Filomat 39:19 (2025), 6603–6628
https://doi.org/10.2298/FIL2519603D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The solutions of the Sylvester-like quaternion matrix equation $AX^{\varepsilon} + X^{\delta}B = 0$

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Abstract. In this paper, we discuss the Sylvester-like quaternion matrix equation $AX^{\varepsilon} + X^{\delta}B = 0$, where $\varepsilon \in \{\mathbb{I}, \mathbb{C}\}, \delta \in \{\dagger, *\}$ and $\mathbb{I},\mathbb{C},\dagger,*$ denote the identity mapping, involutive automorphism, involutive anti-automorphism and transpose, involutive automorphism and anti-automorphism and transpose, respectively. Firstly, we transform the given equation into the new equation $\widetilde{A}Y^{\varepsilon} + Y^{\delta}\widetilde{B} = 0$ with complex coefficient matrices $\widetilde{A}, \widetilde{B}$ and unknown quaternion matrix Y by utilizing the regularity of the matrix pencil $(A, B^{\varepsilon\delta})$, where $\widetilde{A} = PAQ$ and $\widetilde{B} = Q^{\delta\varepsilon}BP^{\delta\varepsilon}$ with P, Q being two nonsingular quaternion matrices. Secondly, we decouple the transformed equation into some systems of small-scale equations in terms of Kronecker canonical form of $(\widetilde{A}, \widetilde{B}^{\varepsilon\delta})$. Moreover, we also show that the solution can be gotten in terms of P, Q, the Kronecker canonical form of $(\widetilde{A}, \widetilde{B}^{\varepsilon\delta})$ and the two nonsingular quaternion matrices which transform $(\widetilde{A}, \widetilde{B}^{\varepsilon\delta})$ into its Kronecker canonical form. Thirdly, we determine the dimension of the solution space of the equation in terms of the sizes of the blocks arising in the Kronecker canonical form. Moreover, we give the necessary and sufficient condition for the existence of the unique solution. Finally, we also present a concrete example to demonstrate the process of calculating the solution of the considered matrix equation.

1. Introduction

Throughout this paper, we will adopt the following notations. Let $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex fields, respectively. For the complex $z = a + b\mathrm{i}$, we will use $\mathrm{Re}(z)$, $\mathrm{Im}(z)$ and $\bar{z}$ to denote its real part a, imaginary part b and complex conjugate $a - b\mathrm{i}$, respectively. Let $h = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$ denote a quaternion, where $a, b, c, d \in \mathbb{R}$ and $\mathrm{i}, \mathrm{j}, \mathrm{k}$ are three imaginary units with

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1, \mathrm{ij} = -\mathrm{ji} = \mathrm{k}, \mathrm{jk} = -\mathrm{kj} = \mathrm{i}, \mathrm{ki} = -\mathrm{ik} = \mathrm{j}.$$

2020 *Mathematics Subject Classification*. Primary 15A06; Secondary 15A22, 15A24, 15A30, 65F05.
Keywords. Sylvester-like matrix equation, Quaternion, Involutive automorphism and anti-automorphism, Solution space, Kronecker canonical form.
Received: 07 November 2024; Revised: 18 May 2025; Accepted: 31 May 2025
Communicated by Dijana Mosić
The work of Liqiang Dong was supported by Natural Science Basic Research Program of Shaanxi (Program No.2023-JC-QN-0067) and Chinese Universities Scientific Fund (2452020194). The work of Jicheng Li was supported by National Natural Science Foundation of China (12171384).
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Let $\mathbb{H}$ denote the set of all quaternions. $\overline{h} = a - b\mathrm{i} - c\mathrm{j} - d\mathrm{k}$ denotes the quaternion conjugate of h. Obviously, $h = a + b\mathrm{i} + (c + d\mathrm{i})\mathrm{j}$. Let $\mathrm{C}(h)$ and $\mathrm{C_j}(h)$ denote the first complex $a + b\mathrm{i}$ and the second complex $c + d\mathrm{i}$ of h, respectively. Then, $h = \mathrm{C}(h) + \mathrm{C_j}(h)\mathrm{j}$. Let $\mathbb{F}^{m\times n}$ denote the set of all $m \times n$ matrices whose elements belong to $\mathbb{F}$, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Let I_n denote the identity matrix of order n, and it is also abbreviated as I if the size is clear from the contents. Let $\bar{A}, A^T, A^*$ and A^{-1} denote the conjugate, transpose, transpose conjugate and the inverse matrix of the complex matrix A, respectively. We use $\mathbb{I}$ to denote the identity mapping. e_i denotes the vector whose i-th component is 1 and the remaining components are zero, and its dimension can be obtained from the contents. For a set S, we use $|S|$ to denote its cardinality.

An involutive automorphism of $\mathbb{F}$ is a bijection $a \mapsto a^{\mathbb{C}}$ of $\mathbb{F}$ onto itself, satisfying

$$(a + b)^{\mathbb{C}} = a^{\mathbb{C}} + b^{\mathbb{C}}, (ab)^{\mathbb{C}} = a^{\mathbb{C}}b^{\mathbb{C}}, (a^{\mathbb{C}})^{\mathbb{C}} = a, \ for\ all\ a, b \in \mathbb{F}.$$

An involutive anti-automorphism of $\mathbb{F}$ is a bijection $a \mapsto a^{\diamond}$ of $\mathbb{F}$ onto itself, satisfying

$$(a + b)^{\diamond} = a^{\diamond} + b^{\diamond}, (ab)^{\diamond} = b^{\diamond}a^{\diamond}, (a^{\diamond})^{\diamond} = a, \ for\ all\ a, b \in \mathbb{F}.$$

The above definitions can be seen in [3–5]. Obviously, the complex conjugate is not only an involutive automorphism, but also an involutive anti-automorphism of $\mathbb{C}$. Besides, the quaternion conjugate is an involutive anti-automorphism of $\mathbb{H}$.

For each quaternion matrix A over $\mathbb{F}$, we define

$$A^{\dagger} := (A^{\diamond})^T, A^* := ((A^{\mathbb{C}})^{\diamond})^T.$$

In this paper, we will consider the Sylvester-like quaternion matrix equation

$$AX^{\varepsilon} + X^{\delta}B = 0, \tag{1}$$

where $\varepsilon, \delta \in \{\mathbb{I}, \mathbb{C}, \dagger, *\}$, and the sizes of A and B will be determined according to the contexts and the specific situations.

We have three important observations about the notations $\varepsilon, \delta \in \{\mathbb{I}, \mathbb{C}, \dagger, *\}$. Firstly, if $\mathbb{F} = \mathbb{R}$, then both the involutive automorphism and anti-automorphism are identity mappings, and $\{\mathbb{I}, \mathbb{C}, \dagger, *\} = \{\mathbb{I}, T\}$, thus $\varepsilon, \delta \in \{\mathbb{I}, T\}$. Secondly, if $\mathbb{F} = \mathbb{C}$, then the complex conjugate is not only an involutive automorphism, but also an involutive anti-automorphism. So $A^{\mathbb{C}} := \overline{A}$ is the complex conjugate of A, $A^{\dagger} = A^* := (\overline{A})^T$ is the complex conjugate and transpose of A. Finally, if $\mathbb{F} = \mathbb{H}$, then, for $h = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$, each involutive automorphism is either the identity mapping $\mathbb{I}$, or $h \mapsto \mathrm{C}(h) - \mathrm{C_j}(h)\mathrm{j} = a + b\mathrm{i} - c\mathrm{j} - d\mathrm{k}$, and each involutive anti-automorphism is either $h \mapsto \overline{\mathrm{C}(h)} + \mathrm{C_j}(h)\mathrm{j} = a - b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$, or $h \mapsto \overline{h} = \overline{\mathrm{C}(h)} - \mathrm{C_j}(h)\mathrm{j} = a - b\mathrm{i} - c\mathrm{j} - d\mathrm{k}$, see [3, 4]. Based on the above observations, for $h = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$, we define

$$h^{\mathbb{I}} := a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k},$$
$$h^{\mathbb{C}} := a + b\mathrm{i} - c\mathrm{j} - d\mathrm{k},$$
$$h^{\diamond} := a - b\mathrm{i} + c\mathrm{j} + d\mathrm{k},$$

and

$$A^{\dagger} := (A^{\diamond})^T, A^* := ((A^{\mathbb{C}})^{\diamond})^T.$$

Then $\overline{h} = (h^{\mathbb{C}})^{\diamond} = a - b\mathrm{i} - c\mathrm{j} - d\mathrm{k}$ and $A^* = (\overline{A})^T$. The above statements can be found in [4].

Hodges [7] firstly studied the matrix equation $A^TX + X^TA = 0$ in which simultaneously involves X and its transpose X^T in 1957. From then on, more and more scholars were devoted to exploring its variants in which simultaneously involves X and its transpose X^T in the real field or its conjugate transpose X^* in the complex field, see [1, 9, 11, 13–15, 17–19, 21–25]. More specifically, Terán et al. [13, 18] researched $AX + X^TB = C$ in the real field. Moreover, Terán et al. [17, 19, 23] also studied $AX + X^*B = C$ in the complex field and gave the dimensions of the solution spaces of $AX+X^TB=0$ and $AX+X^*B=0$, respectively. Chiang et al. [1] presented solvability conditions and stable numerical methods for $AX\pm X^*B^*=C$ and its special cases $AX\pm X^*A^*=C$ and its generalization $AXB^*+CX^*D^*=E$. Besides, Wang et al. [9, 15, 24, 25] devised two iterative algorithms for the minimal Frobenius norm least squares solution of $AXB+CX^TD=0$. Liang et al. [11] extended the corresponding results to the systems $A_iXB_i+C_iX^TD_i=M_i, i=1,2$. Terán et al. [21, 22] gave the necessary and sufficient conditions for the existence of a unique solution of $AXB+CX^*D=0$ in the complex field. Besides, Song et al. [14] devised an iterative algorithm for the generalization $\sum_{i=1}^{r}A_iXB_i+\sum_{j=1}^{s}C_jX^*D_j=E$ in the complex field.

With the proposal of quaternions, many scholars were beginning to extend corresponding results about matrix equations from the complex field to the set of quaternions, see [2, 3, 5, 8, 10, 12, 26–29]. More specifically, Yuan et al. [8, 26–29] studied the quaternion matrix equation $AXB+CXD=E$. Li et al. [12] proposed an efficient algorithm for the reflexive solution of the quaternion matrix equation $AXB+CX^*D=E$. Jiang et al. [10] characterized the existence of the solutions and also drew the closed-form solutions for the quaternion matrix equation $X-A\widetilde{X}B=C$ in sense of j-conjugate. Futorny et al. [5] gave Roth's solvability criteria for the quaternion matrix equations $AX-X^{\mathbb{C}}B=C$ and $X-AX^{\mathbb{C}}B=C$ in sense of involutive automorphism. Besides, Dmytryshyn et al. [2] extended Roth's criteria to the systems $A_iX_{i'}M_i-N_iX_{i''}^{\varepsilon_i}B_i=C_i, i=1,2,\cdots,s$ with unknown $X_1,X_2,\cdots,X_t$ over a field of characteristic not 2 in which $i',i''\in\{1,2,\cdots,t\}$ and $X_{i''}^{\varepsilon_i}\in\{X_{i''},X_{i''}^T,X_{i''}^*\}$. Besides, Dmytryshyn et al. [3] also extended the criterion to the generalized systems

$$A_iX_{i'}^{\varepsilon_i}M_i-N_iX_{i''}^{\delta_i}B_i=C_i, i',i''\in\{1,2,\cdots,s\}, \tag{2}$$

where $\varepsilon_i,\delta_i\in\{\mathbb{I},\mathbb{C},\dagger,*\}$. Evidently, all of the aforementioned systems are special cases of (2). Though they gave the necessary and sufficient conditions for solvability, the uniqueness of solution has not been presented. So it is necessary to consider the dimension of the solution space of the corresponding homogeneous systems. Based on this motivation, Terán et al. [18, 19] considered the dimensions of the solution spaces of $AX+X^TA=0$ and $AX+X^*A=0$. Moreover, Terán et al. [16, 20] also gave the dimensions of the solution spaces of $AX+X^TB=0$ and $AX+BX^T=0$ in the real field, $AX+X^*B=0$ and $AX+BX^*=0$ in the complex field, respectively. Dong et al. [4] gave the dimension of the solution space of the quaternion matrix equation $AX^{\varepsilon}+BX^{\delta}=0$ for $\varepsilon\in\{\mathbb{I},\mathbb{C}\},\delta\in\{\dagger,*\}$. Wimmer et al. [6, 23] also pointed out that the aforementioned homogeneous equations played an important part in the symmetric or hermitian solution of a linear matrix equation. To the best of our knowledge, the solution space of the quaternion matrix equation $AX^{\varepsilon}+X^{\delta}B=0$ has not been researched yet. In this paper, we mainly concentrate on it. The results of others will be presented in follow-up researches.

Next, we discuss (1) for several different combinations of ε,δ. The first combination: if $\varepsilon=\delta=\mathbb{I}$ or $\varepsilon=\delta=\mathbb{C}$, then (1) becomes $AX+XB=0$ or $AX^{\mathbb{C}}+X^{\mathbb{C}}B=0$. By letting $Y=X^{\mathbb{C}}$, $AX^{\mathbb{C}}+X^{\mathbb{C}}B=0$ has the form $AY+YB=0$. Thus, (1) can be transformed into the matrix equation of the form $AX+XB=0$. The second combination: if $\varepsilon=\delta=\dagger$ or $\varepsilon=\delta=*$, then (1) becomes $A(X^{\diamond})^T+(X^{\diamond})^TB=0$ or $A((X^{\mathbb{C}})^{\diamond})^T+((X^{\mathbb{C}})^{\diamond})^TB=0$. By letting $Y=(X^{\diamond})^T$ or $Y=((X^{\mathbb{C}})^{\diamond})^T$, the two equations have the same form $AY+YB=0$. Thus, (1) can also be transformed into the matrix equation of the form $AX+XB=0$. The third combination: if $\varepsilon=\mathbb{I},\delta=\mathbb{C}$ or $\varepsilon=\mathbb{C},\delta=\mathbb{I}$, then (1) becomes $AX+X^{\mathbb{C}}B=0$ or $AX^{\mathbb{C}}+XB=0$. Obviously, (1) can be transformed into the matrix equation of the form $AX+X^{\mathbb{C}}B=0$. The fourth combination: if $\varepsilon=\dagger,\delta=*$ or $\varepsilon=*,\delta=\dagger$, then (1) becomes $A(X^{\diamond})^T+((X^{\diamond})^{\mathbb{C}})^TB=0$ or $A((X^{\diamond})^{\mathbb{C}})^T+(X^{\diamond})^TB=0$. By letting $Y=(X^{\diamond})^T$ or $Y=((X^{\diamond})^{\mathbb{C}})^T$, the two equations can be transformed into $AY+Y^{\mathbb{C}}B=0$. Thus (1) can also be transformed into the matrix equation of the form $AX+X^{\mathbb{C}}B=0$.

Based on the above discussions, the quaternion matrix equation (1) can be transformed into the quaternion matrix equation of the form $AX+XB=0$ or $AX+X^{\mathbb{C}}B=0$ or

$$AX^{\varepsilon}+X^{\delta}B=0, \tag{3}$$

where $A\in\mathbb{H}^{m\times n}, B\in\mathbb{H}^{n\times m}, X\in\mathbb{H}^{n\times m}, \varepsilon\in\{\mathbb{I},\mathbb{C}\},\delta\in\{\dagger,*\}$. The quaternion matrix equation $AX+XB=0$ has been studied by many scholars, see [4, 26–29]. The equation $AX+X^{\mathbb{C}}B=0$ will be researched in other papers. In this paper, we mainly concentrate on (3).

The remaining parts are organized as follows. Section 2 presents some preliminary works. In Section 3, we transform (3) into a quaternion matrix equation with complex coefficients, and also give the relationship between the original solution and that of the transformed equation. Besides, Section 3 also decouples the transformed equation into some subsystems of small-scale equations in terms of Kronecker canonical form. In Section 4, we give our main results about the dimension of the solution space of (3). More specifically, Subsection 4.1 and 4.2 give the dimensions of the solution spaces of the subsystems in terms of single blocks and pairs of blocks arising in Kronecker canonical form, respectively. Section 5 gives the necessary and sufficient condition for the existence of a unique solution. Moreover, Section 6 presents a concrete example to demonstrate the process of calculating the solution of the matrix equation $AX^{\varepsilon}+X^{\delta}B=C$. Finally, Section 7 gives our conclusions and lines of future work.

2. Preliminaries

Lemma 2.1. *[4, 8] If $A\in\mathbb{H}^{n\times n}$, then A is similar to a complex Jordan normal form matrix $J\in\mathbb{C}^{n\times n}$ with diagonal elements of the form $a+b\mathrm{i}$ with $a\in\mathbb{R}, b\geq 0$.*

Definition 2.2. *[8] Let $A,B\in\mathbb{H}^{n\times n}$. If there exists $\lambda_0\in\mathbb{R}$ such that $A+\lambda_0 B$ is a nonsingular quaternion matrix, then (A,B) is called a linear regular matrix pencil.*

Theorem 2.3. *[8] Let $A,B\in\mathbb{H}^{n\times n}$. If (A,B) is a linear regular matrix pencil, then there exists two nonsingular quaternion matrices $P,Q\in\mathbb{H}^{n\times n}$ such that PAQ and PBQ are two complex matrices.*

Lemma 2.4. *[4] Let $A\in\mathbb{H}^{m\times s}, B\in\mathbb{H}^{s\times n}, \varepsilon\in\{\mathbb{I},\mathbb{C}\}, \delta\in\{\dagger,*\}$. Then*

(1) $(AB)^{\varepsilon}=A^{\varepsilon}B^{\varepsilon}, (AB)^{\delta}=B^{\delta}A^{\delta}$;

(2) $(A^{\varepsilon})^{\delta}=(A^{\delta})^{\varepsilon}$; In following statements, we define $A^{\varepsilon\delta}:=(A^{\varepsilon})^{\delta}$, $A^{\delta\varepsilon}:=(A^{\delta})^{\varepsilon}$;

(3) $(A^{\varepsilon})^{\varepsilon}=A, (A^{\delta})^{\delta}=A$;

(4) For the square quaternion matrix A, A is nonsingular if and only if A^{δ} is nonsingular if and only if A^{ε} is nonsingular; Furthermore,

$$(A^{\delta})^{-1}=(A^{-1})^{\delta}, (A^{\varepsilon})^{-1}=(A^{-1})^{\varepsilon}.$$

In following statements, we will use $A^{-\delta}$ and $A^{-\varepsilon}$ to denote $(A^{\delta})^{-1}$ and $(A^{\varepsilon})^{-1}$, respectively.

Lemma 2.5. *[3] Let $X\in\mathbb{H}^{m\times n}$ be such that $X=AXB$, where $A\in\mathbb{H}^{m\times m}, B\in\mathbb{H}^{n\times n}$. If at least one of A and B is nilpotent, then $X=0$.*

Lemma 2.6. *Let $A,B\in\mathbb{H}^{n\times n}, \varepsilon\in\{\mathbb{I},\mathbb{C}\}, \delta\in\{\dagger,*\}$. If the matrix product of A^{ε} and B^{δ} is commutative and at least one of A and B is nilpotent, then the quaternion matrix equation $X^{\varepsilon}+AX^{\delta}B=0$ has the unique trivial solution $X=0$.*

Proof. It is obvious that the solution X must satisfy $X^{\varepsilon}=-AX^{\delta}B$. Taking $(\cdot)^{\varepsilon}$ and then $(\cdot)^{\delta}$, we can get

$$X^{\delta}=-B^{\varepsilon\delta}X^{\varepsilon}A^{\varepsilon\delta}.$$

Substituting this into the original identity, we get $X^{\varepsilon}=AB^{\varepsilon\delta}X^{\varepsilon}A^{\varepsilon\delta}B$. Taking $(\cdot)^{\varepsilon}$, we have

$$X=A^{\varepsilon}B^{\delta}XA^{\delta}B^{\varepsilon}.$$

If A (or B) is nilpotent, then A^{ε} (or B^{δ}) is also evidently nilpotent. Furthermore, $A^{\varepsilon}B^{\delta}$ is also nilpotent because of $A^{\varepsilon}B^{\delta}=B^{\delta}A^{\varepsilon}$. The result is immediate from Lemma 2.5. □

Definition 2.7. *[4, 16] If there exists two nonsingular complex matrices $M\in\mathbb{C}^{m\times m}, N\in\mathbb{C}^{n\times n}$ such that $M(A+\lambda B)N=C+\lambda D$, where $A,B,C,D\in\mathbb{C}^{m\times n}, \lambda\in\mathbb{C}$, then the two matrix pencils $A+\lambda B$ and $C+\lambda D$ are called strictly equivalent.*

Theorem 2.8 (Kronecker canonical form). *[4, 16–20] Each complex matrix pencil $A + \lambda B$ with $A, B \in \mathbb{C}^{m\times n}, \lambda \in \mathbb{C}$ is strictly equivalent to a direct sum of blocks of the following matrix pencils:*

(1) Right singular blocks:

$$L_\phi = \begin{pmatrix} \lambda & 1 & & & & \\ & \lambda & 1 & & & \\ & & \ddots & \ddots & & \\ & & & \lambda & 1 & \\ & & & & \lambda & 1 \end{pmatrix}_{\phi\times(\phi+1)}.$$

(2) Left singular blocks: L_η^T, where L_η is a right singular block.

(3) Finite blocks: $J_k(\mu) + \lambda I_k$, where $J_k(\mu)$ is a Jordan block of size $k \times k$ associated with $\mu \in \mathbb{C}$, i.e.,

$$J_k(\mu) = \begin{pmatrix} \mu & 1 & & & \\ & \mu & 1 & & \\ & & \ddots & \ddots & \\ & & & \mu & 1 \\ & & & & \mu \end{pmatrix}_{k\times k}.$$

(4) Infinite blocks: $N_u := I_u + \lambda J_u(0)$.

The pencil is uniquely determined, up to permutation of blocks, and is called as the Kronecker canonical form (KCF) of $A + \lambda B$.

Let

$$A_\phi = \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 0 & 1 & \\ & & & 0 & 1 \end{pmatrix}_{\phi\times(\phi+1)}, \qquad B_\phi = \begin{pmatrix} 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & 1 & 0 & \\ & & & 1 & 0 \end{pmatrix}_{\phi\times(\phi+1)}.$$

A_ϕ and B_ϕ can be partitioned into $A_\phi = \begin{pmatrix} J_\phi(0) & e_\phi \end{pmatrix}$ and $B_\phi = \begin{pmatrix} I_\phi & 0_\phi \end{pmatrix}$, respectively. Obviously, $L_\phi = A_\phi + \lambda B_\phi$. If there exists some block $J_k(-\mu) + \lambda I_k$ with $k > 0$ in the KCF of $A + \lambda B$, then $\mu \in \mathbb{C}$ is called as an eigenvalue of $A + \lambda B$.

3. Transforming $AX^\varepsilon + X^\delta B = 0$ into a quaternion matrix equation with complex coefficients

Theorem 3.1. *Assume that $A \in \mathbb{H}^{m\times n}, B \in \mathbb{H}^{n\times m}, \varepsilon \in \{\mathbb{I}, \mathbb{C}\}, \delta \in \{\dagger, *\}$. Let $(A, B^{\varepsilon\delta})$ be a regular quaternion matrix pencil, and P, Q be two nonsingular quaternion matrices such that $PAQ = \widetilde{A}$ and $Q^{\delta\varepsilon}BP^{\delta\varepsilon} = \widetilde{B}$ are two complex matrices. Then Y is a solution of the quaternion matrix equation $\widetilde{A}Y^\varepsilon + Y^\delta\widetilde{B} = 0$ if and only if $X = Q^\varepsilon Y P^{-\delta}$ is a solution of the quaternion matrix equation $AX^\varepsilon + X^\delta B = 0$. As a consequence, the solution spaces of both equations are isomorphic via $Y \mapsto Q^\varepsilon Y P^{-\delta} = X$.*

Proof. Let $X = Q^\varepsilon Y P^{-\delta}$, and then taking $(\cdot)^\varepsilon$ and $(\cdot)^\delta$, respectively, we obtain

$$X^\varepsilon = QY^\varepsilon P^{-\delta\varepsilon}, X^\delta = P^{-1}Y^\delta Q^{\varepsilon\delta} = P^{-1}Y^\delta Q^{\delta\varepsilon}.$$

By $\widetilde{A} = PAQ$ and $\widetilde{B} = Q^{\delta\varepsilon}BP^{\delta\varepsilon}$, we have

$$\begin{aligned}\widetilde{A}Y^{\varepsilon} + Y^{\delta}\widetilde{B} &= PAQY^{\varepsilon} + Y^{\delta}Q^{\delta\varepsilon}BP^{\delta\varepsilon}\\ &= PAQY^{\varepsilon}P^{-\delta\varepsilon}P^{\delta\varepsilon} + PP^{-1}Y^{\delta}Q^{\delta\varepsilon}BP^{\delta\varepsilon}\\ &= P(AQY^{\varepsilon}P^{-\delta\varepsilon} + P^{-1}Y^{\delta}Q^{\delta\varepsilon}B)P^{\delta\varepsilon}\\ &= P(AX^{\varepsilon} + X^{\delta}B)P^{\delta\varepsilon}.\end{aligned}$$

Since P and $P^{\delta\varepsilon}$ are nonsingular, $\widetilde{A}Y^{\varepsilon} + Y^{\delta}\widetilde{B} = 0$ if and only if $AX^{\varepsilon} + X^{\delta}B = 0$. The mapping $Y \mapsto Q^{\varepsilon}YP^{-\delta}$ is clearly linear and invertible, so it is an isomorphism. □

Theorem 3.1 points out that (3) can be transformed into a quaternion matrix equation $\widetilde{A}Y^{\varepsilon} + Y^{\delta}\widetilde{B} = 0$ with complex coefficient matrices $\widetilde{A}, \widetilde{B}$. Let $\widehat{A} + \lambda\widehat{B}^{\varepsilon\delta} = S(\widetilde{A} + \lambda\widetilde{B}^{\varepsilon\delta})T$ be the Kronecker canonical form of $(\widetilde{A}, \widetilde{B}^{\varepsilon\delta})$, where S, T are two nonsingular complex matrices. By taking again advantage of Theorem 3.1, we can obtain that Z is the solution of $\widehat{A}Z^{\varepsilon} + Z^{\delta}\widehat{B} = 0$ if and only if $Y = T^{\varepsilon}ZS^{-\delta}$ is the solution of $\widetilde{A}Y^{\varepsilon} + Y^{\delta}\widetilde{B} = 0$. In other words, there exists a bijection between the solution of $\widehat{A}Z^{\varepsilon} + Z^{\delta}\widehat{B} = 0$ and that of $\widetilde{A}Y^{\varepsilon} + Y^{\delta}\widetilde{B} = 0$. This bijection is $Z \mapsto T^{\varepsilon}ZS^{-\delta}$. Thus, the solution spaces of $\widehat{A}Z^{\varepsilon} + Z^{\delta}\widehat{B} = 0$ and $\widetilde{A}Y^{\varepsilon} + Y^{\delta}\widetilde{B} = 0$ are isomorphic via $Z \mapsto T^{\varepsilon}ZS^{-\delta} = Y$. Then $X = Q^{\varepsilon}YP^{-\delta} = Q^{\varepsilon}T^{\varepsilon}ZS^{-\delta}P^{-\delta} = (QT)^{\varepsilon}Z(P^{-1}S^{-1})^{\delta}$. Thus, after getting the solution Z of $\widehat{A}Z^{\varepsilon} + Z^{\delta}\widehat{B} = 0$, we can obtain the solution X of (3) in terms of the two nonsingular quaternion matrices P, Q and the two nonsingular complex matrices S, T. Briefly, we can obtain the solution of (3) in terms of four nonsingular matrices P, Q, S, T and the KCF of $(\widetilde{A}, \widetilde{B}^{\varepsilon\delta})$.

Theorem 3.2. *Let $\widehat{A} = A_1 \oplus A_2 \oplus \cdots \oplus A_d$ and $\widehat{B} = B_1 \oplus B_2 \oplus \cdots \oplus B_d$ be two block-diagonal matrices in $\mathbb{C}^{m\times n}$. Let $X = [X_{ij}]_{i,j=1}^{d}$ be partitioned conformally with the partition of $\widehat{A}$ and $\widehat{B}$. Then the quaternion matrix equation $\widehat{A}X^{\varepsilon} + X^{\delta}\widehat{B} = 0$ is equivalent to following several subsystems of the quaternion matrix equations:*
(i) d matrix equations:

$$A_iX_{ii}^{\varepsilon} + X_{ii}^{\delta}B_i = 0, \quad for \quad i = 1, 2, \cdots, d, \tag{4}$$

together with
(ii) $\frac{d(d-1)}{2}$ subsystems of 2 matrix equations with 2 unknown matrices X_{ij} and X_{ji}:

$$\begin{cases} A_iX_{ij}^{\varepsilon} + X_{ji}^{\delta}B_i = 0,\\ A_jX_{ji}^{\varepsilon} + X_{ij}^{\delta}B_j = 0,\end{cases} \quad for \quad i, j = 1, 2, \cdots, d \text{ and } i < j. \tag{5}$$

Theorem 3.2 is a generalization of Lemma 2.3 in [16]. Similar to the notations in [4, 16], we will denote the vector space of solutions of the equations (4) by $\mathcal{S}(A_i + \lambda B_i^{\varepsilon\delta})$, i.e.,

$$\mathcal{S}(A_i + \lambda B_i^{\varepsilon\delta}) := \{X_{ii} \in \mathbb{H}^{n\times n} | A_iX_{ii}^{\varepsilon} + X_{ii}^{\delta}B_i = 0\}.$$

Besides, we will also denote the solution spaces of the subsystems (5) by $\mathcal{S}(A_i + \lambda B_i^{\varepsilon\delta}, A_j + \lambda B_j^{\varepsilon\delta})$, i.e.,

$$\mathcal{S}(A_i + \lambda B_i^{\varepsilon\delta}, A_j + \lambda B_j^{\varepsilon\delta}) := \{(X_i, X_j) \in \mathbb{H}^{n\times n} \times \mathbb{H}^{n\times n} | A_iX_i^{\varepsilon} + X_j^{\delta}B_i = 0, A_jX_j^{\varepsilon} + X_i^{\delta}B_j = 0\}.$$

4. The dimension of the solution space of $AX^{\varepsilon} + X^{\delta}B = 0$

Theorem 4.1. *Let $A \in \mathbb{H}^{m\times n}, B \in \mathbb{H}^{n\times m}$ be two quaternion matrices, and let $(A, B^{\varepsilon\delta})$ be a regular matrix pencil. Let P, Q be two nonsingular quaternion matrices such that $PAQ = \widetilde{A}$ and $Q^{\delta\varepsilon}BP^{\delta\varepsilon} = \widetilde{B}$ are two complex matrices. Let* *the KCF of the matrix pencil* $\widetilde{A} + \lambda\widetilde{B}^{\varepsilon\delta}$ *be*

$$\begin{aligned}\widehat{A} + \lambda\widehat{B}^{\varepsilon\delta} =& L_{\phi_1} \oplus L_{\phi_2} \oplus \cdots \oplus L_{\phi_p} \\ & \oplus L_{\eta_1}^T \oplus L_{\eta_2}^T \oplus \cdots \oplus L_{\eta_q}^T \\ & \oplus N_{u_1} \oplus N_{u_2} \oplus \cdots \oplus N_{u_r} \\ & \oplus (J_{k_1}(\mu_1) + \lambda I_{k_1}) \oplus (J_{k_2}(\mu_2) + \lambda I_{k_2}) \oplus \cdots \oplus (J_{k_s}(\mu_s) + \lambda I_{k_s}).\end{aligned}$$

Then the real dimension of the solution space of (3) depends only on $\widehat{A} + \lambda\widehat{B}^{\varepsilon\delta}$*, and can be computed as*

$$d^\star_{Total} = d^\star_{right} + d^\star_{fin} + d^\star_{right,right} + d^\star_{fin,fin} + d^\star_{right,left} + d^\star_{right,\infty} + d^\star_{right,fin} + d^\star_{\infty,fin},$$

whose summands are given by:

1. *The real dimension due to the equations (4) corresponding to the right singular blocks:*

$$d^\star_{right} = 4\sum_{i=1}^{p} \phi_i.$$

2. *The real dimension due to the equations (4) corresponding to the finite blocks:*

$$d^\star_{fin} = \begin{cases} \sum_{i\in I_1}(2k_i) + \sum_{i\in I_2} k_i + \sum_{i\in I_3}(2k_i - 1) + \sum_{i\in I_4}(2k_i + 1) + \sum_{i\in I_5}(k_i - 1) \\ = 2(\sum_{i\in I_1} k_i + \sum_{i\in I_3} k_i + \sum_{i\in I_4} k_i) + \sum_{i\in I_2} k_i + \sum_{i\in I_5} k_i - |I_3| + |I_4| - |I_5|, \quad for\ \varepsilon = \mathbb{C}, \delta = * \ or\ \varepsilon = \mathbb{I}, \delta = \dagger, \\ \sum_{i\in I_1}(2k_i) + \sum_{i\in I_2} k_i + \sum_{i\in I_3}(2k_i + 1) + \sum_{i\in I_4}(2k_i - 1) + \sum_{i\in I_5}(k_i - 1) \\ = 2(\sum_{i\in I_1} k_i + \sum_{i\in I_3} k_i + \sum_{i\in I_4} k_i) + \sum_{i\in I_2} k_i + \sum_{i\in I_5} k_i + |I_3| - |I_4| - |I_5|, \quad for\ \varepsilon = \mathbb{C}, \delta = \dagger \ or\ \varepsilon = \mathbb{I}, \delta = *, \end{cases}$$

where

$$\begin{aligned} I_0 &= \{1, 2, \cdots, s\}, \\ I_1 &= \{i \in I_0 |\ \mu_i = \pm 1\ and\ k_i\ is\ even\}, \\ I_2 &= \{i \in I_0 |\ \mu_i^\delta \mu_i^\varepsilon = 1\ with\ \mathrm{Im}(\mu_i) \neq 0,\ and\ k_i\ is\ even\}, \\ I_3 &= \{i \in I_0 |\ \mu_i = 1,\ and\ k_i\ is\ odd\}, \\ I_4 &= \{i \in I_0 |\ \mu_i = -1,\ and\ k_i\ is\ odd\}, \\ I_5 &= \{i \in I_0 |\ \mu_i^\delta \mu_i^\varepsilon = 1\ with\ \mathrm{Im}(\mu_i) \neq 0,\ and\ k_i\ is\ odd\}. \end{aligned}$$

3. *The real dimension due to the subsystems (5) involving a pair of right singular blocks:*

$$d^\star_{right,right} = 4\sum_{i,j=1;i<j}^{p} (\phi_i + \phi_j).$$

4. *The real dimension due to the subsystems (5) involving a pair of finite blocks:*

$$d^\star_{fin,fin} = 2\sum_{i,j} \min\{k_i(k_i + 1), k_j(k_j + 1)\},$$

where the sum is taken over all pairs $J_{k_i}(\mu_i) + \lambda I_{k_i}, J_{k_j}(\mu_j) + \lambda I_{k_j}$ *of blocks in* $\widehat{A} + \lambda\widehat{B}$ *such that* $i < j$ *and* $\mu_i^\delta \mu_j^\varepsilon = 1$.

5. The real dimension due to the subsystems (5) involving a right singular block and a left singular block:

$$d^{\star}_{right,left} = 4\sum_{i,j}(\eta_j - \phi_i - 1),$$

where the sum is taken over all pairs $L_{\phi_i}, L^T_{\eta_j}$ of blocks in $\widehat{A} + \lambda\widehat{B}$ such that $\eta_j - \phi_i > 1$.

6. The real dimension due to the subsystems (5) involving a right singular block and an infinite block:

$$d^{\star}_{right,\infty} = 4\phi\sum_{i=1}^{r} u_i.$$

7. The real dimension due to the subsystems (5) involving a right singular block and a finite block:

$$d^{\star}_{right,fin} = 4\phi\sum_{i=1}^{s} k_i.$$

8. The real dimension due to the subsystems (5) involving an infinite block and a finite block:

$$d^{\star}_{\infty,fin} = 2\sum_{i,j}\min\{u_i(u_i+1), k_j(k_j+1)\},$$

where the sum is taken over all pairs $N_{u_i}, J_{k_j}(\mu_j) + \lambda I_{k_j}$ of blocks in $\widehat{A} + \lambda\widehat{B}$ with $\mu_j = 0$.

The proof of Theorem 4.1 can be completed by Lemmas 4.2-4.16 below in Subsections 4.1-4.2.

4.1. Dimensions of the solution spaces of the equations corresponding to single blocks

Lemma 4.2 (Right singular block). *The real dimension of the solution space of*

$$A_\phi X^\varepsilon + X^\delta B^{\varepsilon\delta}_\phi = 0 \tag{6}$$

is

$$\dim \mathcal{S}(L_\phi) = 4\phi.$$

The solution can be completely determined by the first row elements $x_{1i} \in \mathbb{H}, i = 1, 2, \cdots, \phi$ of X.

Proof. Set $X = [x_{ij}] \in \mathbb{H}^{(\phi+1)\times\phi}$. Then (6) becomes

$$\begin{pmatrix} x^\varepsilon_{21} & x^\varepsilon_{22} & \cdots & x^\varepsilon_{2\phi} \\ x^\varepsilon_{31} & x^\varepsilon_{32} & \cdots & x^\varepsilon_{3\phi} \\ \vdots & \vdots & \ddots & \vdots \\ x^\varepsilon_{\phi+1,1} & x^\varepsilon_{\phi+1,2} & \cdots & x^\varepsilon_{\phi+1,\phi} \end{pmatrix} + \begin{pmatrix} x^\delta_{11} & x^\delta_{21} & \cdots & x^\delta_{\phi1} \\ x^\delta_{12} & x^\delta_{22} & \cdots & x^\delta_{\phi2} \\ \vdots & \vdots & \ddots & \vdots \\ x^\delta_{1\phi} & x^\delta_{2\phi} & \cdots & x^\delta_{\phi\phi} \end{pmatrix} = 0_{\phi\times\phi}. \tag{7}$$

This equation (7) is equivalent to $x^\varepsilon_{ij} + x^\delta_{j,i-1} = 0, i = 2, 3, \cdots, \phi+1; j = 1, 2, \cdots, \phi$. Iterating this identity, we get $x^\varepsilon_{ij} = -x^\delta_{j,i-1}, x^\varepsilon_{j,i-1} = -x^\delta_{i-1,j-1}, x_{j,i-1} = -x^{\delta\varepsilon}_{i-1,j-1}$. Further, we can get

$$x^\varepsilon_{ij} = -(-x^{\delta\varepsilon}_{i-1,j-1})^\delta = x^\varepsilon_{i-1,j-1},$$

i.e., $x_{ij} = x_{i-1,j-1}, i = 2, 3, \cdots, \phi+1; j = 1, 2, \cdots, \phi$, which implies that X is a Toeplitz matrix, and $x_{j1} =$

$-x^{\delta\varepsilon}_{1,j-1}, j = 2, 3, \cdots, \phi + 1$. More specifically,

$$x_{j1} = \begin{cases} -x^{\diamond}_{1,j-1}, & if \ \ \varepsilon = \mathbb{I}, \delta = \dagger, \ or \ \ \varepsilon = \mathbb{C}, \delta = *, \\ -x^{\mathbb{C}\diamond}_{1,j-1}, & if \ \ \varepsilon = \mathbb{I}, \delta = *, \ or \ \ \varepsilon = \mathbb{C}, \delta = \dagger, \end{cases} \quad for \quad j = 2, 3, \cdots, \phi + 1.$$

Thus X can be determined by its first row elements $x_{1i} \in \mathbb{H}, i = 1, 2, \cdots, \phi$. On the other hand, since (7) consists of ϕ^2 equations and X has $(\phi + 1)\phi$ elements, the variables $x_{1i} \in \mathbb{H}, i = 1, 2, \cdots, \phi$ can be taken as free variables. Hence X has the following form

$$X = \begin{cases} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1,\phi-1} & x_{1\phi} \\ -x^{\diamond}_{11} & x_{11} & x_{12} & \cdots & x_{1,\phi-1} \\ -x^{\diamond}_{12} & -x^{\diamond}_{11} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & x_{12} \\ -x^{\diamond}_{1,\phi-1} & -x^{\diamond}_{1,\phi-2} & \cdots & -x^{\diamond}_{11} & x_{11} \\ -x^{\diamond}_{1\phi} & -x^{\diamond}_{1,\phi-1} & -x^{\diamond}_{1,\phi-2} & \cdots & -x^{\diamond}_{11} \end{pmatrix}, & if \ \varepsilon = \mathbb{I}, \delta = \dagger, \ or \ \varepsilon = \mathbb{C}, \delta = *, \\ \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1,\phi-1} & x_{1\phi} \\ -x^{\mathbb{C}\diamond}_{11} & x_{11} & x_{12} & \cdots & x_{1,\phi-1} \\ -x^{\mathbb{C}\diamond}_{12} & -x^{\mathbb{C}\diamond}_{11} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & x_{12} \\ -x^{\mathbb{C}\diamond}_{1,\phi-1} & -x^{\mathbb{C}\diamond}_{1,\phi-2} & \cdots & -x^{\mathbb{C}\diamond}_{11} & x_{11} \\ -x^{\mathbb{C}\diamond}_{1\phi} & -x^{\mathbb{C}\diamond}_{1,\phi-1} & -x^{\mathbb{C}\diamond}_{1,\phi-2} & \cdots & -x^{\mathbb{C}\diamond}_{11} \end{pmatrix}, & if \ \varepsilon = \mathbb{I}, \delta = *, \ or \ \varepsilon = \mathbb{C}, \delta = \dagger, \end{cases}$$

with $x_{1i} \in \mathbb{H}, i = 1, 2, \cdots, \phi$ arbitrary, and this is the general solution of (6). Then the result follows. □

Lemma 4.3 (Left singular block). *The real dimension of the solution space of*

$$A^T_\eta X^\varepsilon + X^\delta (B^T_\eta)^{\varepsilon\delta} = 0 \tag{8}$$

is

$$\dim \mathcal{S}(L^T_\eta) = 0.$$

The solution of (8) is $X = 0$.

Proof. Multiplying A_η from the left side of (8) and utilizing the property

$$A_\eta A^T_\eta = I_\eta, \tag{9}$$

we can get

$$X^\varepsilon + A_\eta X^\delta (B^T_\eta)^{\varepsilon\delta} = 0. \tag{10}$$

Taking $(\cdot)^\varepsilon$ and then $(\cdot)^\delta$, we can obtain $X^\delta + B^T_\eta X^\varepsilon A^{\varepsilon\delta}_\eta = 0$. Substituting this into (10), we can get $X^\varepsilon - A_\eta B^T_\eta X^\varepsilon A^{\varepsilon\delta}_\eta (B^T_\eta)^{\varepsilon\delta} = 0$, i.e.,

$$X^\varepsilon - A_\eta B^T_\eta X^\varepsilon (B^T_\eta A_\eta)^{\varepsilon\delta} = 0, \tag{11}$$

Utilizing the properties

$$A_\eta B_\eta^T = J_\eta(0), B_\eta^T A_\eta = J_{\eta+1}(0), \tag{12}$$

the equation (11) becomes $X^\varepsilon - J_\eta(0)X^\varepsilon(J_{\eta+1}(0))^{\varepsilon\delta} = 0$. Obviously, $J_\eta(0)$ is nilpotent. Lemma 2.6 implies $X = 0$, and the result follows.
□

Lemma 4.4 (Infinite block). *The real dimension of the solution space of*

$$X^\varepsilon + X^\delta J_u(0)^{\varepsilon\delta} = 0 \tag{13}$$

is

$$\dim \mathcal{S}(N_u) = 0.$$

The solution of (13) is $X = 0$.

Proof. Obviously, $J_u(0)$ is nilpotent. Lemma 2.6 implies $X = 0$, and the result follows.
□

Lemma 4.5 below presents the solution of matrix equation $J_k(\mu)X^\varepsilon + X^\delta = 0$, see Lemma 4.4 in [4], and its detailed proof can be found in Lemmas 8.1-8.4 in [4].

Lemma 4.5 (Finite block). *[4] The real dimension of the solution space of*

$$J_k(\mu)X^\varepsilon + X^\delta = 0 \tag{14}$$

is

$$\dim \mathcal{S}(J_k(\mu) + \lambda I_k) = \begin{cases} 0, & if \ \ \mu^\delta\mu^\varepsilon \neq 1, \\ 2k, & if \ \ \mu = \pm 1, \ and\ k\ is\ even, \\ k, & if \ \ \mu^\delta\mu^\varepsilon = 1\ with\ \mathrm{Im}(\mu) \neq 0, \ and\ k\ is\ even, \\ 2k-1, & if \ \ \mu = 1, \ for\ \varepsilon = \mathbb{C}, \delta = *\ or\ \varepsilon = \mathbb{I}, \delta = \dagger, \ and\ k\ is\ odd, \\ 2k+1, & if \ \ \mu = -1, \ for\ \varepsilon = \mathbb{C}, \delta = *\ or\ \varepsilon = \mathbb{I}, \delta = \dagger, \ and\ k\ is\ odd, \\ 2k+1, & if \ \ \mu = 1, \ for\ \varepsilon = \mathbb{C}, \delta = \dagger\ or\ \varepsilon = \mathbb{I}, \delta = *, \ and\ k\ is\ odd, \\ 2k-1, & if \ \ \mu = -1, \ for\ \varepsilon = \mathbb{C}, \delta = \dagger\ or\ \varepsilon = \mathbb{I}, \delta = *, \ and\ k\ is\ odd, \\ k-1, & if \ \ \mu^\delta\mu^\varepsilon = 1\ with\ \mathrm{Im}(\mu) \neq 0, \ and\ k\ is\ odd. \end{cases}$$

Remark 4.6. *The detailed procedure for the solution of the quaternion matrix equation $J_k(\mu)X^\varepsilon + X^\delta = 0$ can be found in Algorithms 1-4 in [4].*

4.2. Dimensions of the solution spaces for the subsystems involving pairs of blocks

Lemma 4.7 (Two right singular blocks). *The real dimension of the solution space of the system of matrix equations*

$$A_\phi X^\varepsilon + Y^\delta(B_\varphi)^{\varepsilon\delta} = 0 \tag{15}$$

$$A_\varphi Y^\varepsilon + X^\delta(B_\phi)^{\varepsilon\delta} = 0 \tag{16}$$

is

$$\dim S(L_\phi, L_\varphi) = 4(\phi + \varphi).$$

The solution is determined by the first row elements $x_{1i} \in \mathbb{H}, i = 1, 2, \cdots, \varphi$ and the first column elements $x_{i1} \in \mathbb{H}, i = 2, 3, \cdots, \phi + 1$ of X.

Proof. Set $X = [x_{ij}] \in \mathbb{H}^{(\phi+1)\times\varphi}$, $Y = [y_{ij}] \in \mathbb{H}^{(\varphi+1)\times\phi}$. Then (15)-(16) are equivalent to

$$\begin{pmatrix} x^{\varepsilon}_{21} & x^{\varepsilon}_{22} & \cdots & x^{\varepsilon}_{2\varphi} \\ x^{\varepsilon}_{31} & x^{\varepsilon}_{32} & \cdots & x^{\varepsilon}_{3\varphi} \\ \vdots & \vdots & \ddots & \vdots \\ x^{\varepsilon}_{\phi+1,1} & x^{\varepsilon}_{\phi+1,2} & \cdots & x^{\varepsilon}_{\phi+1,\varphi} \end{pmatrix} + \begin{pmatrix} y^{\delta}_{11} & y^{\delta}_{21} & \cdots & y^{\delta}_{\varphi 1} \\ y^{\delta}_{12} & y^{\delta}_{22} & \cdots & y^{\delta}_{\varphi 2} \\ \vdots & \vdots & \ddots & \vdots \\ y^{\delta}_{1\phi} & y^{\delta}_{2\phi} & \cdots & y^{\delta}_{\varphi\phi} \end{pmatrix} = 0_{\phi\times\varphi} \tag{17}$$

and

$$\begin{pmatrix} y^{\varepsilon}_{21} & y^{\varepsilon}_{22} & \cdots & y^{\varepsilon}_{2\phi} \\ y^{\varepsilon}_{31} & y^{\varepsilon}_{32} & \cdots & y^{\varepsilon}_{3\phi} \\ \vdots & \vdots & \ddots & \vdots \\ y^{\varepsilon}_{\varphi+1,1} & y^{\varepsilon}_{\varphi+1,2} & \cdots & y^{\varepsilon}_{\varphi+1,\phi} \end{pmatrix} + \begin{pmatrix} x^{\delta}_{11} & x^{\delta}_{21} & \cdots & x^{\delta}_{\phi 1} \\ x^{\delta}_{12} & x^{\delta}_{22} & \cdots & x^{\delta}_{\phi 2} \\ \vdots & \vdots & \ddots & \vdots \\ x^{\delta}_{1\varphi} & x^{\delta}_{2\varphi} & \cdots & x^{\delta}_{\phi\varphi} \end{pmatrix} = 0_{\varphi\times\phi}. \tag{18}$$

From (17), we can see that $y_{ij}, i = 1, 2, \cdots, \varphi; j = 1, 2, \cdots, \phi$ depends on the elements of X. Besides, from (18), $y_{\varphi+1,j}, j = 1, 2, \cdots, \phi$ also depends on the elements of X. Hence, Y can be completely determined by X. Obviously, (17) and (18) are equivalent to

$$x^{\varepsilon}_{b+1,a} = -y^{\delta}_{ab}, \quad for \ \ a = 1, 2, \cdots, \varphi; b = 1, 2, \cdots, \phi \tag{19}$$

and

$$x^{\delta}_{ij} = -y^{\varepsilon}_{j+1,i}, \quad for \ \ i = 1, 2, \cdots, \phi; j = 1, 2, \cdots, \varphi. \tag{20}$$

From (19)-(20), we can derive that X is a Toeplitz matrix. To see this fact, let x_{ij} be any element of X such that $x_{i+1,j+1}$ is well-defined. Then i can be taken integers from 1 to ϕ and j can be taken integers from 1 to $\varphi - 1$. By applying (20), we can get

$$x_{ij} = -y^{\delta\varepsilon}_{j+1,i}, \quad for \ \ i = 1, 2, \cdots, \phi; j = 1, 2, \cdots, \varphi - 1. \tag{21}$$

Since $j + 1 = 2, 3, \cdots, \varphi$ and $i = 1, 2, \cdots, \phi$, by (19) we have

$$-y^{\delta\varepsilon}_{j+1,i} = x_{i+1,j+1}, \quad for \ \ i = 1, 2, \cdots, \phi; j = 1, 2, \cdots, \varphi - 1. \tag{22}$$

Hence $x_{ij} = x_{i+1,j+1}$ for $i = 1, 2, \cdots, \phi; j = 1, 2, \cdots, \varphi - 1$, so X is definitely a Toeplitz matrix. This shows that

$X \in \mathbb{H}^{(\phi+1)\times\varphi}$ is the form

$$X = \begin{cases} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1,\varphi-1} & x_{1\varphi} \\ x_{21} & x_{11} & x_{12} & \cdots & x_{1,\varphi-1} \\ \vdots & x_{21} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \ddots & x_{21} & x_{11} \\ x_{\phi-2,1} & x_{\phi-3,1} & \cdots & \ddots & x_{21} \\ x_{\phi-1,1} & x_{\phi-2,1} & \cdots & \cdots & \vdots \\ x_{\phi 1} & x_{\phi-1,1} & \cdots & x_{\phi+2-\varphi,1} & x_{\phi+1-\varphi,1} \\ x_{\phi+1,1} & x_{\phi 1} & \cdots & x_{\phi+3-\varphi,1} & x_{\phi+2-\varphi,1} \end{pmatrix}, & \phi+1 \geq \varphi, \\ \begin{pmatrix} x_{11} & x_{12} & \cdots & \cdots & \cdots & \cdots & x_{1,\varphi-1} & x_{1\varphi} \\ x_{21} & x_{11} & x_{12} & \cdots & \cdots & \cdots & \cdots & x_{1,\varphi-1} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ x_{\phi 1} & \cdots & x_{21} & x_{11} & x_{12} & \cdots & x_{1,\varphi-\phi} & x_{1,\varphi-\phi+1} \\ x_{\phi+1,1} & x_{\phi 1} & \cdots & x_{21} & x_{11} & x_{12} & \cdots & x_{1,\varphi-\phi} \end{pmatrix}, & \phi+1 < \varphi, \end{cases} \tag{23}$$

for any $x_{1i} \in \mathbb{H}, i = 1, 2, \cdots, \varphi$ and $x_{i1} \in \mathbb{H}, i = 2, 3, \cdots, \phi+1$. Thus X can be determined by its first row and column, a total of $\phi + \varphi$ quaternions. Besides, it is straightforward to verify that any X of form (23) determines a unique matrix $Y \in \mathbb{H}^{(\varphi+1)\times\phi}$ of the form

$$Y = \begin{cases} \begin{pmatrix} -x_{21}^{\varepsilon\delta} & -x_{31}^{\varepsilon\delta} & \cdots & -x_{\phi 1}^{\varepsilon\delta} & -x_{\phi+1,1}^{\varepsilon\delta} \\ -x_{11}^{\varepsilon\delta} & -x_{21}^{\varepsilon\delta} & -x_{31}^{\varepsilon\delta} & \cdots & -x_{\phi 1}^{\varepsilon\delta} \\ -x_{12}^{\varepsilon\delta} & -x_{11}^{\varepsilon\delta} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -x_{1,\phi-1}^{\varepsilon\delta} & \cdots & -x_{12}^{\varepsilon\delta} & -x_{11}^{\varepsilon\delta} & -x_{21}^{\varepsilon\delta} \\ -x_{1\phi}^{\varepsilon\delta} & \cdots & \ddots & -x_{12}^{\varepsilon\delta} & -x_{11}^{\varepsilon\delta} \\ -x_{1,\phi+1}^{\varepsilon\delta} & \cdots & \cdots & \ddots & -x_{12}^{\varepsilon\delta} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ -x_{1,\varphi-1}^{\varepsilon\delta} & -x_{1,\varphi-2}^{\varepsilon\delta} & \cdots & -x_{1,\varphi-\phi+1}^{\varepsilon\delta} & -x_{1,\varphi-\phi}^{\varepsilon\delta} \\ -x_{1\varphi}^{\varepsilon\delta} & -x_{1,\varphi-1}^{\varepsilon\delta} & \cdots & -x_{1,\varphi-\phi+2}^{\varepsilon\delta} & -x_{1,\varphi-\phi+1}^{\varepsilon\delta} \end{pmatrix}, & \varphi+1 \geq \phi, \\ \begin{pmatrix} -x_{21}^{\varepsilon\delta} & -x_{31}^{\varepsilon\delta} & \cdots & \cdots & \cdots & \cdots & -x_{\phi 1}^{\varepsilon\delta} & -x_{\phi+1,1}^{\varepsilon\delta} \\ -x_{11}^{\varepsilon\delta} & -x_{21}^{\varepsilon\delta} & -x_{31}^{\varepsilon\delta} & \cdots & \cdots & \cdots & \cdots & -x_{\phi 1}^{\varepsilon\delta} \\ -x_{12}^{\varepsilon\delta} & -x_{11}^{\varepsilon\delta} & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ -x_{1,\varphi-1}^{\varepsilon\delta} & -x_{1,\varphi-2}^{\varepsilon\delta} & \cdots & -x_{11}^{\varepsilon\delta} & -x_{21}^{\varepsilon\delta} & \cdots & \ddots & -x_{\phi-\varphi+2,1}^{\varepsilon\delta} \\ -x_{1\varphi}^{\varepsilon\delta} & -x_{1,\varphi-1}^{\varepsilon\delta} & \cdots & -x_{12}^{\varepsilon\delta} & -x_{11}^{\varepsilon\delta} & -x_{21}^{\varepsilon\delta} & \cdots & -x_{\phi-\varphi+1,1}^{\varepsilon\delta} \end{pmatrix}, & \varphi+1 < \phi, \end{cases}$$

More specifically, for $\varepsilon = \mathbb{I}, \delta = \dagger$, or $\varepsilon = \mathbb{C}, \delta = *$, we have

$$Y = \left\{ \begin{array}{ll} \begin{pmatrix} -x^{\diamond}_{21} & -x^{\diamond}_{31} & \cdots & -x^{\diamond}_{\phi 1} & -x^{\diamond}_{\phi+1,1} \\ -x^{\diamond}_{11} & -x^{\diamond}_{21} & -x^{\diamond}_{31} & \cdots & -x^{\diamond}_{\phi 1} \\ -x^{\diamond}_{12} & -x^{\diamond}_{11} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -x^{\diamond}_{1,\phi-1} & \cdots & -x^{\diamond}_{12} & -x^{\diamond}_{11} & -x^{\diamond}_{21} \\ -x^{\diamond}_{1\phi} & \cdots & -x^{\diamond}_{13} & -x^{\diamond}_{12} & -x^{\diamond}_{11} \\ -x^{\diamond}_{1,\phi+1} & \cdots & -x^{\diamond}_{14} & -x^{\diamond}_{13} & -x^{\diamond}_{12} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -x^{\diamond}_{1,\varphi-1} & -x^{\diamond}_{1,\varphi-2} & \cdots & -x^{\diamond}_{1,\varphi-\phi+1} & -x^{\diamond}_{1,\varphi-\phi} \\ -x^{\diamond}_{1\varphi} & -x^{\diamond}_{1,\varphi-1} & \cdots & -x^{\diamond}_{1,\varphi-\phi+2} & -x^{\diamond}_{1,\varphi-\phi+1} \end{pmatrix}, & \varphi+1 \geq \phi, \\ \begin{pmatrix} -x^{\diamond}_{21} & -x^{\diamond}_{31} & \cdots & \cdots & \cdots & \cdots & -x^{\diamond}_{\phi 1} & -x^{\diamond}_{\phi+1,1} \\ -x^{\diamond}_{11} & -x^{\diamond}_{21} & -x^{\diamond}_{31} & \cdots & \cdots & \cdots & \cdots & -x^{\diamond}_{\phi 1} \\ -x^{\diamond}_{12} & -x^{\diamond}_{11} & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ -x^{\diamond}_{1,\varphi-1} & -x^{\diamond}_{1,\varphi-2} & \cdots & -x^{\diamond}_{11} & -x^{\diamond}_{21} & \cdots & \ddots & -x^{\diamond}_{\phi-\varphi+2,1} \\ -x^{\diamond}_{1\varphi} & -x^{\diamond}_{1,\varphi-1} & \cdots & -x^{\diamond}_{12} & -x^{\diamond}_{11} & -x^{\diamond}_{21} & \cdots & -x^{\diamond}_{\phi-\varphi+1,1} \end{pmatrix}, & \varphi+1 < \phi. \end{array} \right.$$

For $\varepsilon = \mathbb{I}, \delta = *$ or $\varepsilon = \mathbb{C}, \delta = \dagger$, we have

$$Y = \left\{ \begin{array}{ll} \begin{pmatrix} -x^{\mathbb{C}\diamond}_{21} & -x^{\mathbb{C}\diamond}_{31} & \cdots & -x^{\mathbb{C}\diamond}_{\phi 1} & -x^{\mathbb{C}\diamond}_{\phi+1,1} \\ -x^{\mathbb{C}\diamond}_{11} & -x^{\mathbb{C}\diamond}_{21} & -x^{\mathbb{C}\diamond}_{31} & \cdots & -x^{\mathbb{C}\diamond}_{\phi 1} \\ -x^{\mathbb{C}\diamond}_{12} & -x^{\mathbb{C}\diamond}_{11} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -x^{\mathbb{C}\diamond}_{1,\phi-1} & \cdots & -x^{\mathbb{C}\diamond}_{12} & -x^{\mathbb{C}\diamond}_{11} & -x^{\mathbb{C}\diamond}_{21} \\ -x^{\mathbb{C}\diamond}_{1\phi} & \cdots & -x^{\mathbb{C}\diamond}_{13} & -x^{\mathbb{C}\diamond}_{12} & -x^{\mathbb{C}\diamond}_{11} \\ -x^{\mathbb{C}\diamond}_{1,\phi+1} & \cdots & -x^{\mathbb{C}\diamond}_{14} & -x^{\mathbb{C}\diamond}_{13} & -x^{\mathbb{C}\diamond}_{12} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -x^{\mathbb{C}\diamond}_{1,\varphi-1} & -x^{\mathbb{C}\diamond}_{1,\varphi-2} & \cdots & -x^{\mathbb{C}\diamond}_{1,\varphi-\phi+1} & -x^{\mathbb{C}\diamond}_{1,\varphi-\phi} \\ -x^{\mathbb{C}\diamond}_{1\varphi} & -x^{\mathbb{C}\diamond}_{1,\varphi-1} & \cdots & -x^{\mathbb{C}\diamond}_{1,\varphi-\phi+2} & -x^{\mathbb{C}\diamond}_{1,\varphi-\phi+1} \end{pmatrix}, & \varphi+1 \geq \phi, \\ \begin{pmatrix} -x^{\mathbb{C}\diamond}_{21} & -x^{\mathbb{C}\diamond}_{31} & \cdots & \cdots & \cdots & \cdots & -x^{\mathbb{C}\diamond}_{\phi 1} & -x^{\mathbb{C}\diamond}_{\phi+1,1} \\ -x^{\mathbb{C}\diamond}_{11} & -x^{\mathbb{C}\diamond}_{21} & -x^{\varepsilon\delta}_{31} & \cdots & \cdots & \cdots & \cdots & -x^{\mathbb{C}\diamond}_{\phi 1} \\ -x^{\mathbb{C}\diamond}_{12} & -x^{\mathbb{C}\diamond}_{11} & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ -x^{\mathbb{C}\diamond}_{1,\varphi-1} & -x^{\mathbb{C}\diamond}_{1,\varphi-2} & \cdots & -x^{\mathbb{C}\diamond}_{11} & -x^{\mathbb{C}\diamond}_{21} & \cdots & \cdots & -x^{\mathbb{C}\diamond}_{\phi-\varphi+2,1} \\ -x^{\mathbb{C}\diamond}_{1\varphi} & -x^{\mathbb{C}\diamond}_{1,\varphi-1} & \cdots & -x^{\mathbb{C}\diamond}_{12} & -x^{\mathbb{C}\diamond}_{11} & -x^{\mathbb{C}\diamond}_{21} & \cdots & -x^{\mathbb{C}\diamond}_{\phi-\varphi+1,1} \end{pmatrix}, & \varphi+1 < \phi. \end{array} \right.$$

Now, it can be easily verified that X and Y of the above form is the general solution of the subsystem

(15)-(16). In particular, the real dimension of the solution space is $4(\phi+\varphi)$, as stated. The results follow. □

Lemma 4.8 (Two left singular blocks). *The real dimension of the solution space of the system of matrix equations*

$$A_\eta^T X^\varepsilon + Y^\delta (B_\gamma^T)^{\varepsilon\delta} = 0 \tag{24}$$
$$A_\gamma^T Y^\varepsilon + X^\delta (B_\eta^T)^{\varepsilon\delta} = 0 \tag{25}$$

is

$$\dim \mathcal{S}(L_\eta^T, L_\gamma^T) = 0.$$

The solution is $(X, Y) = (0, 0)$.

Proof. Multiplying A_η and A_γ from the left side of (24) and (25), respectively, and utilizing (9) we get

$$X^\varepsilon + A_\eta Y^\delta (B_\gamma^T)^{\varepsilon\delta} = 0, \tag{26}$$
$$Y^\varepsilon + A_\gamma X^\delta (B_\eta^T)^{\varepsilon\delta} = 0. \tag{27}$$

From (26), we have $X^\varepsilon = -A_\eta Y^\delta (B_\gamma^T)^{\varepsilon\delta}$, i.e., $X^\delta = -B_\gamma^T Y^\varepsilon A_\eta^{\varepsilon\delta}$. Substituting this into (27), we can get $Y^\varepsilon = A_\gamma B_\gamma^T Y^\varepsilon A_\eta^{\varepsilon\delta} (B_\eta^T)^{\varepsilon\delta}$, i.e., $Y = (A_\gamma B_\gamma^T)^\varepsilon Y (B_\eta^T A_\eta)^\delta$. Using the properties

$$A_\gamma B_\gamma^T = J_\gamma(0), B_\eta^T A_\eta = J_{\eta+1}(0), \tag{28}$$

we have $Y = J_\gamma(0)^\varepsilon Y J_{\eta+1}(0)^\delta$. Obviously, $Y = 0$ can be drawn by Lemma 2.5, hence $X = 0$. The result follows. □

Lemma 4.9 (Two infinite blocks). *The real dimension of the solution space of the system of matrix equations*

$$X^\varepsilon + Y^\delta J_u(0)^{\varepsilon\delta} = 0 \tag{29}$$
$$Y^\varepsilon + X^\delta J_t(0)^{\varepsilon\delta} = 0 \tag{30}$$

is

$$\dim \mathcal{S}(N_u, N_t) = 0.$$

The solution is $(X, Y) = (0, 0)$.

Proof. From (29) we obtain $X^\varepsilon = -Y^\delta J_u(0)^{\varepsilon\delta}$, i.e., $X^\delta = -J_u(0) Y^\varepsilon$. Substituting this into (30) we get

$$Y^\varepsilon = J_u(0) Y^\varepsilon J_t(0)^{\varepsilon\delta}, \tag{31}$$

which implies $Y = 0$ by Lemma 2.5, further $X = 0$. Then the unique solution of (29)-(30) is the trivial solution $(X, Y) = (0, 0)$, and the result follows. □

Lemma 4.10 below presents the solution of the systems of matrix equations $J_k(\mu)X^\varepsilon + X^\delta = 0$ and $J_l(\nu)Y^\varepsilon + X^\delta = 0$, which can also be seen in Lemma 4.8 in [4].

Lemma 4.10 (Two finite blocks). *The real dimension of the solution space of the system of matrix equations*

$$J_k(\mu) X^\varepsilon + Y^\delta = 0 \tag{32}$$
$$J_l(\nu) Y^\varepsilon + X^\delta = 0 \tag{33}$$

is

$$\dim \mathcal{S}(J_k(\mu)+\lambda I_k, J_l(\nu)+\lambda I_l) = \begin{cases} 2\min\{k(k+1), l(l+1)\}, & if\ \ \mu^\delta\nu^\varepsilon = 1, \\ 0, & otherwise. \end{cases}$$

For $\mu^\delta\nu^\varepsilon = 1$, if $l \geq k$, the solution can be completely determined by the elements $y_{ij} \in \mathbb{H}, i = 1,2,\cdots,k; j = i, i+1,\cdots,k$ of Y; otherwise, the solution can be completely determined by the elements $y_{ij} \in \mathbb{H}, j-i \geq k-l$. For $\mu^\delta\nu^\varepsilon \neq 1$, the solution is $(X,Y) = (0,0)$.

Lemma 4.11 (Right singular and left singular blocks). *The real dimension of the solution space of the system of matrix equations*

$$A_\phi X^\varepsilon + Y^\delta (B_\eta^T)^{\varepsilon\delta} = 0 \tag{34}$$
$$A_\eta^T Y^\varepsilon + X^\delta (B_\phi)^{\varepsilon\delta} = 0 \tag{35}$$

is

$$\dim S(L_\phi, L_\eta^T) = \begin{cases} 0, & if\ \ \eta-\phi < 0, \\ 4(\eta-\phi+1), & if\ \ \eta-\phi \geq 0. \end{cases}$$

If $\eta-\phi \geq 0$, then the solution can be completely determined by the elements $x_{1i} \in \mathbb{H}, i = \phi+1,\phi+2,\cdots,\eta,\eta+1$ of X. Otherwise, the solution is $(X,Y) = (0,0)$.

Proof. Note that $X = [x_{ij}] \in \mathbb{H}^{(\phi+1)\times(\eta+1)}, Y = [y_{ij}] \in \mathbb{H}^{\eta\times\phi}$. Equations (34)-(35) are equivalent to

$$\begin{pmatrix} x_{21}^\varepsilon & x_{22}^\varepsilon & \cdots & x_{2,\eta+1}^\varepsilon \\ x_{31}^\varepsilon & x_{32}^\varepsilon & \cdots & x_{3,\eta+1}^\varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ x_{\phi+1,1}^\varepsilon & x_{\phi+1,2}^\varepsilon & \cdots & x_{\phi+1,\eta+1}^\varepsilon \end{pmatrix} + \begin{pmatrix} y_{11}^\delta & y_{21}^\delta & \cdots & y_{\eta1}^\delta & 0 \\ y_{12}^\delta & y_{22}^\delta & \cdots & y_{\eta2}^\delta & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ y_{1\phi}^\delta & y_{2\phi}^\delta & \cdots & y_{\eta\phi}^\delta & 0 \end{pmatrix} = 0_{\phi\times(\eta+1)} \tag{36}$$

and

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ y_{11}^\varepsilon & y_{12}^\varepsilon & \cdots & y_{1\phi}^\varepsilon \\ y_{21}^\varepsilon & y_{22}^\varepsilon & \cdots & y_{2\phi}^\varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ y_{\eta1}^\varepsilon & y_{\eta2}^\varepsilon & \cdots & y_{\eta\phi}^\varepsilon \end{pmatrix} + \begin{pmatrix} x_{11}^\delta & x_{21}^\delta & \cdots & x_{\phi1}^\delta \\ x_{12}^\delta & x_{22}^\delta & \cdots & x_{\phi2}^\delta \\ \vdots & \vdots & \ddots & \vdots \\ x_{1\eta}^\delta & x_{2\eta}^\delta & \cdots & x_{\phi\eta}^\delta \\ x_{1,\eta+1}^\delta & x_{2,\eta+1}^\delta & \cdots & x_{\phi,\eta+1}^\delta \end{pmatrix} = 0_{(\eta+1)\times\phi}. \tag{37}$$

Equation (37) implies that Y can be completely determined by X. Furthermore, (36)-(37) are equivalent to the systems of equations

$$x_{2,\eta+1} = x_{3,\eta+1} = \cdots = x_{\phi,\eta+1} = x_{\phi+1,\eta+1}, \tag{38}$$
$$y_{ij} = -x_{j+1,i}^{\varepsilon\delta},\ \ for\ \ i = 1,2,\cdots,\eta; j = 1,2,\cdots,\phi, \tag{39}$$
$$x_{11} = x_{21} = \cdots = x_{\phi1} = 0, \tag{40}$$
$$y_{ij} = -x_{j,i+1}^{\delta\varepsilon},\ \ for\ \ i = 1,2,\cdots,\eta; j = 1,2,\cdots,\phi. \tag{41}$$

Combining (39) and (41), we can obtain

$$x_{ij} = x_{i+1,j-1},\ \ for\ \ i = 1,2,\cdots,\phi; j = 1,2,\cdots,\eta+1,$$

which shows that all elements of X sitting on the same anti-diagonal, $\mathcal{L}_s = \{x_{ij} : i + j = s\}$, are equal. Furthermore, (38) and (40) imply that all elements in the first and $(\eta + 1)$-th columns of X are zero except for $x_{1,\eta+1}$ and $x_{\phi+1,1}$.

If $\eta < \phi$, every anti-diagonal contains one of these elements equal to zero. This in turn implies $X = 0, Y = 0$, so $(X, Y) = (0, 0)$ is the unique solution.

If $\eta \geq \phi$, then there are anti-diagonals of X which do not contain any of these zero elements from the first and $(\eta + 1)$-th columns. More specifically, these anti-diagonals are those which have an element in the first and last rows. Because there are $\eta - \phi + 1$ anti-diagonals like this, X depends on $\eta - \phi + 1$ free quaternion variables. On the other hand, it can be verified that if $X \in \mathbb{H}^{(\phi+1)\times(\eta+1)}$ has the form

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 & x_{1,\phi+1} & x_{1,\phi+2} & \cdots & x_{1\eta} & x_{1,\eta+1} \\ 0 & 0 & 0 & x_{1,\phi+1} & x_{1,\phi+2} & \cdots & x_{1\eta} & x_{1,\eta+1} & 0 \\ 0 & 0 & \iddots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ 0 & x_{1,\phi+1} & x_{1,\phi+2} & \cdots & x_{1\eta} & x_{1,\eta+1} & 0 & 0 & 0 \\ x_{1,\phi+1} & x_{1,\phi+2} & \cdots & x_{1\eta} & x_{1,\eta+1} & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{42}$$

and Y is defined by (39), then X and Y satisfy (38)-(41) for all values of $x_{1i} \in \mathbb{H}, i = \phi + 1, \phi + 2, \cdots, \eta, \eta + 1$. As a consequence, the general solution of (34)-(35) depends on exactly $\eta - \phi + 1$ free quaternion variables. Moreover, (42) and (39) give the expressions of the general solution (X, Y).

□

Lemma 4.12 (Right singular and infinite blocks). *The real dimension of the solution space of the system of matrix equations*

$$A_\phi X^\varepsilon + Y^\delta J_u(0)^{\varepsilon\delta} = 0 \tag{43}$$

$$Y^\varepsilon + X^\delta (B_\phi)^{\varepsilon\delta} = 0 \tag{44}$$

is

$$\dim S(L_\phi, N_u) = 4u.$$

The solution can be completely determined by the first row elements $x_{1i} \in \mathbb{H}, i = 1, 2, \cdots, u$ of X.

Proof. Note that $X \in \mathbb{H}^{(\phi+1)\times u}$ and $Y \in \mathbb{H}^{u\times\phi}$. From (43), we can obtain

$$\begin{pmatrix} x^\varepsilon_{21} & x^\varepsilon_{22} & \cdots & x^\varepsilon_{2u} \\ x^\varepsilon_{31} & x^\varepsilon_{32} & \cdots & x^\varepsilon_{3u} \\ \vdots & \vdots & \ddots & \vdots \\ x^\varepsilon_{\phi+1,1} & x^\varepsilon_{\phi+1,2} & \cdots & x^\varepsilon_{\phi+1,u} \end{pmatrix}_{\phi\times u} + \begin{pmatrix} y^\delta_{21} & y^\delta_{31} & \cdots & y^\delta_{u1} & 0 \\ y^\delta_{22} & y^\delta_{32} & \cdots & y^\delta_{u2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ y^\delta_{2\phi} & y^\delta_{3\phi} & \cdots & y^\delta_{u\phi} & 0 \end{pmatrix}_{\phi\times u} = 0_{\phi\times u},$$

which implies that $x_{2u} = x_{3u} = \cdots = x_{\phi+1,u} = 0$ and

$$x^\varepsilon_{ij} = -y^\delta_{j+1,i-1} \ for \ i = 2, 3, \cdots, \phi + 1; j = 1, 2, \cdots, u - 1. \tag{45}$$

From (44), we also have

$$\begin{pmatrix} y^\varepsilon_{11} & y^\varepsilon_{12} & \cdots & y^\varepsilon_{1\phi} \\ y^\varepsilon_{21} & y^\varepsilon_{22} & \cdots & y^\varepsilon_{2\phi} \\ \vdots & \vdots & \ddots & \vdots \\ y^\varepsilon_{u1} & y^\varepsilon_{u2} & \cdots & y^\varepsilon_{u\phi} \end{pmatrix}_{u\times\phi} + \begin{pmatrix} x^\delta_{11} & x^\delta_{21} & \cdots & x^\delta_{\phi1} \\ x^\delta_{12} & x^\delta_{22} & \cdots & x^\delta_{\phi2} \\ \vdots & \vdots & \ddots & \vdots \\ x^\delta_{1u} & x^\delta_{2u} & \cdots & x^\delta_{\phi u} \end{pmatrix}_{u\times\phi} = 0_{u\times\phi},$$

which implies that

$$x_{ij}^{\delta} = -y_{ji}^{\varepsilon}, \quad for \ \ i = 1, 2, \cdots, \phi; j = 1, 2, \cdots, u. \tag{46}$$

This shows that Y can be completely determined by X. Now, utilizing (45)-(46), we can obtain

$$x_{ij} = x_{i-1,j+1}, \quad for \ \ i = 2, 3, \cdots, \phi+1; j = 1, 2, \cdots, u-1. \tag{47}$$

Formula (47) shows that any elements of X sitting on the same anti-diagonal are equal. Besides, the anti-diagonals below the main anti-diagonal of X are equal to zero since $x_{2u} = \cdots = x_{\phi+1,u} = 0$. Thus $X \in \mathbb{H}^{(\phi+1)\times u}$ depends on u free quaternion parameters $x_{1i} \in \mathbb{H}, i = 1, 2, \cdots, u$ and has the form

$$X = \begin{cases} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1,u-1} & x_{1u} \\ x_{12} & \cdots & x_{1,u-1} & x_{1u} & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ x_{1,u-1} & x_{1u} & 0 & 0 & 0 \\ x_{1u} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & if \ \phi+1 \geq u, \\ \begin{pmatrix} x_{11} & x_{12} & \cdots & \cdots & x_{1,\phi+1} & x_{1,\phi+2} & \cdots & x_{1u} \\ x_{12} & \cdots & \cdots & x_{1,\phi+1} & x_{1,\phi+2} & \cdots & x_{1u} & 0 \\ \vdots & \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ x_{1\phi} & x_{1,\phi+1} & x_{1,\phi+2} & \cdots & x_{1u} & 0 & 0 & 0 \\ x_{1,\phi+1} & x_{1,\phi+2} & \cdots & x_{1u} & 0 & 0 & 0 & 0 \end{pmatrix}, & if \ \phi+1 < u. \end{cases}$$

Moreover, given any matrix X which has the above form and let Y be defined by (46), we have that (X, Y) is the solution of (43)-(44). Then, the general solution depends on exactly u free quaternion variables.

□

Lemma 4.13 (Right singular and finite blocks). *The real dimension of the solution space of the system of matrix equations*

$$A_{\phi}X^{\varepsilon} + Y^{\delta} = 0 \tag{48}$$

$$J_k(\mu)Y^{\varepsilon} + X^{\delta}(B_{\phi})^{\varepsilon\delta} = 0 \tag{49}$$

is

$$\dim S(L_{\phi}, J_k(\mu) + \lambda I_k) = 4k.$$

The solution can be completely determined by the $(\phi+1)$*-th row elements* $x_{\phi+1,i}, i = 1, 2, \cdots, k$ *of* X.

Proof. Note that $X \in \mathbb{H}^{(\phi+1)\times k}, Y \in \mathbb{H}^{k\times\phi}$. (48)-(49) can be reformed as

$$\begin{pmatrix} x_{21}^{\varepsilon} & x_{22}^{\varepsilon} & \cdots & x_{2k}^{\varepsilon} \\ x_{31}^{\varepsilon} & x_{32}^{\varepsilon} & \cdots & x_{3k}^{\varepsilon} \\ \vdots & \vdots & \ddots & \vdots \\ x_{\phi+1,1}^{\varepsilon} & x_{\phi+1,2}^{\varepsilon} & \cdots & x_{\phi+1,k}^{\varepsilon} \end{pmatrix} + \begin{pmatrix} y_{11}^{\delta} & y_{21}^{\delta} & \cdots & y_{k1}^{\delta} \\ y_{12}^{\delta} & y_{22}^{\delta} & \cdots & y_{k2}^{\delta} \\ \vdots & \vdots & \ddots & \vdots \\ y_{1\phi}^{\delta} & y_{2\phi}^{\delta} & \cdots & y_{k\phi}^{\delta} \end{pmatrix} = 0_{\phi\times k} \tag{50}$$

and

$$\mu\begin{pmatrix} y_{11}^{\varepsilon} & y_{12}^{\varepsilon} & \cdots & y_{1\phi}^{\varepsilon} \\ y_{21}^{\varepsilon} & y_{22}^{\varepsilon} & \cdots & y_{2\phi}^{\varepsilon} \\ \vdots & \vdots & \ddots & \vdots \\ y_{k1}^{\varepsilon} & y_{k2}^{\varepsilon} & \cdots & y_{k\phi}^{\varepsilon} \end{pmatrix} + \begin{pmatrix} y_{21}^{\varepsilon} & y_{22}^{\varepsilon} & \cdots & y_{2\phi}^{\varepsilon} \\ y_{31}^{\varepsilon} & y_{32}^{\varepsilon} & \cdots & y_{3\phi}^{\varepsilon} \\ \vdots & \vdots & \ddots & \vdots \\ y_{k1}^{\varepsilon} & y_{k2}^{\varepsilon} & \cdots & y_{k\phi}^{\varepsilon} \\ 0 & 0 & \cdots & 0 \end{pmatrix} + \begin{pmatrix} x_{11}^{\delta} & x_{21}^{\delta} & \cdots & x_{\phi 1}^{\delta} \\ x_{12}^{\delta} & x_{22}^{\delta} & \cdots & x_{\phi 2}^{\delta} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1k}^{\delta} & x_{2k}^{\delta} & \cdots & x_{\phi k}^{\delta} \end{pmatrix} = 0_{k\times\phi}. \tag{51}$$

By inspecting columns in (51), we can obtain

$$\mu\begin{pmatrix} y_{1j}^{\varepsilon} \\ y_{2j}^{\varepsilon} \\ \vdots \\ y_{k-1,j}^{\varepsilon} \\ y_{kj}^{\varepsilon} \end{pmatrix} + \begin{pmatrix} y_{2j}^{\varepsilon} \\ y_{3j}^{\varepsilon} \\ \vdots \\ y_{kj}^{\varepsilon} \\ 0 \end{pmatrix} + \begin{pmatrix} x_{j1}^{\delta} \\ x_{j2}^{\delta} \\ \vdots \\ x_{j,k-1}^{\delta} \\ x_{jk}^{\delta} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, j = 1, 2, \cdots, \phi.$$

By utilizing (50), the identity can be rewritten as

$$\mu\begin{pmatrix} x_{j+1,1}^{\delta} \\ x_{j+1,2}^{\delta} \\ \vdots \\ x_{j+1,k-1}^{\delta} \\ x_{j+1,k}^{\delta} \end{pmatrix} + \begin{pmatrix} x_{j+1,2}^{\delta} \\ x_{j+1,3}^{\delta} \\ \vdots \\ x_{j+1,k}^{\delta} \\ 0 \end{pmatrix} = \begin{pmatrix} x_{j1}^{\delta} \\ x_{j2}^{\delta} \\ \vdots \\ x_{j,k-1}^{\delta} \\ x_{jk}^{\delta} \end{pmatrix}, j = 1, 2, \cdots, \phi,$$

and taking $(\cdot)^{\delta}$, we have

$$\begin{pmatrix} x_{j+1,1} \\ x_{j+1,2} \\ \vdots \\ x_{j+1,k-1} \\ x_{j+1,k} \end{pmatrix}\mu^{\delta} + \begin{pmatrix} x_{j+1,2} \\ x_{j+1,3} \\ \vdots \\ x_{j+1,k} \\ 0 \end{pmatrix} = \begin{pmatrix} x_{j1} \\ x_{j2} \\ \vdots \\ x_{j,k-1} \\ x_{jk} \end{pmatrix}, j = 1, 2, \cdots, \phi.$$

From the recursion relation, we can see that the $(j+1)$-th row elements of X can be determined in terms of the j-th row elements, vice versa. Thus X can be uniquely determined by k free quaternion parameters $x_{\phi+1,i} \in \mathbb{H}, i = 1, 2, \cdots, k$. From (50), we can see that Y is completely determined by X.

Moreover, if we set

$$c_j = x_{\phi+1,j}, j = 1, 2, \cdots, k, \tag{52}$$

then we can prove that X satisfies

$$x_{ij} = \sum_{l=0}^{\phi+1-i} \binom{\phi+1-i}{l} c_{j+l}(\mu^{\delta})^{\phi+1-i-l}, \tag{53}$$

where we assume that $c_{j+l} = 0$ if $j + l > k$.

To prove (53), we can proceed by induction on $i = \phi + 1, \phi, \cdots, 2, 1$ and downwards. The base case for $i = \phi + 1$ is just (52). Now, assume that (53) holds for some i with $1 \leq i \leq \phi + 1$ and $j = 1, 2, \cdots, k$. In order to prove that X satisfies (53) for $i - 1$ and $j = 1, 2, \cdots, k$, we only need to prove that (53) satisfies the recursion relation $x_{i-1,j} = x_{ij}\mu^{\delta} + x_{i,j+1}$. By (53), $x_{ij}\mu^{\delta} + x_{i,j+1}$ becomes

$$\sum_{l=0}^{\phi+1-i} \binom{\phi+1-i}{l} c_{j+l}(\mu^{\delta})^{\phi+1-i-l}\mu^{\delta} + \sum_{l=0}^{\phi+1-i} \binom{\phi+1-i}{l} c_{j+1+l}(\mu^{\delta})^{\phi+1-i-l},$$

i.e.,

$$\sum_{l=0}^{\phi+1-i} \binom{\phi+1-i}{l} c_{j+l}(\mu^{\delta})^{\phi+2-i-l} + \sum_{l=1}^{\phi+2-i} \binom{\phi+1-i}{l-1} c_{j+l}(\mu^{\delta})^{\phi+2-i-l}. \tag{54}$$

By the binomial identity

$$\binom{m-1}{n-1} + \binom{m-1}{n} = \binom{m}{n},$$

for any positive integers m, n, the formula (54) equals to

$$\begin{aligned} & c_j(\mu^{\delta})^{\phi+2-i} + \sum_{l=1}^{\phi+1-i} \binom{\phi+2-i}{l} c_{j+l}(\mu^{\delta})^{\phi+2-i-l} + c_{j+\phi+2-i} \\ = & \sum_{l=0}^{\phi+2-i} \binom{\phi+2-i}{l} c_{j+l}(\mu^{\delta})^{\phi+2-i-l}, \end{aligned}$$

which is just the left side of (53), and thus the proof is completed.
□

Lemma 4.14 (Left singular and infinite blocks). *The real dimension of the solution space of the system of matrix equations*

$$A_\eta^T X^{\varepsilon} + Y^{\delta} J_u(0)^{\varepsilon\delta} = 0 \tag{55}$$

$$Y^{\varepsilon} + X^{\delta}(B_\eta^T)^{\varepsilon\delta} = 0 \tag{56}$$

is

$$\dim S(L_\eta^T, N_u) = 0.$$

The solution is $(X, Y) = (0, 0)$.

Proof. By (56), we have $Y^{\varepsilon} = -X^{\delta}(B_\eta^T)^{\varepsilon\delta}$, i.e., $Y^{\delta} = -B_\eta^T X^{\varepsilon}$. Substituting into (55), we have

$$A_\eta^T X^{\varepsilon} - B_\eta^T X^{\varepsilon} J_u(0)^{\varepsilon\delta} = 0$$

and then premultiplying A_η, and using the properties (9) and (12), we can get $X^{\varepsilon} = J_u(0) X^{\varepsilon} J_u(0)^{\varepsilon\delta}$, i.e.,

$$X = J_u(0)^{\varepsilon} X J_u(0)^{\delta}.$$

Since $J_\eta(0)^{\varepsilon}$ and $J_u(0)^{\delta}$ are nilpotent, Lemma 2.5 implies $X = 0$, and this in turn implies $Y = 0$.
□

Lemma 4.15 (Left singular and finite blocks). *The real dimension of the solution space of the system of matrix equations*

$$A_\eta^T X^{\varepsilon} + Y^{\delta} = 0 \tag{57}$$

$$J_k(\mu)Y^{\varepsilon} + X^{\delta}(B_{\eta}^{T})^{\varepsilon\delta} = 0 \tag{58}$$

is

$$\dim S(L_{\eta}^{T}, J_k(\mu) + \lambda I_k) = 0.$$

The solution is $(X, Y) = (0, 0)$.

Proof. By (57), we have $Y^{\delta} = -A_{\eta}^{T}X^{\varepsilon}$, i.e., $Y^{\varepsilon} = -X^{\delta}(A_{\eta}^{T})^{\delta\varepsilon}$. Substituting into (58), we have

$$J_k(\mu)X^{\delta}(A_{\eta}^{T})^{\delta\varepsilon} - X^{\delta}(B_{\eta}^{T})^{\varepsilon\delta} = 0,$$

and taking $(\cdot)^{\delta}$, we have

$$(A_{\eta}^{T})^{\varepsilon}XJ_k(\mu)^{\delta} - (B_{\eta}^{T})^{\varepsilon}X = 0,$$

and then premultiplying B_{η}^{ε}, and utilizing the properties

$$B_{\eta}A_{\eta}^{T} = J_{\eta}(0)^{T}, B_{\eta}B_{\eta}^{T} = I_{\eta},$$

we get

$$X = (J_{\eta}(0)^{T})^{\varepsilon}XJ_k(\mu)^{\delta}.$$

Obviously, $J_{\eta}(0)$ is nilpotent, so is $(J_{\eta}(0)^{T})^{\varepsilon}$. Lemma 2.5 implies $X = 0$, which in turn implies $Y = 0$.
□

Lemma 4.16 (Infinite and finite blocks). *The real dimension of the solution space of the system of matrix equations*

$$X^{\varepsilon} + Y^{\delta} = 0 \tag{59}$$
$$J_k(\mu)Y^{\varepsilon} + X^{\delta}J_u(0)^{\varepsilon\delta} = 0 \tag{60}$$

is

$$\dim S(N_u, J_k(\mu) + \lambda I_k) = \begin{cases} 2\min\{u(u+1), k(k+1)\}, & if \;\; \mu = 0, \\ 0, & if \;\; \mu \neq 0. \end{cases}$$

For $\mu = 0$, if $k \geq u$, the solution can be completely determined by the elements $y_{ij} \in \mathbb{H}, i = 1, 2, \cdots, u; j = i, i+1, \cdots, u$; otherwise, the solution can be completely determined by the elements $y_{ij} \in \mathbb{H}, j - i \geq u - k$. For $\mu \neq 0$, the solution is $(X, Y) = (0, 0)$.

Proof. By (59), we have $X^{\varepsilon} = -Y^{\delta}$, i.e., $X^{\delta} = -Y^{\varepsilon}$. Substituting into (60), we get

$$J_k(\mu)Y^{\varepsilon} - Y^{\varepsilon}J_u(0)^{\varepsilon\delta} = 0.$$

Taking $(\cdot)^{\varepsilon}$, we have

$$J_k(\mu^{\varepsilon})Y - YJ_u(0)^{\delta} = 0. \tag{61}$$

The solution of this Sylvester is as follows.
(1) If $\mu \neq 0$, then $\mu^{\varepsilon} \neq 0$. Thus $Y = 0, X = 0$.
(2) If $\mu = 0$, then (61) becomes

$$J_k(0)Y - YJ_u(0)^{T} = 0, \tag{62}$$

Let $R \in \mathbb{H}^{u\times u}$ be a nonsingular matrix such that $R^{-1}J_u(0)^T R = J_u(0)$, a Jordan canonical form. Then (62) becomes

$$J_k(0)Y - YRJ_u(0)R^{-1} = 0, \tag{63}$$

equivalently,

$$J_k(0)\widetilde{Y} - \widetilde{Y}J_u(0) = 0, \tag{64}$$

where $\widetilde{Y} = YR$. The remaining proofs are similar to Lemma 4.12 in [4].

□

Remark 4.17. *Similar to Remark 4.1 in [4], the reverse matrix $R = \begin{pmatrix} e_u & e_{u-1} & \cdots & e_2 & e_1 \end{pmatrix}$ can be selected as the required nonsingular matrix $R \in \mathbb{H}^{u\times u}$. Then $Y \in \mathbb{H}^{k\times u}$ has the form*

$$Y = \begin{cases} \begin{pmatrix} \widetilde{y}_{1u} & \widetilde{y}_{1,u-1} & \cdots & \widetilde{y}_{12} & \widetilde{y}_{11} \\ \widetilde{y}_{2u} & \widetilde{y}_{2,u-1} & \cdots & \widetilde{y}_{22} & 0 \\ \vdots & \vdots & \iddots & 0 & \vdots \\ \widetilde{y}_{u-1,u} & \widetilde{y}_{u-1,u-1} & 0 & \cdots & 0 \\ \widetilde{y}_{uu} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & if \;\; k \geq u, \\ \begin{pmatrix} \widetilde{y}_{1u} & \widetilde{y}_{1,u-1} & \cdots & \cdots & \widetilde{y}_{1,u-k+2} & \widetilde{y}_{1,u-k+1} & 0 \\ \widetilde{y}_{2u} & \widetilde{y}_{2,u-1} & \cdots & \cdots & \widetilde{y}_{2,u-k+2} & 0 & 0 \\ \vdots & \vdots & \iddots & \iddots & 0 & \vdots & \vdots \\ \widetilde{y}_{k-2,u} & \widetilde{y}_{k-2,u-1} & \widetilde{y}_{k-2,u-2} & 0 & \cdots & \cdots & 0 \\ \widetilde{y}_{k-1,u} & \widetilde{y}_{k-1,u-1} & 0 & \cdots & \cdots & \cdots & 0 \\ \widetilde{y}_{ku} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}, & if \;\; k < u. \end{cases}$$

Remark 4.18. *If the solution of $AX^\varepsilon + X^\delta B = 0$ is restricted in the complex field instead of the set of quaternions, then all of the results presented in Theorem 4.1 and Lemmas 4.2-4.16 can be reduced to the corresponding results stated in Theorem 4 and Lemmas 21-34 in [20], respectively. That is to say, the related theories of the complex matrix equation $AX + X^* B = 0$ can be generalized to the quaternion matrix equation $AX^\varepsilon + X^\delta B = 0$. This is our main contributions.*

5. Uniqueness of the solution of $AX^\varepsilon + X^\delta B = 0$

Theorem 5.1 below presents necessary and sufficient condition for the unique solution of the equation (3) in terms of the KCF and eigenvalues of the matrix pencil $\widetilde{A} + \lambda\widetilde{B}^{\varepsilon\delta}$.

Theorem 5.1. *Let $A \in \mathbb{H}^{m\times n}, B \in \mathbb{H}^{n\times m}$ be two quaternion matrices, and let $(A, B^{\varepsilon\delta})$ be a regular matrix pencil. Let P, Q be two nonsingular quaternion matrices such that $\widetilde{A} = PAQ, \widetilde{B} = Q^{\varepsilon\delta}BP^{\varepsilon\delta}$ are two complex matrices. Then the quaternion matrix equation $AX^\varepsilon + X^\delta B = 0$ has the unique trivial solution $X = 0$ if and only if the following two conditions hold:*

(a) The KCF of the matrix pencil $\widetilde{A} + \lambda\widetilde{B}^{\varepsilon\delta}$ has no right singular blocks.

(b) If $\mu^\varepsilon \in \mathbb{C} \cup \{\infty\}$ is an eigenvalue of $\widetilde{A} + \lambda\widetilde{B}^{\varepsilon\delta}$, then $1/\mu^\delta$ is not an eigenvalue of $\widetilde{A} + \lambda\widetilde{B}^{\varepsilon\delta}$.

Note that, in particular, it must be $m \geq n$, and $\widetilde{A} + \lambda\widetilde{B}^{\varepsilon\delta}$ has not eigenvalues of module one.

Proof. The matrix equation $AX^\varepsilon + X^\delta B = 0$ has the unique trivial solution $X = 0$ if and only if the dimension of the solution space is zero. Inspecting Theorem 4.1, the dimension is zero if and only if the two conditions (a)-(b) in the statements hold.

□

By Theorem 3.1, $AX^{\varepsilon} + X^{\delta}B = 0$ has the unique solution $X = 0$ if and only if $\widetilde{A}Y^{\varepsilon} + Y^{\delta}\widetilde{B} = 0$ has the unique solution $Y = 0$. We intend to point out that $\widetilde{A}Y^{\varepsilon} + Y^{\delta}\widetilde{B} = 0$ may have a unique solution $Y = 0$ with $\widetilde{A} + \lambda\widetilde{B}^{\varepsilon\delta}$ being singular. By the condition (a) in Theorem 5.1, the KCF of $\widetilde{A} + \lambda\widetilde{B}^{\varepsilon\delta}$ has not right singular blocks, but it may have left singular blocks. For instance, consider the quaternion matrix equation $\widetilde{A}Y^{\varepsilon} + Y^{\delta}\widetilde{B} = 0$, where

$$\widetilde{A} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \widetilde{B} = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

Though $\widetilde{A} + \lambda\widetilde{B}^{\varepsilon\delta}$ is a left singular block, the matrix equation $\widetilde{A}Y^{\varepsilon} + Y^{\delta}\widetilde{B} = 0$ has the unique trivial solution $Y = 0$. Incredibly, the operator $Y \mapsto \widetilde{A}Y^{\varepsilon} + Y^{\delta}\widetilde{B}$ is not invertible. In fact, this operator maps $\mathbb{H}$ to $\mathbb{H}^{2\times 1}$. In order for this operator to be invertible, the dimension of the original space must be the same as that of the mapped space. In general, this holds if and only if $m = n$. Then, the condition (a) in Theorem 5.1 implies that $\widetilde{A} + \lambda\widetilde{B}^{\varepsilon\delta}$ is regular. This also leads to the following result.

Theorem 5.2. *Let $A \in \mathbb{H}^{m\times n}, B \in \mathbb{H}^{n\times m}, C \in \mathbb{H}^{m\times m}$ be three quaternion matrices, and let $(A, B^{\varepsilon\delta})$ be a regular matrix pencil. Let P, Q be two nonsingular quaternion matrices such that $\widetilde{A} = PAQ, \widetilde{B} = Q^{\varepsilon\delta}BP^{\varepsilon\delta}$ are two complex matrices. Then Y is a solution of the quaternion matrix equation $\widetilde{A}Y^{\varepsilon} + Y^{\delta}\widetilde{B} = PCP^{\varepsilon\delta}$ if and only if $X = Q^{\varepsilon}YP^{-\delta}$ is a solution of the quaternion matrix equation $AX^{\varepsilon} + X^{\delta}B = C$. As a consequence, the solutions of the two equations are one-to-one via $Y \mapsto Q^{\varepsilon}YP^{-\delta} = X$. Besides, the quaternion matrix equation $AX^{\varepsilon} + X^{\delta}B = C$ has a unique solution if and only if the following two conditions hold:*

(a) The matrix pencil $\widetilde{A} + \lambda\widetilde{B}^{\varepsilon\delta}$ is regular.

(b) If $\mu^{\varepsilon} \in \mathbb{C} \cup \{\infty\}$ is an eigenvalue of $\widetilde{A} + \lambda\widetilde{B}^{\varepsilon\delta}$, then $1/\mu^{\delta}$ is not an eigenvalue of $\widetilde{A} + \lambda\widetilde{B}^{\varepsilon\delta}$.

Proof. Let $X = Q^{\varepsilon}YP^{-\delta}$. Similarly to the proof of Theorem 3.1, we have

$$X^{\varepsilon} = QY^{\varepsilon}P^{-\delta\varepsilon}, X^{\delta} = P^{-1}Y^{\delta}Q^{\varepsilon\delta} = P^{-1}Y^{\delta}Q^{\delta\varepsilon}.$$

By computations, we have

$$\begin{aligned} \widetilde{A}Y^{\varepsilon} + Y^{\delta}\widetilde{B} &= PAQY^{\varepsilon} + Y^{\delta}Q^{\delta\varepsilon}BP^{\delta\varepsilon} \\ &= PAQY^{\varepsilon}P^{-\delta\varepsilon}P^{\delta\varepsilon} + PP^{-1}Y^{\delta}Q^{\delta\varepsilon}BP^{\delta\varepsilon} \\ &= P(AQY^{\varepsilon}P^{-\delta\varepsilon} + P^{-1}Y^{\delta}Q^{\delta\varepsilon}B)P^{\delta\varepsilon} \\ &= P(AX^{\varepsilon} + X^{\delta}B)P^{\delta\varepsilon}. \end{aligned}$$

Since P and $P^{\delta\varepsilon}$ are nonsingular, $\widetilde{A}Y^{\varepsilon} + Y^{\delta}\widetilde{B} = PCP^{\varepsilon\delta}$ if and only if $AX^{\varepsilon} + X^{\delta}B = C$. The mapping $Y \mapsto Q^{\varepsilon}YP^{-\delta}$ is clearly linear and invertible, so it is a one-to-one correspondence. The remaining proof can be obtained from Theorem 5.1. □

Theorem 5.2 above not only presents the necessary and sufficient condition for existence of the unique solution of $AX^{\varepsilon} + X^{\delta}B = C$ in terms of the properties of the matrix pencil $\widetilde{A} + \lambda\widetilde{B}^{\varepsilon\delta}$, but also presents a necessary and sufficient condition for the invertibility of the operators $Y \mapsto \widetilde{A}Y^{\varepsilon} + Y^{\delta}\widetilde{B}$ and $X \mapsto AX^{\varepsilon} + X^{\delta}B$.

Remark 5.3. *Similar to Remark 4.18, if the solutions of $AX^{\varepsilon} + X^{\delta}B = 0$ and $AX^{\varepsilon} + X^{\delta}B = C$ are restricted in the complex fields instead of the set of quaternions, then the results presented in Theorems 5.1-5.2 can be reduced to the corresponding results about the complex matrix equations $AX + X^{*}B = 0$, $AX + X^{*}B = C$ and the operator $X \mapsto AX + X^{*}B$ stated in [20]. That is to say, the related theories about the complex equation $AX + X^{*}B = 0$ and $AX + X^{*}B = C$ and the complex operator $X \mapsto AX + X^{*}B$ can be generalized to the set of quaternions. This is our another main contribution.*

6. Numerical examples

In this section, we will give a specific example to display the process of calculating the solution of the nonhomogeneous quaternion matrix equation $AX^{\varepsilon} + X^{\delta}B = C$.

Example 6.1. *Consider the Sylverter-like quaternion matrix equation* $AX^{\varepsilon} + X^{\delta}B = C$ *with*

$$A = \begin{pmatrix} \frac{1}{4}i - \frac{1}{2}k & \frac{1}{4}k \\ \frac{1}{2}j + \frac{1}{4}k & -\frac{1}{4} - \frac{1}{4}j - \frac{1}{4}k \end{pmatrix}, B = \begin{pmatrix} \frac{1}{4} + \frac{1}{4}i & -\frac{1}{4}j - \frac{1}{4}k \\ -\frac{1}{4}j + \frac{1}{4}k & \frac{1}{4} - \frac{1}{4}i \end{pmatrix}, C = \begin{pmatrix} j - k & i + j + k \\ i - j - k & j + k \end{pmatrix},$$

where $\varepsilon = \mathbb{C}, \delta = *$.

By the definitions of $\mathbb{C}$ and $*$, we have

$$B^{\varepsilon\delta} = \begin{pmatrix} \frac{1}{4} - \frac{1}{4}i & -\frac{1}{4}j + \frac{1}{4}k \\ -\frac{1}{4}j - \frac{1}{4}k & \frac{1}{4} + \frac{1}{4}i \end{pmatrix}.$$

Let $\lambda_0 = 1$, the matrix

$$A + \lambda_0 B^{\varepsilon\delta} = \begin{pmatrix} \frac{1}{4} - \frac{1}{2}k & -\frac{1}{4}j + \frac{1}{2}k \\ \frac{1}{4}j & \frac{1}{4}i - \frac{1}{4}j - \frac{1}{4}k \end{pmatrix}$$

is a nonsingular quaternion matrix. Thus $(A, B^{\varepsilon\delta})$ is a regular quaternion matrix pencil. Then there exist two nonsingular quaternion matrices P, Q such that $\widetilde{A} = PAQ$ and $\widetilde{B^{\varepsilon\delta}} = PB^{\varepsilon\delta}Q$ are two complex matrices.

Let

$$P = \begin{pmatrix} i + j & -1 - k \\ \frac{1}{2} - \frac{1}{2}i - \frac{1}{2}j - \frac{1}{2}k & \frac{3}{2} - \frac{1}{2}i - \frac{1}{2}j + \frac{1}{2}k \end{pmatrix}, Q = \begin{pmatrix} 1 + i - j - k & 1 - i + j + k \\ 1 + i + j - k & 1 - i + j + k \end{pmatrix}.$$

Then P, Q are obviously two nonsingular quaternion matrices with

$$P^{-1} = \begin{pmatrix} \frac{1}{2} - \frac{1}{2}i - \frac{1}{2}j & \frac{1}{2} - \frac{1}{2}i \\ \frac{1}{2}i & \frac{1}{2} + \frac{1}{2}i \end{pmatrix}.$$

Besides,

$$\begin{aligned} \widetilde{A} = PAQ &= \begin{pmatrix} i + j & -1 - k \\ \frac{1}{2} - \frac{1}{2}i - \frac{1}{2}j - \frac{1}{2}k & \frac{3}{2} - \frac{1}{2}i - \frac{1}{2}j + \frac{1}{2}k \end{pmatrix} \begin{pmatrix} \frac{1}{4}i - \frac{1}{2}k & \frac{1}{4}k \\ \frac{1}{2}j + \frac{1}{4}k & -\frac{1}{4} - \frac{1}{4}j - \frac{1}{4}k \end{pmatrix} \begin{pmatrix} 1 + i - j - k & 1 - i + j + k \\ 1 + i + j - k & 1 - i + j + k \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2}k & \frac{1}{2}k \\ \frac{1}{2}j & -\frac{1}{4} + \frac{1}{4}i - \frac{1}{4}j - \frac{1}{4}k \end{pmatrix} \begin{pmatrix} 1 + i - j - k & 1 - i + j + k \\ 1 + i + j - k & 1 - i + j + k \end{pmatrix} \\ &= \begin{pmatrix} -i & \\ & i \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \widetilde{B^{\varepsilon\delta}} = PB^{\varepsilon\delta}Q &= \begin{pmatrix} i + j & -1 - k \\ \frac{1}{2} - \frac{1}{2}i - \frac{1}{2}j - \frac{1}{2}k & \frac{3}{2} - \frac{1}{2}i - \frac{1}{2}j + \frac{1}{2}k \end{pmatrix} \begin{pmatrix} \frac{1}{4} - \frac{1}{4}i & -\frac{1}{4}j + \frac{1}{4}k \\ -\frac{1}{4}j - \frac{1}{4}k & \frac{1}{4} + \frac{1}{4}i \end{pmatrix} \begin{pmatrix} 1 + i - j - k & 1 - i + j + k \\ 1 + i + j - k & 1 - i + j + k \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}j + \frac{1}{2}k & -\frac{1}{2}j - \frac{1}{2}k \\ -\frac{1}{2}j - \frac{1}{2}k & \frac{1}{2} + \frac{1}{2}k \end{pmatrix} \begin{pmatrix} 1 + i - j - k & 1 - i + j + k \\ 1 + i + j - k & 1 - i + j + k \end{pmatrix} \\ &= \begin{pmatrix} 1 + i & \\ & 1 - i \end{pmatrix} \end{aligned}$$

are two complex matrices. Evidently, the complex matrix pencil $\widetilde{A}+\lambda\widetilde{B}^{\varepsilon\delta}$ is regular. By a simple computation, we can obtain the two eigenvalues of the matrix pencil $\widetilde{A}+\lambda\widetilde{B}^{\varepsilon\delta}$ are $\lambda_1 = \frac{1}{2}+\frac{1}{2}i$ and $\lambda_2 = \frac{1}{2}-\frac{1}{2}i$. Let $\mu_1 = \frac{1}{2}+\frac{1}{2}i$ and $\mu_2 = \frac{1}{2}-\frac{1}{2}i$. Then $\mu_1^{\varepsilon} = \frac{1}{2}+\frac{1}{2}i$ and $\mu_2^{\varepsilon} = \frac{1}{2}-\frac{1}{2}i$. Obviously, $1/\mu_1^{\delta} = 1+i$ and $1/\mu_2^{\delta} = 1-i$ are not eigenvalues of the matrix pencil $\widetilde{A}+\lambda\widetilde{B}^{\varepsilon\delta}$. So the two conditions of Theorem 5.2 hold. Then $AX^{\varepsilon}+X^{\delta}B = C$ has a unique solution.

Let

$$\begin{aligned}\widetilde{C} = PCP^{\varepsilon\delta} &= \begin{pmatrix} i+j & -1-k \\ \frac{1}{2}-\frac{1}{2}i-\frac{1}{2}j-\frac{1}{2}k & \frac{3}{2}-\frac{1}{2}i-\frac{1}{2}j+\frac{1}{2}k \end{pmatrix}\begin{pmatrix} j-k & i+j+k \\ i-j-k & j+k \end{pmatrix}\begin{pmatrix} -i+j & \frac{1}{2}+\frac{1}{2}i-\frac{1}{2}j-\frac{1}{2}k \\ -1-k & \frac{3}{2}+\frac{1}{2}i-\frac{1}{2}j+\frac{1}{2}k \end{pmatrix}\\ &= \begin{pmatrix} -2-3i+j+2k & -1+2i-2j-k \\ \frac{1}{2}+\frac{7}{2}i-\frac{3}{2}j-\frac{3}{2}k & \frac{3}{2}-\frac{1}{2}i+\frac{5}{2}j+\frac{3}{2}k \end{pmatrix}\begin{pmatrix} -i+j & \frac{1}{2}+\frac{1}{2}i-\frac{1}{2}j-\frac{1}{2}k \\ -1-k & \frac{3}{2}+\frac{1}{2}i-\frac{1}{2}j+\frac{1}{2}k \end{pmatrix}\\ &= \begin{pmatrix} -4 & -1-i-3j+k \\ 5-i-j-k & 4i+4j \end{pmatrix}.\end{aligned}$$

By a simple computation, the quaternion matrix equation $\widetilde{A}Y^{\varepsilon}+Y^{\delta}\widetilde{B} = \widetilde{C}$ has a unique solution

$$Y = \begin{pmatrix} -4+8i & 3-i+j-3k \\ 1-i+3j+3k & -4i-4j \end{pmatrix}.$$

By Theorem 5.2, the quaternion matrix equation $AX^{\varepsilon}+X^{\delta}B = C$ has a unique solution

$$\begin{aligned}X = Q^{\varepsilon}YP^{-\delta} &= \begin{pmatrix} 1+i+j+k & 1-i-j-k \\ 1+i-j+k & 1-i-j-k \end{pmatrix}\begin{pmatrix} -4+8i & 3-i+j-3k \\ 1-i+3j+3k & -4i-4j \end{pmatrix}\begin{pmatrix} \frac{1}{2}+\frac{1}{2}i+\frac{1}{2}j & -\frac{1}{2}i \\ \frac{1}{2}+\frac{1}{2}i & \frac{1}{2}-\frac{1}{2}i \end{pmatrix}\\ &= \begin{pmatrix} -6+2i+10j-14k & -2-10i+6j+2k \\ -6+2i+18j+2k & -4i \end{pmatrix}\begin{pmatrix} \frac{1}{2}+\frac{1}{2}i+\frac{1}{2}j & -\frac{1}{2}i \\ \frac{1}{2}+\frac{1}{2}i & \frac{1}{2}-\frac{1}{2}i \end{pmatrix}\\ &= \begin{pmatrix} -5-i-j-13k & -5-i+9j+9k \\ -11-5i+7j-7k & -1+i-j+9k \end{pmatrix}.\end{aligned} \tag{65}$$

In fact, substituting (65) into the left side of the quaternion matrix equation $AX^{\varepsilon}+X^{\delta}B = C$, we can verify that

$$\begin{aligned}AX^{\varepsilon}+X^{\delta}B &= \begin{pmatrix} \frac{1}{4}i-\frac{1}{2}k & \frac{1}{4}k \\ \frac{1}{2}j+\frac{1}{4}k & -\frac{1}{4}-\frac{1}{4}j-\frac{1}{4}k \end{pmatrix}X^{\varepsilon}+X^{\delta}\begin{pmatrix} \frac{1}{4}+\frac{1}{4}i & -\frac{1}{4}j-\frac{1}{4}k \\ -\frac{1}{4}j+\frac{1}{4}k & \frac{1}{4}-\frac{1}{4}i \end{pmatrix}\\ &= \begin{pmatrix} \frac{1}{4}i-\frac{1}{2}k & \frac{1}{4}k \\ \frac{1}{2}j+\frac{1}{4}k & -\frac{1}{4}-\frac{1}{4}j-\frac{1}{4}k \end{pmatrix}\begin{pmatrix} -5-i+j+13k & -5-i-9j-9k \\ -11-5i-7j+7k & -1+i+j-9k \end{pmatrix}\\ &\quad+\begin{pmatrix} -5+i+j+13k & -11+5i-7j+7k \\ -5+i-9j-9k & -1-i+j-9k \end{pmatrix}\begin{pmatrix} \frac{1}{4}+\frac{1}{4}i & -\frac{1}{4}j-\frac{1}{4}k \\ -\frac{1}{4}j+\frac{1}{4}k & \frac{1}{4}-\frac{1}{4}i \end{pmatrix}\\ &= \begin{pmatrix} 5+i-4j & -2-6i+3j \\ -1+4i+3j-k & 5-3j+2k \end{pmatrix}+\begin{pmatrix} -5-i+5j-k & 2+7i-2j+k \\ 1-3i-4j & -5+4j-k \end{pmatrix}\\ &= \begin{pmatrix} j-k & i+j+k \\ i-j-k & j+k \end{pmatrix}.\end{aligned} \tag{66}$$

The matrix on the right side of (66) is exactly equal to the matrix C. This confirms that the matrix X in (65) is the solution of the considered equation in Example 6.1.

7. Conclusions

In this paper, we presented the procedure for the dimension of the solution space of the quaternion matrix equation $AX^{\varepsilon} + X^{\delta}B = 0$ for $\varepsilon \in \{\mathbb{I}, \mathbb{C}\}, \delta \in \{\dagger, *\}$. Firstly, based on the regularity of $(A, B^{\varepsilon\delta})$, we transformed the target equation into a quaternion matrix equation with complex coefficients. Secondly, we decoupled the transformed equation into several subsystems in terms of the Kronecker canonical form of the transformed matrix pencil. Moreover, we also pointed out that we can compute the solution in terms of the Kronecker canonical form and its four related nonsingular matrices. Thirdly, we presented the dimension of the solution space in terms of the sizes of the blocks of the Kronecker canonical form. Finally, we also gave the necessary and sufficient condition for the existence of the unique solution of $AX^{\varepsilon} + X^{\delta}B = 0$ and $AX^{\varepsilon} + X^{\delta}B = C$, respectively.

However, as declared in Introduction, the quaternion matrix equation $AX + X^{\mathbb{C}}B = 0$ has not been studied at present. Thus, it still be an unsettled problem to determine the dimension of the solution space of $AX + X^{\mathbb{C}}B = 0$, and the necessary and sufficient condition for existence and uniqueness of solution of $AX + X^{\mathbb{C}}B = 0$ and $AX + X^{\mathbb{C}}B = C$, respectively. Furthermore, the dimension of the solution space of the more general quaternion matrix equation $AX^{\varepsilon}B + CX^{\delta}D = 0$ has not been completely addressed. In the future, we will also explore the necessary and sufficient condition for the existence of the unique solution of $AX^{\varepsilon}B + CX^{\delta}D = 0$ and $AX^{\varepsilon}B + CX^{\delta}D = E$, respectively.

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