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On existence and uniqueness of a solution of mixed problem for a class of non-classical equations

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Abstract. Existence and uniqueness of the solution of initial boundary value problem for a class of equations with complex-valued coefficients is treated in this work. These equations behave like parabolic ones, though in time they may transform from parabolic to Schrodinger type or even to antiparabolic type. Note that for the equations of corresponding spectral problems the arguments of the roots of characteristic polynomials in the sense of Birkhoff are not constant.

1. Introduction

As is known, the second order parabolic equations and the current parameters of heat, diffusion and other processes can be used to forecast the future parameters, while the antiparabolic equations create conditions for the study of past processes based on the current parameters. For the equations of completely antiparabolic type, the classical initial (boundary value or initial boundary value) problems are ill-posed problem, which either makes it impossible to study the heat or diffusion processes taking place in the past based on the current parameters or makes it difficult to do so. The results obtained in this work are important for finding out to which time interval and in which part of the considered domain it is possible to return based on the current parameters, with an aim to study the past parameters of heat/diffusion process.

Unique solvability and well-posed problem of linear initial boundary value problems have been considered by many researchers (see [1, 5, 8, 9, 18, 21]).

Different methods are used to consider such problems depending on their statements. For example, Fourier's separation of variables method, Laplace transform, method of freezing the coefficients, heat potential method, a priori estimates method, contour integral method, residue method, finite-difference method, etc.

Also, there are cases where none of the above methods is applicable for one reason or another.

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Note that the residue method and the contour integral method are among the most universal methods used in the solution of one-dimensional and multidimensional mixed problems [1], [8].

In this work, we use the residue method and the contour integral method to treat the unique solvability of one-dimensional initial boundary value problem with the new and previously unexplored singularities.

Definition 1.1. ([5]) The equation of the form

$$\frac{\partial U}{\partial t} = a\,(t,\ x)\,\frac{\partial^2 U}{\partial x^2} \tag{1}$$

is called parabolic (uniformly parabolic) in the sense of Petrovski in the domain Q of the space t, x if the inequality $Re\,a\,(t,x) > 0 (Re\,a\,(t,x) \geq \delta > 0)$ holds for every point $(t,x) \in Q$.

Initial boundary value problems for the equations of the form (3) have been considered only when they are parabolic [1–3], or when they are of Schrodinger type [4, 7, 11, 20–22], i.e. when

$$Re\,a\,(t,x) = 0. \tag{2}$$

At the same time, it was shown in [10] that if the equation (1) is *antiparabolic* (i.e. if $Re\,a\,(t,x) < 0$ for $(t,x) \in Q$), then, for the boundary and initial-conditions with their right-hand sides having only finite smoothness, the initial boundary value problem for this equation is not correct.

Definition 1.2. The equation (1) is called *generalized parabolic* in some domain $Q_T = \{(t,x) : 0 < t < T \leq \infty,$ $0 < x < 1\}$ if $Re \int_0^t a\,(\tau,x)\,d\tau > 0$ for every $(t,x) \in Q_T$.

Note that the parabolic equation is generalized parabolic in the considered domain. However, not every generalized parabolic equation is parabolic: being parabolic till some moment of time $t_0 > 0$, generalized parabolic equation may then transform into Schrodinger or even antiparabolic type. For example, let $a\,(t,x) = a_1\,(t,x) + i\,a_2\,(t,x)$, where $a_j\,(t,x)$, $(j = 1,\ 2)$ are real continuous functions, with $a_2\,(t,x) \neq 0$. Then, if $a_1\,(t,x) > 0$ for $0 \leq t < t_0$ and $a_1\,(t,x) \equiv 0$ for $t \geq t_0$, then the equation (1) is generalized parabolic, but not parabolic, because after passing the moment of time t_0 it degenerates to Schrodinger type. And if $a_1\,(t,x) > 0$ for $0 \leq t < t_0$ and $a_1\,(t_0,x) = 0$, $a_1\,(t,x) < 0$ for $t_0 \leq t < T$ with $\int_0^{t_0} a_1\,(\tau,x)\,d\tau > \left|\int_{t_0}^{T} a_1\,(\tau,x)\,d\tau\right|$, then the equation (1) is generalized parabolic. At the same time, it is parabolic for $0 \leq t < t_0$ Schrodinger for $t = t_0$ and antiparabolic for $t_0 \leq t < T$. As an example, we can consider the function

$$a_1\,(t,x) = \begin{cases} t_0 - t, & for\ 0 \leq t < t_0 < T, \\ \frac{t_0^2(t_0-t)}{2(T-t_0)^2}, & for\quad t_0 \leq t \leq T. \end{cases}$$

It was shown in [12, 13] that the initial boundary value problems may be ill-posed for the equations well-posed in the sense of Petrovski, and well-posed problem for the ill-posed problem equations.

Note that our initial boundary value problem also has such properties.

2. Problem statement

In this work, we treat the unique solvability of the initial boundary value problem

$$M\left(t, \frac{\partial}{\partial t}\right) u = L\left(x, \frac{\partial}{\partial x}\right) u, 0 < t < T,\ \ 0 < x < 1, \tag{3}$$

$$u(0,x) = \varphi(x), \tag{4}$$

$$u(t,0) = u(t,1) = 0, \tag{5}$$

where $M\left(t, \frac{\partial}{\partial t}\right) = \frac{1}{P(t)}\frac{\partial}{\partial t}, L\left(x, \frac{\partial}{\partial x}\right) = \frac{1}{(x+b)^2} \cdot \frac{\partial^2}{\partial x^2}, b = b_1 + ib_2, p(t) = p_1(t) + ip_2(t)$, are complex-valued functions $p_j(t) \in C[0, 1]$ $(j = 1, 2)$, $p_1(t) \neq 0$, $\varphi(x)$ is a given $u(x)$ is a sought for function.

It is known [5] that the equation (3) is parabolic in the sense of Petrovski in the domain $D = \{(t, x) : 0 \leq t \leq T, 0 \leq x \leq 1\}$ if the real part of the root γ of characteristic equation

$$\frac{1}{P(t)}\gamma - \frac{1}{(x+b)^2}\sigma^2 = 0$$

at every point $(t, x) \in D$ satisfies the inequality

$$Re\gamma(t, x, \sigma) < 0$$

for any real $\sigma \neq 0$.

For solvability, the following conditions must be satisfied:

$$1^0. Re\left(\int_0^t P(\tau)\,d\tau\right) > 0, \ \ Reb < -1, \ Im\,b > 0;$$

$2^0. Re(1 + b)^2 + r(0)\,Im(1 + b)^2 > 0$ if $Im\left[\overline{P(t)} \cdot \int_0^t P(\tau)\,d\tau\right] \geq 0$ and $Re(1 + b)^2 + r(T)\,Im(1 + b)^2 > 0$ if $Im\left[\overline{p} \cdot \int_0^t p(\tau)\,d\tau\right] < 0$, where

$$r(t) = Im\left(\int_0^t P(\tau)\,d\tau\right)\cdot\left(Re\int_0^t P(\tau)\,d\tau\right)^{-1}, \ \ t \in (0, \ T);$$

$3^0. \varphi(x) \in C^2[0, \ 1], \varphi(0) = \varphi(1) = 0.$

It can be verified that if the inequalities $ReP(t) > 0$, $Imb < -1$, $Imb > 0$ hold, then the equation (3) is parabolic in the sense of Petrovski if and only if either

$$Im\left[\overline{P}(t) \cdot (P'(t))\right] \leq 0, \ \ Re(1 + b)^2 + \omega(0)\,Im(1 + b)^2 > 0 \tag{6}$$

or

$$Im\left[\overline{P}(t) \cdot (P'(t))\right] > 0, \ \ Re(1 + b)^2 + \omega(T)\,Im(1 + b)^2 > 0, \tag{7}$$

where $\omega(t) = mP(t)\,(ReP(t))^{-1}$.

Note that despite conditions 1^0 and 2^0 holding for the equation

$$(x - 2 + i)^2\,\frac{\partial u}{\partial t} = (2t + 1 + i(2t - 1))\,\frac{\partial^2 u}{\partial x^2},$$

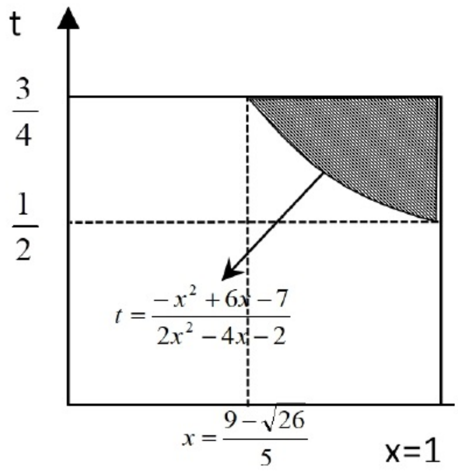
the second of inequalities (6) does not, so this equation is not parabolic in the sense of Petrovski. It is not difficult to show that this equation is not parabolic even in the sense of Shilov. In some parts of the rectangle below it is antiparabolic (for example, in the set of points satisfying the inequalities $\left(2x^2 - 4x - 2\right)^{-1}\left(-x^2 + 6x - 7\right) < t \leq \frac{3}{4}$, $\frac{9-\sqrt{26}}{5} \leq x \leq 1$, see the shaded area).

3. Classical solvability of initial boundary value problem

It is easy to show that the non-homogeneous spectral problem (more precisely, non-homogeneous boundary value problem with a spectral parameter) corresponding to the initial boundary value problem (3)-(5) has the following form:

$$y'' - \mu^2(x + b)^2 y = -\varphi(x)(x + b)^2, \tag{8}$$

$$y(0) = 0, y(1) = 0. \tag{9}$$



Note that the important property [6, 18] of the spectral problem (8)-(9) is that the arguments of the roots $\pm(x+b)$ of the characteristic equation in the sense of Birkhoff are not constant in [0, 1]. As is known [6, 18], this fact significantly complicates both obtaining the asymptotics of the fundamental system of special solutions to the equation $y'' - \mu^2 (x+b)^2 y = 0$ and the study of the scattering of eigenvalues of the problem (8),(9). In fact, these matters are basic for the solution of one-dimensional initial boundary value problems, and in general case (where $\theta_{1,2}(x)$ are the functions from rather general class, such that $\arg \theta_j (x) \neq const\ (j = 1, 2)$), they have never been studied before [6, 18].

So, the presence of seemingly simple coefficient $(x+b)^2$ in the equation (3) is due to the absence of corresponding spectral theory for more general case and the desire to use the recent result [14, 16] for the problem (8), (9), the only one in this field so far. As far as we know, there has been no research dedicated to this problem in more or less general statement.

Green's function of this spectral problem is analytic in the whole of λ-complex plane, except for the countable set of values $\mu = \mu_k$ $(k = 0, \pm 1, \pm 2, ...)$, which are the poles of this function. The poles of Green's function of this spectral problem have the following asymptotic representation [3]:

$$\mu_k = \frac{\pi k \sqrt{-1}}{1+2b} + O\left(\frac{1}{k}\right), \ (|k| \to \infty). \tag{10}$$

Let

$$S_i = \left\{\mu \backslash \ Re(\mu b) \cdot Re(\mu(1+b)) \leq 0, \ (-1)^i Re\mu > 0\right\}; \ (i = 1, 2),$$

$$S_i = \left\{\mu \backslash \ Re(\mu b) < 0, \ (-1)^i Re(\mu(1+b)) \leq 0\right\}; \ (i = 3, 4),$$

$$\chi(\mu) = -(Re\mu)^{-1} \cdot Re\mu b, \ (\mu \in S_i, \ i = 1, 2).$$

Hence, $0 \leq \chi(\mu) \leq 1$ for $\lambda \in S_i$ $(i = 1, 2)$.

As seen from the asymptotic representation of eigenvalues (10) of Green's function $G(x, \xi, \mu)$, distant poles μ_k lie in the sectors $\mu \in S_i$ $(i = 1, 2)$, and only finite number of them can get in the sectors $\mu \in S_i$ $(i = 3, 4)$.

The following estimates have been obtained for Green's function and its derivatives [15, 16] outside δ-neighborhoods of the poles:

$$\left|\frac{\partial^k G(x, \xi, \mu)}{\partial x^k}\right| \leq c|\mu|^{k-1}, \ k = 0, 1, 2; \ \mu \in S_3 \bigcup S_4, \ |\lambda| > R, \tag{11}$$

$$\left|\frac{\partial^k G(x,\,\xi,\,\mu)}{\partial x^k}\right| \le c e^{(-1)^i \chi_0^2(\mu) Re\mu},\ k=0,1,2;\ \mu\in S_i,\ |\mu|>R,\ (i=1,2),$$

where R is a sufficiently big, and δ is a sufficiently small positive number, $\chi_0(\mu) = \min(\chi(\mu);\ 1-\chi(\mu))$.

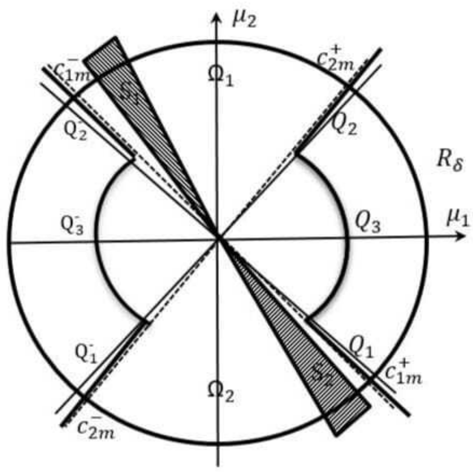
The following theorem is true:

Theorem 3.1. *Let the conditions* 1^0, 2^0, 3^0 *be satisfied. Then the problem* (3)-(5) *has a classical solution* $u(t,\,x) \in C^{1,2}((0;T]\times[0;1]) \bigcap C([0;T]\times[0;1])$ *which can be represented as*

$$u(t,\,x) = \frac{1}{\pi i}\int_Q \mu e^{\mu^2\int_0^t P(\tau)d\tau}\cdot\left(\int_0^1 G(x,\,\xi,\mu)(\xi+b)^2\varphi(\xi)\,d\xi\right)d\mu \tag{12}$$

for $t>0$, *where*

$$Q=\bigcup_{j=1}^{3} Q_j,$$



$$Q_j=\left\{\mu:\ \mu=r\left(1+\tilde{p}_j\right),\ r\ge R\right\}\ (j=1,2),$$

$$Q_3=\{\mu:\ \mu=R(1+i\eta),\ \tilde{p}_1\le\eta\le\tilde{p}_2\},$$

$$\tilde{p}_j=M_j\left(t_j\right)+(-1)^j\delta,\ M_j\left(t_j\right)=-r(t)+(-1)^j\sqrt{r^2(t)+1},\ (j=1,2) \tag{13}$$

$t_1=0,\ t_2=0$ *if* $Im\left[\overline{P(t)}\cdot\int_0^t P(\tau)\,d\tau\right]\ge 0$ *and* $t_1=T,\ t_2=T$ *if* $Im\left[\overline{P(t)}\cdot\int_0^t P(\tau)\,d\tau\right]<0$, *R is a sufficiently big, and* δ *is a sufficiently small positive number.*

Let us first prove the following auxiliary facts.

Lemma 3.2. *Let $Re\left(\int_0^t P(\tau)\,d\tau\right) > 0$. Then, for $t \in [t_0,\ T]$ ($\forall\ t_0 \in (0, T)$), the following estimate is true on the beams $\mu = \rho\left(1 + i\tilde{p}_j\right)$ $(\rho \geq 0,\ j = 1, 2)$:*

$$Re\left(\mu^2 \int_0^t P(\tau)\,d\tau\right) \leq c\left|\mu\right|^2, \tag{14}$$

where $c < 0$.

Proof. From the conditions of lemma it follows that there exists a number $\delta_1 > 0$ such that for $t \in [t_0,\ T]$

$$Re\left(\int_0^t P(\tau)\,d\tau\right) > \delta_1. \tag{15}$$

Further, from the equality

$$\begin{aligned} Re\left(\mu^2 \int_0^t P(\tau)\,d\tau\right) &= \left(\mu_1^2 - \mu_2^2\right)\int_0^t p_1(\tau)\,d\tau - 2\mu_1\mu_2 \int_0^t p_2(\tau)\,d\tau = \\ &= -Re\left(\int_0^t P(\tau)\,d\tau\right)\left[\mu_2^2 - \mu_1^2 + 2\mu_1\mu_2 \cdot \frac{Im\left(\int_0^t P(\tau)\,d\tau\right)}{Re\left(\int_0^t P(\tau)\,d\tau\right)}\right] = \\ &= -Re\left(\int_0^t P(\tau)\,d\tau\right)\left[\left(Im\mu - M_1(t)\,Re\mu\right)\left(Im\mu - M_2(t)\,Re\mu\right)\right] = \\ &= -Re\left(\int_0^t P(\tau)\,d\tau\right)\cdot \prod_{k=1}^{2}\left[Im\mu - M_k(t)\,Re\mu\right], \end{aligned} \tag{16}$$

for $\mu = \rho\left(1 + i\tilde{p}_j\right)$ $(\rho \geq 0,\ j = 1, 2)$ we obtain

$$\begin{aligned} Re\left(\mu^2 \int_0^t P(\tau)\,d\tau\right) &= -\rho^2 Re\left(\int_0^t P(\tau)\,d\tau\right)\prod_{k=1}^{2}\left[\tilde{p}_j - M_k(t)\right] = \\ &= -\rho^2 Re\left(\int_0^t P(\tau)\,d\tau\right)\prod_{k=1}^{2}\left[M_j\left(t_j\right) + (-1)^j\,\delta - M_m(t)\right]. \end{aligned} \tag{17}$$

But, from the expressions (13) for the functions $M_k(t)$ we can see that if $Im\left[\overline{P}(t)\cdot \int_0^t P(\tau)\,d\tau\right] > 0$, then $M_k'(t) > 0$, $(k = 1, 2)$. Hence, by the inequality $M_k'(t) > 0$, $(k = 1, 2)$, we have $M_k(0) \leq M_k(t) \leq M_k(T)$. Then $M_k'(t) > 0$. Consequently, $M_k(0) \leq M_k(t) \leq M_k(T))$, $k = 1, 2$. Having estimated the second term in (17), we obtain the following inequalities:

$$\begin{aligned} &M_1(t_1) - \delta - M_1(t) = M_1(0) - \delta - M_1(t) \leq M_1(0) - \delta - M_1(0) \leq -\delta, (m = 1,\ j = 1) \\ &M_1(t_1) - \delta - M_2(t) = M_1(0) - \delta - M_2(t) \leq M_1(0) - \delta - M_2(0) \leq -\delta, (m = 2,\ j = 1) \\ &M_2(t_2) + \delta - M_1(t) = M_1(0) + \delta - M_1(t) \geq M_2(0) + \delta - M_1(T) \geq \delta,\ \ (m = 1,\ j = 2) \\ &M_2(t_2) + \delta - M_2(t) = M_1(0) + \delta - M_2(t) \geq M_2(0) + \delta - M_2(T) \geq \delta,\ \ (m = 2,\ j = 2) \end{aligned} \tag{18}$$

Taking into account the inequalities (15), (18), from (17) we get

$$Re\left(\mu^2 \int_0^t P(\tau)\,d\tau\right) \leq -\delta_1\delta^2\rho^2 \leq c\cdot\left|\mu\right|^2, \tag{19}$$

where $c = -\frac{\delta_1 \delta^2}{\max\limits_j \sqrt{1+\tilde{p}_j^2}}$.

And if $Im\left[\overline{P(t)} \cdot \int_0^t P(\tau)\, d\tau\right] \le 0$, then, by $M_k'(t) \le 0$, we obtain $M_k(T) \le M_k(t) \le M_k(0)$, $k = 1, 2$. Therefore,

$$
\begin{aligned}
&M_1(t_1) - \delta - M_1(t) = M_1(T) - \delta - M_1(t) \le M_1(T) - \delta - M_1(T) \le -\delta \ \ (k = 1,\ j = 1),\\
&M_1(t_1) - \delta - M_2(t) = M_1(T) - \delta - M_2(t) \le M_1(T) - \delta - M_2(T) \le -\delta \ \ (k = 2,\ j = 1),\\
&M_2(t_2) + \delta - M_1(t) \ge M_2(0) + \delta - M_1(0) \ge \delta \ \ (k = 1,\ j = 2),\\
&M_2(t_2) + \delta - M_2(t) \ge M_2(0) + \delta - M_2(0) \ge \delta \ (k = 2,\ j = 2).
\end{aligned} \tag{20}
$$

Also, by (15), (20), from (17) we obtain

$$
Re\left(\mu^2 \int_0^t P(\tau)\, d\tau\right) = -\rho^2 Re\left(\int_0^t P(\tau)\, d\tau\right)(K_1(t_1) - \delta - K_1(t))(K_1(t_1) - \delta - K_2(t)) \le
$$

$$
\le -\delta_1 \delta^2 \rho^2 \le c \cdot |\mu|^2
$$

for j=1, and

$$
Re\left(\mu^2 \int_0^t P(\tau)\, d\tau\right) = -\rho^2 Re\left(\int_0^t P(\tau)\, d\tau\right)(M_2(t_2) + \delta - M_1(t))(M_2(t_2) + \delta - M_2(t)) \le
$$

$$
\le -\delta_1 \delta^2 \rho^2 \le c \cdot |\mu|^2
$$

for j=2, where $c = -\frac{\delta_1 \delta^2}{\max\limits_j \sqrt{1+\tilde{p}_j^2}}$. This estimate has also the form (19). □

Lemma 3.3. *Let $Re\left(\int_0^t P(\tau)\, d\tau\right) > 0$. Then for every λ from the sectors*

$$
\begin{aligned}
\Omega_1 &= \{\mu :\ \arg(1 + i\tilde{p}_2) \le \arg \mu \le \pi + \arg(1 + i\tilde{p}_1)\},\\
\Omega_2 &= \{\mu :\ \arg(1 + i\tilde{p}_2) - \pi \le \arg \mu \le \arg(1 + i\tilde{p}_1)\}
\end{aligned}
$$

and $t \in [t_0, T]$ (for $\forall\, t_0 \in (0, T)$) the estimate of the form (14) *is true.*

Proof. Denote

$$
r = |\mu|, \beta = \arg \mu, \beta_j = \arg(1 + i\tilde{p}_j).
$$

Then, using this notation, we can rewrite the function $Re\left(\mu^2 \int_0^t p(\tau)\, d\tau\right)$ as follows:

$$
Re\left(\mu^2 \int_0^t P(\tau)\, d\tau\right) = Re\left(|\mu|^2 e^{2i\beta} \cdot \int_0^t P(\tau)\, d\tau\right) = |\mu|^2 \cdot Re\left(e^{2i\beta} \cdot \int_0^t P(\tau)\, d\tau\right) = r^2 v(\beta, t),
$$

where $v(\beta, t) = Re e^{2i\beta}\left(\int_0^t P(\tau)\, d\tau\right)$. By Lemma 3.2 there exists $\varepsilon > 0$ such that

$$
v(\beta_j, t) \le -\varepsilon \quad (j = 1,\ 2)
$$

for $t \in [t_0, T]$, $(t_0 \in (0, T))$. Hence,

$$
v(\beta_1 + \pi, t) = v(\beta_1, t) \le -\varepsilon,
$$

$$
v(\beta_2 - \pi, t) = v(\beta_2, t) \le -\varepsilon.
$$

Therefore we have to prove that the function $v(\beta,\ t)$ has no zero inside the intervals $[\beta_2,\ \beta_1+\pi]$ and $[\beta_2-\pi,\ \beta_1]$.

But, as at the ends of these intervals this function is negative, it can have inside these intervals both multiple zeros and at least two different zeros.

If the function $w(\beta,\ t)$ vanishes at the point β_0, i.e.

$$v(\beta_0,\ t)=\frac{dv(\beta_0,\ t)}{d\beta_0}=0$$

for $\beta_0\in(\beta_2,\ \beta_1+\pi)$ (or for $\beta_0\in(\beta_2-\pi,\ \beta_1)$), then we have

$$Ree^{2i\beta_0}\left(\int_0^t P(\tau)\,d\tau\right)=0, Re2ie^{2i\beta_0}\left(\int_0^t P(\tau)\,d\tau\right)=-2Ime^{2i\beta_0}\left(\int_0^t P(\tau)\,d\tau\right)=0.$$

Taking into account these relations, we obtain $e^{2i\beta_0}\left(\int_0^t P(\tau)\,d\tau\right)=0$. But this is impossible because of the condition $Re\left(\int_0^t P(\tau)\,d\tau\right)>0$.

Let us consider another case. Assume

$$v(\beta_0',\ t)=v(\beta_0'',\ t)=0,\quad (\beta_0'<\beta_0''),$$

where $\beta_0',\beta_0''\in(\beta_2,\ \beta_1+\pi)$(or $\beta_0',\beta_0''\in(\beta_2-\pi,\ \beta_1)$). It is not difficult to show that the function $v(\beta,\ t)$ is a solution of the following differential equation:

$$\frac{d^2v}{d\beta^2}+4v=0.$$

Then it is clear that the distance between two neighboring zeros of an arbitrary solution of this equation is equal to $\frac{\pi}{2}$. Consequently,

$$\frac{\pi}{2}\le\beta_0''-\beta_0'<\pi+\beta_1-\beta_2.$$

Hence

$$\beta_2-\beta_1<\frac{\pi}{2}. \tag{21}$$

On the other hand, as the difference $\beta_2-\beta_1$ is an angle between the vectors $\{1,\ \tilde p_1\}$ and $\{1,\ \tilde p_2\}$, the scalar product of these vectors can be found as follows:

$$h(\delta)=1+\tilde p_1\tilde p_2=1+[M_1(t_1)-\delta]\,[M_2(t_2)+\delta].$$

This expression implies that $f(\delta)$ is a decreasing function, because

$$h'(\delta)=-2\delta+M_1(t_1)-M_2(t_2)<0.$$

Therefore we have

$$h(\delta)<h(0)=1+M_1(t_1)M_2(t_2)=\begin{cases}1+M_1(0)M_2(T), & if\quad Im\left[\overline{P(t)}\cdot\int_0^t P(\tau)\,d\tau\right]>0,\\ 1+M_1(T)M_2(0), & if\quad Im\left[\overline{P(t)}\cdot\int_0^t P(\tau)\,d\tau\right]\le 0,\end{cases} \tag{22}$$

for $\delta>0$.

Since the function $M_j(t)$ increases as $Im\left[\overline{P(t)}\cdot\int_0^t P(\tau)\,d\tau\right]>0$ and does not increase as $Im\left[\overline{P(t)}\cdot\int_0^t P(\tau)\,d\tau\right]\le 0$, with $M_1(0)M_2(0)=-1$, from (22) we obtain

$$h(\delta)<h(0)\le 1+M_1(0)M_2(0)=0.$$

The negativity of the scalar product $h(\delta)$ implies that the angle $\beta_2-\beta_1$ (for $\delta>0$) between two vectors $\{1,\ \tilde p_1\}$ and $\{1,\ \tilde p_2\}$ is obtuse. And this contradicts the inequality (21). □

Lemma 3.4. *Let the conditions* 1^0,2^0 *be satisfied. Then the contour* Γ *can be chosen in such a way that*

$$Q\bigcap S_j = \emptyset \quad (j = 1, 2) \tag{23}$$

and the domain

$$R_\delta = \{\mu : \quad \mu = r(1 + i\eta), \;\; r \geq R, \;\; p_1 \leq \eta \leq p_2\} \tag{24}$$

does not contain the poles μ_k *of Green's function* $G(x, \xi, \mu)$.

Proof. By the definitions of sectors S_j $(j = 1, 2)$, it is clear that to prove (23) we have to investigate the sign of the function

$$I(\mu) = Re\mu bRe\mu(1 + b) \tag{25}$$

for $\mu \in Q$. Let $\mu = \rho\left(1 + ip_j\right) \quad (r \geq R)$ in (25). Then we obtain the following expression for the function $J(\mu)$:

$$K_j(\delta) = J\left[\rho(1 + i\tilde{p}_j)\right] = \rho^2(b_1 - b_2\tilde{p}_j)(1 + b_1 - b_2\tilde{p}_j) = \rho^2\left[\left(b_1 - b_2\tilde{p}_j + \frac{1}{2}\right)^2 - \frac{1}{4}\right]. \tag{26}$$

From (25) and (26) it follows

$$K_j(0) = \rho^2\left[\left(b_1 - b_2M_j(t_j) + \frac{1}{2}\right)^2 - \frac{1}{4}\right].$$

Consequently, if $b_1 - b_2M_j(t_j) \notin [-1, \; 0]$, then $K_j(0) > 0$. But, in this case there can be found $\delta_0 > 0$ such that $K_j(\delta) > 0$ for $\delta \in (0, \; \delta_0)$. This contradicts the definition of the sectors S_1 and S_2. Therefore let us assume that $-1 \leq b_1 - b_2M_j(t_j) \leq 0$.

From the condition 1^0 and the expression for the function $M_1(t)$ it follows that the last inequality is impossible for $j = 1$. Consequently, let us assume that

$$-1 \leq b_1 - b_2M_2(t_2) \leq 0.$$

Hence we have

$$b_1 - b_2\omega(t_2) \leq b_2\sqrt{\omega^2(t_2) + 1} \leq b_1 + 1 - b_2\omega(t_2). \tag{27}$$

Two cases are possible here:

1) $b_1 - b_2r(t_2) \geq 0,$

2) $b_1 - b_2r(t_2) < 0.$

In first case, from (27) we have

$$Reb^2 - r(t_2)Imb^2 \leq 0.$$

And in second case we obtain $r(t_2) > \frac{b_1}{b_2}$. Consequently, the following inequality is true:

$$Reb^2 - r(t_2)Imb^2 < Reb^2 - 2b_1^2 = -|b|^2 < 0.$$

These two inequalities contradict the conditions 1^0,2^0, the expression for the function $r(t)$ and the number t_2.

Now, assuming $\mu = R\,(1 + i\eta)$ $(p_1 \le \eta \le p_2)$ in (25), we get

$$K\,(\eta) = I\,[R(1 + i\eta)] = R^2(b_1 - b_2\eta)(b_1 + 1 - b_2\eta) = R^2\left[\left(b_1 - b_2\eta + \frac{1}{2}\right)^2 - \frac{1}{4}\right].$$

As we have seen above, there exists $\delta > 0$ such that $K\left(p_j\right) > 0$ $(j = 1,\ 2)$ for $\eta = p_j$. Therefore, it suffices to consider only the stationary point $\eta_0 = \frac{1}{b_2}\left(b_1 + \frac{1}{2}\right)$.

But, since $f(\delta) = \frac{1}{b_2}\left(b_1 + \frac{1}{2}\right) - \tilde{p}_2$, we obtain

$$f(0) = \frac{1}{b_2}\left(b_1 + \frac{1}{2}\right) - M_2(t_2) > \frac{1}{b_2}\left(-b_2M_2(t_2) + b_1\right) = \frac{1}{b_2}\left(-b_{.2}\sqrt{r^2(t_2) + 1} + b_1 - b_2r(t_2)\right) =$$

$$= \frac{-r(t_2)Imb^2 + Reb^2}{b_2[b_{.2}\sqrt{r^2(t_2) + 1} - b_2r(t_2) + b_1]} > 0.$$

Then the number $\delta_0 > 0$ can be chosen in such a way that $f(\delta) > 0$ for $\delta \in (0,\ \delta_0)$. And this means that the stationary point η_0 does not lie inside the interval $[\tilde{p}_1,\ \tilde{p}_2]$. Hence it follows that $K(\eta) > 0$ for $\eta \in [\tilde{p}_1,\ \tilde{p}_2]$. Thus, the first part of the lemma is proved.

First assertion of this lemma implies

$$R_\delta \subset \left(S_3 \bigcup S_4\right). \tag{28}$$

As the sectors S_3 and S_4 can only contain a finite number of poles μ_k, it follows that for sufficiently big R>0 the following relation is true:

$$\{\mu_k\} \bigcap R_\delta = \emptyset.$$

□

Now let us assume that the numbers R, δ (in definition of the contour Γ) are chosen in accordance with the requirements of Lemma 3.4. Using Lemmas 3.2-3.4, let us prove our theorem.

Proof. Denote

$$Q^- = \bigcup_{j=1}^{3} Q_j^-,$$

where

$$Q_j^- = \left\{\lambda:\ \lambda = -r\left(1 + \tilde{p}_j\right),\ r \ge R\right\}\ (j = 1, 2),$$

$$Q_3^- = \{\lambda:\ \lambda = -R\,(1 + i\eta),\ \tilde{p}_1 \le \eta \le \tilde{p}_2\}\ .$$

Let us choose the positive directions on the contours Q and Q^- as follows:

$Q_1 \to Q_3 \to Q_2$ and $Q_1^- \to Q_3^- \to Q_2^-$.

Consider a positive integer m_0 satisfying the inequality

$$m_0 > \frac{2\pi R}{|1 + 2b|}\sqrt{1 + \max_j \tilde{p}_j^2}$$

and denote the numerical sequence

$$r_m = \frac{(4m + 4m_0 + 1)\,\pi}{2\,|1 + 2b|} \quad (n = 0, 1, ...) \tag{29}$$

by $\{r_m\}$.

Depending on the choice of n_0, we see that the circles

$$O_m = \left\{\mu : \quad \mu = r_m e^{i\beta}, \ (0 \le \beta \le 2\pi)\right\}$$

intersect the contours Q and Q^- only at the points lying on $Q_j^\pm$ $(j = 1, 2)$, and, moreover,

$$c_{jn}^\pm = Q_j^\pm \bigcap O_m = \pm \frac{r_m}{\sqrt{1+\tilde{p}_j^2}} \left(1 + i\tilde{p}_j\right) = \pm r_m e^{i\beta_j}.$$

On the other hand, from (29) it follows that for sufficiently large $R > 0$ the inequality

$$\left|r_m e^{i\beta} - \mu_k\right| \ge \frac{\pi}{4\,|1+2b|} \quad (\pm k,\ m = 0, 1, ...; 0 \le \beta \le 2\pi)$$

holds.

Denote some arcs of the circles O_n as follows:

$$\overset{\smile}{c_{1m}^+ c_{2m}^+} = \left\{\mu : \quad \mu = r_m e^{i\beta}, \ \beta_1 \le \beta \le \beta_2\right\},$$

$$\overset{\smile}{c_{2m}^+ c_{1m}^-} = \left\{\mu : \quad \mu = r_m e^{i\beta}, \ \beta_2 \le \beta \le \beta_1 + \pi\right\},$$

$$\overset{\smile}{c_{2n}^- c_{1m}^+} = \left\{\mu : \quad \mu = r_m e^{i\beta}, \ \beta_2 - \pi \le \beta \le \beta_1 + \pi\right\}.$$

Also, denote by Ω_n and Ω_m^+ the following closed contours:

$$\Sigma_m = Q^{m,+} \bigcup \overset{\smile}{c_{2m}^+ c_{1m}^-} \bigcup Q^{m,-} \bigcup \overset{\smile}{c_{2m}^- c_{1m}^+},$$

$$\Sigma_m^+ = Q^{m,+} \bigcup \overset{\smile}{c_{2m}^+ c_{1m}^+},$$

where

$$Q^{m,\pm} = \left\{\pm\mu : \quad \mu \in Q, \ \left|\mu\right| \le r_m\right\}.$$

Let us formally perform the operations $x \to +0$, $x \to 1-0$ under the sign of integration:

$$u(t, x) = -\frac{1}{\pi i} \int_Q \mu\, e^{\mu^2 \int_0^t P(\tau) d\tau} d\mu \int_0^1 G(x, \xi, \mu)(\xi + b)^2 \varphi(\xi)\, d\xi. \tag{30}$$

Using the properties of Green's function $G(x, \xi, \mu)$, we have

$$u(t, 0) = 0, \ u(t, 1) = 0 \tag{31}$$

for $t \in (0, T]$.

Also, formally bringing the derivatives $\frac{\partial}{\partial t}, \frac{\partial^2}{\partial x^2}$ in (30) under the sign of integration, we obtain

$$(x + b)^2 u_t - P(t)\, u_{xx} = \frac{1}{\pi i} P(t) \cdot \ (x + b)^2 \varphi(x) \int_Q \mu e^{\mu^2 \int_0^t P(\tau) d\tau} d\mu \tag{32}$$

for $(t, x) \in (0, T] \times [0, 1]$.

By condition 3^0 and the equality

$$\int_0^1 G(x, \xi, \mu)(\xi + b)^2 \varphi(\xi) d\xi = \frac{\varphi(x)}{\mu^2} + \frac{1}{\mu^2} \int_0^1 G(x, \xi, \mu) \varphi''(\xi) d\xi,$$

we can rewrite (30) as follows:

$$\int_0^1 G(x,\xi,\mu)(\xi+b)^2\varphi(\xi)d\xi = \frac{\varphi(x)}{\mu^2} + \frac{1}{\mu^2}\int_0^1 G(x,\xi,\mu)\varphi''(\xi)d\xi$$

$$u(t,x) = u_1(t,x) + u_2(t,x), \tag{33}$$

where

$$u_1(t,x) = \frac{1}{\pi i}\varphi(x)\int_Q \frac{1}{\mu} e^{\mu^2 \int_0^t P(\tau)d\tau} d\mu, \tag{34}$$

$$u_2(t,x) = \frac{1}{\pi i}\int_Q \frac{1}{\mu} e^{\mu^2 \int_0^t P(\tau)d\tau} d\mu \int_0^1 G(x,\xi,\mu)\varphi''(\xi)d\xi, \tag{35}$$

$$u_2(0,x) = \frac{1}{\pi i}\int_Q \frac{1}{\mu} d\mu \int_0^1 G(x,\xi,\mu)\varphi''(\xi)d\xi. \tag{36}$$

Now let us calculate the integrals over the contour Q using the formulas (32), (34), (36). Assume

$$\gamma_k(Q) = \int_Q \mu^{2k-1} e^{\mu^2 \int_0^t P(\tau)d\tau} d\mu \quad (k = 0,\ 1). \tag{37}$$

Hence,

$$\gamma_k(Q) = \lim_{m\to\infty} \gamma_k(Q^{m,+}) = \frac{1}{2}\cdot \lim_{m\to\infty}\left[\gamma_k(Q^{m,+}) + \gamma_k(Q^{m,-})\right].$$

By Lemma 3.3 we obtain

$$\lim_{n\to\infty} \gamma_k\left(\overset{\smile}{c^+_{2m}c^-_{1m}}\right) = 0, \lim_{m\to\infty} \gamma_k\left(\overset{\smile}{c^-_{2m}c^+_{1m}}\right) = 0. \tag{38}$$

Consequently, the function $\gamma_k(Q)$ can be expressed as follows:

$$\gamma_k(Q) =$$

$$= \frac{1}{2}\lim_{m\to\infty}\left[\gamma_k(Q^{m,+}) + \gamma_k\left(\overset{\smile}{c^+_{2n}c^-_{1m}}\right) + \gamma_k(Q^{m,-}) + \gamma_k\left(c^-_{2m}c^+_{1m}\right)\right] = \frac{1}{2}\cdot\lim_{m\to\infty}\gamma_k(\Sigma_m) \tag{39}$$

for $t > 0$ and $k = 0, 1$. But, since $\gamma_k(\Sigma_m)$ is an integral of the function $\mu^{2k-1}e^{\mu^2 \int_0^t P(\tau)d\tau}$ over the closed contour Σ_m, we have

$$\gamma_k(\Sigma_m) = \begin{cases} 2\pi i, & for\ \ k = 0, \\ 0, & for\ \ k = 1. \end{cases}$$

Taking into account the formulas (32), (34), (37), (39), we arrive at the following conclusion:

$$(x+b)^2 u_t - P(t)\ u_{xx} = 0 \tag{40}$$

and

$$u_1(t,x) = \varphi(x), \tag{41}$$

for $(t,x) \in (0,\ T] \times [0,\ 1]$.

By the estimates (3.4) for Green's function of the problem (8)-(9) and the relation $\overset{\smile}{c^+_{2n}c^-_{1n}} \subset R_\delta \subset (S_3 \bigcup S_4)$ (see Lemma 3.4), we get

$$\lim_{m\to\infty}\frac{1}{\pi i}\int_{\overset{\smile}{c^+_{2m}c^-_{1m}}}\frac{1}{\mu}\,d\lambda\int_0^1 G(x,\xi,\mu)\varphi''(\xi)d\xi = 0$$

uniformly with respect to $x \in [0, 1]$. Thus, from (36) we obtain

$$u_2(0,x) = \frac{1}{\pi i}\lim_{m\to\infty}\int_{Q^{m+}\bigcup c^+_{2m}c^-_{1m}}\frac{1}{\mu}\,d\lambda\int_0^1 G(x,\xi,\lambda)\varphi''(\xi)d\xi, \tag{42}$$

i.e.

$$u_2(0,x) = 0. \tag{43}$$

Also, by the formulas (33), (41), (43), we get

$$\lim_{t\to+0} u(t,x) = \lim_{t\to+0}\left[u_1(t,x) + u_2(t,x)\right] = \lim_{t\to+0}\left[\varphi(x) + u_2(t,x)\right] = \varphi(x) + u_2(0,x) = \varphi(x). \tag{44}$$

Consequently, the function $U(t,x)$ defined by the formula (12), belongs to the space $C^{1,2}((0,\,T]\times[0,\,1])$ (see (37)). Note that this function satisfies the equation (3) for $0 < t \le T,\ 0 \le x \le 1$ (see (44)) and the boundary conditions (5) for $0 < t \le T$ (see (35)). It also satisfies (44) for $0 \le x \le 1$.

Then it is clear that if this function is defined for $t = 0,\ 0 \le x \le 1$ by the equality $u(0,x) = \varphi(x)$, then it belongs to $C^{1,2}((0,\,T]\times[0,\,1])$, satisfies the equation (3) for $0 < t \le T,\ 0 \le x \le 1$, the initial conditions (4) for $0 \le x \le 1$ and the boundary conditions (5) for $0 \le t \le T$ (for $t = 0$ due to the condition $\varphi(0) = \varphi(1) = 0$). □

Remark 3.5. The proved theorem covers not only initial boundary value problems for parabolic equations, but also those for nonparabolic equations. So, even though the conditions of theorem hold for the equation (3), this equation is not parabolic.

4. Conclusion

In the paper, the existence and uniqueness conditions for the solution of the problem in the form of (3)-(5), which changes its type from parabolic to antiparabolic and where the arguments of the roots of the characteristic equation in the sense of Birkhoff are not constant, have been found, and an explicit analytical expression for the solution has been obtained.

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