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On the structure of standard von Neumann algebras

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Abstract. Let $\mathcal{H}$ be a complex separable Hilbert space. A von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is said to be standard if there is a conjugation C on $\mathcal{H}$ such that $C\mathcal{M}C = \mathcal{M}'$. In this paper we determine which type I von Neumann algebras are standard.

1. Introduction

The Tomita-Takesaki Modular theory is one of the most important developments in the study of von Neumann algebras, revealing a very precise and intimate connection between a von Neumann algebra and its commutant, along with a canonical one-parameter group of outer automorphisms. This theory was first developed by M. Tomita [16], and was carefully and thoroughly explained, clarified, and refined by M. Takesaki [14].

Throughout this paper, we denote by $\mathcal{H}$ a complex separable Hilbert space, and by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. Recall that a map C on $\mathcal{H}$ is called an *antiunitary operator* if C is conjugate linear, invertible and $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$; if, in addition, $C^{-1} = C$, then C is called a *conjugation*.

Theorem 1.1 (Tomita-Takesaki). *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra with a cyclic and separating vector. Then there exists a conjugation C on $\mathcal{H}$ and a collection $\{\Delta^{it} : t \in \mathbb{R}\}$ of unitary operators in $\mathcal{B}(\mathcal{H})$ satisfying*

(i) $C\mathcal{M}C = \mathcal{M}'$;
(ii) $\{\Delta^{it} : t \in \mathbb{R}\}$ *defines a one parameter group of automorphisms for* $\mathcal{M}$.

This theory made possible the great advances in the 1970s by Connes et al. on the classification of factors [4]. Inspired by that, Haagerup [9] proposed the definition of a standard form of a von Neumann algebra.

Definition 1.2. *A von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is said to be* standard *if there exsits a conjugation C on $\mathcal{H}$ such that $C\mathcal{M}C = \mathcal{M}'$; if, in addition, there exsit a self-dual cone $\mathcal{P} \subseteq \mathcal{H}$ such that*

(i) $C\xi = \xi$ *for any* $\xi \in \mathcal{P}$;

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(ii) $TCTC\mathcal{P} \subseteq \mathcal{P}$ *for any* $T \in \mathcal{M}$*;*

(iii) $CTC = T^*$ *for any* $T \in \mathcal{Z}(\mathcal{M})$ *(the center of* $\mathcal{M}$*).*

Then the quadruple $(\mathcal{M}, \mathcal{H}, C, \mathcal{P})$ *is called a* standard form *of* $\mathcal{M}$*.*

By [9], any von Neumann algebra is isomorphic to a von Neumann algebra $\mathcal{M}$ in a standard form $(\mathcal{M}, \mathcal{H}, C, \mathcal{P})$; and if $\mathcal{M}_1 \subseteq \mathcal{B}(\mathcal{H})$ is a standard von Neumann algebra, then one can choose C_1 on $\mathcal{H}$ and $\mathcal{P}_1 \subseteq \mathcal{H}$ such that $(\mathcal{M}_1, \mathcal{H}, C_1, \mathcal{P}_1)$ is a standard form of $\mathcal{M}_1$. Hence it is natural to ask the following question.

Question 1.3. *Which von Neumann algebras are standard?*

Clearly, $\mathcal{M}$ is standard if and only if $\mathcal{M}'$ is standard. By Theorem 1.1, if $\mathcal{M}$ is a von Neumann algebra with a cyclic and separating vector, then $\mathcal{M}$ is standard. Note that a von Neumann algebra may not be standard, for example, if $\mathcal{M} = \mathcal{B}(\mathbb{C}^n)$ for $n \geq 2$, then $C\mathcal{M}C = \mathcal{M} \neq \mathbb{C}I_n = \mathcal{M}'$ for any conjugation C on $\mathbb{C}^n$.

Definition 1.4. *Let* $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ *be a von Neumann algebra. If* C *is a conjugation on* $\mathcal{H}$*, then we call* $C\mathcal{M}C$ *a transpose of* $\mathcal{M}$*.*

The notion "transpose" for von Neumann algebras is in fact a generalization of that for matrices. Assume that $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra and C is a conjugation on $\mathcal{H}$. Then there exists an orthonormal basis (ONB, for short) $\{e_n\}$ of $\mathcal{H}$ such that $Ce_n = e_n$ for all n (see [5, Lemma 1]). Choose an operator $A \in C\mathcal{M}C$ with $A = CTC$ for $T \in \mathcal{M}$. Note that T^* has the matrix representation $[a_{i,j}]$ with respect to $\{e_n\}$, where $a_{i,j} = \langle T^* e_j, e_i \rangle$. And

$$\begin{aligned} \langle Ae_i, e_j \rangle &= \langle CTCe_i, e_j \rangle = \langle CTe_i, e_j \rangle \\ &= \langle Ce_j, Te_i \rangle = \langle e_j, Te_i \rangle \\ &= \langle T^* e_j, e_i \rangle. \end{aligned}$$

Thus the matrix representation of A with respect to $\{e_n\}$ is exactly the transpose of T^*'s matrix $[a_{i,j}]$. So, given a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$, a transpose of $\mathcal{M}$ is obtained from $\mathcal{M}$ by transposing the matrix of each $T \in \mathcal{M}$ with respect to some ONB.

Remark 1.5. *(i) A von Neumann algebra may have more than one transpose; however, any two transposes of a von Neumann algebra are unitarily equivalent. In fact, if* $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ *is a von Neumann algebra and* C, D *are conjugations on* $\mathcal{H}$*, then* $U := CD$ *is unitary and*

$$U^*(C\mathcal{M}C)U = (DC)(C\mathcal{M}C)(CD) = D\mathcal{M}D;$$

that is, $C\mathcal{M}C$ *and* $D\mathcal{M}D$ *are unitarily equivalent.*

(ii) Each von Neumann algebra $\mathcal{M}$ *is anti-isomorphic to its transpose* $\mathcal{M}^t = C\mathcal{M}C$*. In fact,* $X \mapsto CX^*C$ *is the corresponding anti-isomorphism. Connes [2, 3] constructed von Neumann factors of type* II_1 *and type* III *which are not anti-isomorphic to themselves. This shows that a von Neumann algebra may not be a transpose of itself. However, each type* I *von Neumann algebra is a transpose of itself (see Remark 2.3).*

Here we provide more examples of standard von Neumann algebras.

Lemma 1.6. *Let* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *be von Neumann algebras. If* $\mathcal{M}_1' \cong \mathcal{M}_2^t$*, where* $\cong$ *denotes unitary equivalence, then* $\mathcal{M}_1 \oplus \mathcal{M}_2$ *is standard.*

Proof. Without loss of generality, we assume that $\mathcal{M}_1, \mathcal{M}_2 \subseteq \mathcal{B}(\mathcal{H})$ are von Neumann algebras with $\mathcal{M}_1' \cong \mathcal{M}_2^t$. Then there exists a conjugation C_0 and a unitary operator U on $\mathcal{H}$ such that $U\mathcal{M}_1'U^* = C_0\mathcal{M}_2C_0$.

Let $D = C_0U$. Then D is an antiunitary operator on $\mathcal{H}$. Set

$$C = \begin{bmatrix} 0 & D^{-1} \\ D & 0 \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H} \end{matrix}.$$

Then C is a conjugation on $\mathcal{H} \oplus \mathcal{H}$, and one can check that $C(\mathcal{M}_1 \oplus \mathcal{M}_2)C = \mathcal{M}_1' \oplus \mathcal{M}_2'$, i.e., $\mathcal{M}_1 \oplus \mathcal{M}_2$ is standard. $\square$

By the preceding lemma, we have $\mathcal{A}' \oplus \mathcal{A}^t$ is standard for any von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$. Hence, it is natural to ask whether each standard von Neumann algebra $\mathcal{M}$ is of the form $\mathcal{A}' \oplus \mathcal{A}^t$ for some von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$. In general, this is not the case. In fact, if $\mathcal{M} = \{A \oplus A : A \in \mathcal{B}(\mathbb{C}^2)\}$, then one can check that $\mathcal{M}$ is standard (see Corollary 2.7); however, $\mathcal{M}$ can not be written as $\mathcal{A}' \oplus \mathcal{A}^t$.

The main result of this paper is the following theorem, which provides a canonical decomposition of standard type I von Neumann algebras.

Theorem 1.7. *If $\mathcal{M}$ is a von Neumann algebra of type* I, *then $\mathcal{M}$ is standard if and only if $\mathcal{M}$ is unitarily equivalent to a direct sum of von Neumann algebras of the form (some of the summands may be absent)*

(i) $\mathcal{A} \otimes \mathcal{B}(\mathcal{K}) \otimes \mathbb{C}I$, *where $\mathcal{A}$ is a maximal abelian von Neumann algebra and I is the identity operator on some Hilbert space $\mathcal{K}$; and*

(ii) $\mathcal{A}' \oplus \mathcal{A}^t$, *where $\mathcal{A}$ is a von Neumann algebra.*

It is well known that every von Neumann algebra $\mathcal{M}$ can be uniquely decomposed into the direct sum of type I, type II and type III von Neumann algebras, that is, $\mathcal{M} = \mathcal{M}_{\mathrm{I}} \oplus \mathcal{M}_{\mathrm{II}} \oplus \mathcal{M}_{\mathrm{III}}$. We will show that $\mathcal{M}$ is standard if and only if each of $\mathcal{M}_{\mathrm{I}}$, $\mathcal{M}_{\mathrm{II}}$ and $\mathcal{M}_{\mathrm{III}}$ is standard (see Proposition 3.3). Furthermore, we will show that if G is a locally compact group, then the group von Neumann algebra $\mathcal{R}(G)$ is standard (see Examples 3.7).

Our approach to Question 1.3 is inspired by the study of complex symmetric operators. Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is called a *complex symmetric operator* (CSO) if $CT^*C = T$ for some conjugation C on $\mathcal{H}$. The study of complex symmetric operators was initiated by Garcia and Putinar [5, 6] and has received much attention in the last decade (see [7, 8, 10] for references). In [8, Thm. 1.6], Guo and Zhu obtained a decomposition theorem of CSOs, which describes the block structure of complex symmetric operators and inspires the present study.

2. Proof of Theorem 1.7

This section is devoted to the proof of Theorem 1.7. We first recall some basic definitions and results concerning the classification of von Neumann algebras. More details can be found in [15].

A von Neumann algebra $\mathcal{M}$ is said to be of type I if every nonzero central projection in $\mathcal{M}$ majorizes a nonzero abelian projection in $\mathcal{M}$. If there is no nonzero finite projection in $\mathcal{M}$, that is, $\mathcal{M}$ is purely infinite, then it is said to be of type III. If $\mathcal{M}$ has no nonzero abelian projection and every nonzero central projection in $\mathcal{M}$ majorizes a nonzero finite projection of $\mathcal{M}$, then it is said to be of type II. If $\mathcal{M}$ is finite and of type II, then it is said to be of type II_1. If $\mathcal{M}$ is of type II and has no nonzero central finite projection, then $\mathcal{M}$ is said to be of type II_∞.

Lemma 2.1 ([15, Cor. 2.24]). *Let $\mathcal{M}$ be a von Neumann algebra. Then*

(i) *$\mathcal{M}$ is of type* I *if and only if $\mathcal{M}'$ is of type* I;

(ii) *$\mathcal{M}$ is of type* II *if and only if $\mathcal{M}'$ is of type* II;

(iii) *$\mathcal{M}$ is of type* III *if and only if $\mathcal{M}'$ is of type* III.

Suppose that $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra of type I. Then there is an orthogonal family of central projections $\{Z_\alpha\}$ such that $\sum_\alpha Z_\alpha = I$ and Z_α is the greatest α-homogeneous central projection in $\mathcal{M}$. Since $\mathcal{M}'$ is of type I, one can see that $\mathcal{M}'$ can also be decomposed by an orthogonal family of central projections $\{Z'_\alpha\}$ such that $\sum_\alpha Z'_\alpha = I$ and Z'_α is the greatest α-homogeneous central projection in $\mathcal{M}'$. Note that the center $\mathcal{Z}(\mathcal{M})$ of $\mathcal{M}$ coincides with the center of $\mathcal{M}'$. Therefore, $\{Z_\alpha\}$ and $\{Z'_\alpha\}$ are both in $\mathcal{Z}(\mathcal{M})$. Put

$$Z_{\alpha,\beta} = Z_\alpha Z'_\beta.$$

We can obtain a finer partition $\{Z_{\alpha,\beta}\}$ of the identity. Therefore, Takesaki [15] obtained the following description of the spatial type of a von Neumann algebra of type I.

Lemma 2.2 ([15, Thm. 1.31]). *A von Neumann algebra $\mathcal{M}$ of type* I *has the unique decomposition*

$$\mathcal{M} \cong \Sigma^{\oplus}_{\alpha,\beta}(\mathcal{A}_{\alpha,\beta} \otimes \mathcal{B}(\mathcal{H}_\alpha) \otimes \mathbb{C}I_\beta),$$

and

$$\mathcal{M}' \cong \Sigma^{\oplus}_{\alpha,\beta}(\mathcal{A}_{\alpha,\beta} \otimes \mathbb{C}I_\alpha \otimes \mathcal{B}(\mathcal{H}_\beta)),$$

where $\mathcal{A}_{\alpha,\beta}$ is a maximal abelian von Neumann algebra and $\mathcal{H}_\alpha$ (resp. $\mathcal{H}_\beta$) is an α-dimensional (resp. β-dimensional) Hilbert space.

Throughout the rest of this paper, we denote $\mathcal{M}_{\alpha,\beta} = \mathcal{A}_{\alpha,\beta} \otimes \mathcal{B}(\mathcal{H}_\alpha) \otimes \mathbb{C}I_\beta$, where $\mathcal{A}_{\alpha,\beta}$ is a maximal abelian von Neumann algebra, $\mathcal{H}_\alpha$ is an α-dimensional Hilbert space, and I_β is the identity operator on some β-dimensional Hilbert space.

Remark 2.3. $\mathcal{M}^t_{\alpha,\beta} = \mathcal{M}_{\alpha,\beta}$. *In fact, without loss of generality, we can assume $\mathcal{A}_{\alpha,\beta} = L^\infty(X,\mu)$. Let C_1, C_2, C_3 be conjugations on $L^2(X,\mu)$, $\mathcal{H}_\alpha$, $\mathcal{H}_\beta$, respectively. Then $C := C_1 \otimes C_2 \otimes C_3$ is a conjugation on $\mathcal{H} := L^2(X,\mu) \otimes \mathcal{H}_\alpha \otimes \mathcal{H}_\beta$ and*

$$C\mathcal{M}_{\alpha,\beta}C = \mathcal{M}_{\alpha,\beta}.$$

Proposition 2.4. $\mathcal{M}'_{\alpha,\beta} \cong \mathcal{M}_{\xi,\eta}$ *if and only if* $\mathcal{A}_{\alpha,\beta} \cong \mathcal{A}_{\xi,\eta}$, $\alpha = \eta$ *and* $\beta = \xi$.

Proof. "$\Longleftarrow$". If $\mathcal{A}_{\alpha,\beta} \cong \mathcal{A}_{\xi,\eta}$ and $\alpha = \eta, \beta = \xi$, then $\mathcal{A}_{\alpha,\beta} \cong \mathcal{A}_{\beta,\alpha}$. It follows that

$$\mathcal{M}'_{\alpha,\beta} = \mathcal{A}_{\alpha,\beta} \otimes \mathbb{C}I_\alpha \otimes \mathcal{B}(\mathcal{H}_\beta) \cong \mathcal{M}_{\beta,\alpha}.$$

"$\Longrightarrow$". Without loss of generality, we can assume that

$$\mathcal{A}_{\alpha,\beta} = L^\infty(X_{\alpha,\beta}, \mu_{\alpha,\beta})$$

and

$$\mathcal{A}_{\xi,\eta} = L^\infty(X_{\xi,\eta}, \mu_{\xi,\eta}).$$

Note that

$$\mathcal{M}_{\alpha,\beta} = L^\infty(X_{\alpha,\beta}, \mu_{\alpha,\beta}) \otimes \mathcal{B}(\mathcal{H}_\alpha) \otimes \mathbb{C}I_\beta,$$

$$\mathcal{M}_{\xi,\eta} = L^\infty(X_{\xi,\eta}, \mu_{\xi,\eta}) \otimes \mathcal{B}(\mathcal{H}_\xi) \otimes \mathbb{C}I_\eta,$$

and

$$\mathcal{M}'_{\alpha,\beta} = L^\infty(X_{\alpha,\beta}, \mu_{\alpha,\beta}) \otimes \mathbb{C}I_\alpha \otimes \mathcal{B}(\mathcal{H}_\beta).$$

Since $\mathcal{M}'_{\alpha,\beta} \cong \mathcal{M}_{\xi,\eta}$ and the unitary isomorphism " $\cong$ " preserves the center of $\mathcal{M}'_{\alpha,\beta}$ and the center of $\mathcal{M}_{\xi,\eta}$, it follows immediately that $L^\infty(X_{\alpha,\beta}, \mu_{\alpha,\beta}) \cong L^\infty(X_{\xi,\eta}, \mu_{\xi,\eta})$. Hence we have $\mathcal{A}_{\alpha,\beta} \cong \mathcal{A}_{\xi,\eta}$ and $\alpha = \eta, \beta = \xi$. □

Corollary 2.5. *If $\mathcal{A}_{\alpha,\beta} \cong \mathcal{A}_{\beta,\alpha}$, then $\mathcal{M}_{\alpha,\beta} \oplus \mathcal{M}_{\beta,\alpha}$ is standard.*

Corollary 2.6. *$\mathcal{M}_{\alpha,\beta}$ is standard if and only if $\alpha = \beta$.*

Corollary 2.7. *Let $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$ be Hilbert spaces with* $\dim \mathcal{H}_\alpha = \alpha$ *and* $\dim \mathcal{H}_\beta = \beta$. *Let $\mathcal{N}_{\alpha,\beta} = \mathcal{B}(\mathcal{H}_\alpha) \otimes \mathbb{C}I_\beta$. Then $\mathcal{N}_{\alpha,\beta}$ is standard if and only if $\alpha = \beta$.*

Now we are going to give the proof of Theorem 1.7.

Proof. [Proof of Theorem 1.7] Without loss of generality, we assume that $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra of type I. The sufficiency is obvious by Lemma 1.6 and Corollary 2.6.

For the necessity, note that $\mathcal{M}$ is a von Neumann algebra of type I and, by Lemma 2.2, $\mathcal{M}$ has the unique decomposition

$$\mathcal{M} \cong \Sigma^{\oplus}_{(\alpha,\beta)\in\Lambda}\mathcal{M}_{\alpha,\beta},$$

where $\Lambda \subseteq \{1, 2, \cdots, \infty\} \times \{1, 2, \cdots, \infty\}$.

Since $\mathcal{M}$ is standard, there exists a conjugation C on $\mathcal{H}$ such that $C\mathcal{M}C = \mathcal{M}'$. Let Z_α be the greatest α-homogeneous central projection in $\mathcal{M}$. Then $CZ_\alpha C$ is the greatest α-homogeneous central projection in $C\mathcal{M}C = \mathcal{M}'$. Hence, $Z'_\alpha = CZ_\alpha C$ and

$$CZ_{\alpha,\beta}C = CZ_\alpha Z'_\beta C = Z'_\alpha Z_\beta = Z_\beta Z'_\alpha = Z_{\beta,\alpha}.$$

It follows that

$$C\{Z_{\alpha,\beta}\}C = \{Z_{\alpha,\beta}\}.$$

Let $Z_{\tau(\alpha,\beta)}$ denote $CZ_{\alpha,\beta}C$. Then $\tau : \Lambda \to \Lambda$ is an involutive map. The rest of this proof is divided into two cases.

Case 1. $\tau(\alpha, \beta) = (\alpha, \beta)$.

It follows that $CZ_{\alpha,\beta} = Z_{\alpha,\beta}C$. Hence $C = C_{\alpha,\beta} \oplus D$ for some conjugation $C_{\alpha,\beta}$ on the underlying space of $\mathcal{M}_{\alpha,\beta}$ and some conjugation D on its orthogonal complement; moreover,

$$C_{\alpha,\beta}\mathcal{M}_{\alpha,\beta}C_{\alpha,\beta} = CZ_{\alpha,\beta}\mathcal{M}C = Z_{\alpha,\beta}C\mathcal{M}C = Z_{\alpha,\beta}\mathcal{M}' = \mathcal{M}'_{\alpha,\beta},$$

that is, $\mathcal{M}_{\alpha,\beta}$ is standard. By Corollary 2.6, we have $\alpha = \beta$. Hence $\mathcal{M}_{\alpha,\beta} = \mathcal{A} \otimes \mathcal{B}(\mathcal{K}) \otimes \mathbb{C}I$, where $\mathcal{A}$ is a maximal abelian von Neumann algebra and I is the identity operator on $\mathcal{K}$.

Case 2. $\tau(\alpha, \beta) = (\xi, \eta) \neq (\alpha, \beta)$.

In this case, we have $CZ_{\alpha,\beta}C = Z_{\xi,\eta}$ and $CZ_{\xi,\eta}C = Z_{\alpha,\beta}$. Let $Z = Z_{\alpha,\beta} \oplus Z_{\xi,\eta}$. Then $CZC = Z$, $CZ = ZC$ and C has the decomposition $C = C_0 \oplus C_1$ relative to the decomposition $\mathcal{H} = \text{ran } Z \oplus \text{ran } (I - Z)$; moreover,

$$C_0 = \begin{bmatrix} 0 & D^{-1} \\ D & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_{\alpha,\beta} \\ \mathcal{H}_{\xi,\eta} \end{matrix},$$

where D is antiunitary. Note that

$$C_0(\mathcal{M}_{\alpha,\beta} \oplus \mathcal{M}_{\xi,\eta})C_0 = CZ\mathcal{M}C = ZC\mathcal{M}C = \mathcal{M}'_{\alpha,\beta} \oplus \mathcal{M}'_{\xi,\eta}$$

and

$$C_0(\mathcal{M}_{\alpha,\beta} \oplus \mathcal{M}_{\xi,\eta})C_0 = D^{-1}\mathcal{M}_{\xi,\eta}D \oplus D\mathcal{M}_{\alpha,\beta}D^{-1}.$$

It follows that $\mathcal{M}'_{\alpha,\beta} = D^{-1}\mathcal{M}_{\xi,\eta}D$ and $E\mathcal{M}'_{\alpha,\beta}E = (ED^{-1})\mathcal{M}_{\xi,\eta}(DE)$ for any conjugation E on the underlying space of $\mathcal{M}_{\alpha,\beta}$. Since DE is unitary, we conclude that $\mathcal{M}'_{\alpha,\beta} \cong \mathcal{M}^t_{\xi,\eta}$. That is, $\mathcal{M}_{\alpha,\beta} \oplus \mathcal{M}_{\xi,\eta} \cong (\mathcal{M}'_{\alpha,\beta})' \oplus (\mathcal{M}'_{\alpha,\beta})^t$. This completes the proof. □

Corollary 2.8. *Let $\mathcal{M}$ be a finite dimensional von Neumann algebra. Then $\mathcal{M}$ is standard if and only if $\mathcal{M} = \oplus_{i\in\Lambda}\mathcal{M}_i$, where each $\mathcal{M}_i$ satisfies either (i) $\mathcal{M}_i \cong M_n(\mathbb{C})^{(n)}$, or (ii) $\mathcal{M}_i \cong (M_n(\mathbb{C})^{(k)} \oplus M_k(\mathbb{C})^{(n)})$, where $n \neq k$.*

3. More standard von Neumann algebras

This section is devoted to constructing standard von Neumann algebras. We first recall the type decomposition theorem for von Neumann algebras.

Lemma 3.1 ([15, Thm. 1.19]). *Every von Neumann algebra $\mathcal{M}$ can be uniquely decomposed into the direct sum of type* I, *type* II_1, *type* II_∞ *and type* III *von Neumann algebras.*

The decomposition is constructed as follows. Let $\{E_i\}$ be a maximal family of centrally orthogonal abelian projections in $\mathcal{M}$ and $E = \sum_i E_i$. Put Z_{I} be the smallest central projection in $\mathcal{M}$ majorizing E. Then $\mathcal{M}Z_{\mathrm{I}}$ is of type I, and there is no nonzero abelian projection in $\mathcal{M}(I - Z_{\mathrm{I}})$. Let $\{F_j\}$ be a maximal family of centrally orthogonal finite projections in $\mathcal{M}(I - Z_{\mathrm{I}})$ and $F = \sum_j F_j$. Then F is finite. Put Z_{II} be the smallest central projection in $\mathcal{M}$ majorizing F. Then $\mathcal{M}Z_{\mathrm{II}}$ has no nonzero abelian projection and every nonzero central projection Z in $\mathcal{M}Z_{\mathrm{II}}$ majorizes a finite projection $ZF \neq 0$. Hence $\mathcal{M}Z_{\mathrm{II}}$ is of type II. By the maximality of $\{F_j\}$, $Z_{\mathrm{III}} := I - Z_{\mathrm{I}} - Z_{\mathrm{II}}$ does not majorize a nonzero finite projection, so $\mathcal{M}Z_{\mathrm{III}}$ is of type III. Note that $Z_{\mathrm{I}} + Z_{\mathrm{II}} + Z_{\mathrm{III}} = I$. Let $\{Z_k\}$ be a maximal family of orthogonal central finite projections in $\mathcal{M}Z_{\mathrm{II}}$. Put $Z_{\mathrm{II}_1} = \sum_k Z_k$ and $Z_{\mathrm{II}_\infty} = Z_{\mathrm{II}} - Z_{\mathrm{II}_1}$. It follows that $\mathcal{M}Z_{\mathrm{II}_1}$ is of type II_1 and $\mathcal{M}Z_{\mathrm{II}_\infty}$ is of type II_∞. Thus we obtain the direct sum decomposition

$$\mathcal{M} = \mathcal{M}Z_{\mathrm{I}} \oplus \mathcal{M}Z_{\mathrm{II}_1} \oplus \mathcal{M}Z_{\mathrm{II}_\infty} \oplus \mathcal{M}Z_{\mathrm{III}}.$$

Note that $Z_{\mathrm{II}} = Z_{\mathrm{II}_1} + Z_{\mathrm{II}_\infty}$. Then we have

$$\mathcal{M} = \mathcal{M}Z_{\mathrm{I}} \oplus \mathcal{M}Z_{\mathrm{II}} \oplus \mathcal{M}Z_{\mathrm{III}}.$$

Denote $\mathcal{M}_{\mathrm{I}} = \mathcal{M}Z_{\mathrm{I}}$, $\mathcal{M}_{\mathrm{II}} = \mathcal{M}Z_{\mathrm{II}}$ and $\mathcal{M}_{\mathrm{III}} = \mathcal{M}Z_{\mathrm{III}}$.

Remark 3.2. *Assume a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ has the direct sum decomposition*

$$\mathcal{M} = \mathcal{M}_{\mathrm{I}} \oplus \mathcal{M}_{\mathrm{II}} \oplus \mathcal{M}_{\mathrm{III}}$$

respective to the central projections Z_{I}, Z_{II} and Z_{III} constructed above. If C is a conjugation on $\mathcal{H}$, then $CZ_{\mathrm{I}}C$, $CZ_{\mathrm{II}}C$ and $CZ_{\mathrm{III}}C$ are the central projections for which $C\mathcal{M}C$ can be uniquely decomposed into the direct sum of type I, *type* II *and type* III *von Neumann algebras, that is,*

$$C\mathcal{M}C = C\mathcal{M}_{\mathrm{I}}C \oplus C\mathcal{M}_{\mathrm{II}}C \oplus C\mathcal{M}_{\mathrm{III}}C.$$

Thus we reduce the problem of whether a von Neumann algebra is standard to several special cases.

Proposition 3.3. *If $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra, then $\mathcal{M}$ is standard if and only if each of $\mathcal{M}_{\mathrm{I}}$, $\mathcal{M}_{\mathrm{II}}$ and $\mathcal{M}_{\mathrm{III}}$ is standard.*

Proof. The sufficiency is obvious. It suffices to prove the necessity. In fact, if $\mathcal{M}$ is standard, then there exists a conjugation C on $\mathcal{H}$ such that $C\mathcal{M}C = \mathcal{M}'$. Denote $\mathcal{N} = C\mathcal{M}C$. It follows from

$$\mathcal{M} = \mathcal{M}_{\mathrm{I}} \oplus \mathcal{M}_{\mathrm{II}} \oplus \mathcal{M}_{\mathrm{III}}$$

that

$$\mathcal{M}' = \mathcal{M}_{\mathrm{I}}' \oplus \mathcal{M}_{\mathrm{II}}' \oplus \mathcal{M}_{\mathrm{III}}';$$

moreover, $\mathcal{M}_{\mathrm{I}}'$ is of type I, $\mathcal{M}_{\mathrm{II}}'$ is of type II, and $\mathcal{M}_{\mathrm{III}}'$ is of type III.

On the other hand, by Remark 3.2, $\mathcal{N} = C\mathcal{M}C$ is uniquely decomposed into

$$\mathcal{N} = C\mathcal{M}_{\mathrm{I}}C \oplus C\mathcal{M}_{\mathrm{II}}C \oplus C\mathcal{M}_{\mathrm{III}}C.$$

Since $C\mathcal{M}C = \mathcal{M}'$, by the uniqueness of the decomposition of $\mathcal{N}$, we have $\mathcal{M}_{\mathrm{I}}' = C\mathcal{M}_{\mathrm{I}}C$. Hence $\mathcal{M}_{\mathrm{I}}$ is standard. Similarly, $\mathcal{M}_{\mathrm{II}}$ and $\mathcal{M}_{\mathrm{III}}$ are standard. □

Now we are going to consider a special class of von Neumann algebras.

Proposition 3.4. *Let $\mathcal{M} = \oplus_{i\in\Lambda}\mathcal{M}_i$, where each $\mathcal{M}_i$ is a factor von Neumann algebra. Then the following are equivalent:*

(i) $\mathcal{M}$ is standard;

(ii) there is an involutive map $\tau : \Lambda \to \Lambda$ such that $\mathcal{M}_i' \cong \mathcal{M}_{\tau(i)}^t$ for all i and, in particular, $\mathcal{M}_i$ is standard whenever $\tau(i) = i$;

(iii) there is a partition $\Lambda = \cup_{j\in\Gamma}\Lambda_j$ *of* Λ *such that* $\operatorname{card}\Lambda_j \leq 2$ *and* $\oplus_{i\in\Lambda_j}\mathcal{M}_i$ *is standard for all* $j \in \Gamma$.

Proof. Without loss of generality, we assume that $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$.

$(i) \Longrightarrow (ii)$. Since $\mathcal{M}$ is standard, there exists a conjugation C on $\mathcal{H}$ such that $C\mathcal{M}C = \mathcal{M}'$. Since $\mathcal{M} = \oplus_{i\in\Lambda}\mathcal{M}_i$, we have $\mathcal{M}' = \oplus_{i\in\Lambda}\mathcal{M}_i'$ and $\mathcal{Z}(\mathcal{M}) = \oplus_{i\in\Lambda}(\mathcal{M}_i \cap \mathcal{M}_i')$. Let P_i denote the identity of $\mathcal{M}_i$. Note that $C\mathcal{Z}(\mathcal{M})C = \mathcal{Z}(\mathcal{M})$. Then, for each $i \in \Lambda$, there is a unique $\tau(i) \in \Lambda$ such that $CP_iC = P_{\tau(i)}$. Since $X \mapsto CXC$ is involutive, one can see that $\tau : \Lambda \to \Lambda$ is an involutive map.

Now we fix some $i \in \Lambda$. Note that $CP_iC = P_{\tau(i)}$ and $CP_{\tau(i)}C = P_i$. Let $P = P_i \oplus P_{\tau(i)}$. Then $CPC = P$, $CP = PC$ and C has the decomposition $C = C_0 \oplus C_1$ relative to the decomposition $\mathcal{H} = \operatorname{ran} P \oplus \operatorname{ran}(I - P)$; moreover,

$$C_0 = \begin{bmatrix} 0 & D^{-1} \\ D & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_i \\ \mathcal{H}_{\tau(i)} \end{matrix},$$

where D is antiunitary. Note that

$$C_0(\mathcal{M}_i \oplus \mathcal{M}_{\tau(i)})C_0 = CP\mathcal{M}C = PC\mathcal{M}C = \mathcal{M}_i' \oplus \mathcal{M}_{\tau(i)}'$$

and

$$C_0(\mathcal{M}_i \oplus \mathcal{M}_{\tau(i)})C_0 = D^{-1}\mathcal{M}_{\tau(i)}D \oplus D\mathcal{M}_iD^{-1}.$$

Hence $\mathcal{M}_i' = D^{-1}\mathcal{M}_{\tau(i)}D$ and $E\mathcal{M}_i'E = (ED^{-1})\mathcal{M}_{\tau(i)}(DE)$ for any conjugation E on the underlying space of $\mathcal{M}_i$. Since DE is unitary, we conclude that $\mathcal{M}_i' \cong \mathcal{M}_{\tau(i)}^t$. If $\tau(i) = i$, then $\mathcal{M}_i' \cong \mathcal{M}_i^t$, that is, $\mathcal{M}_i$ is standard.

$(ii) \Longrightarrow (iii)$. Clearly, τ induces a partition $\Lambda = \cup_{j\in\Gamma}\Lambda_j$ of Λ, where each Λ_j has the form $\{k, \tau(k)\}$ for some $k \in \Lambda$. By Lemma 1.6, $\oplus_{i\in\Lambda_j}\mathcal{M}_i$ is standard for each $j \in \Gamma$.

$(iii) \Longrightarrow (i)$. Since $\mathcal{M} = \oplus_{j\in\Gamma}(\oplus_{i\in\Lambda_j}\mathcal{M}_i)$, it follows immediately that $\mathcal{M}$ is standard. □

Now we are going to give an example of standard von Neumann algebra.

The group measure space construction, as a special case of the W^*-crossed product, is the earliest non-type I factor construction and was used for the first time by Murray and von Neumann [11–13].

Recall the definition of the W^*-crossed product as follows (see [1, 15] for further details). Given a von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ and a group G, denote by $Aut(\mathcal{A})$ the set of all automorphisms of $\mathcal{A}$. A homomorphism $\alpha : G \to Aut(\mathcal{A})$ is called an *action* of G on $\mathcal{A}$. Suppose that G is countable and discrete. Consider $\mathcal{R} := \mathcal{H} \otimes l^2(G)$ as the square summable functions from G into $\mathcal{H}$.

Define the representations $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{R})$ and $u : G \to \mathcal{B}(\mathcal{R})$ as follows, for $\xi \in \mathcal{R}, a \in \mathcal{A}, g, h \in G$,

$$(\pi(a)\xi)(g) = \alpha_g^{-1}(a)(\xi(g))$$

and

$$(u(g)\xi)(h) = \xi(g^{-1}h).$$

It is clear that G gets mapped into the unitary group of $\mathcal{B}(\mathcal{R})$. A straightforward computation yields that

$$u(g)\pi(a)u(g)^* = \pi(\alpha_g(a))$$

for any $a \in \mathcal{A}, g \in G$.

In fact,

$$\begin{aligned} u(g)\pi(a)u(g)^*\xi(h) &= \pi(a)u(g)^*\xi(g^{-1}h) \\ &= \alpha_{g^{-1}h}^{-1}(a)(u(g)^*\xi(g^{-1}h)) \\ &= \alpha_h^{-1}\alpha_g(a)(\xi(h)) \\ &= \pi(\alpha_g(a))\xi(h) \end{aligned}$$

for any $\xi \in \mathcal{R}$ and $h \in G$.

Definition 3.5. *The von Neumann algebra generated by $\pi(\mathcal{A})$ and $u(G)$ is called the* crossed product *of $\mathcal{A}$ by G with respect to α, denoted by $\mathcal{R}(\mathcal{A}, G, \alpha)$. When $\mathcal{A} = \mathbb{C}$, we denote it by $\mathcal{R}(G)$ and call it the* group von Neumann algebra *of G.*

We know that if G is a discrete group and only have infinite conjugacy classes (ICC, for short), then $\mathcal{R}(G)$ is a type II_1 factor [1, III.3.3.7].

Let G be a locally compact group with the left Haar measure μ and a modular function Δ on G. Let u (resp. v) be the left (resp. right) regular representation of G on $L^2(G, \mu)$ defined by

$$[u(g)f](h) = f(g^{-1}h)$$

and

$$[v(g)f](h) = \Delta(g)^{\frac{1}{2}} f(hg)$$

for $f \in L^2(G)$ and $g \in G$.

Assume $\mathcal{R}(G)$ and $\mathcal{L}(G)$ are the von Neumann algebras generated by $\{u(g) : g \in G\}$ and $\{v(g) : g \in G\}$ respectively. Then $\mathcal{R}(G)' = \mathcal{L}(G)$.

Remark 3.6. *(i) For $g \in G$ and $f \in L^2(G)$, we have*

$$\int_G f(gh)d\mu(h) = \int_G f(h)d\mu(h)$$

and

$$\Delta(g) \int_G f(hg)d\mu(h) = \int_G f(h)d\mu(h).$$

(ii) Let $\nu(E) = \mu(E^{-1})$ for $E \subseteq G$. Then ν is a right Haar measure and $d\nu(h) = \Delta(h^{-1})d\mu(h)$. It follows that

$$\int_G f(h)d\nu(h) = \int_G f(h^{-1})d\mu(h)$$

and

$$\int_G f(h^{-1})\Delta(h^{-1})d\mu(h) = \int_G f(h)d\mu(h).$$

Example 3.7. *Assume that G is a locally compact group. Then $\mathcal{R}(G)$ is standard. In fact, let C be the conjugation on $L^2(G)$ defined by*

$$Cf(g) = \Delta(g^{-1})^{\frac{1}{2}} \overline{f(g^{-1})}.$$

Then, for $f \in L^2(G)$ and $g, h \in G$,

$$\begin{aligned}
(Cu(g)Cf)(h) &= \Delta(h^{-1})^{\frac{1}{2}} \overline{(u(g)Cf)(h^{-1})} \\
&= \Delta(h^{-1})^{\frac{1}{2}} \overline{(Cf)(g^{-1}h^{-1})} \\
&= \Delta(h^{-1})^{\frac{1}{2}} \Delta(hg)^{\frac{1}{2}} f(hg) \\
&= \Delta(g)^{\frac{1}{2}} f(hg) \\
&= (v(g)f)(h).
\end{aligned}$$

That is, $Cu(g)C = v(g)$ for any $g \in G$. It follows that $\mathcal{R}(G)$ is standard.

Example 3.8. *Let G be a discrete ICC group. Then $\mathcal{R}(G)$ is a type II_1 factor which is standard. Since*

$$(\mathcal{R}(G) \otimes \mathcal{B}(\mathcal{H}))' = \mathcal{L}(G) \otimes \mathbb{C}I$$

is not isomorphic to $\mathcal{R}(G) \otimes \mathcal{B}(\mathcal{H})$. It follows that $\mathcal{R}(G) \otimes \mathcal{B}(\mathcal{H})$ is a type II_∞ factor which is not standard.

We end this section with some remarks on standard von Neumann algebras of type II and type III.

Note that there exist standard type II (resp. type III) von Neumann algebras, since $\mathcal{A}' \oplus \mathcal{A}^t$ is a standard type II (resp. type III) von Neumann algebra if $\mathcal{A}$ is a von Neumann algebra of type II (resp. type III). However, due to the complexity of the structure of type II and type III von Neumann algebras, it is difficult for us to determine whether each standard type II (resp. type III) von Neumann algebra is of the form $\mathcal{A}' \oplus \mathcal{A}^t$.

By the following lemma, in order to characterize the structure of standard type II von Neumann algebras, it suffices to consider finite von Neumann algebras. Recall that a von Neumann algebra $\mathcal{M}$ is said to be *finite*, if the identity of $\mathcal{M}$ is a finite projection in $\mathcal{M}$. If every nonzero central projection $Z \in \mathcal{M}$ is infinite, then $\mathcal{M}$ is said to be *properly infinite*. If $Z_{\mathrm{III}} = 0$ in Lemma 3.1, then $\mathcal{M}$ is said to be *semifinite*.

Lemma 3.9 ([15, Prop.1.40]). *If $\mathcal{M}$ is a properly infinite but semifinite von Neumann algebra, then there exists an orthogonal family $\{z_\alpha\}$, indexed by infinite cardinals no greater than* card $\mathcal{M}$, *of central projections with $\sum_\alpha z_\alpha = I$, and a family $\{\mathcal{N}_\alpha\}$ of finite von Neumann algebras such that*

$$\mathcal{M}_{z_\alpha} \cong \mathcal{N}_\alpha \otimes \mathcal{B}(\mathcal{H}_\alpha),$$

where $\dim \mathcal{H}_\alpha = \alpha$ *and z_α may be zero. This family $\{z_\alpha\}$ is unique while $\mathcal{N}_\alpha$ is not unique. If $\mathcal{M}$ is σ-finite, then for any finite projection f with $z(f) = I$,*

$$\mathcal{M} \cong \mathcal{M}_f \otimes \mathcal{B}(\mathcal{H}_0),$$

where $\dim \mathcal{H}_0 = \aleph_0$.

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