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Coefficient inequality for a novel bi-univalent function subclass associated with Krawtchouk polynomials

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Abstract. In this research, we present and study a new subclass of bi-univalent functions related to the Krawtchouk polynomials that meet subordination requirements seen in the open unit disk, a symmetric domain. We derive estimates for the Fekete-Szegö inequality $\left|a_3 - \gamma a_2^2\right|$ and the Taylor-Maclaurin coeffcients $|a_2|$, $|a_3|$ for this new subclass.

1. Introduction

Let $\mathcal{A}$ stand for the class of functions, where each member of the class has the form

$$\Upsilon(\xi) = \xi + \sum_{k=2}^{\infty} a_k \xi^k, \quad (\xi \in \mathfrak{D}), \tag{1}$$

which are analytic in $\mathfrak{D} = \{\xi \in \mathbb{C} : |\xi| < 1\}$.

The symbol $\mathcal{S}$ designates a subclass of $\mathcal{A}$ consisting of members that are univalent in $\mathfrak{D}$. For any univalent function $\Upsilon \in \mathcal{A}$, the Koebe one-quarter theorem [7] ensures the existence of a disk in the image of $\mathfrak{D}$ with a radius of $1/4$. Consequently, an inverse function Υ^{-1} is satisfied for every univalent function Υ

$$\Upsilon^{-1}(\Upsilon(\xi)) = \xi, \ (\xi \in \mathfrak{D}) \text{ and } \Upsilon\left(\Upsilon^{-1}(\omega)\right) = \omega, \ (|\omega| < r_0(\Upsilon), \ r_0(\Upsilon) \geq \frac{1}{4}).$$

In $\mathfrak{D}$, we say that $\Upsilon \in \mathcal{A}$ is bi-univalent if Υ and Υ^{-1} are univalent. Λ represents the class of bi-univalent functions defined on the unit disk $\mathfrak{D}$. Due to the fact that $\Upsilon \in \Lambda$ has the summary of the Maclaurin series by (1), a calculation reveals that $\varrho = \Upsilon^{-1}$ has the expansion

$$\varrho(\omega) = \Upsilon^{-1}(\omega) = \omega - a_2\omega^2 + \left(2{a_2}^2 - a_3\right)\omega^3 + \dots. \tag{2}$$

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We understand that the class Λ is not empty. For instance, the functions

$$\Upsilon_1(\xi) = \frac{\xi}{\xi - 1}, \ \Upsilon_2(\xi) = \frac{1}{2}\log\frac{1+\xi}{1-\xi}, \ \Upsilon_3(\xi) = -\log(1-\xi)$$

with their respective inverses

$$\Upsilon_1^{-1}(\omega) = \frac{\omega}{1+\omega}, \ \Upsilon_2^{-1}(\omega) = \frac{e^{2\omega}-1}{e^{2\omega}+1}, \ \Upsilon_3^{-1}(\omega) = \frac{e^{\omega}-1}{e^{\omega}}$$

belong to Λ. Also, the Koebe function does not belong to Λ.

The analysis of the subclasses of the analytic and bi-univalent functions was actually revived in a pioneering work by Srivastava et al. [20],[21]. In their subsequent research, Srivastava et al. [22] obtained sharp inequalities for a class of novel convex functions defined by Gregory polynomials. They further advanced the field by solving coefficient bounds, the Fekete-Szegö problem, and the second Hankel determinant for symmetric function classes of analytic and bi-univalent functions involving Euler polynomials [23]. Additionally, Srivastava et al. [24] introduced new general subclasses of m-fold symmetric bi-univalent functions using the m-fold Ruscheweyh derivative operator, providing estimates on initial coefficients and Fekete-Szegö inequalities for these classes. Srivastava et al. [25] also derived the estimates on the initial Taylor-Maclaurin coefficients for functions in analytic and bi-concave function classes connected with the combination of the binomial series and the confluent hypergeometric function.

Assume that the analytic functions in $\mathfrak{D}$ are Υ and ϱ. We say that Υ is subordinate to ϱ and denoted by

$$\Upsilon(\xi) \prec \varrho(\xi) \quad (\xi \in \mathfrak{D}),$$

if there exists a Schwarz function ω, which is analytic in $\mathfrak{D}$ with $\omega(0) = 0$ and $|\omega(\xi)| < 1$ $(\xi \in \mathfrak{D})$ such that

$$\Upsilon(\xi) = \varrho(\omega(\xi)) \quad (\xi \in \mathfrak{D}).$$

If ϱ is a univalent function in $\mathfrak{D}$, then

$$\Upsilon(\xi) \prec \varrho(\xi) \Leftrightarrow \Upsilon(0) = \varrho(0) \quad \text{and} \quad \Upsilon(\mathfrak{D}) \subset \varrho(\mathfrak{D}).$$

In [16], Loewner's approach is used to find the Fekete-Szegö inequality for the coefficients of $\Upsilon \in S$:

$$\left|a_3 - \gamma a_2^2\right| \leq 1 + 2\exp\left(\frac{-2\gamma}{1-\gamma}\right) \text{ for } 0 \leq \gamma < 1.$$

As $\gamma \to 1^-$, the inequality $\left|a_3 - a_2^2\right| \leq 1$ is obtained. The coefficient functional

$$F_\gamma(\Upsilon) = a_3 - \gamma a_2^2$$

for normalized analytic functions Υ in the open unit disk $\mathfrak{D}$ is crucial in geometric function theory. The Fekete-Szegö problem involves maximizing the absolute value of this functional.

The Fekete-Szegö inequalities [9], introduced in 1933, have intrigued scholars studying univalent functions [8], [13], [17], [29], and similarly, bi-univalent functions have yielded such inequalities. Recent studies continue to explore this topic, with notable contributions from [1], [4], [31]. For instance, Ali et al. [2] explored the second Hankel determinant and Fekete-Szegö functional using the q-Salagean derivative operator; Srivastava et al. [26] estimated Fekete-Szegö inequalities and Hankel determinants for certain analytic functions involving the Hohlov operator; Srivastava et al. [27] obtained coefficient estimates for subclasses related to Gegenbauer polynomials; and Srivastava et al. [28] studied a new subclass of normalized analytic functions using quantum calculus and solved Fekete-Szegö type problems.

For $q \in (0,1)$. The q-derivative (or q-difference) operator, introduced by Jackson [11], [12], is defined as

$$\partial_q \Upsilon(\xi) = \begin{cases} \frac{\Upsilon(\xi)-\Upsilon(q\xi)}{(1-q)\xi}, & \text{if} \quad \xi \neq 0 \\ \Upsilon'(0), & \text{if} \quad \xi = 0 \end{cases}. \tag{3}$$

We note that

$$\lim_{q\to 1} \partial_q \Upsilon(\xi) = \Upsilon'(\xi)$$

if Υ is differentiable at ξ. From (3), we decude that for function $\Upsilon \in \mathcal{A}$

$$\partial_q \Upsilon(\xi) = 1 + \sum_{k=2}^{\infty} [k]_q \, a_k \xi^{k-1}, \tag{4}$$

where $[k]_q$ is given by

$$[k]_q = \frac{1-q^k}{1-q}, \quad [0]_q = 0 \tag{5}$$

and the q-factorial is given by

$$[k]_q! = \begin{cases} 1, & k = 0 \\ \prod_{r=1}^{k} [r]_q, & k \in \mathbb{N} \end{cases}. \tag{6}$$

As $q \to 1^-$, we obtain $[k]_q \to k$. If we choose the function $l(\xi) = \xi^k$, while $q \to 1^-$, we can thus have

$$\partial_q l(\xi) = \partial_q \xi^k = [k]_q \, \xi^{k-1} = l'(\xi),$$

where the ordinary derivative is denoted by l'.

Classical orthogonal polynomials of a discrete variable are crucial in applied and computational mathematics, probability theory, statistics, physics, and technology. The study of Krawtchouk polynomials and their generalizations, as part of orthogonal polynomials of a discrete variable, has seen significant success across these fields [19].

The most common types of orthogonal polynomials found in applications are the classical varieties (Hermite, Laguerre, and Jacobi polynomials). We include [1]-[6], [10], [14], [30] for a recent relationship between geometric function theory and the classical orthogonal polynomials.

The Krawtchouk polynomials [15] are defined of for any prime power ρ, positive integer and $k = 0, 1, 2, ...n$, which are described by

$$\mathcal{K}_k(x; n, \rho) = \mathcal{K}_k(x) = \sum_{j=0}^{k} (-1)^j (\rho - 1)^{k-j} \binom{x}{j} \binom{n-x}{k-j}, \tag{7}$$

In [15], the generating series of Krawtchouk polynomials is given as below:

$$(1 + (\rho - 1)\xi)^{n-x} (1-\xi)^x = \sum_{k=0}^{\infty} \mathcal{K}_k(x; n, \rho) \, \xi^k. \tag{8}$$

Here ξ is a formal variable.

Krawtchouk polynomials [15] are discrete orthogonal polynomials associated with the binomial distribution, introduced by Mykhailo Kravchuk.

From (8), the following expression is obtained for $\rho = 2$ and $n \geq 2$:

$$\Phi(x, n; \xi) = (1+\xi)^{n-x} (1-\xi)^x = \sum_{k=0}^{\infty} \mathcal{K}_k(x; n) \, \xi^k, \ \xi \in \mathfrak{D}, \ x = 1, 2, ..., \tag{9}$$

The Taylor-Maclaurin series expansion for the function $\Phi(x, n; \xi)$ is as follows:

$$\Phi(x, n; \xi) = 1 + \mathcal{K}_1(x; n) \, \xi^2 + \mathcal{K}_2(x; n) \, \xi^3 + \mathcal{K}_3(x; n) \, \xi^4 + \cdots + \mathcal{K}_{k-1}(x; n) \, \xi^n + \ldots, \tag{10}$$

where

$$\mathcal{K}_0(x;n) = 1,\ \mathcal{K}_1(x;n) = -2x+n,\ \mathcal{K}_2(x;n) = 2x^2-2nx+\binom{n}{2}, \tag{11}$$
$$\mathcal{K}_3(x;n) = -\frac{4}{3}x^3-2nx^2-\left(n^2-n+\frac{2}{3}\right)x+\binom{n}{3}.$$

In this work, using Krawtchouk polynomials, we define and investigate a new subclass of bi univalent functions.

2. Coefficient Bounds of the Class $\mathcal{M}_\Lambda(x,n,q;\Phi)$

We present new subclasses of bi-univalent functions that are subordinate to the Krawtchouk polynomials.

Definition 2.1. *We assert that Υ belonging to Λ is said to be in the class $\mathcal{M}_\Lambda(x,n,q;\Phi)$, for $x=0,1,2,\ldots$ and $n\geq 2$, if the following subordinations hold:*

$$\partial_q\Upsilon(\xi) \prec \Phi(x,n;\xi) \tag{12}$$

and

$$\partial_q\varrho(\omega) \prec \Phi(x,n;\omega), \tag{13}$$

$\xi,\omega\in\mathfrak{D}$, Φ is given by (11), and $\varrho=\Upsilon^{-1}$ is given by (2).

Lemma 2.2. *([18, p.172])Suppose $\omega(\xi)=\sum_{k=1}^{\infty}\omega_k\xi^k$, $\xi\in\mathfrak{D}$, is an analytic function in $\mathfrak{D}$ such that $|\omega(\xi)|<1$, $\xi\in\mathfrak{D}$. Then,*

$$|\omega_1|\leq 1,\ |w_k|\leq 1-|\omega_1|^2,\ k=1,2,3,\ldots.$$

Theorem 2.3. *Let $\Upsilon\in\Lambda$ given by (1) be in the class $\mathcal{M}_\Lambda(x,n,q;\Phi)$. Then,*

$$|a_2|\leq\frac{|\mathcal{K}_1(x;n)|\sqrt{\mathcal{K}_1(x;n)}}{\sqrt{\left|[3]_q\mathcal{K}_1^2(x;n)-[2]_q^2\mathcal{K}_2(x;n)\right|}}, \tag{14}$$

and

$$|a_3|\leq\frac{|\mathcal{K}_1(x;n)|}{[3]_q}+\frac{\mathcal{K}_1^2(x;n)}{[2]_q^2}. \tag{15}$$

Proof. Let $\Upsilon\in\mathcal{M}_\Lambda(x,n,q;\Phi)$ and $\varrho=\Upsilon^{-1}$. We have the following from the definition in formulas (12) and (13)

$$\partial_q\Upsilon(\xi)=\Phi(x,n;\upsilon(\xi)) \tag{16}$$

and

$$\partial_q\varrho(\omega)=\Phi(x,n;\nu(\omega)), \tag{17}$$

where the analytical υ and ν functions have the form

$$\upsilon(\xi)=c_1\xi+c_2\xi^2+\ldots, \tag{18}$$

$$\nu(\omega)=d_1\omega+d_2\omega^2+\ldots, \tag{19}$$

and $\upsilon(0) = 0 = \nu(0)$, $|\upsilon(\xi)| < 1$, $|\nu(\omega)| < 1$, $\xi, \omega \in \mathfrak{D}$. It follows that, from Lemma 2.2, that

$$\left|c_j\right| \leq 1, \ \left|d_j\right| \leq 1, \text{where } j \in \mathbb{N}. \tag{20}$$

If we replace (18) and (19) in (16) and (17), respectively, we obtain

$$\partial_q \Upsilon(\xi) = 1 + \mathcal{K}_1(x;n)\,\upsilon(\xi) + \mathcal{K}_2(x;n)\,\upsilon^2(\xi) + \ldots, \tag{21}$$

and

$$\partial_q \varrho(\omega) = 1 + \mathcal{K}_1(x;n)\,\nu(\omega) + \mathcal{K}_2(x;n)\,\nu^2(\omega) + \ldots. \tag{22}$$

In view of (1) and (2), from (21) and (22), we obtain

$$\begin{aligned} & 1 + [2]_q a_2 \xi + [3]_q a_3 \xi^2 + \ldots \\ = \ & 1 + \mathcal{K}_1(x;n)\, c_1 \xi + \left[\mathcal{K}_1(x;n)\, c_2 + \mathcal{K}_2(x;n)\, c_1^2\right] \xi^2 + \ldots \end{aligned}$$

and

$$\begin{aligned} & 1 - [2]_q a_2 \omega + [3]_q \left(2a_2^2 - a_3\right) \omega^2 + \ldots \\ = \ & 1 + \mathcal{K}_1(x;n)\, d_1 \omega + \left[\mathcal{K}_1(x;n)\, d_2 + \mathcal{K}_2(x;n)\, d_1^2\right] \omega^2 + \ldots. \end{aligned}$$

It gives rise to the following relationships:

$$[2]_q a_2 = \mathcal{K}_1(x;n)\, c_1, \tag{23}$$

$$[3]_q a_3 = \mathcal{K}_1(x;n)\, c_2 + \mathcal{K}_2(x;n)\, c_1^2, \tag{24}$$

and

$$-[2]_q a_2 = \mathcal{K}_1(x;n)\, d_1, \tag{25}$$

$$[3]_q \left(2a_2^2 - a_3\right) = \mathcal{K}_1(x;n)\, d_2 + \mathcal{K}_2(x;n)\, d_1^2. \tag{26}$$

From (23) and (25), it follows that

$$c_1 = -d_1, \tag{27}$$

and

$$2\,[2]_q^2 a_2^2 = \mathcal{K}_1^2(x;n)\left(c_1^2 + d_1^2\right). \tag{28}$$

By adding (24) to (26), yields

$$2\,[3]_q a_2^2 = \mathcal{K}_1(x;n)(c_2 + d_2) + \mathcal{K}_2(x;n)\left(c_1^2 + d_1^2\right). \tag{29}$$

We get that by replacing the value of $\left(c_1^2 + d_1^2\right)$ from (28) on the right side of (29),

$$a_2^2 = \frac{\mathcal{K}_1(x;n)(c_2 + d_2)\mathcal{K}_1^2(x;n)}{2\,[3]_q \mathcal{K}_1^2(x;n) - 2\,[2]_q^2 \mathcal{K}_2(x;n)}. \tag{30}$$

From (20) for c_2 and d_2 we get (14). Furthermore, by deducting (26) from (24), we arrive at

$$2\,[3]_q \left(a_3 - a_2^2\right) = \mathcal{K}_1(x;n)(c_2 - d_2) + \mathcal{K}_2(x;n)\left(c_1^2 - d_1^2\right). \tag{31}$$

Then, in view of (28) and (31), we obtain

$$a_3 = \frac{\mathcal{K}_1^2(x;n)}{2\,[2]_q^2}\left(c_1^2 + d_1^2\right) + \frac{\mathcal{K}_1(x;n)}{2\,[3]_q}(c_2 - d_2).$$

Once again applying (20), for the coefficients c_1, d_1, c_2, d_2, we deduce (15).
The proof is thus finished. □

Based on Theorem 2.3, we can derive the subsequent outcome for $q = 1$.

Corollary 2.4. *Let $\Upsilon \in \Lambda$ given by (1) belong to the class $\mathcal{M}_\Lambda(x, n, 1; \Phi)$. Then,*

$$|a_2| \leq \frac{|\mathcal{K}_1(x;n)| \sqrt{\mathcal{K}_1(x;n)}}{\sqrt{\left|3\mathcal{K}_1^2(x;n) - 4\mathcal{K}_2(x;n)\right|}}$$

and

$$|a_3| \leq \frac{|\mathcal{K}_1(x;n)|}{3} + \frac{\mathcal{K}_1^2(x;n)}{4}.$$

For functions in the class $\mathcal{M}_\Lambda(x, n, q; \Phi)$, we prove the following Fekete–Szegö inequality using the values of a_2^2 and a_3.

Theorem 2.5. *Let $\Upsilon \in \Lambda$ given by (1) be in the class $\mathcal{M}_\Lambda(x, n, q; \Phi)$. Then,*

$$\left|a_3 - \gamma a_2^2\right| \leq \begin{cases} \frac{|\mathcal{K}_1(x;n)|}{[3]_q}, & \text{if } \left|h(\gamma)\right| \leq \frac{1}{2[3]_q}, \\ 2|\mathcal{K}_1(x;n)|\left|h(\gamma)\right|, & \text{if } \left|h(\gamma)\right| \geq \frac{1}{2[3]_q}, \end{cases}$$

where

$$h(\gamma) = \frac{(1-\gamma)\mathcal{K}_1^2(x;n)}{2[3]_q \mathcal{K}_1^2(x;n) - 2[2]_q^2 \mathcal{K}_2(x;n)}.$$

Proof. From (30) and (31),

$$\begin{aligned} a_3 - \gamma a_2^2 &= \mathcal{K}_1(x;n)\left[\frac{c_2}{2[3]_q} - \frac{d_2}{2[3]_q} + \frac{(1-\gamma)\mathcal{K}_1^2(x;n)\, c_2}{2[3]_q \mathcal{K}_1^2(x;n) - 2[2]_q^2 \mathcal{K}_2(x;n)} \right. \\ &\quad \left. + \frac{(1-\gamma)\mathcal{K}_1^2(x;n)\, d_2}{2[3]_q \mathcal{K}_1^2(x;n) - 2[2]_q^2 \mathcal{K}_2(x;n)}\right] \\ &= \mathcal{K}_1(x;n)\left[\left(h(\gamma) + \frac{1}{2[3]_q}\right)c_2 + \left(h(\mu) - \frac{1}{2[3]_q}\right)d_2\right], \end{aligned}$$

where

$$h(\gamma) = \frac{(1-\gamma)\mathcal{K}_1^2(x;n)}{2[3]_q \mathcal{K}_1^2(x;n) - 2[2]_q^2 \mathcal{K}_2(x;n)}.$$

Therefore, given (20), we conclude that the required inequality holds.

The proof is thus finished. □

Based on Theorem 2.5, we can derive the subsequent outcome for $q = 1$.

Corollary 2.6. *Let $\Upsilon \in \Lambda$ given by (1) belong to the class $\mathcal{M}_\Lambda(x, n, 1; \Phi)$. Then,*

$$\left|a_3 - \gamma a_2^2\right| \leq \begin{cases} \frac{|\mathcal{K}_1(x;n)|}{3}, & \text{if } \left|h(\gamma)\right| \leq \frac{1}{6}, \\ 2|\mathcal{K}_1(x;n)|\left|h(\gamma)\right|, & \text{if } \left|h(\gamma)\right| \geq \frac{1}{6}, \end{cases}$$

where

$$h(\gamma) = \frac{(1-\gamma)\mathcal{K}_1^2(x;n)}{2\left(3\mathcal{K}_1^2(x;n) - 4\mathcal{K}_2(x;n)\right)}.$$

3. Coefficient Bounds of the Class $\mathcal{S}^*_\Lambda(x,n,q;\Phi)$

Definition 3.1. *We assert that* Υ *belonging to* Λ *is said to be in the class* $\mathcal{S}^*_\Lambda(x,n,q;\Phi)$ *, for* $x=0,1,2,...$ *and* $n\geq 2$*, if the following subordinations hold:*

$$\frac{\xi\partial_q\Upsilon(\xi)}{\Upsilon(\xi)} \prec \Phi(x,n;\xi) \tag{32}$$

and

$$\frac{\omega\partial_q\varrho(\omega)}{\varrho(\omega)} \prec \Phi(x,n;\omega) \tag{33}$$

$\xi,\omega\in\mathfrak{D}$*,* Φ *is given by (11), and* $\varrho=\Upsilon^{-1}$ *is given by (2).*

Theorem 3.2. *Let* $\Upsilon\in\Lambda$ *given by (1) be in the class* $\mathcal{S}^*_\Lambda(x,n,q;\Phi)$*. Then,*

$$|a_2|\leq\frac{|\mathcal{K}_1(x;n)|\sqrt{\mathcal{K}_1(x;n)}}{q\sqrt{\left|\mathcal{K}_1^2(x;n)-\mathcal{K}_2(x;n)\right|}} \tag{34}$$

and

$$|a_3|\leq\frac{|\mathcal{K}_1(x;n)|}{q(1+q)}+\frac{\mathcal{K}_1^2(x;n)}{q^2}. \tag{35}$$

Proof. Let $\Upsilon\in\mathcal{S}^*_\Lambda(x,n,q;\Phi)$ and $\varrho=\Upsilon^{-1}$. We have the following from the definition in formulas (32) and (33)

$$\frac{\xi\partial_q\Upsilon(\xi)}{\Upsilon(\xi)}=\Phi(x,n;\upsilon(\xi)) \tag{36}$$

and

$$\frac{\omega\partial_q\varrho(\omega)}{\varrho(\omega)}=\Phi(x,n;\nu(\omega)) \tag{37}$$

where the analytical υ and ν functions have the form

$$\upsilon(\xi)=c_1\xi+c_2\xi^2+\ldots, \tag{38}$$

$$\nu(\omega)=d_1\omega+d_2\omega^2+\ldots, \tag{39}$$

and $\upsilon(0)=0=\nu(0)$, $|\upsilon(\xi)|<1$, $|\nu(\omega)|<1$, $\xi,\omega\in\mathfrak{D}$. It follows that, from Lemma 2.2, that

$$\left|c_j\right|\leq 1,\ \left|d_j\right|\leq 1, \text{where } j\in\mathbb{N}. \tag{40}$$

If we replace (38) and (39) in (36) and (37), respectively, we obtain

$$\frac{\xi\partial_q\Upsilon(\xi)}{\Upsilon(\xi)}=1+\mathcal{K}_1(x;n)\upsilon(\xi)+\mathcal{K}_2(x;n)\upsilon^2(\xi)+\ldots, \tag{41}$$

and

$$\frac{\omega\partial_q\varrho(\omega)}{\varrho(\omega)}=1+\mathcal{K}_1(x;n)\nu(\omega)+\mathcal{K}_2(x;n)\nu^2(\omega)+\ldots. \tag{42}$$

In view of (1) and (2), from (41) and (42), we obtain

$$\begin{aligned} & 1+qa_2\xi+q\left((1+q)a_3-a_2^2\right)\xi^2+... \\ = & \ 1+\mathcal{K}_1(x;n)c_1\xi+\left[\mathcal{K}_1(x;n)c_2+\mathcal{K}_2(x;n)c_1^2\right]\xi^2+... \end{aligned}$$

and

$$\begin{aligned} & 1-qa_2\omega+q\left((1+2q)a_2^2-(1+q)a_3\right)\omega^2+... \\ = & \ 1+\mathcal{K}_1(x;n)d_1\omega+\left[\mathcal{K}_1(x;n)d_2+\mathcal{K}_2(x;n)d_1^2\right]\omega^2+.... \end{aligned}$$

It gives rise to the following relationships:

$$qa_2=\mathcal{K}_1(x;n)c_1, \tag{43}$$

$$q\left[(1+q)a_3-a_2^2\right]=\mathcal{K}_1(x;n)c_2+\mathcal{K}_2(x;n)c_1^2, \tag{44}$$

and

$$-qa_2=\mathcal{K}_1(x;n)d_1, \tag{45}$$

$$q\left[(1+2q)a_2^2-(1+q)a_3\right]=\mathcal{K}_1(x;n)d_2+\mathcal{K}_2(x;n)d_1^2. \tag{46}$$

From (43) and (45), it follows that

$$c_1=-d_1, \tag{47}$$

and

$$2q^2a_2^2=\mathcal{K}_1^2(x;n)\left(c_1^2+d_1^2\right). \tag{48}$$

By adding (44) to (46), yields

$$2q^2a_2^2=\mathcal{K}_1(x;n)(c_2+d_2)+\mathcal{K}_2(x;n)\left(c_1^2+d_1^2\right). \tag{49}$$

We determine that, by replacing the value of $\left(c_1^2+d_1^2\right)$ from (48) on the right side of (49),

$$a_2^2=\frac{\mathcal{K}_1(x;n)(c_2+d_2)\mathcal{K}_1^2(x;n)}{2q^2\left(\mathcal{K}_1^2(x;n)-\mathcal{K}_2(x;n)\right)}. \tag{50}$$

From (40) for c_2 and d_2 we get (34). In addition, if we subtract (46) from (44), we obtain

$$2q(1+q)\left(a_3-a_2^2\right)=\mathcal{K}_1(x;n)(c_2-d_2)+\mathcal{K}_2(x;n)\left(c_1^2-d_1^2\right) \tag{51}$$

Then, in view of (48) and (51), we obtain

$$a_3=\frac{\mathcal{K}_1^2(x;n)}{2q^2}\left(c_1^2+d_1^2\right)+\frac{\mathcal{K}_1(x;n)}{2q(1+q)}(c_2-d_2).$$

Once again applying (40), for the coefficients c_1, d_1, c_2, d_2, we deduce (35).
The proof is thus finished. □

Based on Theorem 3.2, we can derive the subsequent outcome for $q=1$.

Corollary 3.3. *Let $\Upsilon \in \Lambda$ given by (1) belong to the class $\mathcal{S}^*_\Lambda(x,n,1;\Phi)$. Then*

$$|a_2| \leq \frac{|\mathcal{K}_1(x;n)|\sqrt{\mathcal{K}_1(x;n)}}{\sqrt{\left|\mathcal{K}_1^2(x;n)-\mathcal{K}_2(x;n)\right|}}$$

and

$$|a_3| \leq \frac{|\mathcal{K}_1(x;n)|}{2}+\mathcal{K}_1^2(x;n).$$

For functions in the class $\mathcal{S}^*_\Lambda(x,n,q;\Phi)$, we prove the following Fekete–Szegö inequality using the values of a_2^2 and a_3.

Theorem 3.4. *Let $\Upsilon \in \Lambda$ given by (1) be in the class $\mathcal{S}^*_\Lambda(x,n,q;\Phi)$. Then,*

$$\left|a_3-\gamma a_2^2\right| \leq \begin{cases} \frac{|\mathcal{K}_1(x;n)|}{q(1+q)}, & \text{if} \quad |\psi(\gamma)| \leq \frac{1}{1+q}, \\ \frac{|\mathcal{K}_1(x;n)||\psi(\gamma)|}{q}, & \text{if} \quad |\psi(\gamma)| \geq \frac{1}{1+q}, \end{cases}$$

where

$$\psi(\gamma)=\frac{(1-\gamma)\mathcal{K}_1^2(x;n)}{q\left(\mathcal{K}_1^2(x;n)-\mathcal{K}_2(x;n)\right)}.$$

Proof. From (50) and (51),

$$\begin{aligned} a_3-\gamma a_2^2 &= \frac{\mathcal{K}_1(x;n)}{2q}\left[\frac{c_2}{1+q}-\frac{d_2}{1+q}+\frac{(1-\gamma)\mathcal{K}_1^2(x;n)\,c_2}{q\left(\mathcal{K}_1^2(x;n)-\mathcal{K}_2(x;n)\right)}\right. \\ &\qquad \left.+\frac{(1-\gamma)\mathcal{K}_1^2(x;n)\,d_2}{q\left(\mathcal{K}_1^2(x;n)-\mathcal{K}_2(x;n)\right)}\right] \\ &= \frac{\mathcal{K}_1(x;n)}{2q}\left[\left(\psi(\gamma)+\frac{1}{1+q}\right)c_2+\left(\psi(\gamma)-\frac{1}{1+q}\right)d_2\right], \end{aligned}$$

where

$$\psi(\gamma)=\frac{(1-\gamma)\mathcal{K}_1^2(x;n)}{q\left(\mathcal{K}_1^2(x;n)-\mathcal{K}_2(x;n)\right)}.$$

Therefore, given (40), we conclude that the required inequality holds.
The proof is thus finished. □

Based on Theorem 3.4, we can derive the subsequent outcome for $q=1$.

Corollary 3.5. *Let $\Upsilon \in \Lambda$ given by (1) belong to the class $\mathcal{S}^*_\Lambda(x,n,1;\Phi)$. Then,*

$$\left|a_3-\gamma a_2^2\right| \leq \begin{cases} \frac{|\mathcal{K}_1(x;n)|}{2}, & \text{if} \quad |\psi(\gamma)| \leq \frac{1}{2}, \\ |\mathcal{K}_1(x;n)||\psi(\gamma)|, & \text{if} \quad |\psi(\gamma)| \geq \frac{1}{2}, \end{cases}$$

where

$$\psi(\gamma)=\frac{(1-\gamma)\mathcal{K}_1^2(x;n)}{\mathcal{K}_1^2(x;n)-\mathcal{K}_2(x;n)}.$$

4. Coefficient Bounds of the Class $C_\Lambda(x,n,q;\Phi)$

Definition 4.1. *We assert that Υ belonging to Λ is said to be in the class $C_\Lambda(x,n,q;\Phi)$, for $x=0,1,2,...$ and $n\geq 2$, if the following subordinations hold:*

$$1+\frac{\xi\partial_q^2\Upsilon(\xi)}{\partial_q\Upsilon(\xi)}\prec\Phi(x,n;\xi) \tag{52}$$

and

$$1+\frac{\omega\partial_q^2\varrho(\omega)}{\partial_q\varrho(\omega)}\prec\Phi(x,n;\omega), \tag{53}$$

$\xi,\omega\in\mathfrak{D}$, Φ is given by (11), and $\varrho=\Upsilon^{-1}$ is given by (2).

Theorem 4.2. *Let $\Upsilon\in\Lambda$ given by (1) be in the class $C_\Lambda(x,n,q;\Phi)$. Then,*

$$|a_2|\leq\frac{|\mathcal{K}_1(x;n)|\sqrt{\mathcal{K}_1(x;n)}}{\sqrt{[2]_q\left|[3]_q\mathcal{K}_1^2(x;n)-[2]_q\mathcal{K}_1^2(x;n)-[2]_q\mathcal{K}_2(x;n)\right|}}, \tag{54}$$

and

$$|a_3|\leq\frac{|\mathcal{K}_1(x;n)|}{[2]_q[3]_q}+\frac{\mathcal{K}_1^2(x;n)}{[2]_q^2}. \tag{55}$$

Proof. Let $\Upsilon\in C_\Lambda(x,n,q;\Phi)$ and $\varrho=\Upsilon^{-1}$. We have the following from the definition in formulas (52) and (53)

$$1+\frac{\xi\partial_q^2\Upsilon(\xi)}{\partial_q\Upsilon(\xi)}=\Phi(x,n;\upsilon(\xi)) \tag{56}$$

and

$$1+\frac{\omega\partial_q^2\varrho(\omega)}{\partial_q\varrho(\omega)}=\Phi(x,n;\nu(\omega)), \tag{57}$$

where the analytical υ and ν functions have the form

$$\upsilon(\xi)=c_1\xi+c_2\xi^2+\ldots, \tag{58}$$

$$\nu(\omega)=d_1\omega+d_2\omega^2+\ldots, \tag{59}$$

and $\upsilon(0)=0=\nu(0)$, $|\upsilon(\xi)|<1$, $|\nu(\omega)|<1$, $\xi,\omega\in\mathfrak{D}$.

It follows that, from Lemma 2.2, that

$$\left|c_j\right|\leq 1,\left|d_j\right|\leq 1,\text{where } j\in\mathbb{N}. \tag{60}$$

If we replace (58) and (59) in (56) and (57), respectively, we obtain

$$1+\frac{\xi\partial_q^2\Upsilon(\xi)}{\partial_q\Upsilon(\xi)}=1+\mathcal{K}_1(x;n)\upsilon(\xi)+\mathcal{K}_2(x;n)\upsilon^2(\xi)+\ldots, \tag{61}$$

and

$$1+\frac{\omega\partial_q^2\varrho(\omega)}{\partial_q\varrho(\omega)}=1+\mathcal{K}_1(x;n)\nu(\omega)+\mathcal{K}_2(x;n)\nu^2(\omega)+\ldots. \tag{62}$$

In view of (1) and (2), from (61) and (62), we obtain

$$\begin{aligned} & 1+[2]_q a_2\xi+\left([2]_q [3]_q a_3-[2]_q^2 a_2^2\right)\xi^2+\ldots \\ = \quad & 1+\mathcal{K}_1(x;n)c_1\xi+\left[\mathcal{K}_1(x;n)c_2+\mathcal{K}_2(x;n)c_1^2\right]\xi^2+\ldots \end{aligned}$$

and

$$\begin{aligned} & 1-[2]_q a_2\omega+\left([2]_q\left(2[3]_q-[2]_q\right)a_2^2-[2]_q[3]_q a_3\right)\omega^2+\ldots \\ = \quad & 1+\mathcal{K}_1(x;n)d_1\omega+\left[\mathcal{K}_1(x;n)d_2+\mathcal{K}_2(x;n)d_1^2\right]\omega^2+\ldots. \end{aligned}$$

It gives rise to the following relationships:

$$[2]_q a_2=\mathcal{K}_1(x;n)c_1, \tag{63}$$

$$[2]_q[3]_q a_3-[2]_q^2 a_2^2=\mathcal{K}_1(x;n)c_2+\mathcal{K}_2(x;n)c_1^2, \tag{64}$$

and

$$-[2]_q a_2=\mathcal{K}_1(x;n)d_1, \tag{65}$$

$$[2]_q\left(2[3]_q-[2]_q\right)a_2^2-[2]_q[3]_q a_3=\mathcal{K}_1(x;n)d_2+\mathcal{K}_2(x;n)d_1^2. \tag{66}$$

From (63) and (65), it follows that

$$c_1=-d_1, \tag{67}$$

and

$$2[2]_q^2 a_2^2=\mathcal{K}_1^2(x;n)\left(c_1^2+d_1^2\right). \tag{68}$$

By adding (64) to (66), yields

$$2[2]_q\left([3]_q-[2]_q\right)a_2^2=\mathcal{K}_1(x;n)(c_2+d_2)+\mathcal{K}_2(x;n)\left(c_1^2+d_1^2\right) \tag{69}$$

We determine that, by replacing the value of $\left(c_1^2+d_1^2\right)$ from (68) on the right side of (69),

$$a_2^2=\frac{\mathcal{K}_1(x;n)(c_2+d_2)\mathcal{K}_1^2(x;n)}{2[2]_q\left([3]_q\mathcal{K}_1^2(x;n)-[2]_q\mathcal{K}_1^2(x;n)-[2]_q\mathcal{K}_2(x;n)\right)}. \tag{70}$$

From (60) for c_2 and d_2 we get (54).

In addition, if we subtract (66) from (64), we obtain

$$2[2]_q[3]_q\left(a_3-a_2^2\right)=\mathcal{K}_1(x;n)(c_2-d_2)+\mathcal{K}_2(x;n)\left(c_1^2-d_1^2\right). \tag{71}$$

Then, in view of (68) and (71), we obtain

$$a_3=\frac{\mathcal{K}_1^2(x;n)}{2[2]_q^2}\left(c_1^2+d_1^2\right)+\frac{\mathcal{K}_1(x;n)}{2[2]_q[3]_q}(c_2-d_2).$$

Once again applying (60), for the coefficients c_1, d_1, c_2, d_2, we deduce (55).

The proof is thus finished. □

Based on Theorem 4.2, we can derive the subsequent outcome for $q=1$.

Corollary 4.3. *Let $\Upsilon \in \Lambda$ given by (1) belong to the class $C_{\Lambda}(x, n, 1; \Phi)$. Then,*

$$|a_2| \leq \frac{|\mathcal{K}_1(x;n)| \sqrt{2\mathcal{K}_1(x;n)}}{2\sqrt{\left|\mathcal{K}_1^2(x;n) - 2\mathcal{K}_2(x;n)\right|}}$$

and

$$|a_3| \leq \frac{|\mathcal{K}_1(x;n)|}{6} + \frac{\mathcal{K}_1^2(x;n)}{4}.$$

For functions in the class $C_{\Lambda}(x, n, q; \Phi)$, we prove the following Fekete–Szegö inequality using the values of a_2^2 and a_3.

Theorem 4.4. *Let $\Upsilon \in \Lambda$ given by (1) be in the class $C_{\Lambda}(x, n, q; \Phi)$. Then,*

$$\left|a_3 - \gamma a_2^2\right| \leq \begin{cases} \frac{|\mathcal{K}_1(x;n)|}{[2]_q[3]_q}, & \text{if } \left|\varphi(\gamma)\right| \leq \frac{1}{2[2]_q[3]_q}, \\ 2|\mathcal{K}_1(x;n)|\left|\varphi(\gamma)\right|, & \text{if } \left|\varphi(\gamma)\right| \geq \frac{1}{2[2]_q[3]_q}, \end{cases}$$

where

$$\varphi(\gamma) = \frac{(1-\gamma)\mathcal{K}_1^2(x;n)}{2[2]_q\left([3]_q\mathcal{K}_1^2(x;n) - [2]_q\mathcal{K}_1^2(x;n) - [2]_q\mathcal{K}_2(x;n)\right)}.$$

Proof. From (70) and (71),

$$\begin{aligned} a_3 - \gamma a_2^2 &= \mathcal{K}_1(x;n)\Bigg[\frac{c_2}{2[2]_q[3]_q} - \frac{d_2}{2[2]_q[3]_q} \\ &\quad + \frac{(1-\gamma)\mathcal{K}_1^2(x;n)\,c_2}{2[2]_q\left([3]_q\mathcal{K}_1^2(x;n) - [2]_q\mathcal{K}_1^2(x;n) - [2]_q\mathcal{K}_2(x;n)\right)} \\ &\quad + \frac{(1-\gamma)\mathcal{K}_1^2(x;n)\,d_2}{2[2]_q\left([3]_q\mathcal{K}_1^2(x;n) - [2]_q\mathcal{K}_1^2(x;n) - [2]_q\mathcal{K}_2(x;n)\right)}\Bigg] \\ &= \mathcal{K}_1(x;n)\left[\left(\varphi(\gamma) + \frac{1}{2[2]_q[3]_q}\right)c_2 + \left(\varphi(\gamma) - \frac{1}{2[2]_q[3]_q}\right)d_2\right], \end{aligned}$$

where

$$\varphi(\gamma) = \frac{(1-\gamma)\mathcal{K}_1^2(x;n)}{2[2]_q\left([3]_q\mathcal{K}_1^2(x;n) - [2]_q\mathcal{K}_1^2(x;n) - [2]_q\mathcal{K}_2(x;n)\right)}.$$

Therefore, given (60), we conclude that the required inequality holds.
The proof is thus finished. □

For $q = 1$, we obtain the following result from Theorem 4.4.

Corollary 4.5. *Let $\Upsilon \in \Lambda$ given by (1) belong to the class $C_{\Lambda}(x, n, 1; \Phi)$. Then,*

$$\left|a_3 - \gamma a_2^2\right| \leq \begin{cases} \frac{|\mathcal{K}_1(x;n)|}{6}, & \text{if } \left|\varphi(\gamma)\right| \leq \frac{1}{12}, \\ 2|\mathcal{K}_1(x;n)|\left|\varphi(\gamma)\right|, & \text{if } \left|\varphi(\gamma)\right| \geq \frac{1}{12}, \end{cases}$$

where

$$\varphi(\gamma) = \frac{(1-\gamma)\mathcal{K}_1^2(x;n)}{4\left(\mathcal{K}_1^2(x;n) - 2\mathcal{K}_2(x;n)\right)}.$$

5. Conclusions

In this paper, we introduced and investigated a new subclass of bi-univalent functions in the open unit disk defined by Krawtchouk polynomials and satisfies subordination conditions. Furthermore, we obtain upper bounds for $|a_2|$, $|a_3|$ and Fekete-Szegö inequality $\left|a_3 - \gamma a_2^2\right|$ for functions in this subclass.

Also, the approach presented here has been extended to establish new subfamilies of bi-univalent functions with the other special functions. The related outcomes may be left to the the researchers for practice.

Data availability: The authors declare that this research is purely theoretical and does not associate with any data.

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