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Strong type inequalities for sublinear operator on weighted modular Banach function spaces

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Abstract. In this paper, we prove the boundedness of sublinear operator on weighted modular Banach function space (BFS) under a certain size condition. We establish sufficient conditions on weight functions and on the geometry of modular BFS for the validity of the strong inequality for sublinear operator on weighted modular BFS under certain size condition. We will assume that the BFS is p-convex and the modular defining the BFS satisfies some growth condition. In particular, we obtain the boundedness of the sublinear operator on weighted Musielak-Orlicz spaces. The size condition is satisfied by most of the operators in harmonic analysis, such as the Calderón-Zygmund singular integral operator, Hardy-Littlewood maximal operator, Bochner-Riesz means at the critical index, Carleson maximal operator, Ricci-Stein's oscillatory singular integrals, C. Fefferman's singular multiplier operator, R. Fefferman's singular integral operator and so on. The main result is new in the case of an unweighted setting.

1. Introduction

One of the central problems of harmonic analysis is the problem of the boundedness of the sublinear operators on the BFS. The investigations of the sublinear operators on weighted BFS have a recent history. The goal of these investigations was closely connected with the finding of criterion on the geometry of BFS and on the weights for validity of boundedness of sublinear operators on BFS. The characterization of the mapping properties such as boundedness and compactness of Hardy type operator on variable Lebesgue spaces were considered in [1]-[3], [5], [6], [8], [15], [17], [30] and so on. Moreover, the compactness and measure of noncompactness of Hardy-type operators on ℓ-convex BFS was established in [27]. Also, the boundedness of Hardy type operator on general BFS was studied in [23]. The characterization of the boundedness of the Hardy-Littlewood maximal operator on BFS has been studied in [10], [20], [25] and so on. We observe that the boundedness of Riesz potential on weighted modular BFS was proved in [7]. In particular, the boundedness Hardy-Littlewood maximal operator on variable Lebesgue spaces and on

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Musielak-Orlicz spaces has been studied in [12], [13], [18], [19], [32] and so on. Note that the notion of BFS was introduced in [29]. In particular, the weighted Lebesgue spaces, weighted Lorentz spaces, weighted variable Lebesgue spaces, variable Lebesgue spaces with mixed norm and Musielak-Orlicz spaces are BFS.

In order to study the well known important operators in harmonic analysis uniformly, many researchers introduced the following sublinear operator satisfying some size condition.

Let T be a sublinear operator that satisfies that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin supp f$

$$|Tf(x)| \leq C_0 \int\limits_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n}\,dy, \tag{1}$$

where C_0 is independent of f and x.

We point out that condition (1) was first introduced by Soria and Weiss in [34]. The condition (1) are satisfied by many interesting operators in harmonic analysis, such as the Calderón-Zygmund singular integral operators, Carleson's maximal operators, Hardy-Littlewood maximal operator, C. Fefferman's singular multipliers, R. Fefferman's singular integrals, Ricci-Stein's oscillatory singular integrals, the Bochner-Riesz means and so on (see [28], [34] for details). Note that the boundedness of sublinear operators and its commutators on different function spaces was studied in [1], [16], [23], [28], [34]-[36] and so on.

Inspired by the above, we proved the boundedness of sublinear operators on weighted modular BFS. We give sufficient conditions on weight functions and on the geometry of modular BFS. In this paper, it is assumed that the BFS satisfies the condition of p-convexity. In particular, we can obtain the boundedness of many important linear operators in harmonic analysis on weighted modular BFS. In particular, we obtain the boundedness of the sublinear operator on weighted variable Lebesgue spaces.

The remainder of the paper is structured as follows. In Section 2, we will recall some related definitions and auxiliary lemmas, including some basic notions regarding modular spaces and weighted BFS. Our principal assertions concerning the Hardy operator in the mentioned spaces are formulated and proved in Section 3. We establish sufficient conditions on weight functions and on the modular defining BFS for the validity of two-weight inequality for sublinear operator on weighted modular BFS under certain size condition. As an application, we obtain the boundedness of some classical sublinear operators on weighted modular BFS. Throughout, we use C to stand for an absolute positive constant, which may have different values in different occurrences.

2. Preliminaries

Let (Ω, μ) be a complete σ-finite measure space. By $L_0 = L_0(\Omega, \mu)$ we denote the collection of all real-valued -measurable functions on Ω.

Definition 2.1. ([14], [31] Let X be a real linear space. A function $\rho : X \mapsto [0, \infty]$ is called semimodular on L if the following properties hold:

(a) $\rho(0) = 0$,
(b) $\rho(\lambda x) = \rho(x)$ for all $x \in X$ and $\lambda \in \mathbb{R}$ with $|\lambda| = 1$,
(c) ρ is convex,
(d) ρ is left-continuous,
(e) $\rho(\lambda x) = 0$ for all $\lambda > 0$ implies that $x = 0$.
A semimodular ρ is called a modular if
(f) $\rho(x) = 0$ implies that $x = 0$,
A semimodular ρ is called continuous if
(f) the mapping $\lambda \mapsto \rho(\lambda x)$ is continuous on $[0, \infty)$ for every $x \in X$.

If ρ is semimodular or modular on X, then

$$X_\rho := \left\{x \in X : \lim_{\lambda \to 0} \rho(\lambda x) = 0\right\}$$

is called a semimodular space or a modular space, respectively. The limit as $\lambda \to 0$ exists in $\mathbb{R}$.

Theorem 2.2. ([14], [31]) *Let ρ be semimodular on X. Then X_ρ is a normed real vector space. The norm in semimodular space is called the Luxemburg norm and defined by the formula*

$$\|x\|_{X_\rho} = \|x\|_\rho := \inf\left\{\lambda > 0 : \ \rho\left(\frac{1}{\lambda}x\right) \leq 1\right\}.$$

Now we give the norm-modular unit ball property in semimodular space.

Lemma 2.3. ([14]) *Let ρ be a semimodular on X. Then $\|x\|_\rho \leq 1$ and $\rho(x) \leq 1$ are equivalent. If ρ is continuous, then also $\|x\|_\rho < 1$ and $\rho(x) < 1$ are equivalent, as are $\|x\|_\rho = 1$ and $\rho(x) = 1$.*

Let X_{ρ_1} and Y_{ρ_2} be two modular spaces on linear spaces X and Y, respectively. We recall that the sublinear operator S is said to be bounded from X_{ρ_1} to Y_{ρ_2} if there exists a constant $C > 0$ such that for any $f \in X_{\rho_1}$

$$\|Sf\|_{\rho_2} \leq C\|f\|_{\rho_1}.$$

Let us give a characterization of the bounded sublinear operator in terms of the modular.

By the norm-modular unit ball property in modular spaces, we have the following lemma.

Lemma 2.4. *Let ρ_1 and ρ_2 be two modulars on linear spaces X and Y, respectively. Suppose that $S : X_{\rho_1} \mapsto Y_{\rho_2}$ is a bounded sublinear operator. Then, S is bounded if and only if there exists $C > 0$ such that*

$$\rho_1(f) \leq 1 \implies \rho_2\left(\frac{Sf}{C}\right) \leq 1.$$

Definition 2.5. ([9], [26]) We say that real normed space X is a BFS over (Ω, μ) if:

(P1) the norm $\|f\|_X$ is defined for every μ-measurable function f, and $f \in X$ if and only if $\|f\|_X < \infty$; $\|f\|_X = 0$ if and only if $f = 0$ μ-a.e.,

(P2) $\|f\|_X = \||f|\|_X$ for all $f \in X$,

(P3) $0 \leq f \leq g$ μ-a.e. $\Longrightarrow \|f\|_X \leq \|g\|_X$,

(P4) if $0 \leq f_n \uparrow f$ μ-a.e. $\Longrightarrow \left\|f_n\right\|_X \uparrow \|f\|_X$ (Fatou property),

(P5) if E is a μ-measurable subset of Ω such that $\mu(E) < \infty$, then $\|\chi_E\|_X < \infty$, where χ_E is the characteristic function of the set E,

(P6) for every μ-measurable set $E \subset \Omega$ with $\mu(E) < \infty$, there is a constant $C_E > 0$ such that

$$\int_E f \, d\mu \leq C_E \, \|f\|_X.$$

Given a BFS X we can always consider its associate space X' consisting of those $g \in L_0$ that $fg \in L_1$ for every $f \in X$ with the usual order and the norm $\|g\|_{X'} = \sup\left\{\int_\Omega fg \, d\mu, \ \|f\|_X \leq 1\right\}$. Note that X' is a BFS over (Ω, μ) and a closed norming subspaces.

The definition of $\|g\|_{X'}$ implies that

$$\int_\Omega fg \, d\mu \leq \|f\|_X \, \|g\|_{X'}.$$

So, we have

$$\|f\|_X = \sup_{\|g\|_{X'} \leq 1} \int_\Omega |fg| \, d\mu.$$

Let X be a BFS over (Ω, μ) and let ω be a weight function on Ω. Let $X_\omega = \{f \in L_0 : \ f\omega \in X\}$. This space is a weighted BFS equipped with the norm $\|f\|_{X_\omega} = \|f\omega\|_X$. (For more detail and proofs of results about BFS we refer the reader to [9] and [25].)

Note that the notion of BFS was introduced by W.A.J. Luxemburg in [29].

Let us recall the notion of p-convexity and p-concavity of BFS's.

Definition 2.6. ([33]) Let X be a BFS. Then X is called p-convex for $1 \le p \le \infty$, if there exists a constant $M > 0$ such that for all $n \in \mathbb{N}$ and $f_1, \dots, f_n \in X$

$$\left\| \left(\sum_{k=1}^{n} |f_k|^p \right)^{\frac{1}{p}} \right\|_X \le M \left(\sum_{k=1}^{n} \|f_k\|_X^p \right)^{\frac{1}{p}} \quad \text{if} \quad 1 \le p < \infty,$$

or $\left\| \sup_{1\le k\le n} |f_k| \right\|_X \le M \max_{1\le k\le n} \|f_k\|_X$ if $p = \infty$. Similarly, X is called p-concave for $1 \le p \le \infty$, if there exists a constant $M > 0$ such that for all $n \in \mathbb{N}$ and $f_1, \dots, f_n \in X$

$$\left(\sum_{k=1}^{n} \|f_k\|_X^p \right)^{\frac{1}{p}} \le M \left\| \left(\sum_{k=1}^{n} |f_k|^p \right)^{\frac{1}{p}} \right\|_X \quad \text{if} \quad 1 \le p < \infty,$$

or $\max_{1\le k\le n} \|f_k\|_X \le M \left\| \sup_{1\le k\le n} |f_k| \right\|_X$ if $p = \infty$.

Remark 2.7. Note that the notions of p-convexity and p-concavity, are closely related to the notions of upper p-estimate (strong ℓ_p-composition property), respectively lower p-estimate (strong ℓ_p-decomposition property) as can be found in [26].

We note that the Lebesgue spaces with mixed norm, weighted Lorentz spaces etc are p-convex(p-concave) BFS. Now we reduce more general result connected with Minkowski's integral inequality.

Let X and Y be BFS's on (Ω_1, μ) and (Ω_2, ν) , respectively. By $X[Y]$ and $Y[X]$ we denote the spaces with a mixed norm and consist of all functions $g \in L_0 (\Omega_1 \times \Omega_2, \mu \times \nu)$ such that $\|g(x, \cdot)\|_Y \in X$ and $\|g(\cdot, y)\|_X \in Y$. The norms in this spaces is defined as

$$\|g\|_{X[Y]} = \left\| \|g(x, \cdot)\|_Y \right\|_X, \quad \|g\|_{Y[X]} = \left\| \|g(\cdot, y)\|_X \right\|_Y .$$

Now we give some examples of p-convex and respectively p-concave BFS.

Theorem 2.8. ([33]) *Let X and Y be BFS's and let there exists $1 \le p \le \infty$ such that X is p-convex and Y is p-concave. Then there exists a constant M such that for all $\mu \times \nu$-measurable $f : \Omega_1 \times \Omega_2 \mapsto \mathbb{R}$ the inequality*

$$\|f\|_{X[Y]} \le M \|f\|_{Y[X]}$$

holds.

It is known that $X[Y]$ and $Y[X]$ are BFS's on $\Omega_1 \times \Omega_2$ (see [24]).

Example 2.9. Let $1 \le q \le \infty$ and $X = L_q$. Then the space L_q is p-convex (p-concave) BFS if and only if $1 \le p \le q \le \infty$ ($1 \le q \le p \le \infty$.)

The proof implies from the usual Minkowski inequality in Lebesgue spaces.

Next, we give the definition of the Musielak-Orlicz space.

Definition 2.10. ([14], [31]) Let $\Omega \subset \mathbb{R}^n$ be a Lebesgue measurable set. A real function $\varphi : \Omega \times [0, \infty) \mapsto [0, \infty]$ is called a generalized Φ-function on Ω if it satisfies:

(a) $\varphi(x, \cdot)$ is a Φ-function for all $x \in \Omega$, i.e., $\varphi(x, \cdot) : [0, \infty) \mapsto [0, \infty]$ is convex and satisfies $\varphi(x, 0) = 0$, $\lim_{t\to+0} \varphi(x, t) = 0$;

(b) $x \mapsto \varphi(x, t)$ is measurable for every $t \ge 0$.

If φ is a generalized Φ-function on Ω, we briefly write $\varphi \in \Phi(\Omega)$.

Definition 2.11. ([14], [31]) Let $\varphi \in \Phi$ and let ρ_φ be given by

$$\rho_\varphi(f) := \int_\Omega \varphi(x, |f(x)|)\, dx \quad \text{for all} \quad f \in L_0(\Omega).$$

We put $L_\varphi(\Omega) = \left\{ f \in L_0(\Omega) : \quad \rho_\varphi(\lambda_0 f) < \infty \text{ for some } \lambda_0 > 0 \right\}$ and

$$\|f\|_{L_\varphi(\Omega)} = \inf \left\{ \lambda > 0 : \quad \rho_\varphi\left(\frac{f}{\lambda}\right) \le 1 \right\}$$

The space $L_\varphi(\Omega)$ is called Musielak-Orlicz space.

Let ω be a weight function on Ω. In this work we considered the weighted Musielak-Orlicz spaces. We denote

$$L_{\varphi,\omega}(\Omega) = \left\{ f \in L_0(\Omega) : \quad f\omega \in L_\varphi(\Omega) \right\}.$$

It is obvious that the norm in this spaces is given by

$$\|f\|_{L_{\varphi,\omega}(\Omega)} = \|f\omega\|_{L_\varphi(\Omega)}.$$

The following lemma shows that the Musielak-Orlicz space L_φ is p-convex BFS.

Lemma 2.12. ([5]) *Let $\Omega_1 \subset \mathbb{R}^n$ and $\Omega_2 \subset \mathbb{R}^m$. Let $(x,t) \in \Omega_1 \times [0,\infty)$ and let $\varphi\left(x, t^{1/p}\right) \in \Phi$ for some $1 \le p < \infty$. Suppose that $f : \Omega_1 \times \Omega_2 \mapsto \mathbb{R}$ is a $\mu \times \nu$-measurable function. Then the inequality*

$$\left\| \|f(x,\cdot)\|_{L_p(\Omega_2)} \right\|_{L_\varphi(\Omega_1)} \le 2^{1/p} \left\| \|f(\cdot,y)\|_{L_\varphi(\Omega_1)} \right\|_{L_p(\Omega_2)}$$

holds.

Let X and Y be BFS over the same measure space (Ω, μ). We write $X \hookrightarrow Y$ to denote the fact that X is continuously embedded into Y.

Now we give an embedding between different Musielak-Orlicz spaces.

Theorem 2.13. ([14]) *Let φ, $\psi \in \Phi(\Omega)$. Then $L_\varphi(\Omega) \hookrightarrow L_\psi(\Omega)$ if and only if there exist $C > 0$ and $h \in L_1(\Omega)$ with $\|h\|_{L_1(\Omega)} \le 1$ such that*

$$\psi(x, \frac{t}{C}) \le \varphi(x,t) + h(x)$$

for almost all $x \in \Omega$ and all $t \ge 0$.

Moreover, C is bounded by the embedding constant, whereas the embedding constant is bounded by $2C$.

The following corollary is a consequence of Theorem 2.13.

Corollary 2.14. *Let $\varphi \in \Phi(\Omega)$. Let $1 \le p < \infty$ and $\psi(x,t) = t^p$, $x \in \Omega$. Then $L_\varphi(\Omega) \hookrightarrow L_p(\Omega)$ if and only if there exist $C > 0$ and $h \in L_1(\Omega)$ with $\|h\|_{L_1(\Omega)} \le 1$ such that*

$$\left(\frac{t}{C}\right)^p \le \varphi(x,t) + h(x) \tag{2}$$

for almost all $x \in \Omega$ and all $t \ge 0$.

Moreover, C is bounded by the embedding constant, whereas the embedding constant is bounded by $2C$.

Let $p : \Omega \mapsto [1, \infty)$ be a Lebesgue measurable function. The Musielak-Orlicz space $L_\varphi(\Omega)$ is the variable Lebesgue space if $\varphi(x,t) = t^{p(x)}$, $t \geq 0$ (see, [11] and [22]). In particular, for more information on embedding between different variable Lebesgue spaces we refer to [4] and [14].

Let f be a non-negative locally integrable function on $\mathbb{R}^n$. The multidimensional Hardy operator and its dual operator are defined by

$$Hf(x) = \int\limits_{|y|<|x|} f(y)\,dy; \quad H^\star f(x) = \int\limits_{|y|>|x|} f(y)\,dy, \ \ x \in \mathbb{R}^n.$$

As shown in [6], we have the following two theorems.

Theorem 2.15. *Let v and w are weights on $\mathbb{R}^n$ and let $0 < \alpha < 1$. Suppose that there exists $1 \leq p < \infty$ such that X_w is a p-convex weighted BFS's. Then the inequality*

$$\|Hf\|_{X_w} \leq C\, \|f\|_{L_{p,v}(\mathbb{R}^n)} \tag{3}$$

holds if and only if

$$A(\alpha) = \sup_{t>0} \left(\int\limits_{|y|<t} [v(y)]^{-p'}\, dy \right)^{\frac{\alpha}{p'}} \left\| \chi_{\{|z|>t\}}(\cdot) \left(\int\limits_{|y|<|\cdot|} [v(y)]^{-p'}\, dy \right)^{\frac{1-\alpha}{p'}} \right\|_{X_w} < \infty.$$

Moreover, if $C > 0$ is the best possible constant in (3), then

$$\sup_{0<\alpha<1} \frac{p'\, A(\alpha)}{(1-\alpha)\left[\left(\frac{p'}{1-\alpha}\right)^p + \frac{1}{\alpha(p-1)}\right]^{1/p}} \leq C \leq M \inf_{0<\alpha<1} \frac{A(\alpha)}{(1-\alpha)^{1/p'}}.$$

The similar theorem holds for dual operator of the multidimensional Hardy operator.

Theorem 2.16. *Let v and w are weights on $\mathbb{R}^n$ and $0 < \beta < 1$. Suppose that there exists $1 \leq p < \infty$ such that X_w is a p-convex weighted BFS's. Then the inequality*

$$\|H^\star f\|_{X_w} \leq C\, \|f\|_{L_{p,v}(\mathbb{R}^n)} \tag{4}$$

holds if and only if

$$B(\beta) = \sup_{t>0} \left(\int\limits_{|y|>t} [v(y)]^{-p'}\, dy \right)^{\frac{\beta}{p'}} \left\| \chi_{\{|z|<t\}}(\cdot) \left(\int\limits_{|y|>|\cdot|} [v(y)]^{-p'}\, dy \right)^{\frac{1-\beta}{p'}} \right\|_{X_w} < \infty.$$

Moreover, if $C > 0$ is the best possible constant in (4), then

$$\sup_{0<\beta<1} \frac{p'\, B(\beta)}{(1-\beta)\left[\left(\frac{p'}{1-\beta}\right)^p + \frac{1}{\beta(p-1)}\right]^{1/p}} \leq C \leq M \inf_{0<\beta<1} \frac{B(\beta)}{(1-\beta)^{1/p'}}.$$

3. Main results

In this section, we start with the proof of the main result.

Let $SB(X,Y)$ be the set of the bounded sublinear operators from a Banach space X into a Banach space Y.

Let $\mathbb{Z}$ be the set of integers. For $k \in \mathbb{Z}$ we define $E_k = \left\{x \in \mathbb{R}^n : \ 2^k < |x| \leq 2^{k+1}\right\}$, $E_{k,1} = \left\{x \in \mathbb{R}^n : \ |x| \leq 2^{k-1}\right\}$, $E_{k,2} = \left\{x \in \mathbb{R}^n : \ 2^{k-1} < |x| \leq 2^{k+2}\right\}$ and $E_{k,3} = \left\{x \in \mathbb{R}^n : \ |x| > 2^{k+2}\right\}$. Then $E_{k,2} = E_{k-1} \cup E_k \cup E_{k+1}$ and the multiplicity of the covering $\{E_{k,2}\}_{k\in\mathbb{Z}}$ is equal to 3.

Theorem 3.1. *Let v and w be weight functions defined on $\mathbb{R}^n$. Suppose $\{\omega_k\}_{k\in\mathbb{Z}}$ is a sequence of nonnegative measurable functions on $\mathbb{R}^n$. Let X be a modular BFS on $\mathbb{R}^n$ and let X_v be a corresponding weighted modular BFS. Suppose that there exists $1<p<\infty$ such that X_w is a p-convex BFS and let $X_v \hookrightarrow L_{p,v}(\mathbb{R}^n)$. Suppose that r is a bounded measurable function on $\mathbb{R}^n$ such that $\underline{r}\ge p$. Let T be a sublinear operator satisfying the condition (1) and $T\in SB\left(L_p(\mathbb{R}^n),\ X\right)$. Let $0<\alpha,\beta<1$ and assume that the following conditions are satisfied:*

$$1)\ A_1(\alpha)=\sup_{t>0}\left(\int\limits_{|y|<t}[v(y)]^{-p'}\,dy\right)^{\frac{\alpha}{p'}}\left\|\frac{1}{|x|^n}\left(\int\limits_{|y|<|x|}[v(y)]^{-p'}\,dy\right)^{\frac{1-\alpha}{p'}}\chi_{\{|x|>t\}}\right\|_{X_w}<\infty,$$

$$2)\ B_1(\beta)=\sup_{t>0}\left(\int\limits_{|y|>t}\left[v(y)\,|y|^n\right]^{-p'}\,dy\right)^{\frac{\beta}{p'}}\left\|\left(\int\limits_{|y|>|x|}\left[v(y)\,|y|^n\right]^{-p'}\,dy\right)^{\frac{1-\beta}{p'}}\chi_{\{|x|<t\}}\right\|_{X_w}<\infty,$$

3) there exists $C_1>0$ such that for any $k\in\mathbb{Z}$ the following inequality holds:

$$\operatorname*{ess\,sup}_{x\in E_k} w(x)\le C_1\ \operatorname*{ess\,inf}_{x\in E_{k,2}} v(x)$$

4) there exists $C_2>0$ such that for any $k\in\mathbb{Z}$ and $g_k\in X$ the following inequality holds:

$$\rho\left(\sum_{k\in\mathbb{Z}}\omega_k\,\left|g_k\right|\,\chi_{E_k}\right)\le C_2\sum_{k\in\mathbb{Z}}\operatorname*{ess\,sup}_{x\in E_k}\left[\omega_k(x)\right]^{r(x)}\rho\left(g_k\,\chi_{E_k}\right).$$

Then $T\in SB\left(X_v,\ X_w\right)$.

Proof. Let $f\in X_v$. By the fact that T is a sublinear operator, we can write

$$|Tf(x)|=\sum_{k\in\mathbb{Z}}|Tf(x)|\chi_{E_k}(x)\le\sum_{k\in\mathbb{Z}}\left|Tf_{k,1}(x)\right|\chi_{E_k}(x)+\sum_{k\in\mathbb{Z}}\left|Tf_{k,2}(x)\right|\chi_{E_k}(x)$$

$$+\sum_{k\in\mathbb{Z}}\left|Tf_{k,3}(x)\right|\chi_{E_k}(x)=T_1f(x)+T_2f(x)+T_3f(x),$$

where χ_{E_k} is the characteristic function of the set E_k, $f_{k,i}=f\chi_{E_{k,i}}$, $i=1,2,3$.

First we shall estimate $\left\|T_1 f\right\|_{X_w}$. We observe that for $x\in E_k$ and $y\in E_{k,1}$, we have $|y|\le 2^{k-1}\le\frac{1}{2}|x|$. It is obvious that $E_k\cap supp\, f_{k,1}=\emptyset$ and $|x-y|\ge|x|-|y|\ge\frac{1}{2}|x|$. So, by (1), one has

$$\left|T_1f(x)\right|\le C\sum_{k\in\mathbb{Z}}\left(\int\limits_{\mathbb{R}^n}\frac{\left|f_{k,1}(y)\right|}{|x-y|^n}\,dy\right)\chi_{E_k}(x)=C\sum_{k\in\mathbb{Z}}\left(\int\limits_{E_{k,1}}\frac{\left|f(y)\right|}{|x-y|^n}\,dy\right)\chi_{E_k}(x)$$

$$\le C\left(\int\limits_{|y|<\frac{1}{2}|x|}\frac{\left|f(y)\right|}{|x-y|^n}\,dy\right)\sum_{k\in\mathbb{Z}}\chi_{E_k}(x)=C\left(\int\limits_{|y|<\frac{1}{2}|x|}\frac{\left|f(y)\right|}{|x-y|^n}\,dy\right)$$

$$\le C\left(\int\limits_{|y|<|x|}\frac{\left|f(y)\right|}{|x-y|^n}\,dy\right)\le 2^nC\,\frac{1}{|x|^n}\int\limits_{|y|<|x|}|f(y)|\,dy.$$

So, we have

$$\left\|T_1 f\right\|_{X_w} \le 2^n C \left\| \frac{1}{|\cdot|^n} \int_{|y|<|\cdot|} |f(y)|\, dy \right\|_{X_w} = 2^n C \left\| \int_{|y|<|\cdot|} |f(y)|\, dy \right\|_{X_{\frac{w}{|\cdot|^n}}} .$$

By condition *1*) and Theorem 2.15, we have

$$\left\|T_1 f\right\|_{X_w} \le \frac{2^n\, C\, M\, A_1(\alpha)}{(1-\alpha)^{1/p'}} \, \|f\|_{L_{p,\nu}(\mathbb{R}^n)}. \tag{5}$$

Next we estimate $\left\|T_3 f\right\|_{X_w.}$ We observe that for $x \in E_k$ and $y \in E_{k,3}$, we have $|y| > 2^{k+2} \ge 2\,|x|$. It is obvious that $E_k \cap supp\, f_{k,3} = \emptyset$ and $|x-y| \ge |y| - |x| \ge \frac{1}{2}\,|y|$. Similarly, we can show that

$$\left|T_3 f(x)\right| \le 2^n C \int_{|y|>|x|} \frac{|f(y)|}{|y|^n}\, dy.$$

Thus, one has

$$\left\|T_3 f\right\|_{X_w} \le 2^n C \left\| \int_{|y|>|\cdot|} \frac{|f(y)|}{|y|^n}\, dy \right\|_{X_w} .$$

By condition *2*) and Theorem 2.16, we have

$$\left\|T_3 f\right\|_{X_w} \le \frac{2^n\, C\, M\, B_1(\beta)}{(1-\beta)^{1/p'}} \, \|f\|_{L_{p,\nu}(\mathbb{R}^n)}. \tag{6}$$

Finally, we estimate $\left\|T_2 f\right\|_{X_w.}$ Let $\|f\|_{L_{p,\nu}(\mathbb{R}^n)} \le 1$. So, $\rho_p(f) = \int_{\mathbb{R}^n} [|f(x)|\, \nu(x)]^p\, dx \le 1$. By Lemma 2.4, it suffices to prove that there exists a constant $M > 0$ such that

$$\rho\left(\frac{w\, T_2 f}{M}\right) \le 1.$$

Taking $\omega_k(x) = \dfrac{C\, w\, \left\|f_{k,2}\right\|_{L_p(\mathbb{R}^n)}}{M}$ by the conditions *3*) and *4*), we have

$$\rho\left(\frac{w\, T_2 f}{M}\right) = \rho\left(\frac{w \sum\limits_{k\in\mathbb{Z}} \left|T f_{k,2}\right| \chi_{E_k}}{M}\right) = \rho\left(\sum_{k\in\mathbb{Z}} \frac{C\, w\, \left\|f_{k,2}\right\|_{L_p(\mathbb{R}^n)}}{M} \, \frac{\left|T f_{k,2}\right|}{C\, \left\|f_{k,2}\right\|_{L_p(\mathbb{R}^n)}} \, \chi_{E_k}\right)$$

$$\le C_2 \sum_{k\in\mathbb{Z}} \operatorname*{ess\,sup}_{x\in E_k} \left[\frac{C\, w(x)\, \left\|f_{k,2}\right\|_{L_p(\mathbb{R}^n)}}{M}\right]^{r(x)} \rho\left(\frac{T f_{k,2}}{C\, \left\|f_{k,2}\right\|_{L_p(\mathbb{R}^n)}} \, \chi_{E_k}\right)$$

$$\le C_2 \sum_{k\in\mathbb{Z}} \operatorname*{ess\,sup}_{x\in E_k} \left[\frac{C\, \left\|f\, w(x)\right\|_{L_p(E_{k,2})}}{M}\right]^{r(x)} \rho\left(\frac{T f_{k,2}}{C\, \left\|f_{k,2}\right\|_{L_p(\mathbb{R}^n)}}\right)$$

$$\le C_2 \sum_{k\in\mathbb{Z}} \operatorname*{ess\,sup}_{x\in E_k} \left[\frac{C\, C_1\, \left\|f \operatorname*{ess\,inf}_{x\in E_{k,2}} \nu(x)\right\|_{L_p(E_{k,2})}}{M}\right]^{r(x)}$$

$$\leq C_2 \operatorname*{ess\,sup}_{x\in\mathbb{R}^n}\left(\frac{C\,C_1}{M}\right)^{r(x)} \sum_{k\in\mathbb{Z}} \operatorname*{ess\,sup}_{x\in E_k}\left[\|f\,\nu\|_{L_p(E_{k,2})}\right]^{r(x)}$$

$$= C_2 \operatorname*{ess\,sup}_{x\in\mathbb{R}^n}\left(\frac{C\,C_1}{M}\right)^{r(x)} \sum_{k\in\mathbb{Z}} \left[\|f\|_{L_{p,\nu}(E_{k,2})}\right]^{r}$$

$$\leq C_2 \operatorname*{ess\,sup}_{x\in\mathbb{R}^n}\left(\frac{C\,C_1}{M}\right)^{r(x)} \sum_{k\in\mathbb{Z}} \left[\|f\,\nu\|_{L_p(E_{k,2})}\right]^{p}$$

$$= C_2 \operatorname*{ess\,sup}_{x\in\mathbb{R}^n}\left(\frac{C\,C_1}{M}\right)^{r(x)} \sum_{k\in\mathbb{Z}}\int_{E_{k,2}} [|f(x)|\,\nu(x)]^p\,dx.$$

Next, we have

$$\sum_{k\in\mathbb{Z}}\int_{E_{k,2}} [|f(x)|\,\nu(x)]^p\,dx$$

$$= \left(\sum_{k\in\mathbb{Z}}\int_{E_{k-1}} [|f(x)|\,\nu(x)]^p\,dx + \sum_{k\in\mathbb{Z}}\int_{E_{k}} [|f(x)|\,\nu(x)]^p\,dx + \sum_{k\in\mathbb{Z}}\int_{E_{k+1}} [|f(x)|\,\nu(x)]^p\,dx\right)$$

$$= 3\int_{\mathbb{R}^n} [|f(x)|\,\nu(x)]^p\,dx \leq 3.$$

Finally, one has

$$\rho\left(\frac{w\,T_2 f}{M}\right) \leq 3\,C_2 \operatorname*{ess\,sup}_{x\in\mathbb{R}^n}\left(\frac{C\,C_1}{M}\right)^{r(x)}.$$

Let $\operatorname*{ess\,sup}_{x\in\mathbb{R}^n}\left(\frac{C\,C_1}{M}\right)^{r(x)} \leq \frac{1}{3\,C_2}$. Let's choose M so that

$$M \geq C\,C_1 \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} (3\,C_2)^{\frac{1}{r(x)}}.$$

Therefore, we have

$$\rho\left(\frac{w\,T_2 f}{M}\right) \leq 1.$$

So, by Lemma 2.4, one has

$$\left\|T_2 f\right\|_{X_w} \leq C\|f\|_{L_{p,\nu}(\mathbb{R}^n)}. \tag{7}$$

Combining the estimates (5), (6) and (7), we have

$$\left\|T f\right\|_{X_w} \leq C\|f\|_{L_{p,\nu}(\mathbb{R}^n)}.$$

Since $X_\nu \hookrightarrow L_{p,\nu}(\mathbb{R}^n)$, we have that

$$\left\|T f\right\|_{X_w} \leq C\|f\|_{X_\nu}.$$

□

Let $\underline{p} = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x)$ and $\overline{p} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x)$.

The following corollary is a consequence of Theorem 3.1.

Corollary 3.2. *Let v and w be weight functions defined on $\mathbb{R}^n$. Suppose that p is a bounded measurable function on $\mathbb{R}^n$ such that $1 < \underline{p} \le p(x) \le \overline{p} < \infty$. Let there exists $0 < \delta < 1$ such that $\int\limits_{\mathbb{R}^n} \delta^{\frac{\underline{p}\,p(x)}{p(x)-\underline{p}}}\,dx < \infty$. Let T be a sublinear operator satisfying the condition (1) and $T \in SB\left(L_{\underline{p}}(\mathbb{R}^n),\, L_{p(x)}(\mathbb{R}^n)\right)$. Let $0 < \alpha, \beta < 1$ and assume that the following conditions are satisfied:*

$$1)\ A_1(\alpha) = \sup_{t>0} \left(\int\limits_{|y|<t} [v(y)]^{-\underline{p}'}\,dy \right)^{\frac{\alpha}{\underline{p}'}} \left\| \frac{1}{|x|^n} \left(\int\limits_{|y|<|x|} [v(y)]^{-\underline{p}'}\,dy \right)^{\frac{1-\alpha}{\underline{p}'}} \right\|_{L_{p(\cdot),w}(|x|>t)} < \infty,$$

$$2)\ B_1(\beta) = \left(\int\limits_{|y|>t} [v(y)\,|y|^n]^{-\underline{p}'}\,dy \right)^{\frac{\beta}{\underline{p}'}} \left\| \left(\int\limits_{|y|>|x|} [v(y)\,|y|^n]^{-\underline{p}'}\,dy \right)^{\frac{1-\beta}{\underline{p}'}} \right\|_{L_{p(\cdot),w}(|x|<t)} < \infty,$$

3) there exists $C_1 > 0$ such that for any $k \in \mathbb{Z}$ the following inequality holds:

$$\operatorname*{ess\,sup}_{x\in E_k} w(x) \le C_1 \operatorname*{ess\,inf}_{x\in E_{k,2}} v(x)$$

Then $T \in SB\left(L_{p(x),v}(\mathbb{R}^n),\, L_{p(x),w}(\mathbb{R}^n)\right)$.

Proof. Let $X_w = L_{p(x),w}(\mathbb{R}^n)$, $r(x) = p(x)$ and $p = \underline{p}$. It is well known that the modular on weighted variable Lebesgue space is defined by

$$\rho(f) = \rho_{p(\cdot),w}(f) = \int\limits_{\mathbb{R}^n} (|f(x)|\,w(x))^{p(x)}\,dx.$$

Thus, condition 4) of Theorem 3.1 is fulfilled directly. □

Remark 3.3. It should be noted that Corollary 3.2 was proved in [1]. The boundedness of the Hardy-Littlewood maximal operator on variable Lebesgue spaces was studied in [12], [13], [18], [32] and so on. If $p(x) = p_0$ is constant, then two-weight inequalities for singular integrals defined on homogeneous groups was proved in [21] and [28]. If $p(x) = p_0$ is constant, then various versions of Corollary 3.2 on weighted Lebesgue spaces were proved in [16], [36] and so on. Also, the boundedness of sublinear operators and its commutators on weighted grand Morrey spaces and on generalized mixed Morrey spaces were proved in [23] and [35].

Now we formulate a strong type inequality for sublinear operator satisfying the condition (1) on weighted Orlicz-Musielak space.

Corollary 3.4. *Let $\varphi \in \Phi(\mathbb{R}^n)$ and let v and w be weight functions defined on $\mathbb{R}^n$. Suppose that $\{\omega_k\}_{k\in\mathbb{Z}}$ is a sequence of nonnegative measurable functions on $\mathbb{R}^n$. Let $L_{\varphi,v}(\mathbb{R}^n)$ be a corresponding weighted Musielak-Orlicz space. Suppose that there exists $1 < p < \infty$ such that $\varphi\left(x, t^{\frac{1}{p}}\right) \in \Phi(\mathbb{R}^n)$ and let φ satisfy condition (2) of Corollary 2.14. Suppose that r is a bounded measurable function on $\mathbb{R}^n$ such that $\underline{r} \ge p$. Let T be a sublinear operator satisfying the condition (1) and $T \in SB\left(L_p(\mathbb{R}^n),\, L_\varphi(\mathbb{R}^n)\right)$. Let $0 < \alpha, \beta < 1$ and assume that the following conditions are satisfied:*

$$1)\ A_1(\alpha) = \sup_{t>0} \left(\int\limits_{|y|<t} [v(y)]^{-p'}\,dy \right)^{\frac{\alpha}{p'}} \left\| \frac{1}{|x|^n} \left(\int\limits_{|y|<|x|} [v(y)]^{-p'}\,dy \right)^{\frac{1-\alpha}{p'}} \right\|_{L_{\varphi,w}(|x|>t)} < \infty,$$

$$2)\; B_1(\beta) = \sup_{t>0} \left(\int\limits_{|y|>t} \left[\nu(y)\,|y|^n\right]^{-p'} \, dy \right)^{\frac{\beta}{p'}} \left\| \left(\int\limits_{|y|>|x|} \left[\nu(y)\,|y|^n\right]^{-p'} \, dy \right)^{\frac{1-\beta}{p'}} \right\|_{L_{\varphi,w}(|x|<t)} < \infty,$$

3) there exists $C_1 > 0$ *such that for any* $k \in \mathbb{Z}$ *the following inequality holds:*

$$\operatorname*{ess\,sup}_{x\in E_k} w(x) \le C_1 \operatorname*{ess\,inf}_{x\in E_{k,2}} \nu(x)$$

4) there exists $C_2 > 0$ *such that for any* $k \in \mathbb{Z}$, $x \in E_k$ *and* $g_k \in L_\varphi(\mathbb{R}^n)$ *the following inequality holds:*

$$\varphi\left(x, \sum_{k\in\mathbb{Z}} \omega_k \left|g_k\right| \chi_{E_k}\right) \le C_2 \sum_{k\in\mathbb{Z}} \operatorname*{ess\,sup}_{x\in E_k} \left[\omega_k(x)\right]^{r(x)} \varphi\left(x; \left|g_k\right| \chi_{E_k}\right).$$

Then $T \in SB\left(L_{\varphi,\nu}(\mathbb{R}^n),\, L_{\varphi,w}(\mathbb{R}^n)\right)$.

Proof. Let $X_w = L_{\varphi,w}(\mathbb{R}^n)$. By the Definition 2.11 the modular on weighted Musielak-Orlicz space is defined by

$$\rho(f) = \rho_{\varphi,w}(f) = \int\limits_{\mathbb{R}^n} \varphi\left(x, |f(x)|\, w(x)\right) dx.$$

□

Remark 3.5. We observe that the boundedness of Hardy-Littlewood maximal operator on Musielak-Orlicz spaces was studied in [13] and [19]. Similar results for the multidimensional Hardy operator on weighted Musielak-Orlicz spaces were obtained in [5].

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