Filomat 39:19 (2025), 6665–6679
https://doi.org/10.2298/FIL2519665C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

$C^*(X)$ from a graphical point of view

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Abstract. In this paper, we define a graph called the chain-link graph $\gamma(C^*(X))$ on the ring $C^*(X)$ of all real-valued, bounded, continuous functions defined over a Tychonoff space X. We briefly study some aspects like connectedness, diameter, radius, cycles, chords, dominating sets etc. of $\gamma(C^*(X))$ and some of its subgraphs. We also inspect the relation between the ideals of $C^*(X)$ and the cliques of $\gamma(C^*(X))$ and finally provide a characterization for all maximal cliques of $\gamma(C^*(X))$. In the sequel, we prove that there are at least 2^c many different maximal cliques, which are never graph isomorphic to each other. Moreover, we inquire about the topological and algebraic notions linked to the neighbourhood of a vertex of the graph. We then observe the correspondence between graph isomorphisms on $\gamma(C^*(X))$, ring isomorphisms on $C^*(X)$ and homeomorphims on X when the topology of X is suitably chosen.

1. Introduction

The construction of a graph on an algebraic structure is usually done to impose a discrete setting upon it and then study patterns and behaviour of the known algebraic facts from the graphical point of view. Conversely, sometimes graphical facts are also uncovered from such investigations. In this direction, one of the first works was by Beck [5], who defined a zero divisor graph on a commutative ring and investigated the finite coloring problem there, whereas Azarpanah in [3] first studied the zero divisor graph over the ring $C(X)$ of all real-valued continuous functions defined on a topological space X. Later in [4], Badie studied the comaximal graph $\Gamma'_2C(X)$ on $C(X)$ (while the comaximal graph over a ring was discussed by Amini et. al. in [2]). Also Bose and Das studied the zero-set intersection graph $\Gamma(C(X))$ in [7] which is incidentally the complement $\overline{\Gamma'_2\, C(X)}$ of the comaximal graph in [4]. Lately, the authors in [1], [13], [6] and [12] also worked further on the aforementioned graphs over different rings related to $C(X)$. In each such paper, the definition of adjacency is constructed by using some interrelations between the zero-sets of X. In [7], the adjacency is closely related to the finite intersection property of z-filters on X. As a result, many topological and ring theoretic aspects of X and $C(X)$ could be characterized in graphical sense. In the current paper, we have tried to study $C^*(X)$ by graphical means. But zero-sets are not a reliable equipment for $C^*(X)$. So we set up a suitable definition to formulate a new graph, called the 'chain-link graph' $\gamma(C^*(X))$ with an objective of analysing the topology of X as well as the ring $C^*(X)$ in this new setting. Section 2 is fully

2020 *Mathematics Subject Classification*. Primary 54C30; Secondary 05C38, 05C69.
Keywords. e-ideal, e-filter, cycle, chord, maximal clique, graph isomorphism.
Received: 31 October 2024; Revised: 17 June 2025; Accepted: 15 July 2025
Communicated by Ljubiša D. R. Kočinac
Research of the first author is supported by University Grants Commission (UGC), Government of India, Grant number 211610173470 and DST-FIST, Government of India, Grant number SR/FST/MS-I/2019/41.
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devoted to necessary definitions and required prerequisites. In section 3, we first define the graph and make some quick observations about diameter, connectedness, radius, girth, dominating sets etc. of $\gamma(C^*(X))$. Next, we define some suitable subgraphs to explore the influence of the topology of X, upon the graphical properties of those subgraphs. We then enquire about cycles and chords of $\gamma(C^*(X))$ and its subgraphs. After that we delve into the characterization of cliques and neighbourhoods of $\gamma(C^*(X))$. Finally in section 4, we investigate the graph isomorphisms on $\gamma(C^*(X))$ and find its influence on the structure space of $C^*(X)$. Here we also observe that $\gamma(C^*(X))$ contains at least $2^{\mathfrak{c}}$ many non-isomorphic maximal cliques. This is a consequence of non-homogeneity of βX (refer [15] for details), a question which Rudin addressed in [14] for $\mathbb{N}^*$ and later generalized by Isiwata in [11] for the growth X^* of any non-pseudocompact Tychonoff space X. Both of the results were based on the Continuum Hypothesis. Later, Frolik [9] worked this out independent of the Continuum Hypothesis.

2. Definitions and prerequisites

We first recall some basic definitions, concepts and declare some notations of graph theory and the theory of rings of continuous functions. For undefined terms and notations, the readers are referred to [16], [10] and [8].

Usually X is considered as a completely regular, Hausdorff topological space and the ring of all real-valued continuous functions over X is denoted by $C(X)$, whereas the subring of all bounded functions of $C(X)$ is denoted by $C^*(X)$. For $f \in C(X)$, the set $Z(f) = \{x \in X : f(x) = 0\}$ is called the zero-set of f. The family of all such zero-sets in X is denoted by $Z[X]$. For $f \in C^*(X)$, $\sup_{x\in X}|f(x)|$ is denoted by $\|f\|$. Also, the cardinality of any set A is denoted by $|A|$. If a filter on X consists of only zero-sets, then it is said to be a z-filter, whereas a z-ultrafilter is a maximal z-filter with respect to inclusion. For $f \in C^*(X)$ and $\epsilon > 0$, consider the set $E_\epsilon(f) = \{x \in X : |f(x)| \leq \epsilon\}$. Note that for $g \in C(X)$, the set $\{x \in X : g(x) \geq 0\}$ is the zero-set $Z(g - |g|)$. Thus $E_\epsilon(f) = Z(h - |h|)$, where $h = \underline{\epsilon} - |f| \in C^*(X)$ ($\underline{\epsilon}$ is the constant map $\underline{\epsilon}(x) = \epsilon$, $x \in X$). Now for any $N \subseteq C^*(X)$, $E(N)$ is defined as $\{E_\epsilon(f) : f \in N, \epsilon > 0\}$ and for $\{f\}$, $E(f)$ is the chain $E(\{f\})$. In fact, E is a set operator from the power set of $C^*(X)$ into that of $Z[X]$ and when restricted to the family of ideals of $C^*(X)$, E maps ideals to z-filters. For any family $\mathscr{F} \subseteq Z[X]$, we can define $E^{\leftarrow}(\mathscr{F}) = \{f \in C^*(X) : E(f) \subseteq \mathscr{F}\}$. Clearly $N \subseteq E^{\leftarrow}(E(N))$, where $E^{\leftarrow}(E(N)) = \{f \in C^*(X) : E_\epsilon(f) \in E(N) \text{ for all } \epsilon > 0\}$. For any ideal $I \subseteq C^*(X)$, I is called an e-ideal whenever $I = E^{\leftarrow}(E(I))$. Also the inclusion $E(E^{\leftarrow}(\mathscr{F})) \subseteq \mathscr{F}$ holds and if the equality occurs for some z-filter $\mathscr{F}$, we call it an e-filter on X. In fact, for any ideal I, $E(I)$ is an e-filter and for any z-filter $\mathscr{F}$ on X, $E^{\leftarrow}(\mathscr{F})$ is an e-ideal in $C^*(X)$. For all results, notations, and relations related to this, the reader is referred to Exercise 2L of [10].

Consider a commutative ring $\mathscr{A}$ with unity. For $N \subseteq \mathscr{A}$, the smallest ideal containing N in $\mathscr{A}$ is denoted by $< N >$. The set of all non-units of $\mathscr{A}$ is denoted by $\mathscr{N}(\mathscr{A})$. The structure space $\mathscr{M}(\mathscr{A})$ of $\mathscr{A}$ is the space of all maximal ideals of $\mathscr{A}$, endowed with the hull-kernel topology. The subspace $\mathscr{PM}(\mathscr{A})$ of $\mathscr{M}(\mathscr{A})$ is the set of all principal maximal ideals of $\mathscr{A}$. βX is the Stone-Čech compactification of X, which in this context is basically the index set for all z-ultrafilters on X with the Stone topology defined on it. In fact, both $\mathscr{M}(C(X))$ and $\mathscr{M}(C^*(X))$ are homeomorphic to βX. In the following discussions, we frequently use the following facts from [10]. By 6.6 (b), $C(\beta X)$ and $C^*(X)$ are ring isomorphic. From Theorem 6.5 (IV), $cl_{\beta X}(Z_1 \cap Z_2) = cl_{\beta X}Z_1 \cap cl_{\beta X}Z_2$. Finally, $\upsilon X(\subseteq \beta X)$ denotes the Hewitt real compactification of X which is homeomorphic to the subspace $\mathscr{RM}(C(X))(\subseteq \mathscr{M}(C(X)))$ of all real maximal ideals of $C(X)$.

By a graph $G = (V, E)$, we mean a non-empty set V and a symmetric binary relation E (which can be empty) on V. V is called the vertex set and E is the edge set of the graph G and each element of V and E are called a vertex and an edge of G respectively. Any two vertices $u, v \in V$ are called adjacent or neighbours if and only if $(u, v) \in E$ and we denote this by $u \sim v$. Also, u and v are called the endpoints of the edge (u, v). A subgraph of G is a pair $H = (W, F)$ whenever $\emptyset \neq W \subseteq V$ and $F \subseteq E$. If a subgraph $H = (W, F)$ of a graph $G = (V, E)$ is such that for any $a, b \in W$, $(a, b) \in E$ if and only if $(a, b) \in F$, then H is called a subgraph induced by the vertex set W. Any two graphs $G = (V, E)$ and $G' = (V', E')$ are called graph isomorphic if there is a bijection $\phi : V \longrightarrow V'$ such that $(a, b) \in E$ if and only if $(\phi(a), \phi(b)) \in E'$. The set of all vertices adjacent to a chosen vertex $a \in V$ is called the (open) neighbourhood of a, denoted by $N(a)$. By $N[a] = N(a) \cup \{a\}$, we denote the closed neighbourhood of a. In this paper by neighbourhood, we will always indicate a closed neighbourhood. If all the vertices of a subgraph are pairwise adjacent, then we call it a complete subgraph, while the set of vertices of such a subgraph is called a clique. A clique that is maximal with respect to inclusion is said to be a maximal clique. Here the clique number of G ($|G|$ can be finite or infinite), denoted by $cl(G)$, is defined as $sup\{|C| : C \text{ is a clique in } G\}$. A vertex u is called simplicial if its neighbourhood $N[u]$ is a clique. A set of vertices S in a graph G is called an independent set in G, if the members of S are pairwise non-adjacent. We define the independence number of G (where $|G|$ can be finite or infinite) as $In(G) = sup\{|\mathscr{I}| : \mathscr{I} \text{ is an independent set in } G\}$. Consider a simple subgraph of G such that the vertex set is just a finite sequence of ordered vertices of G and any two vertices are adjacent only if those are consecutive members of the ordered vertex set (e.g. if the vertex set is $\{v_0, v_1, v_2, \ldots, v_n\}$, where all the v_i's are distinct, then (v_1, v_2) is an edge in the path but (v_0, v_2) is not so). Such a subgraph is said to be a path joining v_0 and v_n or a $v_0 - v_n$ path in G. The length of a path is the number of edges in a path, e.g. in the last instance, the path is of length n. In this paper, we use the notation $(v_o, v_1, \ldots, v_n)$ for such a path. If every pair of vertices of a graph G are joined by a path, then G is said to be connected. A maximal (in the sense of inclusion) connected subgraph of the graph G is called a component of G. In a graph, the distance $d(a, b)$ of the vertices a, b is defined as the shortest length of an $a - b$ path (if it exists). Otherwise, $d(a, b) = \infty$. The diameter of a connected graph G is considered as $diam(G) = sup\{d(a, b) : a, b \text{ are any pair of vertices in } G\}$. In a connected graph $G = (V, E)$, the eccentricity of a vertex a is assumed to be $ecc(a) = sup\{d(a, b) : b \in V \setminus \{a\}\}$. The radius of such a graph is defined as $Rad(G) = min\{ecc(a) : a \in V\}$. The subgraph induced by the vertex set $\{a \in V : ecc(a) = Rad(G)\}$ is called the center of G. If there is a subgraph of G which is essentially similar to a path with n vertices (e.g. $\{v_o, v_1, \ldots, v_{n-1}\}$) except for the fact that the initial and the final vertices of the ordered vertex set are equal, then the subgraph is called a cycle of length n or an n-cycle. In this paper, for a cycle with a vertex set $\{v_o, v_1, \ldots, v_n\}$ we use the notation $(v_o, v_1, \ldots, v_n, v_o)$. For two vertices a, b in G, the length of the smallest cycle (if it exists) containing both a, b is denoted by $c(a, b)$. A 3-cycle is called a triangle. If every vertex of G is a vertex of a triangle, we call G triangulated. Similarly, if each edge of G is an edge of some triangle, then G is called hyper-triangulated. The least possible length of a cycle in a graph G is called the girth of G, denoted by $gr(G)$. If for a cycle C in a graph G, there exists an edge (a, b) that is not in C but the endpoints a, b are vertices of C, then (a, b) is said to be a chord of C. If all n-cycles ($n \geq 4$) possess chords, then G is called a chordal graph. In a graph G, two vertices a, b are defined to be orthogonal, i.e. $a \perp b$ if $a \sim b$ and $\{a, b\} = N[a] \cap N[b]$. If every vertex of a graph G is orthogonal, then G is called complemented. In a graph $G = (V, E)$, a set $D \subseteq V$ is called a dominating set if for any $u \in V \setminus D$, there is $d \in D$ such that $d \sim u$. The dominating number of G denoted by $dt(G)$, is defined as $min\{|D| : D \text{ is a dominating set in } G\}$.

3. The chain-link graph $\gamma(C^*(X))$

To study the ring $C^*(X)$ and the topology of X in a graphical setting, we need to define a suitable graph over $C^*(X)$. Although we wish to incorporate the finite intersection property enjoyed by z-filters on X in our setup, the `zero-set intersection route' used in [7] is not very useful for $C^*(X)$. It may happen that for a non-unit $f \in C^*(X)$, the zero-set $Z(f) = \emptyset$ (e.g. the map $j \in C^*(\mathbb{N})$ in 0.1 of [10]). We have seen that to achieve results for $C^*(X)$ similar to that of $C(X)$, the chain $E(f) = \{E_\epsilon(f) : \epsilon > 0\}$, for $f \in C^*(X)$ and the concepts like e-ideals, e-filters etc. were introduced in $C^*(X)$. Incidentally, f is a non-unit if and only if $\emptyset \notin E(f)$. Using this, we compose the necessary definition for the graph on $C^*(X)$, which will henceforth be denoted as $\gamma(C^*(X))$.

3.1. *Introduction of the graph $\gamma(C^*(X))$*

Definition 3.1. ***(The Chain-link graph)*** The vertex set of $\gamma(C^*(X))$ is $\mathscr{N}(C^*(X))$ and there is an edge between two distinct members f and g of $\mathscr{N}(C^*(X))$ if $E(f) \cup E(g)$ has the finite intersection property. Thus two distinct vertices f and g are adjacent in $\gamma(C^*(X))$ (to be denoted by $f \sim g$) if and only if $E_\epsilon(f) \cap E_\delta(g) \neq \emptyset$, for all $\epsilon, \delta > 0$.

The adjacency of edges can be redefined by some alternative statements as follows.

Proposition 3.2. *For $f, g \in \mathcal{N}(C^*(X))$, the following statements are equivalent:*

(1) *$E(f) \cup E(g)$ has the finite intersection property.*

(2) *$E(f)$, $E(g) \subseteq \mathscr{F}$, for some z-filter (or, a z-ultrafilter) $\mathscr{F}$ on X.*

(3) *$E(f)$, $E(g) \subseteq \mathscr{E}$, for some e-filter (or, an e-ultrafilter) $\mathscr{E}$ on X.*

Using the above definition we first point out few obvious observations about $\gamma(C^*(X))$.

Remark 3.3. (*i*) For $|X| = 1$, $C^*(X) \cong \mathbb{R}$, hence $\gamma(C^*(X)) = \{\underline{0}\}$.

For $|X| \geq 2$, $E_\delta(\underline{0}) = X$, for any $\delta > 0$. So the chain $E(\underline{0}) = \{X\}$. Therefore any $f \in \mathcal{N}(C^*(X))$ is adjacent to $\underline{0}$, i.e. $N[\underline{0}] = \mathcal{N}(C^*(X))$. Again for any $f, g \in \mathcal{N}(C^*(X))$, either $f \sim g$ or $f \nsim g$. Accordingly, $d(f, g) = 1$ or $d(f, g) \neq 1$. However, $(f, \underline{0}, g)$ being a path in $\gamma(C^*(X))$ for the case $f \nsim g$, $d(f, g) = 2$. Clearly, $diam(\gamma(C^*(X))) = 2$. [Note that there always exist $f, g \in \mathcal{N}(C^*(X))$ with $f \nsim g$, provided $|X| \geq 2$]. Thus, $\gamma(C^*(X))$ qualifies to be a connected graph (whenever $|X| \geq 2$).

(*ii*) For any non-zero f, $ecc(f) = 2$, but $ecc(\underline{0}) = 1$. Hence $Rad(\gamma(C^*(X))) = 1$. So, $\{\underline{0}\}$ is the center of $\gamma(C^*(X))$.

(*iii*) For $|X| \geq 2$ take any $f \in \mathcal{N}(C^*(X)) \setminus \{\underline{0}\}$, then $(f, 2f, 3f, f)$ is a 3-cycle or a triangle containing f. Also $(f, 2f, \underline{0}, f)$ is a 3-cycle containing $\underline{0}$. Thus, $\gamma(C^*(X))$ is triangulated. Again, any edge (f, g) in $\gamma(C^*(X))$ is an edge of the 3-cycle (f, g, h, f), where $h \in \mathcal{N}(C^*(X))$ can be chosen as $h = 2f$ if $g \neq 2f$ and $h = 3f$ if $g = 2f$. Hence $\gamma(C^*(X))$ is hypertriangulated as well.
From these deductions, we can also conclude that $\gamma(C^*(X))$ is never complemented and $gr(\gamma(C^*(X))) = 3$.

(*iv*) From (*i*), the fact $N[\underline{0}] = \mathcal{N}(C^*(X))$ implies that $\{\underline{0}\}$ is a dominating set and so $dt(\gamma(C^*(X))) = 1$.

Due to the ring structure of $C^*(X)$, $\underline{0} \in \mathcal{N}(C^*(X))$ and $f \in \mathcal{N}(C^*(X))$ implies that $2f, 3f \in \mathcal{N}(C^*(X))$. So the graphical features mentioned in Remark 3.3 of $\gamma(C^*(X))$ are influenced by the ring structure of $C^*(X)$. To flesh out the topological influence of X upon the graphical parameters of $\gamma(C^*(X))$, we construct a vertex induced subgraph of $\gamma(C^*(X))$ as follows.

First define an equivalence relation ρ on $C^*(X)$ by "$f \rho\, g$ if and only if $E(f) = E(g)$". Now by the Axiom of Choice, we can find a choice function from the set of disjoint equivalence classes $\{\rho(f) : f \in (C^*(X))\}$ into $C^*(X)$ given by $\rho(f) \mapsto f_\rho$. Note that $f \in \mathcal{N}(C^*(X))$ if and only if $\rho(f) \subseteq \mathcal{N}(C^*(X))$. Thus, when we restrict the choice function to the classes $\{\rho(f) : f \in \mathcal{N}(C^*(X))\}$, the image $\{f_\rho : f \in \mathcal{N}(C^*(X))\} \subseteq \mathcal{N}(C^*(X))$. Interestingly, $\rho(\underline{0}) = \{\underline{0}_\rho\}$, i.e. $\underline{0} = \underline{0}_\rho$. We consider the set $\mathcal{N}'_\rho(C^*(X)) = \{f_\rho \in \mathcal{N}(C^*(X)) : f \in \mathcal{N}(C^*(X)) \setminus \{\underline{0}\}\}$ and denote the subgraph of $\gamma(C^*(X))$ induced by the vertex set $\mathcal{N}'_\rho(C^*(X))$ as $\gamma'_\rho(C^*(X))$. We also take another subgraph of $\gamma(C^*(X))$ into consideration, the subgraph $\gamma'(C^*(X))$ induced by the vertex set $\mathcal{N}'(C^*(X)) = \mathcal{N}(C^*(X)) \setminus \{\underline{0}\}$.

Example 3.4. Let $|X| = 3$ and $X = \{a, b, c\}$. Since X is finite, the definition of adjacency in $\gamma'_\rho(C^*(X))$ effectively converts into $f_\rho \sim g_\rho$ if and only if $Z(f_\rho) \cap Z(g_\rho) \neq \emptyset$. For a function f in $C^*(X)$ with $Z(f) = \{a\}$, there are three possible chains of zero-sets $E(f) = \{\{a\},\ \{a, b\},\ X\}$ or $\{\{a\},\ \{a, c\},\ X\}$ or $\{\{a\},\ X\}$. Consequently, in $\gamma'_\rho(C^*(X))$ we get three members corresponding to each chain, which we denote by $f_\rho^{a,1}$, $f_\rho^{a,2}$ and $f_\rho^{a,3}$. Proceeding similarly for b and c, we get $f^{b,1}$, $f^{b,2}$, $f^{b,3}$ and $f^{c,1}$, $f^{c,2}$, $f^{c,3}$ respectively. Thus $\mathcal{N}'_\rho(C^*(X)) = \{f_\rho^{x,i} : i = 1, 2, 3;\ x = a, b, c\} \cup \{f_\rho^{ab}, f_\rho^{bc}, f_\rho^{ca}\}$. Note that for $Z(f) = \{a, b\}$ there is only one chain $\{\{a, b\},\ X\}$ and hence only one member f_ρ^{ab} related to it. We proceed analogously when the zero-sets are $\{b, c\}$ and $\{c, a\}$. The corresponding figure of the graph $\gamma'(C^*(X))$ for $X = \{a, b, c\}$ as discussed is as follows.

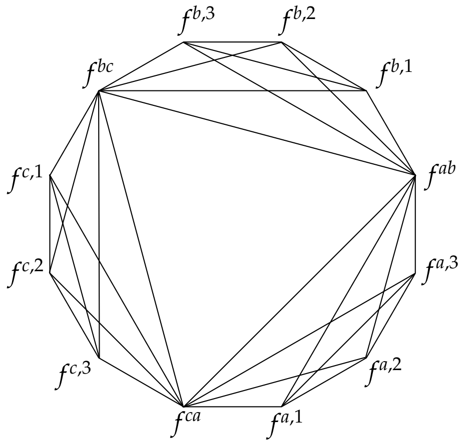

Figure 1: Graph structure of $\gamma'_\rho(C^*(\{a,b,c\}))$

3.2. *Connectedness, diameter and radius of* $\gamma'(C^*(X))$ *and* $\gamma'_\rho(C^*(X))$

If $|X| = 1$, then $\gamma'_\rho(C^*(X)) = \gamma'(C^*(X)) = \emptyset$.

Let $|X| = 2$ and $X = \{a, b\}$. In this scenario, the induced subgraph $\gamma'(C^*(X))$ is disconnected and can be split into two distinct components $A = \{f \in \mathcal{N}'(C^*(X)) : E_\delta(f) = \{a\}, \text{for some } \delta > 0\}$ and $B = \{g \in \mathcal{N}'(C^*(X)) : E_\lambda(g) = \{b\}, \text{for some } \lambda > 0\}$. Similarly, $\gamma'_\rho(C^*(X))$ is disconnected. In fact, $|\mathcal{N}'_\rho(C^*(X))| = 2$. Clearly, if $\gamma'_\rho(C^*(X))$ is connected, then $|X|$ has to be more than 2. In fact, the converse is true as well.

Proposition 3.5. *The subgraph* $\gamma'_\rho(C^*(X))$ *is connected with* $diam(\gamma'_\rho(C^*(X))) = 2$ *if and only if* $|X| \geq 3$.

Proof. Let $|X| \geq 3$. For $f_\rho, g_\rho \in \mathcal{N}'_\rho(C^*(X))$, either $f_\rho \sim g_\rho$ (so that $d(f_\rho, g_\rho) = 1$) or $f_\rho \nsim g_\rho$. In the later case, there are $\epsilon', \delta' > 0$ such that $E_{\epsilon'}(f_\rho) \cap E_{\delta'}(g_\rho) = \emptyset$.

Case I: If for some $\epsilon, \delta > 0$, there is $z \in X \setminus (E_\epsilon(f_\rho) \cup E_\delta(g_\rho))$ and $E_\epsilon(f_\rho) \cap E_\delta(g_\rho) = \emptyset$, then we can find $h_\rho \in \mathcal{N}'_\rho(C^*(X))$ such that $h_\rho(z) = 1$ and $h_\rho(E_\epsilon(f_\rho) \cup E_\delta(g_\rho)) = \{\underline{0}\}$. Thus h_ρ is distinct from f_ρ, g_ρ and (f_ρ, h_ρ, g_ρ) is a path in $\gamma'_\rho(C^*(X))$. Hence $d(f_\rho, g_\rho) = 2$.

Case II: Let for all $\epsilon, \delta > 0, E_\epsilon(f_\rho) \cup E_\delta(g_\rho) = X$. Then $Z(f_\rho) \cup Z(g_\rho) = X$, where both the zero-sets are nonempty. Without loss of generality, suppose that $|Z(f_\rho)| \geq 2$. Let $x, z \in Z(f_\rho)$ and $y \in Z(g_\rho)$. Set $h_\rho \in \mathcal{N}'_\rho(C^*(X))$ such that $h_\rho(z) = 1$ and $h_\rho(\{x, y\}) = \{0\}$. Since, $Z(f_\rho) \cap Z(g_\rho) = \emptyset$, h_ρ is distinct from f_ρ, g_ρ. Also (f_ρ, h_ρ, g_ρ) is a path in $\gamma'_\rho(C^*(X))$. Hence $d(f_\rho, g_\rho) = 2$. Thus, $\gamma'_\rho(C^*(X))$ is connected with $diam(\gamma'_\rho(C^*(X))) = 2$. □

For $|X| \geq 3$ and any arbitrarily chosen vertex f_ρ in $\gamma'_\rho(C^*(X))$, there always exists a non-adjacent vertex $g_\rho \in \mathcal{N}'_\rho(C^*(X))$ such that $d(f_\rho, g_\rho) = 2$, which implies the following result.

Proposition 3.6. *For* $|X| \geq 3$, $ecc(f_\rho) = 2$, *for each vertex* f_ρ *in* $\gamma'_\rho(C^*(X))$. *In other words,* $Rad(\gamma'_\rho(C^*(X))) = 2$.

Remark 3.7. Clearly for $|X| \geq 3$, the center of $\gamma'_\rho(C^*(X))$ is itself. The same can be said for $\gamma'(C^*(X))$ also. As for $|X| \geq 3$, $Rad(\gamma'(C^*(X))) = 2$. So due to the exclusion of the vertex $\underline{0}$, the center of the graphs $\gamma'(C^*(X))$ and $\gamma'_\rho(C^*(X))$ become themselves.

In Proposition 3.5, we see that for $f_\rho \nsim g_\rho$, there is an h_ρ in $\gamma'_\rho(C^*(X))$ such that $h_\rho \sim f_\rho$ and $h_\rho \sim g_\rho$. But this holds regardless of the presence of an edge between f_ρ, g_ρ.

Proposition 3.8. *If* $|X| \geq 3$, *then for any* $f_\rho, g_\rho \in \mathcal{N}'_\rho(C^*(X))$, *there exists an* $h_\rho \in \mathcal{N}'_\rho(C^*(X)) \setminus \{f_\rho, g_\rho\}$ *such that it is adjacent to both* f_ρ *and* g_ρ.

Proof. If $f_\rho \nsim g_\rho$, we have already established the claim in Proposition 3.5. So, we only discuss the case when $f_\rho \sim g_\rho$, i.e. $E_\epsilon(f_\rho) \cap E_\delta(g_\rho) \neq \emptyset$, for all $\epsilon, \delta > 0$.

Case I: Let for each pair of $\epsilon, \delta > 0$, either $E_\epsilon(f_\rho) \subseteq E_\delta(g_\rho)$ or, $E_\epsilon(f_\rho) \supseteq E_\delta(g_\rho)$. Since $f_\rho \neq g_\rho$, without loss of generality, there exists $\epsilon_0 > 0$ such that $E_{\epsilon_0}(f_\rho) \neq E_\delta(g_\rho)$, for all $\delta > 0$. Also, without loss of generality there is $\delta_0 > 0$ such that $E_{\epsilon_0}(f_\rho) \subsetneq E_{\delta_0}(g_\rho) \subsetneq X$ (since $g_\rho \neq \underline{0}$). Then there exist $z \in X \setminus E_{\delta_0}(g_\rho)$ and $y \in E_{\delta_0}(g_\rho) \setminus E_{\epsilon_0}(f_\rho)$ such that we can find $h_\rho \in \mathcal{N}_\rho'(C^*(X))$ with $h_\rho(y) = 1$ and $h_\rho(\{z\} \cup E_{\epsilon_0}(f_\rho)) = \{0\}$. Here $h_\rho \notin \{f_\rho, g_\rho\}$ and $h_\rho \in N[f_\rho] \cap N[g_\rho]$.

Case II: Let there is $\epsilon_1, \delta_1 > 0$ such that $E_{\epsilon_1}(f_\rho) \setminus E_{\delta_1}(g_\rho) \neq \emptyset$ and $E_{\delta_1}(g_\rho) \setminus E_{\epsilon_1}(f_\rho) \neq \emptyset$.

Subcase I: If there exist $\epsilon' \in (0, \epsilon_1)$, $\delta' \in (0, \delta_1)$ such that $z \in X \setminus (E_{\epsilon'}(f_\rho) \cup E_{\delta'}(g_\rho))$, then we can find $h_\rho \in \mathcal{N}_\rho'(C^*(X))$ such that (f_ρ, h_ρ, g_ρ) is a path in $\gamma_\rho'(C^*(X))$ with $h_\rho \notin \{f_\rho, g_\rho\}$ and $h_\rho \in N[f_\rho] \cap N[g_\rho]$.

Subcase II: If there is no such $\epsilon', \delta' > 0$, then for any $\epsilon, \delta > 0$, $E_\epsilon(f_\rho) \cup E_\delta(g_\rho) = X$. Then $Z(f_\rho) \cup Z(g_\rho) = X$, where both the zero-sets are non-empty. In fact, we can find $x \in Z(f_\rho) \setminus Z(g_\rho)$, $y \in Z(g_\rho) \setminus Z(f_\rho)$ and $z \in X \setminus \{x, y\}$. Then there is $h_\rho \in \mathcal{N}_\rho'(C^*(X))$ such that $h_\rho \notin \{f_\rho, g_\rho\}$ and $h_\rho \in N[f_\rho] \cap N[g_\rho]$.

Hence $f_\rho \sim g_\rho$ also implies that $N[f_\rho]$, $N[g_\rho]$ meets at a vertex other than f_ρ *and* g_ρ. □

For $|X| = 2$, we have $|\mathcal{N}_\rho'(C^*(X))| = 2$. Hence $\gamma_\rho'(C^*(X))$ can not be triangulated or hypertriangulated.

For $|X| \geq 3$, choose any $f_\rho \in \mathcal{N}_\rho'(C^*(X))$ and take some $g_\rho \in \mathcal{N}_\rho'(C^*(X)) \setminus \{f_\rho\}$. Then using Proposition 3.8 repeatedly, one can find $h_\rho \in \mathcal{N}_\rho'(C^*(X))$ such that (g_ρ, h_ρ, f_ρ) is a path and there is $h_\rho' \in \mathcal{N}_\rho'(C^*(X))$ such that $(f_\rho, h_\rho', h_\rho, f_\rho)$ is a 3-cycle containing f_ρ. Again for any edge (f_ρ, g_ρ) in $\gamma_\rho'(C^*(X))$, by Proposition 3.8, there is $h_\rho \in \mathcal{N}_\rho'(C^*(X))$ such that $(f_\rho, h_\rho, g_\rho, f_\rho)$ is a 3-cycle containing (f_ρ, g_ρ). So we conclude the following.

Proposition 3.9. *If* $|X| \geq 3$, $\gamma_\rho'(C^*(X))$ *is triangulated as well as hypertriangulated.*

Remark 3.10. Clearly for $|X| \geq 3$, $gr(\gamma_\rho'(C^*(X))) = 3$. Similarly for $|X| \geq 3$, $\gamma'(C^*(X))$ is triangulated (with girth 3) and hypertriangulated as well. Hence $\gamma_\rho'(C^*(X))$ and $\gamma'(C^*(X))$ are never complemented.

3.3. Cycles, chords and dominating sets in $\gamma(C^*(X))$, $\gamma'(C^*(X))$ *and* $\gamma_\rho'(C^*(X))$

We now discuss about the length of the smallest possible cycle containing two vertices of $\gamma(C^*(X))$. For $|X| = 1$, the cases are trivial. So we proceed with $|X| \geq 2$. Using Remark 3.3 and Proposition 3.5, we can show the following.

Proposition 3.11. (1) *If* f, g *are distinct vertices of* $\gamma(C^*(X))$, *then*

- (*i*) $c(f, g) = 3$ *if and only if* $f \sim g$ *(provided* $|X| \geq 2$*).*
- (*ii*) $c(f, g) = 4$ *if and only if* $f \nsim g$ *(provided* $|X| \geq 3$*).*

(2) *If* f_ρ, g_ρ *are distinct vertices of* $\gamma_\rho'(C^*(X))$, *then*

- (*i*) $c(f_\rho, g_\rho) = 3$ *if and only if* $f_\rho \sim g_\rho$ *(provided* $|X| \geq 3$*).*
- (*ii*) $c(f_\rho, g_\rho) = 4$ *or* 5 *if and only if* $f_\rho \nsim g_\rho$ *(provided* $|X| = 3$*).*

Remark 3.12. (2)(*ii*) of Proposition 3.11 follows from Example 3.4 and Figure 1. Further clarification on this is given in Table 1.

Proposition 3.13. *If* f_ρ, g_ρ *are two distinct vertices of* $\gamma_\rho'(C^*(X))$, *then* $c(f_\rho, g_\rho) = 4$ *if and only if* $f_\rho \nsim g_\rho$, *(provided* $|X| \geq 4$*).*

Proof. From Proposition 3.11.(2)(i), $c(f_\rho, g_\rho) \neq 3$ implies $f_\rho \nsim g_\rho$. Conversely, let $f_\rho \nsim g_\rho$. Hence there are $\epsilon, \delta > 0$ such that $E_\epsilon(f_\rho) \cap E_\delta(g_\rho) = \emptyset$. Again from Proposition 3.11.(2)(i), we have $c(f_\rho, g_\rho) \neq 3$. There can be three possibilities.
Case I: Either there are $z, w \in X \setminus (E_{\epsilon'}(f_\rho) \cup E_{\delta'}(g_\rho))$ for some $\epsilon', \delta' > 0$.

Case II: Or, we can find $\epsilon_1, \delta_1 > 0$ such that $X \setminus (E_{\epsilon_1}(f_\rho) \cup E_{\delta_1}(g_\rho)) = \{z\}$. So for $0 < \epsilon < \epsilon_1$ and $0 < \delta < \delta_1$, $E_\epsilon(f_\rho) \cup E_\delta(g_\rho) = X \setminus \{z\}$.

Case III: Otherwise, for any $\epsilon, \delta > 0$ we have $X = E_\epsilon(f_\rho) \cup E_\delta(g_\rho)$.

In each cases, we can find suitable $h^1_\rho \neq h^2_\rho$ in $\mathcal{N}_\rho'(C^*(X))$ distinct from f_ρ, g_ρ such that $(f_\rho, h^1_\rho, g_\rho, h^2_\rho, f_\rho)$ forms a 4-cycle. Hence we have $c(f_\rho, g_\rho) = 4$. □

Next we would like to discuss about the presence of chords in a cycle of length 4 or more. If $|X| \geq 2$ and an n-cycle C ($n \geq 4$) contains $\underline{0}$, then clearly it has chords. Henceforth, we will only consider such n-cycles that has nonzero vertices and $n \geq 4$.

Let $|X| = 2$. Then for each $f \in \mathcal{N}(C^*(X)) \setminus \{\underline{0}\}$, $Z(f) = \{x\}$ for $x \in X$. In fact, for all $f \in C$ (where C is any cycle in $\gamma(C^*(X))$), all $Z(f) = \{x_c\}$ for a fixed $x_c \in X$. Clearly then C possesses chords.

Let $|X| = 3$ and $X = \{a, b, c\}$. Then $\gamma(C^*(X))$ is chordal as the problem turns out to be the same question taken up in [12], Theorem 3.4. So, we can say that $\gamma(C^*(X))$ is chordal if $2 \leq |X| \leq 3$.

If $|X| \geq 4$, choose $x_1, ..., x_4 \in X$ and find pairwise disjoint open sets $U_1, ..., U_4$ in X such that $x_i \in U_i$, for $i = 1, ..., 4$. Then for each $i = 1, ..., 4$, we can find $f_i \in \mathcal{N}(C^*(X))$ such that $f_i(x_i) = 0$ and $f_i(X \setminus U_i) = \{1\}$, provided *image*$(f_i) \subseteq [0, 1]$. Since, $U_i \cap U_j = \emptyset$, for $i \neq j$ and $E_\epsilon(f_i) \subseteq U_i$ for $i, j \in \{1, 2, 3, 4\}$ (where $\epsilon \in (0, 1)$), we get $f_i \nsim f_j$. Set $h_i = f_i f_{i+1}$, for $i = 1, 2, 3$ and $h_4 = f_4 f_1$. Then for all $\epsilon, \delta > 0$, $E_\epsilon(h_1) \cap E_\delta(h_2) \supseteq Z(f_2) \neq \emptyset$, i.e. $h_1 \sim h_2$. Proceeding similarly, $(h_1, h_2, h_3, h_4, h_1)$ is found to be a 4-cycle in $\gamma(C^*(X))$. To show that C has no chords, we need to show that $h_2 \nsim h_4$ and $h_1 \nsim h_3$. If there is $\lambda > 0$ such that $E_\lambda(h_2) \cap E_\lambda(h_4) = \emptyset$, then $h_2 \nsim h_4$. Similarly $h_1 \nsim h_3$ follows. For any $\epsilon > 0$,

$$E_{\frac{\epsilon}{\|g\|}}(f) \cup E_{\frac{\epsilon}{\|f\|}}(g) \subseteq E_\epsilon(fg) \subseteq E_{\sqrt{\epsilon}}(f) \cup E_{\sqrt{\epsilon}}(g) \ldots (*)$$

Then we claim that $E_\lambda(f_2 f_3) \cap E_\lambda(f_4 f_1) = \emptyset$, where $\lambda = \lambda_0 \in (0, 1)$. By (*),

$$E_{\lambda_0}(f_2 f_3) \cap E_{\lambda_0}(f_4 f_1) \subseteq (E_{\sqrt{\lambda_0}}(f_2) \cup E_{\sqrt{\lambda_0}}(f_3)) \cap (E_{\sqrt{\lambda_0}}(f_4) \cup E_{\sqrt{\lambda_0}}(f_1)).$$

Now the right-hand side of the above expression is empty. Hence the claim is true. Thus we have,

Proposition 3.14. *$\gamma(C^*(X))$ is chordal if and only if $|X| = 2$ or 3. Similarly the induced subgraph $\gamma'(C^*(X))$ is chordal if and only if $|X| = 2$ or 3.*

If we proceed analogously in the case of $\gamma'_\rho(C^*(X))$, then we can deduce that $\gamma'_\rho(C^*(X))$ is not chordal for $|X| \geq 4$. Also for $|X| = 2$, there is no cycles in $\gamma'_\rho(C^*(X))$.

For $|X| = 3$, we can summarize the following from Example 3.4 and Figure 1. Cycles of length n, for $4 \leq n \leq 12$ exist in the graph. The different kinds of cycles (upto graph isomorphism) and the number of chords possessed by those (in a respective order) are given below.

Note that a 12 cycle along with its chords is effectively the whole graph $\gamma'_\rho(C^*(X))$. Also for $n \geq 6$, every n-cycle contains the vertices f^{ab}, f^{bc}, f^{ca} and hence always possesses chords. So $\gamma'_\rho(C^*(X))$ is chordal.

From the last discussion, we conclude the following.

Proposition 3.15. *$\gamma'_\rho(C^*(X))$ is a chordal graph if and only if $|X| = 3$.*

Now we aim to find out the dominating sets and the dominating number of $\gamma'(C^*(X))$ and $\gamma'_\rho(C^*(X))$.

Proposition 3.16. *$dt(\gamma'(C^*(X))) = 2$ and $dt(\gamma'_\rho(C^*(X))) = 2$.*

Order of cycle	Types of cycles (upto isomorphism)	Number of chords
4	2	1 or 2
5	3	2, 3 or 5
6	3	6 or 3
7	3	7 or 5
8	3	7, 8 or 9
9	3	9, 10 or 12
10	2	12 or 13
11	1	15
12	1	18

Table 1: Number of cycles and chords of $\gamma'_\rho(C^*(X))$

Proof. There exist $f, g \in \mathcal{N}'(C^*(X))$ such that $\emptyset \neq X \setminus Z(g) \subseteq Z(f) \subsetneq X$. Now for any $h \in \mathcal{N}'(C^*(X))$, if $Z(h) \neq \emptyset$, then $Z(h)$ meets at least one of $Z(g), Z(f)$. Then accordingly $h \sim g$ or $h \sim f$. If $Z(h) = \emptyset$, then we can show that either `$E_\epsilon(h) \cap Z(f) \neq \emptyset$, for all $\epsilon > 0$' or `$E_\epsilon(h) \cap Z(g) \neq \emptyset$, for all $\epsilon > 0$'. Hence $h \sim f$ or $h \sim g$. Thus $\{f, g\}$ is a dominating set in $\gamma'(C^*(X))$ and $dt(\gamma'(C^*(X))) = 2$. For $\gamma'_\rho(C^*(X))$, we proceed similarly. □

We now turn our attention towards the *complete subgraphs* of $\gamma(C^*(X))$.

3.4. *Cliques in $\gamma(C^*(X))$*

In this subsection, we investigate the relation between ideals of $C^*(X)$ and cliques of $\gamma(C^*(X))$. We notice that any ideal in $C^*(X)$ is a clique in $\gamma(C^*(X))$. In fact, for any f, g in an ideal I of $C^*(X)$, $E(f), E(g) \subseteq E(I)$, an e-filter on X. So, any maximal ideal in $C^*(X)$ is a clique in $\gamma(C^*(X))$. In fact, we can improve the result.

Proposition 3.17. *Any maximal ideal M^* in $C^*(X)$ is a maximal clique in $\gamma(C^*(X))$.*

Proof. Assume that for $h \in \mathcal{N}(C^*(X))$, $M^* \cup \{h\}$ is a clique in $\gamma(C^*(X))$. For all $m \in M^*$, $h \sim m$, i.e. $E(h) \cup E(m)$ has the finite intersection property. Therefore $E_\lambda(h) \cap E \neq \emptyset$, for any $\lambda > 0$, $E \in E(M^*)$. Let $\mathscr{U}$ be the unique z-ultrafilter containing the e-ultrafilter $E(M^*)$ on X, i.e. $E^{\leftarrow}(\mathscr{U}) = M^*$. We know that if a zero-set Z meets all the members of the e-ultrafilter $E(E^{\leftarrow}(\mathscr{U}))$, then $Z \in \mathscr{U}$. Evidently $E_\lambda(h) \in \mathscr{U}$ for any $\lambda > 0$, i.e. $E(h) \subseteq \mathscr{U}$. Hence $h \in E^{\leftarrow}(\mathscr{U}) = M^*$, i.e. $M^* \supseteq M^* \cup \{h\}$. Consequently, M^* becomes a maximal clique. □

Since $C^*(X)$ is a *Gelfand ring*, every prime ideal is contained in a unique maximal ideal which is also a maximal clique. So there is always a maximal clique containing any prime ideal. A natural query is whether such a maximal clique is unique. In fact,

Proposition 3.18. *Every prime ideal in $C^*(X)$ is contained in a unique maximal clique of $\gamma(C^*(X))$.*

Proof. Let a prime ideal P be contained in the unique maximal ideal M^* in $C^*(X)$. Take an arbitrary $f \in M'$, where M' is some maximal clique containing P. Then for each $p \in P$, $f \sim p$. It follows that $E(f) \cup E(P)$ possesses the finite intersection property and hence can be embedded in a z-ultrafilter $\mathscr{U}$. Thus $f \in E^{\leftarrow}(\mathscr{U})$ and $P \subseteq E^{\leftarrow}(\mathscr{U})$. Here $E^{\leftarrow}(\mathscr{U})$ is a maximal ideal containing P. But $C^*(X)$ being a *Gelfand ring*, we have $M^* = E^{\leftarrow}(\mathscr{U})$. Consequently $f \in M^*$, in fact $M' \subseteq M^*$. But M^* and M' both being maximal cliques in $\gamma(C^*(X))$, we can conclude that $M' = M^*$. □

Now we turn our attention towards characterising the maximal cliques in $\gamma(C^*(X))$.

Proposition 3.19. *Every maximal clique in $\gamma(C^*(X))$ contains an ideal of $C^*(X)$.*

Proof. Choose any f from a maximal clique $\mathcal{M}$. Consider $< f >$ and take any $fg \in < f >$, for $g \in C^*(X)$. Using the fact that for any $\epsilon > 0$, $E_\epsilon(fg) \supseteq E_{\frac{\epsilon}{\|g\|}}(f)$, for $g \neq \underline{0}$, we can prove that $fg \in \mathcal{M}$. Also for $g = \underline{0}$, $\mathcal{M} \cup \{\underline{0}\}$ is a clique because $\underline{0}$ is adjacent to all other vertices of $\gamma'_\rho(C^*(X))$ (see Remark 3.3.(iv)). Due to maximality of $\mathcal{M}$, $\underline{0} \in \mathcal{M}$. Consequently, $< f > \subseteq \mathcal{M}$. Hence the result follows. □

Remark 3.20. We have seen that there are proper ideals of $C^*(X)$ lying within any maximal clique $\mathcal{M}$ of $\gamma(C^*(X))$. If we consider $\mathscr{I}_\mathcal{M}$ as the family of all such ideals contained in $\mathcal{M}$, then $\mathscr{I}_\mathcal{M}$ forms a partially ordered set under the set inclusion relation. Every ascending chain of ideals in $\mathscr{I}_\mathcal{M}$ has an upper bound in $\mathscr{I}_\mathcal{M}$. So by Zorn's lemma, $\mathscr{I}_\mathcal{M}$ contains a maximal element. Consider $\mathcal{A}_\mathcal{M}$ as the family of all such maximal elements. For any $N \in \mathcal{A}_\mathcal{M}$, let $\widehat{N}$ be the family of all maximal ideals in $C^*(X)$ containing N.

Lemma 3.21. $\bigcup\limits_{N \in \mathcal{A}_\mathcal{M}} N = \mathcal{M}$.

Proof. By construction of $\mathcal{A}_\mathcal{M}$, we can say that $\bigcup\limits_{N \in \mathcal{A}_\mathcal{M}} N \subseteq \mathcal{M}$. By Proposition 3.19, for any $f \in \mathcal{M}$, $< f > \subseteq \mathcal{M}$ and hence there is a $N' \in \mathcal{A}_\mathcal{M}$ such that $f \in < f > \subseteq N' \subseteq \bigcup\limits_{N \in \mathcal{A}_\mathcal{M}} N$. Hence $\mathcal{M} \subseteq \bigcup\limits_{N \in \mathcal{A}_\mathcal{M}} N$. □

Lemma 3.22. *If N_1, $N_2 \in \mathcal{A}_\mathcal{M}$, then $\widehat{N_1} \cap \widehat{N_2} \neq \emptyset$.*

Proof. Let N_1, $N_2 \in \mathcal{A}_\mathcal{M}$. We choose any finite subfamily $\{E_{\lambda_i}(n_i^{(1)}) : i = 1, 2, \ldots, k\} \cup \{E_{\delta_j}(n_j^{(2)}) : j = 1, 2, \ldots, m\}$ from $E(N_1) \cup E(N_2)$, where $n_i^{(1)} \in N_1$ and $n_j^{(2)} \in N_2$, for $i = 1, 2, \ldots, k;\ j = 1, 2, \ldots, m$. Suppose that $\lambda_0 = \min\limits_{1 \leq i \leq k}\{\lambda_i\}$ and $\delta_0 = \min\limits_{1 \leq j \leq m}\{\delta_j\}$. Then $(\bigcap\limits_{i=1}^{k} E_{\lambda_i}(n_i^{(1)})) \bigcap (\bigcap\limits_{j=1}^{m} E_{\delta_j}(n_j^{(2)})) \supseteq E_{{\lambda_0}^2}(n_1) \cap E_{{\delta_0}^2}(n_2)$, where $n_1 = \sum\limits_{1 \leq i \leq k} (n_i^{(1)})^2 \in N_1$ and $n_2 = \sum\limits_{1 \leq j \leq m} (n_j^{(2)})^2 \in N_2$. Then $n_i \in N_i \subseteq \mathcal{M}$ (for i = 1, 2). Thus $n_1 \sim n_2$ and so $E_{{\lambda_0}^2}(n_1) \cap E_{{\delta_0}^2}(n_2) \neq \emptyset$. Consequently, $E(N_1) \cup E(N_2)$ has the finite intersection property and hence $E(N_1) \cup E(N_2) \subseteq \mathscr{U}$, for some z-ultrafilter $\mathscr{U}$ on X. Now the maximal ideal $M^* = E^{\leftarrow}(\mathscr{U})$ contains N_1, N_2. Hence $M^* \in \widehat{N_1} \cap \widehat{N_2}$. □

Lemma 3.23. $\bigcap\limits_{M^* \in \widehat{N}} M^* = N$*, for any* $N \in \mathcal{A}_\mathcal{M}$.

Proof. By construction, $N \subseteq \bigcap\limits_{M^* \in \widehat{N}} M^*$. Now choose any $f \in \bigcap\limits_{M^* \in \widehat{N}} M^*$ and $m \in \mathcal{M}$. By Lemma 3.21, there is $N' \in \mathcal{A}_\mathcal{M}$ such that $m \in N'$. Then by Lemma 3.22, we can choose $M_1^* \in \widehat{N} \cap \widehat{N'}$. Now $M_1^* \in \widehat{N}$ implies that $f \in M_1^*$ and $m \in N' \subseteq M_1^*$. Consequently $E(f)$, $E(m) \subseteq E(M_1^*)$. Therefore $f \sim m$, for any $m \in \mathcal{M}$. So $\mathcal{M} \cup \{f\}$ ($\supseteq \mathcal{M}$) is a clique. Hence $f \in \mathcal{M}$. Clearly, $\bigcap\limits_{M^* \in \widehat{N}} M^* \subseteq \mathcal{M}$ and $\bigcap\limits_{M^* \in \widehat{N}} M^* \in \mathscr{I}_\mathcal{M}$. Then there exists $N_1 \in \mathcal{A}_\mathcal{M}$ such that $N_1 \supseteq \bigcap\limits_{M^* \in \widehat{N}} M^*$, hence $N_1 \supseteq N$. But both $N_1, N \in \mathcal{A}_\mathcal{M}$. So $N_1 = N = \bigcap\limits_{M^* \in \widehat{N}} M^*$. □

Now combining all these results, we can finally give the characterization of maximal cliques in $\gamma(C^*(X))$ in terms of the maximal ideals of $C^*(X)$ as follows.

Theorem 3.24. *Any maximal clique $\mathcal{M}$ in $\gamma(C^*(X))$ is a union of some intersections of families of maximal ideals of $C^*(X)$, i.e. $\mathcal{M} = \bigcup\limits_{N \in \mathcal{A}_\mathcal{M}} (\bigcap\limits_{M^* \in \widehat{N}} M^*)$, provided all the symbols have the same meaning as in Remark 3.20.*

Remark 3.25. In $\gamma(C^*(X))$, the family of cliques can be divided into two categories:

(*a*) The ideal type cliques. When such cliques are maximal, these are exactly the maximal ideals of $C^*(X)$.

(*b*) The non-ideal type cliques. If such cliques are maximal, then the corresponding structures can be described according to Theorem 3.24.

Example 3.26. The following are some examples of maximal cliques in $\gamma(C^*(X))$.

(1) For any Tychonoff space X, the maximal ideals of $C^*(X)$ are $M^{*p} = \{f \in C^*(X) : f^\beta(p) = 0\}$, for any $p \in \beta X$, which form the maximal cliques of ideal type in $\gamma(C^*(X))$.

(2) Choose any $a, b, c, d, e \in \beta\mathbb{N} \setminus \mathbb{N}$. Let $\mathcal{M} = I^{abc} \cup I^{abd} \cup I^{abe} \cup I^{acd} \cup I^{ace} \cup I^{ade} \cup I^{bcd} \cup I^{bce} \cup I^{bde} \cup I^{cde}$, where $I^{xyz} = \{f \in \gamma(C^*(\mathbb{N})) : x, y, z \in \bigcap_{\epsilon>0} cl_{\beta\mathbb{N}} E_\epsilon(f)\}$ (here $x, y, z \in \{a, b, c, d, e\}$). Observe that $\mathcal{M}$ is a clique in $\gamma(C^*(\mathbb{N}))$. For some $f \in \mathscr{N}(C^*(\mathbb{N}))$, let $\{f\} \cup \mathcal{M}$ be also a clique. Now $f \sim g_1$, for $g_1 \in I^{abc}$. For some $\epsilon > 0$, if $a, b, c \notin cl_{\beta\mathbb{N}} E_\epsilon(f)$, then we can find $h \in C^*(X)$ such that $Z_{\beta\mathbb{N}}(h^\beta)$ is a neighbourhood of $\{a, b, c\}$ and $h^\beta(cl_{\beta\mathbb{N}} E_\epsilon(f)) = \{1\}$. Thus $cl_{\beta\mathbb{N}} Z(h)$ is a neighbourhood of $\{a, b, c\}$, so that $h \in I^{abc}$. Hence $\{x \in \beta\mathbb{N} : h^\beta(x) \leq 1/2\} \cap cl_{\beta\mathbb{N}} E_\epsilon(f) = \emptyset$. So $E_{1/2}(h) \cap E_\epsilon(f) = \emptyset$, i.e. $h \nsim f$, which is a contradiction. So $\bigcap_{\epsilon>0} cl_{\beta\mathbb{N}} E_\epsilon(f)$ contains one of a, b, c. Without loss of generality, let $a \in \bigcap_{\epsilon>0} cl_{\beta\mathbb{N}} E_\epsilon(f)$. We can repeat similar argument with $f \sim g_2$, for $g_2 \in I^{bcd}$ and assume $d \in \bigcap_{\epsilon>0} cl_{\beta\mathbb{N}} E_\epsilon(f)$ in this case. Finally, we iterate the process for the last time with some $g_3 \in I^{bce}$ and can assume $e \in \bigcap_{\epsilon>0} cl_{\beta\mathbb{N}} E_\epsilon(f)$. Hence $f \in I^{ade} \subseteq \mathcal{M}$. Thus we can conclude that $\mathcal{M}$ is a maximal clique in $\gamma(C^*(\mathbb{N}))$. Incidentally maximal ideals of $C^*(X)$ can also be written as $M^{*p} = \{f \in C^*(X) : p \in \bigcap_{\epsilon>0} cl_{\beta X} E_\epsilon(f)\}$ for any $p \in \beta X$ (see Remark 4.10.(ii)). Thus we have $\mathcal{M} = \bigcup_{x,y,z \in \{a,b,c,d,e\}} I^{xyz} = \bigcup_{x,y,z \in \{a,b,c,d,e\}} (M^{*x} \cap M^{*y} \cap M^{*z})$. It can be easily checked that $\mathcal{M}$ is not an ideal, i.e. $\mathcal{M}$ is a maximal clique of non-ideal type.

(3) Consider the space of all countable ordinals $\mathbf{W} = W(\omega_1)$ (see 5.11, 5.12 and 5.13 of [10] for details). Since $\mathbf{W}$ is pseudocompact, i.e. $C(\mathbf{W}) = C^*(\mathbf{W})$, the definition of adjacency becomes for $f, g \in \mathscr{N}(C^*(X))$, $f \sim g$ if and only if $Z(f) \cap Z(g) \neq \emptyset$. Now consider the set $\mathcal{M} = I^{1\omega_1} \cup I^{2\omega_1} \cup I^{3\omega_1} \cup I^{4\omega_1} \cup I^{1234}$, where $I^{n\omega_1} = \{f \in C(\mathbf{W}) : f(n) = 0$ and f vanishes on some tail in $W(\omega_1)\}$ for $n = 1, 2, 3, 4$ and $I^{1234} = \{f \in C(\mathbf{W}) : f(m) = 0$ for $m = 1, 2, 3, 4\}$. Clearly, $\mathcal{M}$ is a clique. Let for some $f \in \mathscr{N}(C^*(\mathbf{W}))$, $\{f\} \cup \mathcal{M}$ be also a clique. If f vanishes on some tail in $W(\omega_1)$, for any $g \in I^{1234}$, we have $f \sim g$, i.e. $Z(f) \cap Z(g) \neq \emptyset$. Therefore $n \in Z(f)$ for some $n \in \{1, 2, 3, 4\}$ and hence $f \in I^{n\omega_1} \subseteq \mathcal{M}$. If f does not vanish on any tail in $W(\omega_1)$, then for each $n \in \{1, 2, 3, 4\}$, $f \sim h$ for all $h \in I^{n\omega_1}$. Clearly, $1, 2, 3, 4 \in Z(f)$, i.e. $f \in I^{1234} \subseteq \mathcal{M}$. Consequently $\mathcal{M}$ is a maximal clique in $\gamma(C^*(\mathbf{W}))$. For $C(\mathbf{W})$, $\mathscr{M}(C(\mathbf{W})) = \{M^{\omega_1}\} \cup \{M_x : x \in \mathbf{W}\}$, where $M_x = \{f \in C(\mathbf{W}) : f(x) = 0\}$ for $x \in \mathbf{W}$ and the only free maximal ideal is $M^{\omega_1} = \{f \in C(\mathbf{W}) :$ f vanishes on some tail in $W(\omega_1)\}$. So clearly $\mathcal{M} = \bigcup_{n=1}^{4} (I^{n\omega_1}) \cup I^{1234} = \bigcup_{n=1}^{4} (M_n \cap M^{\omega_1}) \cup (\bigcap_{n=1}^{4} M_n)$. If we choose $f \in I^{1\omega_1} \setminus (I^{2\omega_1} \cup I^{3\omega_1} \cup I^{4\omega_1})$ and $g \in I^{2\omega_1} \setminus (I^{1\omega_1} \cup I^{3\omega_1} \cup I^{4\omega_1})$, then $f^2 + g^2$ does not vanish on $1, 2, 3, 4$, i.e. $f^2 + g^2 \notin \mathcal{M}$. Hence $\mathcal{M}$ is also a maximal clique of non-ideal type in $\gamma(C^*(\mathbf{W}))$.

3.5. Neighbourhoods in $\gamma(C^*(X))$

We observe that there is an intricate connection between neighbourhoods of vertices of $\gamma(C^*(X))$ and a certain kind of families of e-ultrafilters on X. We first describe such a family $\mathscr{E}(f)$ which is essentially the collection of all e-ultrafilters on X containing the chain $E(f)$, for $f \in C^*(X)$. Clearly, for a unit $u \in C^*(X)$, $\mathscr{E}(u) = \emptyset$. Essentially $\mathscr{E}(f) \neq \emptyset$ when and only when $f \in \mathscr{N}(C^*(X))$. Incidentally, we can express Proposition 3.2.(3) mathematically as $f \sim g$ in $\gamma(C^*(X))$ if and only if $\mathscr{E}(f) \cap \mathscr{E}(g) \neq \emptyset$.

Proposition 3.27. *For any $f, g \in \mathscr{N}(C^*(X))$, $N[f] \subseteq N[g]$ if and only if $\mathscr{E}(f) \subseteq \mathscr{E}(g)$.*

Proof. Let $\mathscr{E}(f) \subseteq \mathscr{E}(g)$. Then for any $h \in N[f]$, $E(h), E(f) \subseteq \mathscr{U}$, where $\mathscr{U}$ is an e-ultrafilter on X. Then $\mathscr{U} \in \mathscr{E}(f)$, consequently $\mathscr{U} \in \mathscr{E}(g)$. Thus $E(h), E(g) \subseteq \mathscr{U}$, i.e. $h \sim g$. So $h \in N[g]$. Therefore, $N[f] \subseteq N[g]$.

Conversely, if possible, let $\mathscr{E}(f) \nsubseteq \mathscr{E}(g)$. So we can find a $\mathscr{U} \in \mathscr{E}(f) \setminus \mathscr{E}(g)$. Hence there is an $\epsilon > 0$ such that $E_\epsilon(g)$ does not belong to the unique z-ultrafilter containing $\mathscr{U}$ and there exists some $h \in E^{\leftarrow}(\mathscr{U})$ such that $E_\lambda(h) \cap E_\epsilon(g) = \emptyset$. Then $h \notin N[g]$ but $h \in N[f]$, i.e. $h \in N[f] \setminus N[h]$. Therefore, $N[f] \subseteq N[g]$ implies $\mathscr{E}(f) \subseteq \mathscr{E}(g)$. □

Proposition 3.28. *A vertex $f \in \gamma(C^*(X))$ is simplicial if and only if $\mathscr{E}(f)$ is a singleton.*

Proof. Let $\mathscr{E}(f) = \{\mathscr{U}\}$. Then for any $h_1, h_2 \in N[f]$, we have $E(h_1), E(f), E(h_2) \subseteq \mathscr{U}$. Thus $h_1 \sim h_2$ and so $N[f]$ is a clique.

Conversely, if possible assume that there exist $\mathscr{U}_1, \mathscr{U}_2 \in \mathscr{E}(f)$. Then we can find $E_{\lambda_i}(h_i)$, where $h_i \in E^{\leftarrow}(\mathscr{U}_i)$ and $\lambda_i > 0$ (for $i = 1, 2$), such that $E_{\lambda_1}(h_1) \cap E_{\lambda_2}(h_2) = \emptyset$. Then $h_1 \nsim h_2$. But $E(h_i), E(f) \subseteq \mathscr{U}_i$ (for $i = 1, 2$) which implies that $h_1, h_2 \in N[f]$, i.e. $N[f]$ is not a clique. Contrapositively, $\mathscr{E}(f)$ is a singleton if $N[f]$ is a clique. □

4. Graph isomorphism on $\gamma(C^*(X))$

In this section, we discuss about defining a graph isomorphism on $\gamma(C^*(X))$ and its implications in the context to the topology of X and the ring structure of $C^*(X)$.

4.1. Structures preserved by graph isomorphisms

For any $f \in \mathscr{N}(C^*(X))$, consider the set $I_f^* = \{g \in C^*(X) : \mathscr{E}(f) \subseteq \mathscr{E}(g)\}$. Now we investigate about I_f^* itself and its role in connection with the current topic.

Proposition 4.1. *I_f^* is an e-ideal in $C^*(X)$, hence a clique in $\gamma(C^*(X))$.*

Proof. Clearly, I_f^* is an ideal in $C^*(X)$. Now for any $g \in E^{\leftarrow}E(I_f^*)$, $E(g) \subseteq E(I_f^*)$. For any e-ultrafilter $\mathscr{U} \in \mathscr{E}(f)$, we can show that $E(I_f^*) = \bigcup_{h \in I_f^*} E(h) \subseteq \mathscr{U}$. Hence $E(g) \subseteq \mathscr{U}$, i.e. $\mathscr{U} \in \mathscr{E}(g)$. Thus $\mathscr{E}(f) \subseteq \mathscr{E}(g)$. Therefore $g \in I_f^*$, i.e. $E^{\leftarrow}E(I_f^*) = I_f^*$. □

Lemma 4.2. *For any maximal ideal M^* in $C^*(X)$, $M^* = \bigcup_{f \in M^*} I_f^*$.*

Lemma 4.3. *For any $N \subseteq \mathscr{N}(C^*(X))$, we have $\bigcup_{f \in N} I_f^* \subseteq \{h \in C^*(X) : \bigcap_{\substack{E \in \bigcap \mathscr{E}(f) \\ \forall f \in N}} cl_{\beta X} E \subseteq \bigcap_{\delta > 0} cl_{\beta X} E_\delta(h)\}$.*

Proof. Let $g \in \bigcup_{f \in N} I_f^*$. Then $g \in I_{f'}^*$, for some $f' \in N$. So $\mathscr{E}(f') \subseteq \mathscr{E}(g)$. Then $E(g) \subseteq \bigcap \mathscr{E}(g) \subseteq \bigcap \mathscr{E}(f')$, for $f' \in N$ [here $\bigcap \mathscr{E}(f)$ means $\bigcap_{\mathscr{U} \in \mathscr{E}(f)} \mathscr{U}$]. Thus $E(g) \subseteq \bigcap \{\mathscr{E}(f) : f \in N\}$ [here the right hand side means $\bigcap_{\substack{\mathscr{U} \supseteq E(f) \\ \forall f \in N}} \mathscr{U}$]. Therefore $\bigcap_{\substack{E \in \bigcap \mathscr{E}(f) \\ \forall f \in N}} cl_{\beta X} E \subseteq \bigcap_{\delta > 0} cl_{\beta X} E_\delta(h)$. Hence $g \in \{h \in C^*(X) : \bigcap_{\substack{E \in \bigcap \mathscr{E}(f) \\ \forall f \in N}} cl_{\beta X} E \subseteq \bigcap_{\delta > 0} cl_{\beta X} E_\delta(h)\}$. □

Remark 4.4. Observe that the set $\{E : E \in \bigcap \mathscr{E}(f),\ \text{for each } f \in N\}$ is practically equal to the set $\{E : E \in \bigcap \mathscr{E}(N)\}$, where $\mathscr{E}(N)$ = set of all e-ultrafilters on X containing $E(N) = \bigcup_{f \in N} E(f)$. So we can replace $\bigcap_{\substack{E \in \bigcap \mathscr{E}(f) \\ \forall f \in N}} cl_{\beta X} E$ by $\bigcap_{E \in \bigcap \mathscr{E}(N)} cl_{\beta X} E$ in Lemma 4.3.

From Proposition 3.27, we can conclude the following result.

Proposition 4.5. *If $\varphi : \gamma(C^*(X)) \longrightarrow \gamma(C^*(Y))$ is a graph isomorphism, then $\mathscr{E}(f) \subseteq \mathscr{E}(g)$ in X if and only if $\mathscr{E}(\varphi(f)) \subseteq \mathscr{E}(\varphi(g))$ in Y.*

Lemma 4.6. *If $\varphi : \gamma(C^*(X)) \longrightarrow \gamma(C^*(Y))$ is a graph isomorphism and $N \subseteq \mathscr{N}(C^*(X))$, then $\varphi(\bigcup_{f \in N} I_f^*) = \bigcup_{f \in N} I_{\varphi(f)}^*$.*

Proof. Here it is enough to show that $\varphi(I_f^*) = I_{\varphi(f)}^*$, which follows from Proposition 4.5. □

Lemma 4.7. *If $\varphi : \gamma(C^*(X)) \longrightarrow \gamma(C^*(Y))$ is a graph isomorphism and I is any ideal in $C^*(X)$, then $E(\varphi(I))$ possesses the finite intersection property in Y.*

Proof. For any $f, g \in I$ and $\lambda, \delta > 0$, we have $E_\lambda(f) \cap E_\delta(g) \supseteq E_{\mu^2}(f^2 + g^2)$ (where $\mu = min\{\lambda, \delta\}$). Thus $\mathscr{E}(f) \cap \mathscr{E}(g) \supseteq \mathscr{E}(f^2 + g^2)$. So by Proposition 4.5, $\mathscr{E}(\varphi(f)) \cap \mathscr{E}(\varphi(g)) \supseteq \mathscr{E}(\varphi(f^2 + g^2))$. Since $f^2 + g^2 \in I$ implies that $\varphi(f^2 + g^2) \in \varphi(I)$, $E(\varphi(f)) \cup E(\varphi(g))$ has the finite intersection property. This fact can be extended to arbitrary finite choice of vertices $f_1, f_2, \ldots, f_n \in I$. For $E_{\lambda_i}(\varphi(f_i)) \in E(\varphi(f_i))$ (where $1 \le i \le n$), the union $\bigcup_{i=1}^{n} E(\varphi(f_i))$ possesses the finite intersection property. Hence $\bigcap_{i=1}^{n} E_{\lambda_i}(\varphi(f_i)) \neq \emptyset$. Therefore $E(\varphi(I))$ possesses the finite intersection property in Y. □

Remark 4.8. Note that in the last proof, for an ideal I, due to the clique structure of $\varphi(I)$, any two zero-sets in $E(\varphi(I))$ meet. To extend this for any finite choice of zero-sets, we rely on the above approach.

Theorem 4.9. *If $\varphi : \gamma(C^*(X)) \longrightarrow \gamma(C^*(Y))$ is a graph isomorphism, then for a maximal ideal $M^* \in \mathscr{M}(C^*(X))$ ($\equiv$ the structure space of $C^*(X)$), we have $\varphi(M^*) \in \mathscr{M}(C^*(Y))$. Similarly for $N^* \in \mathscr{M}(C^*(Y))$, we see that $\varphi^{-1}(N^*) \in \mathscr{M}(C^*(X))$.*

Proof. Using Lemma 4.2 and 4.6, $\varphi(M^*) = \varphi(\bigcup_{f\in M^*} I^*_f) = \bigcup_{f\in M^*} I^*_{\varphi(f)}$. Finally Lemma 4.3 implies that

$$\varphi(M^*) \subseteq \{h \in C^*(Y) : \bigcap_{\substack{E\in\bigcap\mathscr{E}(\varphi(f))\\ \forall f\in M^*}} cl_{\beta Y}E \subseteq \bigcap_{\lambda>0} cl_{\beta Y}E_\lambda(h)\} = M' \text{ (say)}.$$

Now M^* being a maximal clique in $\gamma(C^*(X))$, $\varphi(M^*)$ is also so in $\gamma(C^*(Y))$. If M' can be proved to be a proper ideal in $C^*(Y)$, then it becomes a clique. Hence $\varphi(M^*) = M'$. Being an ideal, the maximal clique M' qualifies as a maximal ideal in $C^*(Y)$ (see Remark 3.25). This argument will conclude the proof. Same is true for φ^{-1}.

Now choose any $g_1, g_2 \in M'$, $h \in C^*(Y)$. Note that for any $\lambda > 0$, $E_\lambda(g_1 - g_2) \supseteq E_{\lambda/2}(g_1) \cap E_{\lambda/2}(g_2)$ and $E_\lambda(g_1 h) \supseteq E_{\frac{\lambda}{\|h\|}}(g_1)$ (when $h \neq \underline{0}$). From this, we establish that $g_1 - g_2 \in M'$ and $g_1 h \in M'$ and $\underline{0} \in M'$. Hence M' is an ideal in $C^*(Y)$. To show that it is a proper ideal, we need to prove that the set $E' = \bigcap_{\substack{E\in\bigcap\mathscr{E}(\varphi(f))\\ \forall f\in M^*}} cl_{\beta Y}E \neq \emptyset$. By Remark 4.4, $E' = \bigcap_{E\in\bigcap\mathscr{E}(\varphi(M^*))} cl_{\beta Y}E$. Now M^* being an ideal, from Lemma 4.7, the family of zero-sets $\{E : E \in \bigcap\mathscr{E}(\varphi(M^*))\}$ possesses the finite intersection property in Y. Hence the family $\{cl_{\beta Y}E : E \in \bigcap\mathscr{E}(\varphi(M^*))\}$ possesses the finite intersection property in βY. Thus $E' \neq \emptyset$. This concludes the proof. □

Now we note down some basic observations and draw some connections between the latest findings and the existing literature on the concerned topics.

Remark 4.10. (*i*) In the last Theorem 4.9, it has been shown that for $M^* \in \mathscr{M}(C^*(X))$, $\varphi(M^*) \in \mathscr{M}(C^*(Y))$. Clearly $\varphi(M^*) = M^{*q}$, for some $q \in \beta Y$ and hence $\mathscr{E}(\varphi(M^*)) = \{E(M^{*q})\}$.

(*ii*) Investigating the proof of Theorem 4.9, we can infer that $\varphi(M^*) = \{h \in C^*(Y) : E' \subseteq \bigcap_{\lambda>0} cl_{\beta Y}E_\lambda(h)\}$. Here the only influence M^* has on $\varphi(M^*)$ is that of $E' = \bigcap_{E\in\bigcap\mathscr{E}(\varphi(M^*))} cl_{\beta Y}E$, which becomes $\bigcap_{E\in E(M^{*q})} cl_{\beta Y}E$. Then $E' = \bigcap_{E\in E(M^{*q})} cl_{\beta Y}E = \bigcap_{Z\in Z[O^q]} cl_{\beta Y}Z$ [by [10] 7R, $E(M^{*q}) = Z[O^q]$]. So $E' = \{q\}$. We thus surmise,

$$\varphi(M^*) = \{h \in C^*(Y) : q \in \bigcap_{\lambda>0} cl_{\beta Y}E_\lambda(h)\}. \tag{1}$$

Through some simple calculations, the right hand side set in the above equality turns out to be the well known set $\{h \in C^*(Y) : h^\beta(q) = 0\}$, i.e. M^{*q} (refer to Theorem 7.2 of [10]), which again corroborates the assertion in (i).

Combining all these results, we now frame the theorems that establish a connection between the graph $\gamma(C^*(X))$ and the structure space $\mathscr{M}(C^*(X))$ of $C^*(X)$. In fact, we find a way to retrieve the topology of βX from the graph structure of $\gamma(C^*(X))$.

Theorem 4.11. *Any graph isomorphism $\varphi : \gamma(C^*(X)) \longrightarrow \gamma(C^*(Y))$ induces a bijection $\Phi : \mathscr{M}(C^*(X)) \longrightarrow \mathscr{M}(C^*(Y))$ given by $\Phi(M^*) = \varphi(M^*)$.*

From this, we can comment the following.

Remark 4.12. (*i*) Since $\mathscr{M}(C^*(X))$ and $\mathscr{M}(C^*(Y))$ are equipotent under Φ, we can use the notation $\Phi(M^{*p}) = M^{*p_\varphi}$, where clearly $p_\varphi \in \beta Y$. Thus the graph isomorphism $\varphi : \gamma(C^*(X)) \longrightarrow \gamma(C^*(Y))$ parallelly induces a bijection $\varphi_\beta : \beta X \longrightarrow \beta Y$, defined by $\varphi_\beta(p) = p_\varphi$, for $p \in \beta X$.

(*ii*) If we consider the identity map as a graph isomorphism on $\gamma(C^*(X))$, we see that $M^{*p} = \{h \in C^*(X) : p \in \bigcap_{\lambda>0} cl_{\beta X} E_\lambda(h)\}$, for $p \in \beta X$. This characterization of the maximal ideal M^{*p} in $C^*(X)$ resembles the expression of maximal ideals in $C(X)$ (refer to *The Gelfand Kolmogoroff theorem* 7.3 in [10]).

(*iii*) Here in the proof of Theorem 4.9, the set $\bigcap_{E \in \bigcap \mathscr{E}(\varphi(M^{*p}))} cl_{\beta Y} E$ (which is in fact, $\bigcap_{E \in \bigcap \mathscr{E}(M^{*p_\varphi})} cl_{\beta Y} E = \bigcap_{E \in E(M^{*p_\varphi})} cl_{\beta Y} E$ due to (i)) is proved to be nonempty, precisely the singleton set $\{p_\varphi\}$ in βY, by using the properties of the graph isomorphism φ. Hence the expression in (1) in Remark 4.10. (*ii*) is exactly the required one for maximal ideals of $C^*(X)$, which will be preserved and influenced by a graph isomorphism respective to our graph structure.

Theorem 4.13. *For a graph isomorphism $\varphi : \gamma(C^*(X)) \longrightarrow \gamma(C^*(Y))$, the induced map $\Phi : \mathscr{M}(C^*(X)) \longrightarrow \mathscr{M}(C^*(Y))$ is a homeomorphism from $\mathscr{M}(C^*(X))$ onto $\mathscr{M}(C^*(Y))$.*

Proof. Since Φ is clearly a bijection, to establish that it is a homeomorphism, it is enough to show that Φ and Φ^{-1} are both closed maps. In fact being bijections, it is enough to prove that Φ and Φ^{-1} both assign basic closed sets of $\mathscr{M}(C^*(X))$ to that of $\mathscr{M}(C^*(Y))$ and conversely.

Now for any $f \in \mathscr{N}(C^*(X))$, $\Phi(m(f)) = \{\Phi(M^{*p}) : f \in M^{*p}, p \in \beta X\} = \{\varphi(M^{*p}) : \varphi(f) \in \varphi(M^{*p}), p_\varphi \in \beta Y\} = \{M^{*p_\varphi} : \varphi(f) \in M^{*p_\varphi}, p_\varphi \in \beta Y\} = m(\varphi(f))$.

Similarly for any $q \in \beta Y$ and $g \in \mathscr{N}(C^*(Y))$, $\Phi^{-1}(M^{*q}) = \varphi^{-1}(M^{*q}) = M^{*q_{\varphi^{-1}}}$. Hence $\Phi^{-1}(m(g)) = m(\varphi^{-1}(g))$. In case $f \in C^*(X)$ and $g \in C^*(Y)$ are units, $\Phi(m(f)) = \Phi(\emptyset) = \emptyset$ and $\Phi^{-1}(m(g)) = \emptyset$. Hence the proof. □

We now note down some obvious implications from the above theorem about the corresponding Stone-Čech compactification.

Remark 4.14. Since Φ becomes a homeomorphism between $\mathscr{M}(C^*(X))$ and $\mathscr{M}(C^*(Y))$, the parallel map φ_β also becomes a homeomorphism from βX onto βY. In fact, we are acquainted with the fact that there is a one-one correspondence $M^{*p} \longleftrightarrow p$, which actually induces a homeomorphism $m(f) \longleftrightarrow Z_\beta(f^\beta)$ from $\mathscr{M}(C^*(X))$ onto βX (for $f \in C^*(X)$) (refer to 7.10 and 7M in [10]). For any $f \in C^*(X)$ and $g \in C^*(Y)$, we have $\varphi_\beta(Z_\beta(f^\beta)) = Z_\beta((\varphi(f))^\beta)$ and ${\varphi_\beta}^{-1}(Z_\beta(g^\beta)) = Z_\beta((\varphi^{-1}(g))^\beta)$. Thus φ_β exchanges basic closed sets between βX and βY and the rest follows.

From the above remark and the previous results, we can infer the following conclusion.

Theorem 4.15. *If X, Y are compact or first countable spaces and $\gamma(C^*(X))$, $\gamma(C^*(Y))$ are graph isomorphic, then X, Y are homeomorphic.*

Remark 4.16. One of the above results is parallel to Theorem 9.7(a^*) in [10] : \`If $C^*(X)$, $C^*(Y)$ are ring isomorphic and X, Y are first countable, then X, Y are homeomorphic'. In fact, graph isomorphism leads to ring isomorphism, making both implications of the same weight.

Theorem 4.17. *If $\gamma(C^*(X))$, $\gamma(C^*(Y))$ are graph isomorphic, then $C^*(X)$ and $C^*(Y)$ are ring isomorphic.*

Proof. By Remark 4.14, the graph isomorphism induces a homeomorphism from βX onto βY. From this it follows that $C^*(X)$ and $C^*(Y)$ are ring isomorphic (see [10] 6.6 (b)). $\square$

Now we try to prove the converse of the above theorem. But first we need the following lemma.

Lemma 4.18. *For any $f, g \in \mathscr{N}(C^*(X))$, $f \sim g$ in $\gamma(C^*(X))$ if and only if $f^2 + g^2$ is a non-unit in $C^*(X)$.*

Proof. If $f \sim g$, then for any $\delta > 0$, $E_{\sqrt{\delta/2}}(f) \cap E_{\sqrt{\delta/2}}(g) \subseteq E_\delta(f^2 + g^2)$. The left side being nonempty, $\emptyset \notin E(f^2 + g^2)$, i.e. $f^2 + g^2$ is a non-unit.

Conversely, let f^2+g^2 be a non-unit. For any $\lambda, \epsilon > 0$, $E_\lambda(f) \cap E_\epsilon(g) \supseteq E_{\mu^2}(f^2+g^2) \neq \emptyset$, where $\mu = min\{\lambda, \epsilon\}$. Hence $f \sim g$ in $\gamma(C^*(X))$. $\square$

Remark 4.19. We can add the above result as an equivalent statement in Proposition 3.2 (as statement (4)).

Theorem 4.20. *For any two topological spaces X and Y, if $C^*(X)$ and $C^*(Y)$ are ring isomorphic, then $\gamma(C^*(X))$ and $\gamma(C^*(Y))$ are graph isomorphic.*

Proof. Let $\psi : C^*(X) \longrightarrow C^*(Y)$ be a ring isomorphism. Consider the restriction $\psi|_{\mathscr{N}(C^*(X))}$ on $\mathscr{N}(C^*(X))$ and denote it as ψ_γ. Clearly ψ maps the non-units of $C^*(X)$ to those of the non-units of $C^*(Y)$ and conversely. Hence ψ_γ is a bijection from $\mathscr{N}(C^*(X))$ onto $\mathscr{N}(C^*(Y))$. Again to prove that ψ_γ preserves adjacency, choose any $f, g \in \mathscr{N}(C^*(X))$. Using Lemma 4.18, we have $f \sim g$ in $\gamma(C^*(X))$ if and only if $f^2 + g^2 \in \mathscr{N}(C^*(X))$ if and only if $\psi_\gamma(f^2+g^2) \in \mathscr{N}(C^*(Y))$ if and only if $\psi_\gamma(f)^2+\psi_\gamma(g)^2 \in \mathscr{N}(C^*(Y))$ if and only if $\psi_\gamma(f) \sim \psi_\gamma(g)$ in $\gamma(C^*(Y))$. $\square$

Under a graph automorphism φ on $\gamma(C^*(X))$, M^{*p} goes to M^{*q} in $\mathscr{M}(C^*(X))$ for some $p, q \in \beta X$. Hence the homeomorphism φ_β maps p to q, but that is not always possible due to the non-homogeneity of βX. Thus, we cannot find any graph isomorphism that assigns M^{*p} to M^{*q} for at least two such non-homogeneous points p, q, i.e. $\gamma(C^*(X))$ is partitioned into different equivalence classes under isomorphism relations. Using Frolík's work in [9] and results from [15] (chapter 4) on 'Type of points' on βX, we conclude that

Theorem 4.21. *There are at least $2^{\mathfrak{c}}$ many different (upto isomorphism) maximal cliques of the form M^{*p} in $\gamma(C^*(X))$, for a first countable, non-pseudocompact space X.*

Maximal ideals of $C^*(X)$ being maximal cliques, the last result works for maximal ideals as well. For M^{*p}, similar conclusions can be drawn from algebraic and graphical perspective. But to enquire the reason behind the absence of isomorphism between the two maximal entities, the graphical point of view is clearer than the ring theoretic one. In fact, the neighbourhood $N[M^{*p}] = \cup\{N[f] : f \in M^{*p}\}$ has different graph structure than $N[M^{*q}]$, for two non-homogeneous points p, q *in* βX.

Next we discuss about the relation between the C^*-embeddedness of a subset Y of X and the graphs of the corresponding rings $C^*(X)$ and $C^*(Y)$.

Theorem 4.22. *Let Y be dense in X. Then Y is C^*-embedded in X if and only if $\gamma(C^*(X))$ and $\gamma(C^*(Y))$ are graph isomorphic.*

Proof. If Y is C^*-embedded in X, then by Theorem 6.7 (7) in [10], βY and βX are homeomorphic. Hence $C^*(X)$ and $C^*(Y)$ are ring isomorphic. So by Theorem 4.20, the graphs $\gamma(C^*(X))$ and $\gamma(C^*(Y))$ are isomorphic.

Conversely, if the graphs $\gamma(C^*(X))$ and $\gamma(C^*(Y))$ are isomorphic, then by Remark 4.14, βY and βX are homeomorphic. Now by Theorem 6.7(6) in [10], $Y \subseteq X \subseteq \beta Y$. Clearly then Y is C^*-embedded in X. $\square$

Recalling the result in 4B.1 of [10], we can more generally say that,

Lemma 4.23. *For $M^{*p} \in \mathscr{M}(C^*(X))$ (for $p \in \beta X$), M^{*p} is a principal ideal if and only if p is an isolated point.*

Observe that for an isolated point p in βX (in fact in X) and $p \mapsto p_\varphi$ being a homeomorphism, p_φ is an isolated point of βY (hence of Y). Thus by Lemma 4.23, M^{*p_φ} is a principal maximal ideal. Hence we have the following result.

Proposition 4.24. *If $\varphi : \gamma(C^*(X)) \longrightarrow \gamma(C^*(Y))$ is a graph isomorphism, then $\Phi(\mathscr{PM}(C^*(X))) = \mathscr{PM}(C^*(Y))$, i.e. $\mathscr{PM}(C^*(X))$ and $\mathscr{PM}(C^*(Y))$ are homeomorphic under $\Phi|_{\mathscr{PM}(C^*(X))}$.*

Note that φ_β maps the set of all G_δ, P-points of X onto the same set for Y. Also if $\varphi : \gamma(C^*(X)) \longrightarrow \gamma(C^*(Y))$ is a graph isomorphism for some discrete topological spaces X, Y , then X, Y are homeomorphic. We can generalize this a step further. A point x has a countable base of neighbourhoods in X if and only if it also has a countable base in βX. Also, no point of $\beta X \setminus X$ has a countable base of neighbourhoods in βX. Thus under the homeomorphism φ_β, any point $p \in X$ with countable base of neighbourhoods in X goes to p_φ with countable base of neighbourhoods in Y. If the set of all points with countable base of neighbourhoods in X be S_X^{CNB}, then $\varphi_\beta(S_X^{CNB}) = S_Y^{CNB}$. Thus we have,

Proposition 4.25. *If $\varphi : \gamma(C^*(X)) \longrightarrow \gamma(C^*(Y))$ is a graph isomorphism, then S_X^{CNB} and S_Y^{CNB} are homeomorphic. So if X, Y are chosen as first countable spaces, then X is homeomorphic to Y.*

Finally, regarding the isomorphism between the "zero-set intersection" graph $\Gamma(C(X))$ and the "Chain-link" graph $\gamma(C^*(X)$, we have the following.

Remark 4.26. $\Gamma(C(X))$ and $\gamma(C^*(X))$ are graph isomorphic under any ring automorphism over $C(X)$ if and only if X is pseudocompact. In fact, when a space X is pseudocompact (i.e. $C(X) = C^*(X)$), the definition of adjacency in $\gamma(C^*(X))$ reduces to $f \sim g$ if and only if $Z(f) \cap Z(g) \neq \emptyset$, which is same as that of the definition of adjacency in $\Gamma(C(X))$.

Acknowledgement

The authors are extremely grateful to the learned referees for their valuable suggestions and insights that elevated the quality of this article. It is also important to mention that some suggestions given by the honourable referees need further study which we wish to attend in our future work.

References

[1] S. K. Acharyya, A. D. Ray, P. Nandi, *Zero-Divisor Graph of the Rings $C_{\mathscr{P}}(X)$ and $C_\infty^{\mathscr{P}}(X)$*, Filomat **36** (2022), 5029–5046.
[2] A. Amini, B. Amini, E. Momtahan, M. H. Shirdareh Haghighi, *On a graph of ideals*, Acta Math. Hungar. **134** (2012), 369–384.
[3] F. Azarpanah, M. Motamedi, *Zero-divisor graph of $C(X)$*, Acta Math. Hungar. **108** (2005), 25–36.
[4] M. Badie, *Comaximal graph of $C(X)$*, Comment. Math. Univ. Carolin. **57** (2016), 353–364.
[5] I. Beck, *Coloring of commutative rings*, J. Algebra **116** (1988), 208—226.
[6] R. Bharati, A. Acharyya, A. D. Ray, S. K. Acharyya, *Zero-divisor graph and comaximal graph of rings of continuous functions with countable range*, J. Algebra Appl. (2024), 2450243.
[7] B. Bose, A. Das, *Graph Theoretic Representation of Rings of Continuous Functions*, Filomat **34** (2020), 3417–3428.
[8] R. Engelking, *General Topology*, (Revised and completed edition), Heldermann, Verlag, Berlin, 1989.
[9] Z. Frolík, *Non-homogeneity of $\beta P - P$*, Comment. Math. Univ. Carolin. **8** (1967), 705–709.
[10] L. Gillman, M. Jerison, *Rings of Continuous Functions*, (softcover-1rst edition), D. Van Nostrand Company, Inc., Princeton, New Jersey, 1960.
[11] T. Isiwata, *A Generalization of Rudin's Theorem for the Homogeneity Problem*, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **5** (1957), 300–303.
[12] Y. T. Jamir, S. Dutta, *Some properties of the Zero-set Intersection graph of $C(X)$ and its Line graph*, arXiv preprint: 2405.02056 (2024).
[13] P. Nandi, S. K. Acharyya, A. D. Ray, *Annihilator graph of the ring $C_{\mathscr{P}}(X)$*, arXiv preprint: 2206.05463 (2022).
[14] W. Rudin, *Homogeneity problems in the theory of Čech compactifications*, Duke Math. J. **23** (1956), 409–419.
[15] R. C. Walker, *The Stone-Čech Compactification*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 83. Springer-Verlag, Berlin-New York, 1974.
[16] D. B. West, *Introduction to Graph Theory*, (2nd edition), Prentice Hall, Upper Saddle River, New Jersey, 2001.