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$\mathcal{U}$-chain connectedness

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Abstract. For an open covering $\mathcal{U}$ in a topological space in this paper we define a $\mathcal{U}$-chain connected set, and a pair of $\mathcal{U}$-chain separated sets in the topological space and study its properties.

1. Introduction

The definition of a chain connected set in a topological space is given in [1]. The properties of those sets are studied in [1]-[7]. The notion of a chain connected set in a topological space requires the existence of a chain in every open covering. However, it is possible to introduce analogous notions and to formulate analogous statements also in the case where the existence of the chain is required in one, previously given, open covering. Moreover, the connections between the notions concerning the chain connectedness and the chain connectedness in an open covering i.e., $\mathcal{U}$-chain connectedness are given in Sections 5 and 9.

In this paper by a covering in a topological space we understand an open covering, i.e., a covering that consists of open sets, and by a covering of X, if it is not otherwise stated, we understand a covering of X in X.

Let $\mathcal{U}$ be a covering of the set X and $x, y \in X$. A chain in $\mathcal{U}$ that connects x and y (from x to y) is a finite sequence of sets $U_1, U_2, ..., U_n$ of $\mathcal{U}$ such that $x \in U_1$, $y \in U_n$ and $U_i \cap U_{i+1} \neq \emptyset$ for every $i = 1, 2, ..., n-1$ [1].

2. $\mathcal{U}$-chain connected set in a topological space

Using the notion of a chain we define the notion of $\mathcal{U}$-chain connected set in a topological space, a central notion in this paper.

Let X be a topological space, let $\mathcal{U}$ be a covering of X and let $C \subseteq X$.

Definition 2.1. The set C is **$\mathcal{U}$-chain connected** in X, if for every $x, y \in C$, there exists a chain in $\mathcal{U}$ that connects x and y.

Let $C \subseteq Y \subseteq X$ and $\mathcal{U}_Y = \mathcal{U} \cap Y$.

Theorem 2.2. *1) If the set C is $\mathcal{U}_Y$-chain connected in Y, then C is $\mathcal{U}$-chain connected in X.*
2) If C is $\mathcal{U}$-chain connected in Y, then C is $\mathcal{U}$-chain connected in X.

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Proof. Let C be $\mathcal{U}_Y$-chain connected in Y. Then:

$$\mathcal{U}_Y = \mathcal{U} \cap Y = \{U \cap Y | U \in \mathcal{U}\}.$$

Since C is $\mathcal{U}_Y$-chain connected in Y, it follows that for every two points $x, y \in X$, there exists a chain $U_1 \cap Y, U_2 \cap Y, ..., U_n \cap Y$ of elements of $\mathcal{U}_Y$. Then $U_1, U_2, ..., U_n$ is a chain in X of elements of $\mathcal{U}$, that connects x and y. It follows that C is $\mathcal{U}$-chain connected in X. □

The most important case of the previous theorem is when $Y = C$.
The following example shows that the converse statement does not hold in general.

Example 2.3. Consider the topological space $X = [-2, -1] \cup \{0\} \cup [1, 2]$ and the covering:

$$\mathcal{U} = \{[-2, -1] \cup \{0\}, \{0\} \cup [1, 2]\}.$$

The set $Y = X \backslash \{0\}$ is $\mathcal{U}$-chain connected in X, but is not $\mathcal{U}_Y$-chain connected in Y since does not exist a chain in $\mathcal{U}_Y = \{[-2, -1], [1, 2]\}$ that connects arbitrary element $x \in [-2, -1]$ with arbitrary $y \in [1, 2]$.

The next claim, directly follows from the definition.

Theorem 2.4. *If the set C is $\mathcal{U}$-chain connected in X, then each subset of C is $\mathcal{U}$-chain connected in X*

Definition 2.5. The set C is **$\mathcal{U}$-connected** if C is $\mathcal{U}$-chain connected in C.

Let $\mathcal{V}$ be a covering of X. We say that the covering $\mathcal{U}$ is refinement of $\mathcal{V}$, notation $\mathcal{U} \prec \mathcal{V}$, if for every $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that $U \subseteq V$.

Proposition 2.6. *If C is $\mathcal{U}$-chain connected in X and $\mathcal{U} \prec \mathcal{V}$ then C is $\mathcal{V}$-chain connected in X.*

Theorem 2.7. *Let $f : X \to Y$ be a function, let $\mathcal{U}$ be a covering of X and let $\mathcal{V}$ be a covering of Y such that $f(\mathcal{U}) \prec \mathcal{V}$. If C is $\mathcal{U}$ -chain connected set in X then $f(C)$ is $\mathcal{V}$-chain connected in Y.*

Proof. Let $f(x), f(y) \in f(C)$ and let $\mathcal{V}$ be a covering of Y such that $f(\mathcal{U}) \prec \mathcal{V}$.
Since C is $\mathcal{U}$-chain connected in X, there exists a chain in $\mathcal{U}$ that connects x and y i.e., there exists a finite sequence $U_1, U_2, ..., U_n$ such that $x \in U_1$, $y \in U_n$ and $U_i \cap U_{i+1} \neq \emptyset$; $i = 1, 2, ..., n-1$.
Let $f(U_i) \subseteq V_i$. Since $U_i \cap U_{i+1} \neq \emptyset$ it follows that

$$V_i \cap V_{i+1} \neq \emptyset,$$

$f(x) \in V_1$ and $f(y) \in V_n$ i.e., $V_1, V_2, ..., V_n$ is a chain in $\mathcal{V}$ that connects $f(x)$ and $f(y)$. □

3. $\mathcal{U}$-chain components

Let X be a topological space, let $\mathcal{U}$ be a covering of X, and let $x, y \in X$.
The definition of the next notion is mentioned as a commentary after Corollary 4.3 in [1].

Definition 3.1. The element x is **$\mathcal{U}$-chain related to y in** X, and we denote it by $x \underset{\mathcal{U},X}{\sim} y$, if there exists a chain in $\mathcal{U}$ that connects x and y.

If x is not $\mathcal{U}$-chain related to y in X we use the notation $x \underset{\mathcal{U},X}{\nsim} y$. The $\mathcal{U}$-chain relation in a topological space is an equivalence relation and it depends on the set X, the topology of X, and the covering $\mathcal{U}$ of X.

Definition 3.2. The **$\mathcal{U}$-chain component of the point x in** X, denoted by $V_X(x, \mathcal{U})$ or $V(x, \mathcal{U})$ in X, is the maximal $\mathcal{U}$-chain connected set in X that contains x.

The $\mathcal{U}$-chain component of the point x in X is unique, it is the largest $\mathcal{U}$-chain connected set that contains x and it is a class of equivalence of the $\mathcal{U}$-chain relation in X. The set $V_X(x, \mathcal{U})$ consists of all elements $y \in X$ such that there exists a chain in $\mathcal{U}$ that connects x and y.

Let X be a topological space, let $C \subseteq X$, let $\mathcal{U}$ be a covering of X, and let $x, y \in X$.

Definition 3.3. The element x is **$\mathcal{U}$-chain related to** y **in** X **relatively** C, if $x, y \in C$ and there exists a chain in $\mathcal{U}$ that connects x and y.

The $\mathcal{U}$-chain relation relatively subset C in a topological space X is an equivalence relation and it depends on the sets C and X, the topology of X, and the covering $\mathcal{U}$ of X.

Definition 3.4. The **$\mathcal{U}$-chain component of the point** x **of** C **in** X, denoted by $V_{CX}(x, \mathcal{U})$, is the maximal $\mathcal{U}$-chain connected subset of C in X that contains x.

The $\mathcal{U}$-chain component of the point x of C in X is unique, it is the largest $\mathcal{U}$-chain connected set in C that contains x and it is a class of equivalence of $\mathcal{U}$-chain relation in X relatively C. The set $V_{CX}(x, \mathcal{U})$ consists of all elements $y \in C$ such that there exists a chain in $\mathcal{U}$ that connects x and y.

Let $x, y \in C$. From the properties of the equivalence classes it follows that if $y \in V_{CX}(x, \mathcal{U})$ then $V_{CX}(x, \mathcal{U}) = V_{CX}(y, \mathcal{U})$, and if $V_{CX}(x, \mathcal{U}) \neq V_{CX}(y, \mathcal{U})$ then $V_{CX}(x, \mathcal{U}) \cap V_{CX}(y, \mathcal{U}) = \emptyset$.

From the definitions of $\mathcal{U}$-chain relation, $\mathcal{U}$-chain relation relatively C, and the previous property of the equivalence classes, the next proposition is valid.

Proposition 3.5. *For every $x \in C$, $V_{CX}(x, \mathcal{U}) = C \cap V_{XX}(x, \mathcal{U})$. Each $\mathcal{U}$-chain component of X in X contains at most one $\mathcal{U}$-chain component of C in X.*

Let X be a topological space, $\mathcal{U}$ be a covering of X, $C \subseteq X$ and $\mathcal{U}_C = \mathcal{U} \cap C$.

If $C = X$ then $V_X(x, \mathcal{U}) = V_{XX}(x, \mathcal{U})$. The next proposition is a summary of the previous comments and propositions.

Proposition 3.6. *For every $x \in C$,*

$$V_C(x, \mathcal{U}_C) \subseteq V_{CX}(x, \mathcal{U}) = \bigcup_{y \in V_{CX}(x)} V_C(y, \mathcal{U}_C) \subseteq V_X(x, \mathcal{U}).$$

The proposition shows that every $\mathcal{U}$-chain component of C in X is a union of $\mathcal{U}_C$-chain components of C in C and is a subset of $\mathcal{U}$-chain component of X in X.

The next proposition is a reformulation of the definition of the $\mathcal{U}$-chain connected set by using the notion of $\mathcal{U}$-chain relation.

Proposition 3.7. *The set C is $\mathcal{U}$-chain connected in X if and only if for every $x, y \in C$, $x \underset{\mathcal{U},X}{\sim} y$*

So, C is not $\mathcal{U}$-chain connected in X if and only if there exist $x, y \in C$ such that $x \underset{\mathcal{U},X}{\nsim} y$.

Example 3.8. For the topological space, subspace and covering from Example 2.3, $V_X(1, \mathcal{U}) = X$ and $V_{YX}(x, \mathcal{U}) = X \backslash \{0\}$ for every $x \in Y$.

The next theorem describes all $\mathcal{U}$-chain connected sets in a topological space.

Theorem 3.9. *The set of all $\mathcal{U}$-chain connected subsets of C in X consist of all $\mathcal{U}$-chain components of C in X and their subsets.*

If the set C is $\mathcal{U}$-chain connected in X, then the sets $V_X(x, \mathcal{U})$ match for every $x \in C$. Therefore, we also use the notation $V_X(C, \mathcal{U})$ or $V(C, \mathcal{U})$ in X for $V_X(x, \mathcal{U})$, $x \in C$. So $V_X(C, \mathcal{U})$ is the set that consists of all elements $y \in X$, such that there exists a chain in $\mathcal{U}$ that connects some $x \in C$ and y. Clearly $C \subseteq V_X(C, \mathcal{U})$ and $V_X(C, \mathcal{U}) = V_X(x, \mathcal{U})$ for every $x \in C$.

4. Union of $\mathcal{U}$-chain connected sets. Star of a covering

Now we turn to a union of $\mathcal{U}$-chain connected sets in a topological space.

Let X be a topological space and let $\mathcal{U}$ be a covering of X.

The accuracy of the next statement follows from the properties of the equivalence classes of the $\mathcal{U}$-chain relation.

Lemma 4.1. *1) Let $C, D \subseteq X$. If C and D are $\mathcal{U}$-chain connected sets in X and $V_X(C, \mathcal{U}) \cap V_X(D, \mathcal{U}) \neq \emptyset$, where $V_X(C, \mathcal{U})$ and $V_X(D, \mathcal{U})$ are $\mathcal{U}$-chain components of C and D, respectively, then the union $V_X(C, \mathcal{U}) \cup V_X(D, \mathcal{U})$ is $\mathcal{U}$-chain connected in X and*

$$V_X(C, \mathcal{U}) \cup V_X(D, \mathcal{U}) = V_X(C, \mathcal{U}) = V_X(D, \mathcal{U}).$$

2) Let $C, D \subseteq X$. If C and D are $\mathcal{U}$-chain connected in X and $V_X(C, \mathcal{U}) \cap V_X(D, \mathcal{U}) \neq \emptyset$, where $V_X(C, \mathcal{U})$ and $V_X(D, \mathcal{U})$ are $\mathcal{U}$-chain components of C and D, respectively, then the union $C \cup D$ is $\mathcal{U}$-chain connected in X.

Theorem 4.2. *Let C_i, $i \in I$, be a family of $\mathcal{U}$-chain connected subspaces of X. If there exists $i_0 \in I$ such that for every $i \in I$, $V_X(C_{i_0}, \mathcal{U}) \cap V_X(C_i, \mathcal{U}) \neq \emptyset$, then the union $\bigcup_{i \in I} V_X(C_i, \mathcal{U})$ is $\mathcal{U}$-chain connected in X and $\bigcup_{i \in I} V_X(C_i, \mathcal{U}) = V_X(C_k, \mathcal{U})$ for every $k \in I$.*

Proof. Let C_i, $i \in I$, be a family of $\mathcal{U}$-chain connected subspaces of X. Let $x, y \in \bigcup_{i \in I} V_X(C_i, \mathcal{U})$ i.e., $x \in V_X(C_x, \mathcal{U})$ and $y \in V_X(C_y, \mathcal{U})$ for some $x, y \in I$.

Because $V_X(C_{i_0}, \mathcal{U}) \cap V_X(C_i, \mathcal{U}) \neq \emptyset$, for every $i \in I$, from the previous lemma, it follows that $V_X(C_{i_0}, \mathcal{U}) \cup V_X(C_x, \mathcal{U})$ is $\mathcal{U}$-chain connected in X. Similarly $V_X(C_{i_0}, \mathcal{U}) \cup V_X(C_y, \mathcal{U})$ is $\mathcal{U}$-chain connected in X. Then, since $C_{i_0} \neq \emptyset$, from the previous lemma it follows that $V_X(C_{i_0}, \mathcal{U}) \cup V_X(C_x, \mathcal{U}) \cup V_X(C_y, \mathcal{U})$ is $\mathcal{U}$-chain connected in X i.e., there exists a chain in $\mathcal{U}$ that connects x and y. So $\bigcup_{i \in I} V_X(C_i, \mathcal{U})$ is $\mathcal{U}$-chain connected in X. By the definition of a chain component, $\bigcup_{i \in I} V_X(C_i, \mathcal{U}) = V_X(C_k, \mathcal{U})$, for every $k \in I$. □

Corollary 4.3. *Let C_i, $i \in I$, be a family of $\mathcal{U}$-chain connected subspaces of X. If there exists $i_0 \in I$ such that for every $i \in I$, $C_{i_0} \cap C_i \neq \emptyset$, then the union $\bigcup_{i \in I} C_i$ is $\mathcal{U}$-chain connected in X.*

The star of the element x and the covering $\mathcal{U}$ of X, is the set $st(x, \mathcal{U}) = \cup\{U \in \mathcal{U} | x \in U\}$, the star of degree n, for $n > 1$, of x and $\mathcal{U}$ in X is $st^n(x, \mathcal{U}) = st\left(st^{n-1}(x, \mathcal{U})\right)$, and the infinite star of x and $\mathcal{U}$ in X is $st^\infty = \bigcup_{n=1}^\infty st^n(x, \mathcal{U})$.

Let X be a topological space, let $\mathcal{U}$ be a covering of X and let $C \subseteq X$.

Theorem 4.4. *Set C is $\mathcal{U}$-chain connected in X, if and only if $C \subseteq st^\infty(x, \mathcal{U})$, for every $x \in C$.*

Corollary 4.5. *The topological space X is $\mathcal{U}$-chain connected in X, if and only if $X = st^\infty(x, \mathcal{U})$, for every $x \in X$.*

5. Inheriting a $\mathcal{U}$-chain connectedness from a space to its subspace

Let X be a topological space, $\mathcal{U}$ be a covering of X, $Y \subseteq X$ and $\mathcal{U}_Y = \mathcal{U} \cap Y$.

If the set A is $\mathcal{U}$-chain connected in X, then A is $\mathcal{U}$-chain connected in each super space of X, but the converse statement does not hold in general. The next theorem tell as in which case the converse statement holds.

Theorem 5.1. *If the set A is $\mathcal{U}$-chain connected in X and $V_X(A, \mathcal{U}) \subseteq Y \subseteq X$ then A is $\mathcal{U}_Y$-chain connected in Y.*

Proof. Let A be a $\mathcal{U}$-chain connected set in X. It follows firstly that for arbitrary $x, y \in A$ there exists a chain $U_1, U_2, ..., U_n$ in $\mathcal{U}$ that connects x and y and secondly that for every $z, t \in \bigcup_{i=1}^n U_i$, there exists a chain in $\mathcal{U}$ that connects z and t i.e., $\bigcup_{i=1}^n U_i \subseteq V_X(A, \mathcal{U}) \subseteq Y$. Therefore $U_1, U_2, ..., U_n$ is a chain in $\mathcal{U}_Y$ also that connects x and y i.e., A is $\mathcal{U}_Y$-chain connected in Y. □

If the set A is not subset of $V_X(x, \mathcal{U})$ for every $x \in X$ it follows that there exist $x, y \in A$ such that there is not chain in $\mathcal{U}$ that connects x and y. Then there is not chain in $\mathcal{U}_\mathcal{Y}$ that connects x and y for every $A \subseteq Y \subseteq X$ i.e., A is not $\mathcal{U}_\mathcal{Y}$-chain connected in Y for every $A \subseteq Y \subseteq X$.

6. Product of $\mathcal{U}$-chain connected sets

In this section we will consider the product of $\mathcal{U}$-chain connected sets.

Theorem 6.1. *If C_i are $\mathcal{U}_i$-chain connected sets in X_i, $i = 1, 2, ..., n$; then the product $\prod_{i=1}^{n} V_{X_i}(C_i, \mathcal{U}_i)$ is a $\prod_{i=1}^{n} \mathcal{U}_i$-chain connected set in $\prod_{i=1}^{n} X_i$ and*

$$V_{\prod_{i=1}^{n} X_i}\left(\prod_{i=1}^{n} C_i, \prod_{i=1}^{n} \mathcal{U}_i\right) = \prod_{i=1}^{n} V_{X_i}(C_i, \mathcal{U}_i).$$

Proof. a) Let X and Y be topological spaces, let $\mathcal{U}$ be a coverings of X and let $\mathcal{V}$ be a covering of Y. Firstly we will prove that if C and D are $\mathcal{U}$ and $\mathcal{V}$ chain connected sets in X and Y, respectively, then $V_X(C, \mathcal{U}) \times V_Y(D, \mathcal{V})$ is a $\mathcal{U} \times \mathcal{V}$-chain connected set in $X \times Y$ and

$$V_{X\times Y}(C \times D, \mathcal{U} \times \mathcal{V}) = V_X(C, \mathcal{U}) \times V_Y(D, \mathcal{V}).$$

Let $(c, d) \in V_X(C, \mathcal{U}) \times V_Y(D, \mathcal{V})$. Then $c \in V_X(C, \mathcal{U})$ and $d \in V_Y(D, \mathcal{V})$ i.e., for arbitrary $(e, f) \in C \times D$ there exist a chains $U_1, U_2, ..., U_p$ in $\mathcal{U}$ and $V_1, V_2, ..., V_q$ in $\mathcal{V}$ that connect c and e, and d and f, respectively.

It follows firstly that $U_1 \times V_m$ and $U_1 \times V_{m+1}$ for $m = 1, 2, ..., q - 1$; and $U_r \times V_q$ and $U_{r+1} \times V_q$ for $r = 1, 2, ..., p - 1$; have nonempty intersections, and secondly that:

$$U_1 \times V_1,\ U_1 \times V_2,\ ..., U_1 \times V_q,\ U_2 \times V_q, ..., U_p \times V_q$$

is a chain in $\mathcal{U} \times \mathcal{V}$ that connects (c, d) and (e, f). Similarly, for arbitrary $(g, h) \in V_X(C, \mathcal{U}) \times V_Y(D, \mathcal{V})$ it follows that (g, h) is chain related to (e, f). From transitivity of chain connectedness relation it follows that (g, h) is chain related to (c, d). Hence $V_X(C, \mathcal{U}) \times V_Y(D, \mathcal{V})$ is a $\mathcal{U} \times \mathcal{V}$-chain connected set in $X \times Y$.

Since $C \times D \subseteq V_X(C, \mathcal{U}) \times V_Y(D, \mathcal{V})$, it follows that $C \times D$ is $\mathcal{U} \times \mathcal{V}$-chain connected set in $X \times Y$. Since $V_{X\times Y}(C \times D, \mathcal{U} \times \mathcal{V})$ is the largest $\mathcal{U} \times \mathcal{V}$-chain connected set in $X \times Y$ that contains $C \times D$, it follows that $V_X(C, \mathcal{U}) \times V_Y(D, \mathcal{V}) \subseteq V_{X\times Y}(C \times D, \mathcal{U} \times \mathcal{V})$.

Let (c, d) be arbitrary element of $V_{X\times Y}(C\times D, \mathcal{U}\times\mathcal{V})$. Then for arbitrary $(e, f) \in C\times D$ there exists a chain $U_1 \times V_1,\ U_2 \times V_2, ..., U_n \times V_n$ in $\mathcal{U} \times \mathcal{V}$ that connects (c, d) and (e, f). Then $U_1, U_2, ..., U_n$ is a chain in $\mathcal{U}$ that connects c and e, and $V_1, V_2, ..., V_n$ is a chain in $\mathcal{V}$ that connects d and f. So $e \in V_X(C, \mathcal{U})$ and $f \in V_Y(D, \mathcal{V})$ i.e.,

$$(e, f) \in V_X(C, \mathcal{U}) \times V_Y(D, \mathcal{U}).$$

It follows that $V_{X\times Y}(C \times D, \mathcal{U} \times \mathcal{V}) \subseteq V_X(C, \mathcal{U}) \times V_Y(D, \mathcal{V})$.

b) Now we will prove the finite case. Let X_i, $i = 1, 2, ..., n$; be topological spaces and let $\mathcal{U}_i$, be coverings of X_i, $i = 1, 2, ..., n$; respectively. Let C_i be $\mathcal{U}_i$-chain connected sets in X_i, $i = 1, 2, ..., n$. Then $V_{X_i}(C_i, \mathcal{U}_i)$ are $\mathcal{U}_i$-chain connected sets in X_i for all $i = 1, 2, ..., n$. The theorem we will prove by mathematical induction.

For $n = 1$ the statement is trivial.

For $n = k$, let $\prod_{i=1}^{k} V_{X_i}(C_i, \mathcal{U}_i)$ be $\prod_{i=1}^{k} \mathcal{U}_i$-chain connected set in $\prod_{i=1}^{k} X_i$ and

$$V_{\prod_{i=1}^{k} X_i}\left(\prod_{i=1}^{k} C_i, \prod_{i=1}^{k} \mathcal{U}_i\right) = \prod_{i=1}^{k} V_{X_i}(C_i, \mathcal{U}_i).$$

Then for $n = k + 1$

$$\prod_{i=1}^{k+1} V_{X_i}(C_i, \mathcal{U}_i) = \left(\prod_{i=1}^{k} V_{X_i}(C_i, \mathcal{U}_i)\right) \times V_{X_{k+1}}(C_{k+1}, \mathcal{U}_{k+1}) \overset{ind.}{=}$$
$$V_{\prod_{i=1}^{k} X_i}\left(\prod_{i=1}^{k} C_i, \prod_{i=1}^{k} \mathcal{U}_i\right) \times V_{X_{k+1}}(C_{k+1}, \mathcal{U}_{k+1}) \overset{a}{=}$$
$$V_{(\prod_{i=1}^{k} X_i)\times X_{k+1}}\left(\left(\prod_{i=1}^{k} C_i\right) \times C_{k+1}, \left(\prod_{i=1}^{k} \mathcal{U}_i\right) \times \mathcal{U}_{k+1}\right) = V_{\prod_{i=1}^{k+1} X_i}\left(\prod_{i=1}^{k+1} C_i, \prod_{i=1}^{k+1} \mathcal{U}_i\right).$$

Hence $\prod_{i=1}^{n} V_{X_i}(C_i, \mathcal{U}_i)$ is a chain connected set in $\prod_{i=1}^{n} X_i$ and $V_{\prod_{i=1}^{n} X_i}\left(\prod_{i=1}^{n} C_i, \prod_{i=1}^{n} \mathcal{U}_i\right) = \prod_{i=1}^{n} V_{X_i}(C_i, \mathcal{U}_i)$. □

The next example shows that a product of infinitely many, even countable many $\mathcal{U}_i$-chain connected sets in topological spaces X_i, $i = 1, 2, ...$; respectively, in general is not $\prod_{i=1}^{\infty} \mathcal{U}_i$-chain connected set in product space $\prod_{i=1}^{\infty} X_i$.

Example 6.2. Consider the discrete space $\mathbb{N}$ that consists of the set of natural numbers and an open covering $\mathcal{U} = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, ...\}$. The space $\mathbb{N}$ is $\mathcal{U}$-chain connected in $\mathbb{N}$. Namely for every $x, y \in \mathbb{N}$, $x < y$; there exists a chain

$$\{x, x + 1\}, \{x + 1, x + 2\}, ..., \{y - 1, y\}$$

in $\mathcal{U}$ that connects x and y.

So, the topological spaces $N_i = \mathbb{N}$, for every $i \in \mathbb{N}$; are $\mathcal{U}_i = \mathcal{U}$-chain connected in N_i, but the product space $\prod_{i=1}^{\infty} N_i$ is not $\prod_{i=1}^{\infty} \mathcal{U}_i$-chain connected in $\prod_{i=1}^{\infty} N_i$. Namely for the elements $x = (1, 1, 1, ...)$ and $y = (1, 2, 3, ...)$ of $\prod_{i=1}^{\infty} N_i$, there is no chain in $\prod_{i=1}^{\infty} \mathcal{U}_i$ that connects x and y. If there exists such a finite sequence $U_1, U_2, ..., U_n$ of n elements in $\prod_{i=1}^{\infty} \mathcal{U}_i$ that connects x and y, since $x \in U_1$, it follows that $U_1 = \prod_{i=1}^{\infty}\{1, 2\}$. Since, $U_1 \cap U_2 \neq \emptyset$ for the coordinates z_i, $i \in \mathbb{N}$ of arbitrary element of U_2 it follows that $z_i \leq 3$ for every $i \in \mathbb{N}$. In n-th step, since $y \in U_n$, for every coordinate y_i, $i \in \mathbb{N}$, of y it follows that $y_i \leq n + 1$. But infinitely many coordinates of y are greater than $n + 1$.

7. $\mathcal{U}$-chain separated sets in a topological space

Let X be a topological space, $\mathcal{U}$ be a covering of X and $A, B \subseteq X$, $A, B \neq \emptyset$.

In this section we define a pair of $\mathcal{U}$-chain separated sets in a topological space X and study its properties. Moreover we give a criterion of $\mathcal{U}$-chain connected set by the new notion of $\mathcal{U}$-chain separatedness.

Definition 7.1. The sets A and B are **$\mathcal{U}$-chain separated** in a topological space X, if for every point $x \in A$ and every $y \in B$, there is no chain in $\mathcal{U}$ that connects x and y.

From the definition, it follows that if A and B are $\mathcal{U}$-chain separated in a topological space X, then both C and D, where $C \subseteq A$ and $D \subseteq B$ and $C, D \neq \emptyset$, are $\mathcal{U}$-chain separated in X.

Let X be a topological space, let A and B be nonempty subsets of X, let $A \cup B \subseteq Y \subseteq X$, and let $\mathcal{U}_Y = \mathcal{U} \cap Y$.

The following proposition will show us that a pair of sets, which are $\mathcal{U}$-chain separated in a topological space, are also $\mathcal{U}$-chain separated in every its subspace that contain their union.

Proposition 7.2. *If A and B are $\mathcal{U}$-chain separated in X, then A and B are $\mathcal{U}_Y$-chain separated in Y.*

Proof. Let the sets A and B be $\mathcal{U}$-chain separated in X and let $x \in A$ and $y \in B$. It follows that there is no chain in $\mathcal{U}$ that connects x and y. Then

$$\mathcal{U}_Y = \mathcal{U} \cap Y = \{U \cap Y | U \in \mathcal{U}\},$$

is a covering of Y in Y such that there is no chain in $\mathcal{U}_Y$ that connects x and y. □

Remark 7.3. The most important case of the previous theorem is when $Y = A \cup B$.

The next example shows that the converse statement does not hold in general.

Example 7.4. For the topological space X , the subspace Y and the covering $\mathcal{U}$, from Example 2.3; the sets $A = [-2, -1]$ and $B = [1, 2]$ are $\mathcal{U}_Y$-chain separated in $Y = A \cup B$, where $\mathcal{U}_Y = \mathcal{U} \cap Y = \{[-2, -1], [1, 2]\}$, but they are not $\mathcal{U}$-chain separated in X.

Now let us consider statement that give criterion for $\mathcal{U}$-chain connected set by using the notion of $\mathcal{U}$-chain separatedness.

Let X be a topological space, let $\mathcal{U}$ be a covering of X and let $C \subseteq X$.

Theorem 7.5. *The set C is $\mathcal{U}$-chain connected in X, if and only if C cannot be represented as a union of two $\mathcal{U}$-chain separated sets A and B in X.*

Proof. (⇒) If C can be represented as a union of $\mathcal{U}$-chain separated sets A and B in X, then C is not $\mathcal{U}$-chain connected in X.

(⇐) Let C not be $\mathcal{U}$-chain connected in X. It follows that C has at least two elements for which there is no chain in $\mathcal{U}$ that connects some elements x and y of C. We consider the set $V = V_{CX}(x, \mathcal{U})$. Since C is not $\mathcal{U}$-chain connected in X, it follows that $y \in C\backslash V$. So, C is represented as a union of two $\mathcal{U}$-chain separated sets V and $C\backslash V$. □

Corollary 7.6. *A set C is $\mathcal{U}$-connected, if and only if it cannot be represented as a union of two $\mathcal{U}$-chain separated sets A and B in C.*

Theorem 7.7. *Let $X = A \cup B$, where A and B are $\mathcal{U}$-chain separated sets in X, and C is a $\mathcal{U}$-chain connected set in X. Then $C \subseteq A$ or $C \subseteq B$.*

Proof. If there exists $x, y \in C$ such that $x \in A$ and $y \in B$, because C is a $\mathcal{U}$-chain connected set in X, it follows that there exists a chain in $\mathcal{U}$ that connects x and y. The last claim contradicts the claim that sets A and B are $\mathcal{U}$-chain separated in X. So, $C \subseteq A$ or $C \subseteq B$. □

8. Relation between chain and $\mathcal{U}$-chain connected sets

In this section, the relationship between $\mathcal{U}$-chain connected and chain connected sets will be considered. Moreover criteria for some topological notions using $\mathcal{U}$-chain connectedness will be given. In fact, that is the reason for defining the $\mathcal{U}$-chain connectedness.

Let X be a topological space, let $C \subseteq X$ and let A and B be nonempty subsets of X.

The set C is chain connected in X if for every covering $\mathcal{U}$ of X and every $x, y \in C$, there exists a chain in $\mathcal{U}$ that connects x and y [1].

Theorem 8.1. *1) The set C is chain connected in X if and only if C is $\mathcal{U}$-chain connected in X for every covering $\mathcal{U}$ of X.*

2) The set C is connected if and only if C is $\mathcal{U}$-chain connected in C for every covering $\mathcal{U}$ of C.

3) The set C is connected if and only if C is $\mathcal{U}$-connected for every covering $\mathcal{U}$ of C.

The set C is not connected if there exists a covering $\mathcal{U}$ of C such that C is not $\mathcal{U}$ connected.

Let $x, y \in X$. The element x is chain related to y in X, and we denote it by $x \underset{X}{\sim} y$, if for every covering $\mathcal{U}$ of X there exists a chain in $\mathcal{U}$ that connects x and y. If x is not chain related to y in X we use the notation $x \underset{X}{\nsim} y$. The chain component of element x of X, denoted by $V_X(x)$,is the maximal chain connected set in X that contains x. The chain component of element x of C in X, denoted by $V_{CX}(x)$, is the maximal chain connected subset of C in X that contains x. If $C = X$, $V_X(x) = V_{CX}(x)$ for every $x \in X$ [1].

Proposition 8.2. *Let $x, y \in X$. The element x is chain related to y in X if and only if x is $\mathcal{U}$-chain related to y in X for every covering $\mathcal{U}$ of X.*

Theorem 8.3. *The chain component $V_{CX}(x)$ is an intersection of all $\mathcal{U}_C$-chain components $V_{CX}(x, \mathcal{U})$, where $\mathcal{U}$ is a covering of X and $\mathcal{U}_C = \mathcal{U} \cap C$.*

Corollary 8.4. *The chain component $V_X(x)$ is an intersection of all $\mathcal{U}$-chain components $V_X(x, \mathcal{U})$, where $\mathcal{U}$ is a covering of X.*

At the end we will present statements by using the notion of $\mathcal{U}$-chain separatedness.

The sets A and B are chain separated in X, if there exists a covering $\mathcal{U}$ of X such that for every point $x \in A$ and every $y \in B$, there is no chain in $\mathcal{U}$ that connects x and y [1].

The next Theorem 2) give a criterion for separatedness by using the notion of chain separatedness.

Theorem 8.5. *1) The sets A and B are chain separated in X if there exists a covering $\mathcal{U}$ of X such that A and B are $\mathcal{U}$-chain separated in X.*

2) The sets A and B are separated if and only if there exists an open covering $\mathcal{U}$ of $A \cup B$ such that A and B are $\mathcal{U}$-chain separated in $A \cup B$.

The first two statements of the next proposition give criteria for two more topological notions by usung the notion of $\mathcal{U}$-chain separatedness. The sets A and B are weakly chain separated in X, if for every point $x \in A$ and every $y \in B$, there exists a covering $\mathcal{U} = \mathcal{U}_{xy}$ of X such that there is no chain in $\mathcal{U}$ that connects x and y [3]. The set C is totally weakly chain separated in X if for every two distinct points $x, y \in C$ there exists a covering $\mathcal{U} = \mathcal{U}_{xy}$ of X such that there is no chain in $\mathcal{U}$ that connects x and y [6]. The set C is totally chain separated in X if there exists a covering $\mathcal{U}$ of X such that for every two distinct points $x, y \in C$ there is no chain in $\mathcal{U}$ that connects x and y [6].

Proposition 8.6. *1) The topological space C is the discrete if and only if there exists a covering $\mathcal{U}$ of C such that for every two distinct points $x, y \in C$, $\{x\}$ and $\{y\}$ are $\mathcal{U}$-chain separated in C.*

2) The point $x \in X$ is an isolated point of the T_1 space X if and only if there exists a covering $\mathcal{U}$ of X such that $\{x\}$ and $\{y\}$ are $\mathcal{U}$-chain separated in X for every $y \in X \backslash \{x\}$.

3) The sets A and B are weakly chain separated in X if for every $x \in A$ and every $y \in B$ there exists a covering $\mathcal{U} = \mathcal{U}_{xy}$ of X such that $\{x\}$ and $\{y\}$ are $\mathcal{U}$-chain separated in X.

4) The set C is totally weakly chain separated in X if for every two distinct points $x, y \in C$, there exists a covering $\mathcal{U} = \mathcal{U}_{xy}$ of X such that $\{x\}$ and $\{y\}$ are $\mathcal{U}$-chain separated in X.

5) The set C is totally chain separated in X if there exists a covering $\mathcal{U}$ of X such that for every two distinct points $x, y \in C$, $\{x\}$ and $\{y\}$ are $\mathcal{U}$-chain separated in X.

6) The topological space C is totally separated if and only if for every two distinct points $x, y \in C$, there exists a covering $\mathcal{U} = \mathcal{U}_{xy}$ of C such that $\{x\}$ and $\{y\}$ are $\mathcal{U}$-chain separated in C.

9. Inheriting a chain connectedness from a space to its subspace

Let X be a topological space and $A \subseteq X$. If A is chain connected in X, then A is chain connected in each super space of X, but the converse statement does not hold in general. The next theorem tell as in which case the converse statement holds.

By $C_X(x)$ is denoted the connected component of $x \in X$ in X.

Theorem 9.1. *If the set $A \subseteq C_X(x)$ for some $x \in X$, and $C_X(x) \subseteq Y \subseteq X$, then A is chain connected in Y.*

Proof. Let $A \subseteq C_X(x)$ for some $x \in X$ and let $C_X(x) \subseteq Y \subseteq X$. The set $C_X(x)$ is a connected i.e., chain connected in $C_X(x)$. Since every subset of a chain connected set in a topological space is chain connected in the same space, it follows that A is chain connected in $C_X(x)$. Thus, A is chain connected in each space Y such that $C_X(x) \subseteq Y \subseteq X$. □

The next example show that $C_X(x)$ from the previous theorem, cannot be replaced by $V_X(x)$.

Example 9.2. Consider the topological space:

$$X = \left\{\left(x, \frac{1}{x}\right) \middle| x \in [-1, 1],\ n \in \mathbb{N}\right\} \cup \left\{(x, 0) \middle| x \in [-1, 1] \backslash \{0\}\right\}.$$

The set $Y = ([-1, 1] \backslash \{0\}) \times \{0\}$ is chain connected in X and is a chain component in X i.e. $Y = V_X(x)$ for every $x \in Y$, but it is not chain connected in Y.

The next theorem refers to topological spaces with equal connected components and chain components.

Corollary 9.3. *Let X be a topological space that has one of the following properties:*
1) X is a compact Hausdorff space;
2) The chain components of X are open sets;
3) X is locally connected space;
4) X has a finite number of chain components.
If the set $A \subseteq V_X(x)$ for some $x \in X$, and $V_X(x) \subseteq Y \subseteq X$, then A is chain connected in Y.

Proof. Let $A \subseteq V_X(x)$ for some $x \in X$, and let $V_X(x) \subseteq Y \subseteq X$. For the topological spaces from 1)-4), $C_X(x) = V_X(x)$ for every $x \in X$. Namely, for 1)-3) see [7], for 4), since the chain components are closed sets [2], if they are finite number, then they also are open, and 2) implies the accuracy of the statement. It follows that $A \subseteq C_X(x)$. From the previous theorem it follows that A is chain connected in Y. □

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