| \leq 1$ such that $f(z) = g(w(z))$. If g is univalent, then (*e.g.*, [10, vol.1, p.85])

$$f(z) \prec g(z) \iff [f(0) = g(0) \wedge f(\mathbb{D}) \subset g(\mathbb{D})] .$$

Also, let $f, g \in \mathcal{H}$ defined as $f(z) = \sum_{n=1}^{\infty} r_n z^n$ and $g(z) = \sum_{n=1}^{\infty} s_n z^n$, then the convolution of f and g (denoted by

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$f * g$) is defined as follows:

$$(f * g)(z) = \sum_{n=1}^{\infty} r_n s_n z^n.$$

A function $f \in \mathcal{A}$ is said to be starlike (with respect to the origin 0), denoted by $f \in \mathcal{S}^*$, if $tw \in f(\mathbb{D})$ whenever $w \in f(\mathbb{D})$ and $t \in [0, 1]$. More generally, for a given $\alpha\,(0 \leq \alpha < 1)$, a function $f \in \mathcal{A}$ is said to be starlike function of order α in $\mathbb{D}$, denoted by $f \in \mathcal{S}^*(\alpha)$, if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \qquad (z \in \mathbb{D};\ 0 \leq \alpha < 1).$$

A function $f \in \mathcal{A}$ is said to be convex, denoted by $f \in \mathcal{K}$ if $f(\mathbb{D})$ is starlike with respect to each point in $\mathbb{D}$. More generally, for a given $0 \leq \alpha < 1$, a function $f \in \mathcal{A}$ is called a convex function of order α, denoted by $f \in \mathcal{K}(\alpha)$, if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \qquad (z \in \mathbb{D};\ 0 \leq \alpha < 1).$$

A function $f \in \mathcal{A}$ is said to be close-to-convex, denoted by $f \in \mathcal{C}$, if $\mathbb{C}\backslash f(\mathbb{D})$ can be represented as a union of non-crossing half lines. More generally, for a given $0 \leq \alpha < 1$, a function $f \in \mathcal{A}$ is called a close-to-convex function of order α, denoted by $f \in \mathcal{C}(\alpha)$, if there exists a convex function g in $\mathbb{D}$ (which is not necessarily normalized) and

$$\Re\left(\frac{e^{i\theta} f'(z)}{g'(z)}\right) > \alpha, \qquad (z \in \mathbb{D}; \theta \in \mathbb{R}; 0 \leq \alpha < 1).$$

It is well known that close-to-convex functions are univalent in $\mathbb{D}$, but not necessarily the converse. A function $f \in \mathcal{A}$ is said to be strongly starlike of order β $(0 < \beta \leq 1)$, denoted by $f \in \mathcal{S}_s^*(\beta)$, if $zf'(z)/f(z)$ is subordinate to the function $[(1+z)/(1-z)]^{\beta}$, which is equivalent to

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| \leq \frac{\pi}{2}\beta, \qquad (z \in \mathbb{D};\ 0 < \beta \leq 1).$$

Furthermore, a function $f \in \mathcal{A}$ is strongly convex of order β $(0 < \beta \leq 1)$, denoted by $f \in \mathcal{K}_s(\beta)$, if $zf'(z)$ is strongly starlike of order β. Furthermore, we have the following relationships:

$$\mathcal{S}^*(0) := \mathcal{S}^*, \quad \mathcal{C}(0) := \mathcal{C}, \quad \mathcal{K}(0) := \mathcal{K}, \quad \mathcal{S}_s^*(1) := \mathcal{S}^*, \quad \text{and} \quad \mathcal{K}_s(1) := \mathcal{K}.$$

For more details about the subclasses of $\mathcal{A}$, we refer to [10].

Recently, Peng and Zhong [21] defined a subclass of $\mathcal{A}$ by

$$\Omega = \left\{f \in \mathcal{A} :\ |zf'(z) - f(z)| < \frac{1}{2},\ z \in \mathbb{D}\right\},$$

and proved that functions in Ω are starlike in $\mathbb{D}$, and convex in $\mathbb{D}_{1/2} = \{z : |z| < 1/2\}$. Furthermore, Mahzoon and Kargar [15] proved that functions in Ω are close-to-convex, also proved a sharp result that, if $f \in \mathcal{A}$ and $\left|(f(z)/z)'\right| < 1/2$ $(z \in \mathbb{D},\ z \neq 0)$, then $f \in \Omega$. Earlier, Tuneski [28] defined another subclass of $\mathcal{A}$ by

$$\Lambda_\eta = \{f \in \mathcal{A} :\ |f''(z)| < \eta,\ 0 < \eta \leq 1\},$$

and proved that functions in Λ_η are univalent if $\eta = \alpha$ $(0 < \alpha \leq 1)$, starlike of order α if $\eta = \dfrac{2(1-\alpha)}{2-\alpha}$ $(0 \leq \alpha < 1)$, and convex of order α if $\eta = \dfrac{1-\alpha}{2-\alpha}$ $(0 \leq \alpha < 1)$. An infinite sequence $\{r_n\}_{n\geq 1}$ of complex numbers is called a subordination factor sequence if for every convex univalent function $f(z) = \sum_{n=1}^{\infty} s_n z^n$ $(z \in \mathbb{D})$, we have

$$\left\{\sum_{n=1}^{\infty} r_n s_n z^n : z \in \mathbb{D}\right\} \subseteq f(\mathbb{D}).$$

It is well known that the sequence of complex numbers $\{b_n\}_{n\geq 1}$ is a subordination factor sequence, if

$$\frac{1}{2} + \Re\left(\sum_{n=1}^{\infty} b_n z^n\right) > 0, \quad \forall\, z \in \mathbb{D}.$$

(see [29]).

A unified form of Struve and modified Struve functions can be defined by

$$w_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(p+n+\frac{b+2}{2}\right)}\left(\frac{z}{2}\right)^{2n+p+1}, \quad \left(z \in \mathbb{D}; -p-\frac{b+1}{2} \notin \mathbb{N}\right), \tag{1}$$

where all $b, c, p, z \in \mathbb{C}$. This function unifies certain functions of Bessel's family; for example, the function $w_{p,b,c}$ reduces to the Struve function H_p on taking $b = c = 1$; reduces to the modified Struve function L_p on taking $b = 1$, $c = -1$; and reduces to the Bessel function $J_{1/2}$ on taking $b = -2, p = 1$. Further, for a non-negative integer n, we have $H_{-n-\frac{1}{2},1,1}(z) = (-1)^n J_{n+\frac{1}{2}}(z)$, and $H_{-n-\frac{1}{2},1,-1}(z) = I_{n+\frac{1}{2}}(z)$, where J_m and I_m are Bessel and modified Bessel functions, respectively. These unifying properties of function $w_{p,b,c}$ motivate us to refer to this as the Unified Struve function. In addition, we can find easily that the function $w_{p,b,c}$ is a solution of the Bessel type differential equation

$$z^2 w''(z) + bzw'(z) + (cz^2 - p^2 - (b-1)p)w(z) = \frac{4}{\sqrt{\pi}\,\Gamma\left(p+\frac{1}{2}\right)}\left(\frac{z}{2}\right)^{p+1}. \tag{2}$$

For recent results on geometric properties of unified Struve functions, one can refer to [6, 17].

Clearly, $w_{p,b,c}(z) \notin \mathcal{A}$, thus it is natural to consider the following normalization of the function $w_{p,b,c}$ in $\mathbb{D}$:

$$\begin{aligned} g_{p,b,c}(z) &= 2^p\sqrt{\pi}\Gamma\left(p+\frac{b+2}{2}\right)z^{\frac{1-p}{2}} w_{p,b,c}(\sqrt{z}) = \sum_{n=0}^{\infty} \frac{(-c/4)^n}{\left(\frac{3}{2}\right)_n\left(p+\frac{b+2}{2}\right)_n} z^{n+1} \\ &= z\,{}_1F_2\left(1; \frac{3}{2}, p+\frac{b+2}{2}; -\frac{cz}{4}\right), \qquad \left(z \in \mathbb{D};\, p \in \mathbb{R};\, -p-\frac{b+2}{2} \notin \mathbb{N}\cup\{0\}\right), \end{aligned} \tag{3}$$

where $(a)_n$ is the Pochhammer symbol defined as $a_0 = 1$, $(a)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ with $n \geq 1$, and function ${}_1F_2$ is a special case of well known generalized hypergeometric function. In addition, we define $\widetilde{I}_{p,b,c}(z) = g_{p,b,c}(z)/z$. A further normalization of the function $w_{p,b,c}$ is defined as follows:

$$l_{p,b,c} = z^{\frac{1-p}{2}} 2^p w_{p,b,c}(\sqrt{z}) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{2^{2n+1}\Gamma(n+\frac{3}{2})\Gamma(n+p+\frac{b+2}{2})} z^{n+1}, \qquad \left(z \in \mathbb{D};\, p \in \mathbb{R};\, -p-\frac{b+2}{2} \notin \mathbb{N}\cup\{0\}\right).$$

In view of [16, Chapter 11], we observe that $g_{p,b,c}$ contains many well-known functions as its special case, for example

$$g_{1,-1,c}(z) = {}_0F_1\left(-; -\frac{3}{2}; -\frac{cz}{2}\right), \quad g_{-\frac{1}{2},1,1}(z) = z^{\frac{1}{4}}(\sin\sqrt{z}), \quad g_{-\frac{1}{2},1,-1}(z) = z^{\frac{1}{4}}(\sinh\sqrt{z}),$$

$$g_{\frac{1}{2},1,1}(z) = 2(1-\cos\sqrt{z}), \quad g_{\frac{1}{2},1,-1}(z) = 2(\cosh\sqrt{z}-1), \quad g_{\frac{3}{2},1,1}(z) = 4z^{\frac{1}{4}}\left[\left(1+\frac{2}{z}\right)+\frac{2}{z}\left(\sin\sqrt{z}+\frac{\cos\sqrt{z}}{\sqrt{z}}\right)\right].$$

Furthermore, taking $b \in \mathbb{R}$, $c = -1$, $p + \dfrac{b+2}{2} = 1$, and using duplication formula for Gamma function $\Gamma(2z) = \dfrac{1}{\sqrt{\pi}} 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$, we obtain

$$g_{p,b,-1}(z) = zE_{2,2}(z) = \sqrt{z}\, sinh(\sqrt{z}),$$

where $E_{\alpha,\beta}(z)$ is well known Mittag-Leffler function.

Moreover, suppose that the function $\gamma_{p,b,c}$ defined by $\gamma_{p,b,c}(z) = \widetilde{I}_{p,b,c}(z^2)$, where $p, b, c \in \mathbb{C}$. In particular, we denote the following functions

$$\gamma_{p,1,1}(z) = \mathcal{H}_p(z) = 2^p \sqrt{\pi}\Gamma\left(p + \frac{3}{2}\right) z^{-p-1} H_p(z), \quad z \in \mathbb{D}, \tag{4}$$

$$\gamma_{p,1,-1}(z) = \mathcal{L}_p(z) = 2^p \sqrt{\pi}\Gamma\left(p + \frac{3}{2}\right) z^{-p-1} L_p(z), \quad z \in \mathbb{D}. \tag{5}$$

Recently, many mathematicians have obtained the univalence criteria of several integral operators that preserve the class $\mathcal{S}$. In this context, many results are available in the literature such as, Srivastava et al. [23–26] studied univalency of certain generalized integral operators, Kanas and Srivastava [12] studied univalence criteria for analytic functions defined in $\mathbb{D}$ by using the Loewner chains method, Baricz and Frasin [2] studied some integral operators involving Bessel functions, Frasin [9] studied the convexity and strong convexity of the integral operators defined in [2], Din et al. studied some integral operators involving Struve functions [6], and Park et al. [18] studied some integral operators involving Lommel function. Motivated by the work of the above authors, we contribute to this univalence theory by studying the univalence of integral operators involving Struve functions. Using the Struve functions $g_{p_j,b,c}$, we define the following class preserving operators $\mathcal{J}^{p_1,p_2,\ldots,p_n}_{\alpha_1,\alpha_2,\ldots,\alpha_n;\beta_1,\beta_2,\ldots,\beta_n;n;\gamma}$, $\mathcal{K}^{p_1,p_2,\ldots,p_n}_{\alpha_1,\alpha_2,\ldots,\alpha_n;n}$ and $\mathcal{L}^{p_1,p_2,\ldots,p_n}_{\alpha_1,\alpha_2,\ldots,\alpha_n;\beta_1,\beta_2,\ldots,\beta_n;n;\gamma} : \mathbb{D} \to \mathbb{C}$ as follows:

$$\begin{aligned}\mathcal{J}^{p_1,p_2,\ldots,p_n}_{\alpha_1,\alpha_2,\ldots,\alpha_n;\beta_1,\beta_2,\ldots,\beta_n;n;\gamma}(z) &:= \mathcal{J}_{\alpha_1,\alpha_2,\ldots,\alpha_n;\beta_1,\beta_2,\ldots,\beta_n;n;\gamma}\left[g_{p_1,b,c}, g_{p_2,b,c}, \ldots, g_{p_n,b,c}\right](z) \\ &= \left[\gamma \int_0^z t^{\gamma-1} \prod_{j=1}^n \left(g'_{p_j,b,c}(t)\right)^{\alpha_j} \left(\frac{g_{p_j,b,c}(t)}{t}\right)^{\beta_j} dt\right]^{1/\gamma},\end{aligned} \tag{6}$$

$$\begin{aligned}\mathcal{K}^{p_1,p_2,\ldots,p_n}_{\alpha_1,\alpha_2,\ldots,\alpha_n;n}(z) &:= \mathcal{K}_{\alpha_1,\alpha_2,\ldots,\alpha_n;n}\left[g_{p_1,b,c}, g_{p_2,b,c}, \ldots, g_{p_n,b,c}\right](z) \\ &= \left[\left(1 + \sum_{j=1}^n \alpha_j\right) \int_0^z \prod_{j=1}^n \left(g_{p_j,b,c}(t) e^{\lambda g_{p_j,b,c}(t)}\right)^{\alpha_j} dt\right]^{1/\left(1+\sum_{j=1}^n \alpha_j\right)},\end{aligned} \tag{7}$$

and

$$\begin{aligned}\mathcal{L}^{p_1,p_2,\ldots,p_n}_{\alpha_1,\alpha_2,\ldots,\alpha_n;\beta_1,\beta_2,\ldots,\beta_n;n;\gamma}(z) &:= \mathcal{L}_{\alpha_1,\alpha_2,\ldots,\alpha_n;\beta_1,\beta_2,\ldots,\beta_n;n;\gamma}\left[g_{p_1,b,c}, g_{p_2,b,c}, \ldots, g_{p_n,b,c}\right] \\ &= \left[\gamma \int_0^z t^{\gamma-1} \prod_{j=1}^n \left(g'_{p_j,b,c}(t)\right)^{\alpha_j} \left(e^{g_{p_j,b,c}(t)}\right)^{\beta_j} dt\right]^{1/\gamma}.\end{aligned} \tag{8}$$

In particular, if we consider $\alpha_j = 0$ $(1 \le j \le n)$ in equation (6) , $\alpha_j = \alpha$ $(1 \le j \le n)$, $\lambda = 0$ in equation (7), and $n = 1$ with $\alpha_1 = 0$, $\beta_1 = \beta$ in equation (8), we obtained the operators studied in [6].

In this paper, we improve the following results studied by Orhan and Yagmur [17]:

Lemma 1.1. *[17, Theorem 2.1] Let $b, p, c \in \mathbb{R}$ and $k = p + (b + 2)/2$.*

(i) If $k > \dfrac{7M + 2 + \sqrt{M^2 + 12M + 4}}{24M}|c|$, where M is the solution of the equation $\cos x = x$, then the function $g_{p,b,c}(z)$ is univalent in $\mathbb{D}$.

(ii) If $k > \dfrac{9 + \sqrt{17}}{24}|c|$, then $g_{p,b,c}(z)$ is starlike in $\mathbb{D}$.

(iii) *If* $k > \frac{13}{12}|c|$, *then* $g_{p,b,c}(z)$ *is convex in* $\mathbb{D}$.

(iv) *If* $k > \frac{7}{12}|c|$, *then* $g_{p,b,c}(z)$ *is convex in* $\mathbb{D}_{1/2}$.

Lemma 1.2. *[17, Theorem 3.3] Let* $b, p, c \in \mathbb{R}$, $k = p + (b+2)/2$ *and* $\alpha \in [0, 1)$.

(i) *If* $k > \frac{9 - 7\alpha + \sqrt{\alpha^2 - 14\alpha + 17}}{24(1-\alpha)}|c|$, *then* $g_{p,b,c}(z)$ *is starlike of order* α.

(ii) *If* $k > \frac{13 - 7\alpha}{12(1-\alpha)}|c|$, *then* $g_{p,b,c}(z)$ *is convex of order* α.

Remark 1.3. *The result of Lemma 1.2(i) is corrected version of [17, Theorem 3.3].*

Lemma 1.4. *[17, Theorem 4.1] Let* $b, p, c \in \mathbb{R}$, $k = p + (b+2)/2$ *and* $\alpha \in [1/2, 1)$. *If* $k > \frac{4+\alpha}{4(1-\alpha)}|c|$, *then* $g_{p,b,c}(z)$ *is close-to-convex of order* α.

The following lemmas will be required to get the main results:

Lemma 1.5. *[17] Suppose that* $p, b \in \mathbb{R}$, $c \in \mathbb{C}$ *and* $k = p + \frac{b+2}{2}$, *then* $g_{p,b,c}$ *satisfies the following inequalities:*

(i) $$\left| g'_{p,b,c}(z) - \frac{g_{p,b,c}(z)}{z} \right| \leq \frac{2|c|}{3(4k - |c|)}, \quad \left(z \in \mathbb{D}, k > \frac{|c|}{4} \right),$$

(ii) $$\left| \frac{z g''_{p,b,c}(z)}{g'_{p,b,c}(z)} \right| \leq \frac{6|c|}{12k - 7|c|}, \quad \left(z \in \mathbb{D}, k > \frac{7|c|}{12} \right),$$

(iii) $$\left| \frac{z g'_{p,b,c}(z)}{g_{p,b,c}(z)} - 1 \right| \leq \frac{|c|(6k - |c|)}{3(4k - |c|)(3k - |c|)}, \quad \left(z \in \mathbb{D}, k > \frac{|c|}{3} \right),$$

(iv) $$\left| z g'_{p,b,c}(z) \right| \leq \frac{12k + |c|}{3(4k - |c|)}, \quad \left(z \in \mathbb{D}, k > \frac{|c|}{4} \right).$$

Lemma 1.6. *[13] If* $y \geq x \geq x_0$, *where* $x_0 = 2.089...$ *is the unique root of* $-\ln x + \frac{3}{2x} + \frac{1}{12x^2} = 0$. *Then*

$$\frac{\Gamma(x)}{\Gamma(y)} \leq \frac{x}{y}.$$

Lemma 1.7. *[7] Let* $\{r_n\}_{n\geq 0}$ *be a sequence of real numbers such that* $r_0 = 1$, $r_n - 2r_{n+1} + r_{n+2} \geq 0$ *and* $r_n - r_{n+1} \geq 0$ *for all* $n \in \mathbb{N} \cup \{0\}$, *then the inequality* $\Re\left(1 + \sum_{n=1}^{\infty} r_n z^n\right) > \frac{1}{2}, \quad z \in \mathbb{D}$ *holds.*

Lemma 1.8. *[11] Let* $M(z)$ *be convex and univalent in* $\mathbb{D}$ *with* $M(0) = 1$. *If* $F(z)$ *be analytic in* $\mathbb{D}$ *with* $F(0) = 1$ *and* $F \prec M$ *in* $\mathbb{D}$, *then*

$$\frac{n+1}{z^{n+1}} \int_0^z t^n F(t) dt \prec \frac{n+1}{z^{n+1}} \int_0^z t^n M(t) dt, \quad \forall\, n \in \mathbb{N} \cup \{0\}.$$

Lemma 1.9. *[19] Let $\eta \in \mathbb{C}$ and $\tau \in \mathbb{C}$ be such that $\Re(\eta) > 0$ and $|\tau| \le 1$ ($\tau \ne -1$). If $f \in \mathcal{A}$ satisfies the following inequality:*

$$\left|\tau|z|^{2\eta} + \left(1 - |z|^{2\eta}\right)\frac{zf''(z)}{\eta f'(z)}\right| \le 1 \quad (z \in \mathbb{D}),$$

then the integral operator $F_\eta(z) \in \mathcal{S}$, where F_η is defined by

$$F_\eta(z) = \left(\eta \int_0^z t^{\eta-1} f'(t)dt\right)^{1/\eta}. \tag{9}$$

Lemma 1.10. *[20] Let $\mu \in \mathbb{C}$ be such that $\Re(\mu) > 0$. If $f \in \mathcal{A}$ satisfies the following inequality:*

$$\left(\frac{1 - |z|^{2\Re(\mu)}}{\Re(\mu)}\right)\left|\frac{zf''(z)}{f'(z)}\right| \le 1 \quad (z \in \mathbb{D}),$$

then for all $\eta \in \mathbb{C}$ such that $\Re(\eta) \ge \Re(\mu)$, the integral operator $F_\eta(z) \in \mathcal{S}$, where F_η is given by (9).

2. Improved Geometric Properties

For convenience throughout the sequel, we use the following notations for $\alpha \in [0, 1)$ and $\beta \in (0, 1]$:

$$\begin{aligned}
k &= p + \frac{b+2}{2} \quad (p, b \in \mathbb{R}), \qquad c_0 = \frac{|c|}{4} - 1 \quad (c \in \mathbb{C}),\\
c_1 &= \frac{1}{24}\left[7|c| - 12 + \sqrt{144 + 49|c|^2 + 24|c|}\right], \quad \widetilde{c_1} = \frac{1}{24}\left[7|c| - 12 - \sqrt{144 + 49|c|^2 + 24|c|}\right],\\
c_2 &= \frac{1}{24(1-\alpha)}\left[(7-5\alpha)|c| - 12(1-\alpha) + \sqrt{\left(12(1-\alpha) - (7-5\alpha)|c|\right)^2 + 12(1-\alpha)(2-\alpha)|c|(8+|c|)}\right],\\
\widetilde{c_2} &= \frac{1}{24(1-\alpha)}\left[(7-5\alpha)|c| - 12(1-\alpha) - \sqrt{\left(12(1-\alpha) - (7-5\alpha)|c|\right)^2 + 12(1-\alpha)(2-\alpha)|c|(8+|c|)}\right],\\
c_3 &= \frac{1}{24(1-\alpha)}\left[(11-7\alpha)|c| - 12(1-\alpha) + \sqrt{\left(12(1-\alpha) - (11-7\alpha)|c|\right)^2 + 24(1-\alpha)(2-\alpha)|c|(8+|c|)}\right],\\
\widetilde{c_3} &= \frac{1}{24(1-\alpha)}\left[(11-7\alpha)|c| - 12(1-\alpha) - \sqrt{\left(12(1-\alpha) - (11-7\alpha)|c|\right)^2 + 24(1-\alpha)(2-\alpha)|c|(8+|c|)}\right],\\
c_4 &= \frac{1}{24\beta}\left[(4+3\beta)|c| - 12\beta + \sqrt{(12\beta - (4+3\beta)|c|)^2 + 24\beta|c|(8+|c|)}\right],\\
\widetilde{c_4} &= \frac{1}{24\beta}\left[(4+3\beta)|c| - 12\beta - \sqrt{(12\beta - (4+3\beta)|c|)^2 + 24\beta|c|(8+|c|)}\right].
\end{aligned}$$

Our first result of the paper is provided below.

Theorem 2.1. *Suppose that $p, b \in \mathbb{R}$, $c \in \mathbb{C}$, and c_1 is unique non-negative root of $I_{|c|}(k) = 0$ for all $c \in \mathbb{C}$, where*

$$I_{|c|}(k) = 12k^2 + (12 - 7|c|)k - 4|c|.$$

If $k > c_1$, then $g_{p,b,c}(z) \in \Omega$. In particular $g_{p,b,c}(z)$ is starlike, univalent and close-to-convex in $\mathbb{D}$. Also, $g_{p,b,c}(z)$ is convex in $\mathbb{D}_{1/2}$.

Proof. We require the following inequalities, which can be demonstrated by using mathematical induction:

$$\left(\frac{3}{2}\right)_{n+1} \geq \frac{3}{2}(n+1), \quad (k)_{n+1} \geq k(k+1)^n \quad (n \in \mathbb{N} \cup \{0\}).$$

We have

$$\left|\left(\frac{g_{p,b,c}(z)}{z}\right)'\right| = \left|\sum_{n=0}^{\infty} \frac{(n+1)(-c/4)^{n+1}}{(3/2)_{n+1}(k)_{n+1}} z^n\right| < \frac{|c|}{6k}\sum_{n=0}^{\infty}\left(\frac{|c|}{4(k+1)}\right)^n = \frac{2|c|(k+1)}{3k[4(k+1)-|c|]}, \quad k > max\{0, c_0\},$$

which is less than $1/2$ if $I_{|c|}(k) > 0$.

The equation $I_{|c|}(k) = 0$ has two roots, $c_1 \geq 0$ and $\widetilde{c_1} \leq 0$ for any complex number c. Moreover, if $k > (7|c| - 12)/24$, then $I_{|c|}(k)$ strictly increases. We have $c_1 = max\{0, c_0, c_1\} > (7|c| - 12)/24$. Therefore, $I_{|c|}(k) > 0$ for $k > c_1$. Thus, we can conclude that $g_{p,b,c}(z) \in \Omega$, which completes the proof. □

Remark 2.2. *(i) For $c \in \mathbb{C}$, the inequality $\frac{7M + 2 + \sqrt{M^2 + 12M + 4}}{24M}|c| \geq c_1$ holds, where M is the solution of the equation $\cos x = x$. This indicates that when $c \in \mathbb{C}$, Theorem 2.1 improves Lemma 1.1(i). In particular, for $c = 100$, Lemma 1.1(i) shows that if $k > 61.09$, then $g_{p,b,c}(z)$ is univalent. However, Theorem 2.1 shows that if $k > 57.9$, then $g_{p,b,c}(z)$ is univalent.*

(ii) If $c \in \mathbb{C}$ and $|c| \leq 10.685$, then $\frac{9 + \sqrt{17}}{24}|c| \geq c_1$. This shows that Theorem 2.1 improves Lemma 1.1(ii) for $|c| \leq 10.685$. In particular, for $c = 2$, Lemma 1.1(ii) shows that if $k > 1.094$, then $g_{p,b,c}(z)$ is starlike. However, Theorem 2.1 shows that, if $k > 0.904$, then $g_{p,b,c}(z)$ is starlike.

(iii) Theorem 2.1 improves Lemma 1.1(iv) in $\mathbb{D}_{1/2}$, as for $c \in \mathbb{C}$, we have $\frac{7}{12}|c| \geq c_1$.

Now taking $b = c = 1$ and $b = -c = 1$ in Theorem 2.1, we obtain the following results, respectively:

Corollary 2.3. *If $p \in \mathbb{R}$ such that $p > \frac{-5 + \sqrt{217}}{24} - \frac{3}{2} \approx -1.0945$, then $z\mathcal{H}_p(\sqrt{z}) \in \Omega$. In particular $z\mathcal{H}_p(\sqrt{z})$ is starlike, univalent and close-to-convex in $\mathbb{D}$, and convex in $\mathbb{D}_{1/2}$.*

Corollary 2.4. *If $p \in \mathbb{R}$ such that $p > \frac{-5 + \sqrt{217}}{24} - \frac{3}{2} \approx -1.0945$, then $z\mathcal{L}_p(\sqrt{z}) \in \Omega$. In particular $z\mathcal{L}_p(\sqrt{z})$ is starlike, univalent and close-to-convex in $\mathbb{D}$, and convex in $\mathbb{D}_{1/2}$.*

Remark 2.5. *The Corollary 2.3 improves the result presented in [17, Corollary 2.2 (iii)], which states that if $p > \frac{-27 + \sqrt{17}}{24} \approx -0.9532$, then $z\mathcal{H}_p\sqrt{z}$ is starlike in the unit disk $\mathbb{D}$. Additionally, the findings regarding the univalence of $z\mathcal{H}_p\sqrt{z}$ in $\mathbb{D}$ and its convexity in $\mathbb{D}_{1/2}$ are further improved in comparison to the results [17, Corollary 2.2 (ii) and (vi)]. Furthermore, results established in the Corollary 2.4 for the univalence, starlikeness of $z\mathcal{L}_p(\sqrt{z})$ in $\mathbb{D}$, and convexity in $\mathbb{D}_{1/2}$ are improved than the results obtained by Orhan and Yagmur in [17, Corollary 2.3 (ii), (iii) and (vi)], respectively.*

Theorem 2.6. *Suppose that $p, b \in \mathbb{R}$, $c \in \mathbb{C}\backslash\{0\}$, $\beta \in (0, 1]$ and $\alpha \in [0, 1)$.*

(i) Let c_2 is unique positive root of $J_{|c|,\alpha}(k) = 0$, where

$$J_{|c|,\alpha}(k) = 48(1-\alpha)k^2 + \Big(48(1-\alpha) - (28 - 20\alpha)|c|\Big)k - (2-\alpha)|c|(8 + |c|).$$

If $k \geq c_2$, then $g_{p,b,c}(z)$ is starlike of order α in $\mathbb{D}$.

(ii) Let c_3 is unique positive root of $M_{|c|,\alpha}(k) = 0$, where

$$M_{|c|,\alpha}(k) = 24(1-\alpha)k^2 + \Big(24(1-\alpha) - (22-14\alpha)|c|\Big)k - (2-\alpha)|c|(8+|c|).$$

If $k \geq c_3$, then $g_{p,b,c}(z)$ is convex of order α in $\mathbb{D}$.

(iii) Let c_4 is unique positive root of $N_{|c|,\beta}(k) = 0$, where

$$N_{|c|,\beta}(k) = 24\beta k^2 + (24\beta - 6|c|\beta - 8|c|)k - |c|(8+|c|).$$

If $k \geq c_4$, then $g_{p,b,c}(z)$ is close-to-convex of order β in $\mathbb{D}$.

Proof. By applying the triangle inequality and the following set of inequalities:

$$\left(\frac{3}{2}\right)_{n+1} > \frac{(n+1)(n+2)}{2}, \quad (k)_{n+1} \geq k(k+1)^n \quad (n \in \mathbb{N}),$$

we obtain

$$\begin{aligned} |g''_{p,b,c}(z)| &= \left| -\frac{c}{3k} + \sum_{n=1}^{\infty} \frac{(-c/4)^{n+1}(n+1)(n+2)z^n}{(3/2)_{n+1}(k)_{n+1}} \right| \\ &< \frac{|c|}{3k} + \frac{|c|}{2k}\sum_{n=1}^{\infty}\left(\frac{|c|}{4(k+1)}\right)^n = \frac{8(k+1)|c| + |c|^2}{6k[4(k+1)-|c|]} = \eta \ (say), \quad k > max\{0, c_0\}. \end{aligned}$$

(i) If $\eta \leq \dfrac{2(1-\alpha)}{2-\alpha}$, then $|g''_{p,b,c}(z)| < 2(1-\alpha)/(2-\alpha)$, *i.e.*,

$$J_{|c|,\alpha}(k) = 48(1-\alpha)k^2 + \Big(48(1-\alpha) - (28-20\alpha)|c|\Big)k - (2-\alpha)|c|(8+|c|) \geq 0. \tag{10}$$

The equation $J_{|c|,\alpha}(k) = 0$ has roots $c_2 > 0$ and $\widetilde{c_2} < 0$. Besides, $J_{|c|,\alpha}(k)$ is strictly increasing for $k > ((7-5\alpha)|c| - 12(1-\alpha))/(24(1-\alpha))$ and $c_2 = \max\{0, c_0, c_2\} > ((7-5\alpha)|c| - 12(1-\alpha))/(24(1-\alpha))$. Therefore, $J_{|c|,\alpha}(k) \geq 0$ for $k \geq c_2$. Hence, $g_{p,b,c}(z)$ is starlike of order α for $k \geq c_2$.

(ii) If $\eta \leq \dfrac{1-\alpha}{2-\alpha}$, then $|g''_{p,b,c}(z)| < (1-\alpha)/(2-\alpha)$, *i.e.*,

$$M_{|c|,\alpha}(k) = 24(1-\alpha)k^2 + \Big(24(1-\alpha) - (22-14\alpha)|c|\Big)k - (2-\alpha)|c|(8+|c|) \geq 0. \tag{11}$$

The equation $M_{|c|,\alpha}(k) = 0$ has roots $c_3 > 0$ and $\widetilde{c_3} < 0$. Moreover, $M_{|c|,\alpha}(k)$ is strictly increasing if $k > ((11-7\alpha)|c| - 12(1-\alpha))/(24(1-\alpha))$, and $c_3 = \max\{0, c_0, c_3\} > ((11-7\alpha)|c| - 12(1-\alpha))/(24(1-\alpha))$. Hence, $M_{|c|,\alpha}(k) \geq 0$ for $k \geq c_3$. Therefore, $g_{p,b,c}(z)$ is convex of order α for $k \geq c_3$.

(iii) Suppose $\eta \leq \beta$, then $|g''_{p,b,c}(z)| < \beta$, *i.e.*

$$N_{|c|,\beta}(k) = 24\beta k^2 + (24\beta - 6|c|\beta - 8|c|)k - |c|(8+|c|) \geq 0. \tag{12}$$

The equation $N_{|c|,\alpha}(k) = 0$ has two roots, $c_4 > 0$ and $\widetilde{c_4} < 0$. Furthermore, $N_{|c|,\alpha}(k)$ is strictly increasing if $k > ((4+3\beta)|c| - 12\beta)/12\beta$, and $c_4 = \max\{0, c_0, c_4\} > ((4+3\beta)|c| - 12\beta)/12\beta$. Therefore, $N_{|c|,\alpha}(k) \geq 0$ for $k \geq c_4$. This means that $g_{p,b,c}(z)$ is close-to-convex of order β for $k \geq c_4$. □

Taking $\alpha = 0$ in Theorem 2.6(i), we obtain

Corollary 2.7. *If $p, b \in \mathbb{R}$ and $c \in \mathbb{C}\backslash\{0\}$ such that $k \geq d_1$, where*

$$d_1 = \frac{1}{24}\Big[7|c| - 12 + \sqrt{73|c|^2 + 24|c| + 144}\Big], \tag{13}$$

then $g_{p,b,c}(z)$ is starlike in $\mathbb{D}$.

Remark 2.8. *(i) When $c \in \mathbb{C}\backslash\{0\}$ and $|c| \le 3.463$, the inequality $\frac{9+\sqrt{17}}{24}|c| > d_1$ holds true. This means that Corollary 2.7 improves Lemma 1.1(ii), specifically when $|c| \le 3.463$, $c \in \mathbb{C}\backslash\{0\}$. For instance, if $c = 1.5$, Lemma 1.1(ii) states that $g_{p,b,c}(z)$ is starlike if $k > 0.82$. However, Corollary 2.7 indicates that $g_{p,b,c}(z)$ is starlike if $k > 0.71$.*

(ii) Moreover, we observe that for any non-zero $c \in \mathbb{C}$, the result yielded by Theorem 2.1 is significantly stronger than the result obtained by Corollary 2.7 in terms of the starlikeness of $g_{p,b,c}(z)$.

Taking $\alpha = 1/2$ in Theorem 2.6(*i*), we obtain

Corollary 2.9. *If $p, b \in \mathbb{R}$ and $c \in \mathbb{C}\backslash\{0\}$ such that $k \ge d_2$, where*

$$d_2 = \frac{1}{12}\Big[4.5|c| - 6 + \sqrt{29.25|c|^2 + 18|c| + 36}\Big], \tag{14}$$

then $g_{p,b,c}(z)$ is starlike of order $1/2$ in $\mathbb{D}$.

Remark 2.10. *If $\alpha = 1/2$ and $c \in \mathbb{R}$, Lemma 1.2(i) proves $g_{p,b,c}(z)$ is starlike of order $1/2$ for $k > \frac{11+\sqrt{41}}{24}|c|$. For $c \in \mathbb{C}\backslash\{0\}$ and $|c| \le 2.795$, if $\frac{11+\sqrt{41}}{24}|c| > d_2$, then Corollary 2.9 is a better version of Lemma 1.2(i) with $\alpha = 1/2$.*

Taking $\alpha = 0$ in Theorem 2.6(*ii*), we obtain

Corollary 2.11. *If $p, b \in \mathbb{R}$ and $c \in \mathbb{C}\backslash\{0\}$ such that $k \ge d_3$, where*

$$d_3 = \frac{1}{24}\Big[11|c| - 12 + \sqrt{169|c|^2 + 120|c| + 144}\Big], \tag{15}$$

then $g_{p,b,c}(z)$ is convex in $\mathbb{D}$. In particular $l_{p,b,c}(z)$ is convex in $\mathbb{D}$.

Remark 2.12. *We observe that the inequality $\frac{13}{12}|c| > d_3$ holds true for $c \in \mathbb{C}\backslash\{0\}$. This implies that when $c \in \mathbb{C}\backslash\{0\}$, Corollary 2.11 improves on Lemma 1.1(iii). In particular, if $c = 100$, Lemma 1.1(iii) shows that $g_{p,b,c}(z)$ is convex if $k > 108.33$. However, Corollary 2.11 demonstrates that $g_{p,b,c}(z)$ is convex if $k > 99.69$.*

Taking $\alpha = 1/2$ in Theorem 2.6(*ii*), we obtain

Corollary 2.13. *If $p, b \in \mathbb{R}$ and $c \in \mathbb{C}\backslash\{0\}$ such that $k \ge d_4$, where*

$$d_4 = \frac{1}{12}\Big[7.5|c| - 6 + \sqrt{74.25|c|^2 + 54|c| + 36}\Big], \tag{16}$$

then $g_{p,b,c}(z)$ is convex of order $1/2$ in $\mathbb{D}$.

Remark 2.14. *Taking $\alpha = 1/2$ and $c \in \mathbb{R}$ in Lemma 1.2(ii), we obtain that $g_{p,b,c}(z)$ is convex of order $1/2$ for $k > \frac{19}{12}|c|$. Further, if $c \in \mathbb{C}\backslash\{0\}$, the inequality $\frac{19}{12}|c| > d_4$ holds true. Hence, Corollary 2.13 is an improvement of Lemma 1.2(ii) for $\alpha = 1/2$ and $c \in \mathbb{C}\backslash\{0\}$.*

Taking $\beta = 1/2$ in Theorem 2.6(*iii*), we obtain

Corollary 2.15. *If $p, b \in \mathbb{R}$ and $c \in \mathbb{C}\backslash\{0\}$ such that $k \ge d_3$, where d_3 is given by (15). Then $g_{p,b,c}(z)$ is close-to-convex of order $1/2$ in $\mathbb{D}$.*

Remark 2.16. *(i) For $\gamma = 1/2$ in Lemma 1.4, we note that if $c \in \mathbb{R}$ and $k > \frac{9}{4}|c|$, then $g_{p,b,c}(z)$ is close-to-convex of order $1/2$. Further, if $c \in \mathbb{C}\backslash\{0\}$, the inequality $\frac{9}{4}|c| > d_3$ holds true. This shows that when $c \in \mathbb{C}\backslash\{0\}$, Corollary 2.15 is an improvement of Lemma 1.4 for $\gamma = 1/2$.*

(ii) Theorem 2.6(iii) states that for $\beta \in (0, 1]$, the function $g_{p,b,c}(z)$ is close-to-convex of order β in $\mathbb{D}$. On the other hand, Lemma 1.4 shows that for β in the interval $[1/2, 1)$, $g_{p,b,c}(z)$ is close-to-convex of order β in $\mathbb{D}$. This implies that the result obtained by Theorem 2.6(iii) holds for a larger domain of values of β.

3. Inclusion relation, Strong Starlikeness and Strong Convexity

Theorem 3.1. *Let $p, q, b \in \mathbb{R}$, $c \in \mathbb{C}$ such that $p + \frac{b+2}{2} \geq q + \frac{b+2}{2} \geq max\left\{x_0 - 1, d_3\right\}$, where $x_0 \simeq 2.089$ is the unique root of $-ln(x) + \frac{3}{2x} + \frac{1}{12x^2} = 0$, and d_3 is given by* (15). *Then the following inclusion holds*

$$l_{p,b,c}(\mathbb{D}) \subset l_{q,b,c}(\mathbb{D}),$$

for all $z \in \mathbb{D}$.

Proof. To prove the inclusion relation, firstly we define a sequence $\{r_n\}_{n\geq 1}$ of real numbers by

$$r_n = \frac{\Gamma(n + q + \frac{b+2}{2})}{\Gamma(n + p + \frac{b+2}{2})} \quad (n \in \mathbb{N}),$$

and prove that $\{r_n\}_{n\geq 1}$ is a subordination factor sequence. For this, it is enough to show that the inequality $\frac{1}{2} + \Re\sum_{n=1}^{\infty} r_n z^n > 0$ holds for all $z \in \mathbb{D}$. Now, suppose $r_0 = 1$. Then by using Lemma 1.6, we have

$$r_0 - r_1 = 1 - \frac{\Gamma(1 + q + \frac{b+2}{2})}{\Gamma(1 + p + \frac{b+2}{2})} \geq 1 - \frac{1 + q + \frac{b+2}{2}}{1 + p + \frac{b+2}{2}} \geq 0,$$

for $p + \frac{b+2}{2} \geq q + \frac{b+2}{2} \geq x_0 - 1$. Also, for $n \geq 1$,

$$r_n - r_{n+1} = \frac{\Gamma(n + q + \frac{b+2}{2})}{\Gamma(n + p + \frac{b+2}{2})} - \frac{\Gamma(n + 1 + q + \frac{b+2}{2})}{\Gamma(n + 1 + p + \frac{b+2}{2})} = \frac{\Gamma(n + q + \frac{b+2}{2})}{\Gamma(n + p + \frac{b+2}{2})}\left(1 - \frac{n + q + \frac{b+2}{2}}{n + p + \frac{b+2}{2}}\right) \geq 0.$$

Further, we prove the inequality $r_n - 2r_{n+1} + r_{n+2} \geq 0$ for all $n \geq 0$. By using Lemma 1.6, we have

$$\begin{aligned}
r_0 - 2r_1 + r_2 &= 1 - 2\frac{\Gamma(1 + q + \frac{b+2}{2})}{\Gamma(1 + p + \frac{b+2}{2})} + \frac{\Gamma(2 + q + \frac{b+2}{2})}{\Gamma(2 + p + \frac{b+2}{2})} \\
&\geq 2\sqrt{\frac{(1 + q + \frac{b+2}{2})\Gamma(1 + q + \frac{b+2}{2})}{(1 + p + \frac{b+2}{2})\Gamma(1 + p + \frac{b+2}{2})}} - 2\frac{\Gamma(1 + q + \frac{b+2}{2})}{\Gamma(1 + p + \frac{b+2}{2})} \\
&= 2\sqrt{\frac{\Gamma(1 + q + \frac{b+2}{2})}{\Gamma(1 + p + \frac{b+2}{2})}}\left(\sqrt{\frac{1 + q + \frac{b+2}{2}}{1 + p + \frac{b+2}{2}}} - \sqrt{\frac{\Gamma(1 + q + \frac{b+2}{2})}{\Gamma(1 + p + \frac{b+2}{2})}}\right) \geq 0,
\end{aligned}$$

for $p + \frac{b+2}{2} \geq q + \frac{b+2}{2} \geq x_0 - 1$, and for $n \geq 1$,

$$\begin{aligned}
r_n - 2r_{n+1} + r_{n+2} &\geq \frac{\Gamma(n + q + \frac{b+2}{2})}{\Gamma(n + p + \frac{b+2}{2})}\left(1 + \frac{(n + q + \frac{b+2}{2})(n + q + \frac{b+2}{2} + 1)}{(n + p + \frac{b+2}{2})(n + p + \frac{b+2}{2} + 1)} - 2\frac{n + q + \frac{b+2}{2}}{n + p + \frac{b+2}{2}}\right) \\
&\geq 2\frac{\Gamma(n + q + \frac{b+2}{2})}{\Gamma(n + p + \frac{b+2}{2})}\left(\sqrt{\frac{(n + q + \frac{b+2}{2})(n + q + \frac{b+2}{2} + 1)}{(n + p + \frac{b+2}{2})(n + p + \frac{b+2}{2} + 1)}} - \frac{n + q + \frac{b+2}{2}}{n + p + \frac{b+2}{2}}\right) \geq 0,
\end{aligned}$$

for $p + \frac{b+2}{2} \geq q + \frac{b+2}{2} \geq x_0 - 1$. Thus, by using Lemma 1.7, we conclude that $\{r_n\}_{n\geq 1}$ is a subordination factor sequence. This means that

$$l_{p,b,c}(z) = l_{q,b,c}(z) * \phi(z) \prec l_{q,b,c}(z),$$

where $\phi(z) = \sum_{n=1}^{\infty} r_n z^n,\ z \in \mathbb{D}$. Additionally, Corollary 2.11 shows that the function $l_{q,b,c}(z)$ is convex and univalent in $\mathbb{D}$ for $q + \frac{b+2}{2} \geq d_3$, and $l_{q,b,c}(0) = l_{p,b,c}(0)$. Therefore, the subordination $l_{p,b,c}(z) \prec l_{q,b,c}(z)$ is equivalent to $l_{p,b,c}(\mathbb{D}) \subset l_{q,b,c}(\mathbb{D})$. This proves the result. □

Theorem 3.2. *Let $p, b \in \mathbb{R},\ c \in \mathbb{C}\backslash\{0\}$ such that $k \geq \frac{4}{3}|c|$, then $g_{p,b,c}(z)$ is strongly convex of order α, where*

$$\alpha = \frac{2}{\pi} arcsin\left(\zeta\sqrt{1-\frac{\zeta^2}{4}} + \frac{\zeta}{2}\sqrt{1-\zeta^2}\right), \quad and \quad \zeta = \frac{2|c|}{3k-2|c|}.$$

Proof. By using the following inequalities:

$$(n+1) \leq 4^n, \quad (k)_n \geq k^n, \quad \left(\frac{3}{2}\right)_n > \frac{(n+1)}{2} \quad (\forall\ n \in \mathbb{N}),$$

we obtain

$$|(zg'_{p,b,c}(z))' - 1| = \left|\sum_{n=1}^{\infty} \frac{\left(-\frac{c}{4}\right)^n (n+1)^2 z^n}{\left(\frac{3}{2}\right)_n (k)_n}\right| < \sum_{n=1}^{\infty} \left(\frac{2|c|}{3k}\right)^n = \frac{2|c|}{3k-2|c|} = \zeta\ (say). \tag{17}$$

Since, $k \geq \frac{4}{3}|c|$, therefore $0 < \zeta \leq 1$. So by using equation (17), we get $(zg'_{p,b,c}(z))' \prec 1 + \zeta z$, which implies that

$$\left|arg\,(zg'_{p,b,c}(z))'\right| < arcsin(\zeta),\ z \in \mathbb{D}. \tag{18}$$

We can use Lemma 1.8 by setting $n = 0$, where $F(z) = \left(zg'_{p,b,c}(z)\right)'$ and $M(z) = 1+\zeta z$. This gives $g'_{p,b,c}(z) \prec 1+\frac{\zeta}{2}z$, which is equivalent to

$$\left|arg\left(g'_{p,b,c}(z)\right)\right| < arcsin\left(\frac{\zeta}{2}\right),\ z \in \mathbb{D}. \tag{19}$$

Now using (18) and (19), we obtain

$$\begin{aligned}\left|arg\left(\frac{\left(zg'_{p,b,c}(z)\right)'}{g'_{p,b,c}(z)}\right)\right| &\leq \left|arg\left(zg'_{p,b,c}(z)\right)'\right| + \left|arg\left(g'_{p,b,c}(z)\right)\right| \\ &< arcsin(\zeta) + arcsin\left(\frac{\zeta}{2}\right) = arcsin\left(\zeta\sqrt{1-\frac{\zeta^2}{4}} + \frac{\zeta}{2}\sqrt{1-\zeta^2}\right),\end{aligned}$$

which implies that $g_{p,b,c}(z)$ is strongly convex of order α. □

In the same way, we can obtain the following result:

Theorem 3.3. *Let $p, b \in \mathbb{R},\ c \in \mathbb{C}\backslash\{0\}$ such that $k \geq \frac{3}{4}|c|$, then $g_{p,b,c}(z)$ is strongly starlike of order α, where*

$$\alpha = \frac{2}{\pi} arcsin\left(\zeta\sqrt{1-\frac{\zeta^2}{4}} + \frac{\zeta}{2}\sqrt{1-\zeta^2}\right), \quad and \quad \zeta = \frac{2|c|}{4k-|c|}.$$

4. Univalency of Integral operators

In this section, we determine the univalency of integral operators defined by (6), (7), and (8). Our first result is given below.

Theorem 4.1. *Let $p_j \in \mathbb{R}$ $(1 \le j \le n)$, $b \in \mathbb{R}$, $c \in \mathbb{C}$ and $k_j > 7|c|/12$ with $k_j = p_j + (b+2)/2$ for all $1 \le j \le n$. Also let γ, τ, α_j and β_j be in $\mathbb{C}$ such that $\Re(\gamma) > 0$, $|\tau| \le 1$ $(\tau \ne -1)$, $\alpha_j, \beta_j \ne 0$. Suppose the following inequality holds*

$$|\tau| + \frac{1}{|\gamma|}\left(\frac{6|c|}{12k-7|c|}\sum_{j=1}^{n}|\alpha_j| + \frac{|c|(6k-|c|)}{3(4k-|c|)(3k-|c|)}\sum_{j=1}^{n}|\beta_j|\right) \le 1, \tag{20}$$

where $k = min\{k_1, k_2, \ldots, k_n\}$. Then the integral operator $\mathcal{J}^{p_1,p_2,\ldots,p_n}_{\alpha_1,\alpha_2,\ldots,\alpha_n;\beta_1,\beta_2,\ldots,\beta_n;n;\gamma}$ is in the class $\mathcal{S}$.

Proof. Let us define a function ψ by

$$\psi(z) = \mathcal{J}^{p_1,p_2,\ldots,p_n}_{\alpha_1,\alpha_2,\ldots,\alpha_n;\beta_1,\beta_2,\ldots,\beta_n;n;1}(z) = \int_0^z \prod_{j=1}^{n}\left(g'_{p_j,b,c}(t)\right)^{\alpha_j}\left(\frac{g_{p_j,b,c}(t)}{t}\right)^{\beta_j} dt. \tag{21}$$

Since $g_{p_j,b,c} \in \mathcal{A}$ for all $1 \le j \le n$, therefore $\psi(z) \in \mathcal{A}$. Now differentiating equation (21), we have

$$\frac{z\psi''(z)}{\psi'(z)} = \sum_{j=1}^{n}\alpha_j\frac{zg''_{p_j,b,c}(z)}{g'_{p_j,b,c}(z)} + \sum_{j=1}^{n}\beta_j\left(\frac{zg'_{p_j,b,c}(z)}{g_{p_j,b,c}(z)} - 1\right).$$

Using the inequalities (*ii*) and (*iii*) of Lemma 1.5, we have

$$\left|\frac{z\psi''(z)}{\psi'(z)}\right| \le \sum_{j=1}^{n}\left(|\alpha_j|\left|\frac{zg''_{p_j,b,c}(z)}{g'_{p_j,b,c}(z)}\right| + |\beta_j|\left|\frac{zg'_{p_j,b,c}(z)}{g_{p_j,b,c}(z)} - 1\right|\right) \le \sum_{j=1}^{n}\left(|\alpha_j|\frac{6|c|}{12k_j-7|c|} + |\beta_j|\frac{|c|(6k_j-|c|)}{3(4k_j-|c|)(3k_j-|c|)}\right).$$

Now, define a function $\aleph : \left(\frac{(2+\sqrt{2})|c|}{12}, \infty\right) \to \mathbb{R}$ by $\aleph(k) = \frac{|c|(6k-|c|)}{3(4k-|c|)(3k-|c|)}$. Then it is easy to show that $\aleph$ is strictly decreasing function on $\left((2+\sqrt{2})|c|/12, \infty\right)$. Therefore

$$\frac{|c|(6k_i-|c|)}{3(4k_i-|c|)(3k_i-|c|)} \le \frac{|c|(6k-|c|)}{3(4k-|c|)(3k-|c|)},$$

which implies

$$\left|\frac{z\psi''(z)}{\psi'(z)}\right| \le \sum_{j=1}^{n}\left(|\alpha_j|\frac{6|c|}{12k-7|c|} + |\beta_j|\frac{|c|(6k-|c|)}{3(4k-|c|)(3k-|c|)}\right). \tag{22}$$

Further, using Lemma 1.9, inequality (22) and the triangle inequality, we have

$$\left|\tau|z|^{2\gamma} + (1-|z|^{2\gamma})\frac{z\psi''(z)}{\gamma\psi'(z)}\right| \le |\tau| + \frac{1}{|\gamma|}\left(\frac{6|c|}{12k-7|c|}\sum_{j=1}^{n}|\alpha_j| + \frac{|c|(6k-|c|)}{3(4k-|c|)(3k-|c|)}\sum_{j=1}^{n}|\beta_j|\right) \le 1,$$

if the inequality (20) holds. Hence, the integral operator $\mathcal{J}^{p_1,p_2,\ldots,p_n}_{\alpha_1,\alpha_2,\ldots,\alpha_n;\beta_1,\beta_2,\ldots,\beta_n;n;\gamma} \in \mathcal{S}$. □

Theorem 4.2. *Let $p_j \in \mathbb{R}$ $(1 \le j \le n)$, $b \in \mathbb{R}$, $c \in \mathbb{C}$ and $k_j > 7|c|/12$ with $k_j = p_j + (b+2)/2$ for all $1 \le j \le n$. Also let γ, α_j and β_j be in $\mathbb{C}$ such that $\Re(\gamma) > 0$ and $\alpha_j, \beta_j \ne 0$. Suppose the following inequality holds*

$$\Re(\gamma) \ge \left(\frac{6|c|}{12k-7|c|}\sum_{j=1}^{n}|\alpha_j| + \frac{|c|(6k-|c|)}{3(4k-|c|)(3k-|c|)}\sum_{j=1}^{n}|\beta_j|\right), \tag{23}$$

where $k = min\{k_1, k_2, \ldots, k_n\}$. Then the integral operator $\mathcal{J}^{p_1,p_2,\ldots,p_n}_{\alpha_1,\alpha_2,\ldots,\alpha_n;\beta_1,\beta_2,\ldots,\beta_n;n;\gamma}$ is in the class $\mathcal{S}$..

Proof. Let us define the function ψ as in (21). Then using the inequality (22), we have

$$\frac{1-|z|^{2\Re(\gamma)}}{\Re(\gamma)}\left|\frac{z\psi''(z)}{\psi'(z)}\right| \leq \frac{1}{\Re(\gamma)}\sum_{j=1}^{n}\left(|\alpha_j|\frac{6|c|}{12k-7|c|}+|\beta_j|\frac{|c|(6k-|c|)}{3(4k-|c|)(3k-|c|)}\right)\leq 1,$$

if the inequality (23) holds. Thus in view of Lemma 1.10, the integral operator $\mathcal{J}^{p_1,p_2,\ldots,p_n}_{\alpha_1,\alpha_2,\ldots,\alpha_n;\beta_1,\beta_2,\ldots,\beta_n;n;\gamma} \in \mathcal{S}$. □

By substituting $n = b = c = 1$, $p = -1/2$ in Theorem 4.1, and $n = b = c = 1$, $p = 1/2$ in Theorem 4.2, we obtain the following results respectively:

Corollary 4.3. *Let $\gamma, \tau, \alpha, \beta \in \mathbb{C}$ such that $\Re(\gamma) > 0$, $|\tau| \leq 1$ $(\tau \neq -1)$, $\alpha, \beta \neq 0$. Suppose the inequality $|\tau| + \frac{1}{|\gamma|}\left(\frac{6}{5}|\alpha| + \frac{5}{18}|\beta|\right) \leq 1$ holds, then the integral*

$$\left[4^{-\alpha}\gamma\int_0^z t^{\gamma-\frac{3}{4}(\alpha+\beta)-1}\left(2\sqrt{t}cos\sqrt{t}+sin\sqrt{t}\right)^{\alpha}\left(sin\sqrt{t}\right)^{\beta}dt\right]^{\frac{1}{\gamma}} \in \mathcal{S}.$$

Corollary 4.4. *Let $\gamma, \alpha, \beta \in \mathbb{C}$ such that $\Re(\gamma) > 0$, $\alpha, \beta \neq 0$. Suppose the inequality $\Re(\gamma) \geq \left(\frac{6}{17}|\alpha| + \frac{11}{105}|\beta|\right)$ holds, then the integral*

$$\left[2^{\beta}\gamma\int_0^z t^{\gamma-\beta-\frac{\alpha}{2}-1}\left(sin\sqrt{t}\right)^{\alpha}\left(1-cos\sqrt{t}\right)^{\beta}\right]^{\frac{1}{\gamma}} \in \mathcal{S}.$$

Theorem 4.5. *Let $p_j \in \mathbb{R}$ $(1 \leq j \leq n)$, $b, \lambda \in \mathbb{R}$, $c \in \mathbb{C}$ and $k_j > |c|/3$ with $k_j = p_j + (b+2)/2$ for all $1 \leq j \leq n$. Also let τ and α_j be in $\mathbb{C}$ such that $|\tau| \leq 1$ $(\tau \neq -1)$ and $\Re\left(1+\sum_{j=1}^{n}\alpha_j\right) > 0$. Suppose the following inequality holds*

$$|\tau| + \left(\frac{36|\lambda|k^2 + 3|c|k(2-3|\lambda|) - (1+|\lambda|)|c|^2}{3(12k^2 - 7|c|k + |c|^2)}\right)\frac{\sum_{j=1}^{n}|\alpha_j|}{\left|\left(1+\sum_{j=1}^{n}\alpha_j\right)\right|} \leq 1, \tag{24}$$

where $k = min\{k_1, k_2, \ldots, k_n\}$. Then the integral operator $\mathcal{K}^{p_1,p_2,\ldots,p_n}_{\alpha_1,\alpha_2,\ldots,\alpha_n;n}$ is in the class $\mathcal{S}$.

Proof. Consider the function ξ defined by

$$\xi(z) = \int_0^z \prod_{j=1}^{n}\left(\frac{g_{p_j,b,c}(t)e^{\lambda g_{p_j,b,c}(t)}}{t}\right)^{\alpha_j} dt. \tag{25}$$

Since $g_{p_j,b,c} \in \mathcal{A}$ for all $1 \leq j \leq n$, therefore $\xi(z) \in \mathcal{A}$. Now on differentiating equation (25), we have

$$\frac{z\xi''(z)}{\xi'(z)} = \sum_{j=1}^{n}\alpha_j\left(\frac{zg'_{p_j,b,c}(z)}{g_{p_j,b,c}(z)} + \lambda zg'_{p_j,b,c}(z) - 1\right).$$

Using the inequalities (*iii*) and (*iv*) of Lemma 1.5, we have

$$\left|\frac{z\xi''(z)}{\xi'(z)}\right| \leq \sum_{j=1}^{n}|\alpha_j|\left(\left|\frac{zg'_{p_j,b,c}(z)}{g_{p_j,b,c}(z)} - 1\right| + |\lambda|\left|zg'_{p_j,b,c}(z)\right|\right) \leq \sum_{j=1}^{n}|\alpha_j|\left(\frac{|c|(6k_j-|c|)}{3(4k_j-|c|)(3k_j-|c|)} + |\lambda|\frac{12k_j+|c|}{12k_j-3|c|}\right)$$
$$\leq \sum_{j=1}^{n}|\alpha_j|\left(\frac{|c|(6k-|c|)}{3(4k-|c|)(3k-|c|)} + |\lambda|\frac{12k+|c|}{12k-3|c|}\right) = \left(\frac{36|\lambda|k^2 + 3|c|k(2-3|\lambda|) - (1+|\lambda|)|c|^2}{3(12k^2-7|c|k+|c|^2)}\right)\sum_{j=1}^{n}|\alpha_j|. \tag{26}$$

Therefore, we get

$$\left|\tau|z|^{2\left(1+\sum_{j=1}^{n}\alpha_j\right)}+\frac{\left(1-|z|^{2\left(1+\sum_{j=1}^{n}\alpha_j\right)}\right)z\xi''(z)}{\left(1+\sum_{j=1}^{n}\alpha_j\right)\xi'(z)}\right|\leq|\tau|+\left|\frac{z\xi''(z)}{\left(1+\sum_{j=1}^{n}\alpha_j\right)\xi'(z)}\right|$$
$$\leq|\tau|+\frac{\sum_{j=1}^{n}|\alpha_j|}{\left|\left(1+\sum_{j=1}^{n}\alpha_j\right)\right|}\left(\frac{36|\lambda|k^2+3|c|k(2-3|\lambda|)-(1+|\lambda|)|c|^2}{3(12k^2-7|c|k+|c|^2)}\right)\leq 1,$$

if the inequality (24) holds. Thus in view of Lemma 1.9 with $\eta=1+\sum_{j=1}^{n}\alpha_j$, the integral operator $\mathcal{K}_{\alpha_1,\alpha_2,\ldots,\alpha_n;n}^{p_1,p_2,\ldots,p_n}\in\mathcal{S}$. □

Theorem 4.6. *Let $p_j\in\mathbb{R}$ $(1\leq j\leq n)$, $b,\lambda\in\mathbb{R}$, $c\in\mathbb{C}$ and $k_j>|c|/3$ with $k_j=p_j+(b+2)/2$ for all $1\leq j\leq n$. Also let α_j be in $\mathbb{C}$ such that $\Re(\alpha_j)\geq 0$. Suppose the following inequality holds*

$$\left(\frac{36|\lambda|k^2+3|c|k(2-3|\lambda|)-(1+|\lambda|)|c|^2}{3(12k^2-7|c|k+|c|^2)}\right)\sum_{j=1}^{n}|\alpha_j|\leq 1,\tag{27}$$

where $k=min\{k_1,k_2,\ldots,k_n\}$. Then the integral operator $\mathcal{K}_{\alpha_1,\alpha_2,\ldots,\alpha_n;n}^{p_1,p_2,\ldots,p_n}$ is in the class $\mathcal{S}$.

Proof. Let us define the function ξ as in (25). Then using the inequality (26), we have

$$\left(1-|z|^2\right)\left|\frac{z\xi''(z)}{\xi'(z)}\right|\leq\left(\frac{36|\lambda|k^2+3|c|k(2-3|\lambda|)-(1+|\lambda|)|c|^2}{3(12k^2-7|c|k+|c|^2)}\right)\sum_{j=1}^{n}|\alpha_j|\leq 1,$$

if the inequality (27) holds. Thus in view of Lemma 1.10 with $\mu=1$ and $\eta=1+\sum_{j=1}^{n}\alpha_j$, the integral operator $\mathcal{K}_{\alpha_1,\alpha_2,\ldots,\alpha_n;n}^{p_1,p_2,\ldots,p_n}\in\mathcal{S}$. □

By substituting $n=b=\lambda=1$, $p=-1/2$, $c=-1$ in Theorem 4.5, and $n=b=\lambda=1$, $p=1/2$, $c=-1$ in Theorem 4.6, we obtain the following results respectively:

Corollary 4.7. *Let $\tau,\alpha\in\mathbb{C}$ such that $\Re(1+\alpha)>0$ and $|\tau|\leq 1$ $(\tau\neq -1)$. Suppose the inequality $|\tau|+\dfrac{31}{18}\dfrac{|\alpha|}{|1+\alpha|}\leq 1$ holds, then the integral*

$$\left[(1+\alpha)\int_0^z\left(t^{\frac{1}{4}}\left(sinh\sqrt{t}\right)e^{t^{\frac{1}{4}}sinh\sqrt{t}}\right)^{\alpha}dt\right]^{\frac{1}{1+\alpha}}\in\mathcal{S}.$$

Corollary 4.8. *Let $\alpha\in\mathbb{C}$ such that $\Re(\alpha)>0$. Suppose the inequality $\dfrac{136}{105}|\alpha|\leq 1$ holds, then the integral*

$$\left[(1+\alpha)\int_0^z 2^{\alpha}\left(cosh\sqrt{t}-1\right)^{\alpha}e^{2\alpha\left(cosh\sqrt{t}-1\right)}dt\right]^{\frac{1}{1+\alpha}}\in\mathcal{S}.$$

Theorem 4.9. *Let $p_j\in\mathbb{R}$ $(1\leq j\leq n)$, $b\in\mathbb{R}$, $c\in\mathbb{C}$ and $k_j>7|c|/12$ with $k_j=p_j+(b+2)/2$ for all $1\leq j\leq n$. Also let γ,τ,α_j and β_j be in $\mathbb{C}$ such that $\Re(\gamma)>0$, $|\tau|\leq 1$ $(\tau\neq -1)$, $\alpha_j,\beta_j\neq 0$. Suppose the following inequality holds*

$$|\tau|+\frac{1}{|\gamma|}\sum_{j=1}^{n}\left(\frac{6|c|}{12k-7|c|}|\alpha_j|+\frac{12k+|c|}{12k-3|c|}|\beta_j|\right)\leq 1,\tag{28}$$

where $k=min\{k_1,k_2,\ldots,k_n\}$. Then the integral operator $\mathcal{L}_{\alpha_1,\alpha_2,\ldots,\alpha_n;\beta_1,\beta_2,\ldots,\beta_n;n;\gamma}^{p_1,p_2,\ldots,p_n}$ is in the class $\mathcal{S}$.

Proof. Consider the function Θ defined by

$$\Theta(z) = \int_0^z \prod_{j=1}^{n} \left(g'_{p_j,b,c}(t)\right)^{\alpha_j} \left(e^{g_{p_j,b,c}(t)}\right)^{\beta_j} dt. \tag{29}$$

Since $g_{p_j,b,c} \in \mathcal{A}$ for all $1 \leq j \leq n$, therefore $\Theta(z) \in \mathcal{A}$. Now on differentiating equation (29), we have

$$\frac{z\Theta''(z)}{\Theta'(z)} = \sum_{j=1}^{n} \alpha_j \frac{zg''_{p_j,b,c}(z)}{g'_{p_j,b,c}(z)} + \sum_{j=1}^{n} \beta_j z g'_{p_j,b,c}(z).$$

Using the inequalities (*ii*) and (*iv*) of Lemma 1.5, we have

$$\begin{aligned}\left|\frac{z\Theta''(z)}{\Theta'(z)}\right| &\leq \sum_{j=1}^{n} \left(|\alpha_j| \left|\frac{zg''_{p_j,b,c}(z)}{g'_{p_j,b,c}(z)}\right| + |\beta_j| \left|zg'_{p_j,b,c}(z)\right|\right) \leq \sum_{j=1}^{n} \left(|\alpha_j| \frac{6|c|}{12k_j - 7|c|} + |\beta_j| \frac{12k_j + |c|}{12k_j - 3|c|}\right) \\ &\leq \sum_{j=1}^{n} \left(|\alpha_j| \frac{6|c|}{12k - 7|c|} + |\beta_j| \frac{12k + |c|}{12k - 3|c|}\right).\end{aligned} \tag{30}$$

Therefore, we get

$$\left|\tau|z|^{2\gamma} + (1 - |z|^{2\gamma}) \frac{z\Theta''(z)}{\gamma\Theta'(z)}\right| \leq |\tau| + \frac{1}{|\gamma|} \sum_{j=1}^{n} \left(|\alpha_j| \frac{6|c|}{12k - 7|c|} + |\beta_j| \frac{12k + |c|}{12k - 3|c|}\right) \leq 1,$$

if the inequality (28) holds. Thus in view of Lemma 1.9, the integral operator $\mathcal{L}^{p_1,p_2,\ldots,p_n}_{\alpha_1,\alpha_2,\ldots,\alpha_n;\beta_1,\beta_2,\ldots,\beta_n;n;\gamma} \in \mathcal{S}$. □

Theorem 4.10. *Let $p_j \in \mathbb{R}$ $(1 \leq j \leq n)$, $b \in \mathbb{R}$, $c \in \mathbb{C}$ and $k_j > 7|c|/12$ with $k_j = p_j + (b+2)/2$ for all $1 \leq j \leq n$. Also let γ, α_j and β_j be in $\mathbb{C}$ such that $\Re(\gamma) > 0$ and $\alpha_j, \beta_j \neq 0$. Suppose the following inequality holds*

$$\Re(\gamma) \geq \sum_{j=1}^{n} \left(\frac{6|c|}{12k - 7|c|} |\alpha_j| + \frac{12k + |c|}{12k - 3|c|} |\beta_j|\right), \tag{31}$$

where $p = min\{p_1, p_2, \ldots, p_n\}$. Then the integral operator $\mathcal{L}^{p_1,p_2,\ldots,p_n}_{\alpha_1,\alpha_2,\ldots,\alpha_n;\beta_1,\beta_2,\ldots,\beta_n;n;\gamma}$ is in the class $\mathcal{S}$.

Proof. Let us define the function Θ as in (29). Then using the inequality (30), we have

$$\frac{1 - |z|^{2\Re(\gamma)}}{\Re(\gamma)} \left|\frac{z\Theta''(z)}{\Theta'(z)}\right| \leq \frac{1}{\Re(\gamma)} \sum_{j=1}^{n} \left(|\alpha_j| \frac{6|c|}{12k - 7|c|} + |\beta_j| \frac{12k + |c|}{12k - 3|c|}\right) \leq 1,$$

if the inequality (31) holds. Thus in view of Lemma 1.10, the integral operator $\mathcal{L}^{p_1,p_2,\ldots,p_n}_{\alpha_1,\alpha_2,\ldots,\alpha_n;\beta_1,\beta_2,\ldots,\beta_n;n;\gamma} \in \mathcal{S}$. □

Finally, substituting $n = b = c = 1$, $p = 3/2$ in Theorem 4.9, we obtain the following result:

Corollary 4.11. *Let $\gamma, \tau, \alpha, \beta \in \mathbb{C}$ such that $\Re(\gamma > 0)$, $|\tau| \leq 1$, $(\tau \neq -1)$, $\alpha, \beta \neq 0$. Suppose the inequality $|\tau| + \frac{1}{|\gamma|}\left(\frac{6}{29}|\alpha| + \frac{37}{33}|\beta|\right) \leq 1$ holds, then the integral*

$$\left[\gamma \int_0^z t^{\gamma - \frac{9}{4}\alpha - 1} \left(\sqrt{t}\left(t - 10 sin\sqrt{t} - 6\right) + 2(2t-5) cos\sqrt{t}\right)^{\alpha} e^{4\beta t^{\frac{1}{4}}\left(\left(1+\frac{2}{t}\right) + \frac{2}{t}\left(sin\sqrt{t} + \frac{cos\sqrt{t}}{\sqrt{t}}\right)\right)}\right]^{\frac{1}{\gamma}} \in \mathcal{S}.$$

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