Filomat 39:19 (2025), 6681–6692
https://doi.org/10.2298/FIL2519681A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the topology via kernel sets and primal spaces

Ibtesam Alshammari[a,*], Ahmad Al-Omari[b]

[a]*Department of Mathematics, College of Science, University of Hafr Al Batin, Saudi Arabia*
[b]*Department of Mathematics, Faculty of Science, Al al-Bayt University, Mafraq, Jordan*

Abstract. In this work, we introduce a comprehensive generalization of the kernel of a set within topological spaces equipped with a primal structure. The proposed notion of the primal kernel offers a powerful framework for redefining and analyzing generalized forms of open and closed sets. Leveraging this framework, we establish new, weaker separation axioms and construct a novel topology that is demonstrably incomparable with the classical topology derived from the primal structure. These results not only contribute to the refinement of topological concepts but also highlight the structural richness and potential applications of primal-based topologies.

1. Introduction

Numerous topological structures were actively studied in many mathematical fields, including the social and natural sciences, to find solutions to a wide range of natural issues. The idea of grills was established by Choquet [13] and presented in 1947. Kuratowski [17] subsequently examined and analyzed ideals conceptions in 1966, where the ideal concept is the dual of a filter. Grill structures were used by many researchers, such as general topology [7], fuzzy topology [12], etc. It is important to remember that in [21] and [22], Vaidyanathaswamy introduced the idea of localization theory in set-topology. Furthermore, Janković et al. extensively explored this subject in [15]. Also, Sarkar talked about fuzzy ideals in fuzzy set theory and how to use them to create new fuzzy topologies out of existing ones. In [20], he also examined the ideas of compatibility of fuzzy ideals with fuzzy topologies and fuzzy local functions. Conversely, the concept of soft local functions was first presented by Kandil et al. in [16]. A refinement of the idea of the soft local function, Ameen et al. described the cluster soft closed sets in terms of many types of soft sets [9]. Primal is the dual of the concept of grill, as was recently explored in [1], whereby primal structure was examined. They also investigated the connection between topological spaces and primal topological spaces. Al-Shami et al. [8] developed the soft primal soft topology and looked into its fundamental characteristics, which encourages the quick growth of primal topological space. In addition, some kinds of primal soft operator was given by Al-Omari et al. [3]. Fuzzy primal ideas were also presented by Ameen et al. [10]. An important contribution of Al-Omari et al. [6] was to the construction of operators in primal topological

2020 *Mathematics Subject Classification*. Primary 54A05; Secondary 54C10.
Keywords. kernel set, co-kernel set, primal, g_κ-closed set, λ_κ-closed set, $\overline{\kappa}$-$T_{\frac{1}{2}}$ space, $\overline{\kappa}$-R_0 space.
Received: 01 November 2024; Revised: 22 May 2025; Accepted: 07 July 2025
Communicated by Ljubiša D. R. Kočinac
* Corresponding author: Ibtesam Alshammari
Email addresses: iealshamri@uhb.edu.sa (Ibtesam Alshammari), omarimutah1@yahoo.com (Ahmad Al-Omari)
ORCID iDs: https://orcid.org/0000-0002-2568-845X (Ibtesam Alshammari), https://orcid.org/0000-0002-6696-1301 (Ahmad Al-Omari)

spaces and studying the proximity spaces inspired by primal and others [4, 5]. The idea of κ-sets (Λ-set) in topological spaces was first introduced in 1986 by Maki [19]. A set U that coincides with its kernel (= saturated set), that is, with the intersection of all open sets that include U, is called a κ-set. Using κ-sets and closed sets, Arenas et al. [11] established and researched the concept of λ-closed and λ-open sets in 1997.

In this paper, we analyze the relation between a primal kernel set, $\overline{\kappa}(U)$, and other primal operators. We also investigate and introduce this new operator. Moreover, we examine numerous fundamental characteristics and establish a novel topology ($\overline{\Theta}^{\kappa}$) via the $\overline{\kappa}$-operator. We report on new results on the $\overline{\kappa}$-operator in Section 3. In Section 4, we also apply the notion of the $\overline{\kappa}$-operator to generate a new topology that is incomparable with the previous topology. Additionally, in Section 5, we provide some fundamental results on g_κ-closed sets via primal. Besides, we report results pertaining to λ_κ-closed sets in primal spaces in Section 6.

2. Preliminaries

A topological spaces are denoted by $(\mathbb{T}, \Theta)$ (briefly, $\mathbb{T}$) throughout this article. We designate the interior of V by $Int(V)$ and the closure of V by $cl(V)$ for each $V \subset \mathbb{T}$. The power set of $\mathbb{T}$ will be denoted by $\mathcal{P}(\mathbb{T})$. To represent the family of open sets that contains t, we use the notation $\Theta(t)$. If $F \subseteq \mathbb{T}$, then F is a closed subset of $\mathbb{T}$ when $F \in \Theta^c$. We now have the ideas and conclusions listed below, which are important for this article that follows: The kernel of U is $\kappa(U) = \cap\{V \subseteq \mathbb{T} : U \subseteq V \text{ and } V \in \Theta\}$. It is sometimes referred to as $\Lambda(U)$. If $U = \kappa(U)$, then a subset U of $\mathbb{T}$ is a κ-set [19]. Note that $\kappa(U) = U$ if U is an open set. Moreover, the union of all closed sets included in U is the co-kernel of a set U [19], represented by $co\kappa(U)$ or $\vee(U)$. Note that $co\kappa(U) = U$ if U is a closed set. If $U = co\kappa(U)$, then a subset U is a $co\kappa$-set [19]. The $co\kappa$-sets are the κ-sets' complements. Θ^κ represents the set of all κ-sets in $\mathbb{T}$. $\kappa(U)$ is typically neither an open nor a closed set.

Lemma 2.1. ([19]) *The statement that follows holds for any subsets V and U of a topological space $(\mathbb{T}, \Theta)$:*

1. $V \subseteq \kappa(V)$,
2. *If* $U \subseteq V$, *then* $\kappa(U) \subseteq \kappa(V)$,
3. $\kappa(\kappa(V)) = \kappa(V)$,
4. $\kappa(V)$ *is a* κ*-set,*
5. *If* V *is open, then* V *is a* κ*-set,*
6. $\kappa(\cup_{i\in I}(V_i)) = \cup_{i\in I}\kappa(V_i)$
7. $\kappa(\mathbb{T} \setminus V) = \mathbb{T} \setminus co\kappa(V)$,
8. $\kappa(V \cap U) \subseteq \kappa(V) \cap \kappa(U)$.

Lemma 2.2. ([19]) *Considering $(\mathbb{T}, \Theta)$ as a topological space, we may see the subsequent characteristics:*

1. $\emptyset$ *and* $\mathbb{T}$ *are* κ*-sets.*
2. *Every union of* κ*-sets is a* κ*-set.*
3. *Every intersection of* κ*-sets is a* κ*-set.*

If a subset V of $\mathbb{T}$ and $V = A \cap F$, where F is closed and A is a κ-set, then V is λ-closed [11]. The λ-open is the complement of a λ-closed set. It is clear that every κ-set is λ-closed.

Lemma 2.3. ([11]) *The requirements listed below are identical for a subset V of a topological space $(\mathbb{T}, \Theta)$:*

(a) V *is* λ*-closed.*
(b) $V = L \cap cl(V)$, *where* L *is a* κ*-set.*
(c) $V = \kappa(V) \cap cl(V)$.

Recall that generalized closed, or short g-closed [18], refers to a subset V of a topological space $(\mathbb{T}, \Theta)$, whenever $V \subseteq U$ and U is open in $\mathbb{T}$, then $cl(V) \subseteq U$. A subset B is g-closed iff $\mathbb{T} \setminus B$ is a g-open set of $\mathbb{T}$.

Definition 2.4. ([14]) A topological space $(\mathbb{T}, \Theta)$ is said to be:

(a) R_0-space if the closure of each singleton in an open set is contained within it.
(b) R_1-space if for $t, s \in \mathbb{T}$ with $cl(\{t\}) \neq cl(\{s\})$ there are an open sets U and V that are disjoint such that $cl(\{t\})$ is a subset of U and $cl(\{s\})$ is a subset of V.

Definition 2.5. ([1]) A collection $\mathbb{P}$ of the power set $\mathcal{P}(\mathbb{T})$ of a nonempty set $\mathbb{T}$ is called a primal if the following conditions hold:

1. $\mathbb{T} \notin \mathbb{P}$.
2. $V \in \mathbb{P}$ and $U \subseteq V$, impels that $U \in \mathbb{P}$ (or $U \notin \mathbb{P}$ and $U \subseteq V$, impels that $V \notin \mathbb{P}$).
3. if $V \cap U \in \mathbb{P}$, impels that $V \in \mathbb{P}$ or $U \in \mathbb{P}$ (or $V \notin \mathbb{P}$ and $U \notin \mathbb{P}$ impels that $V \cap U \notin \mathbb{P}$).

The topological space $(\mathbb{T}, \Theta)$ with a primal $\mathbb{P}$ [1] on $\mathbb{T}$ is denoted by a *PTS* and is a primal topological space $(\mathbb{T}, \Theta, \mathbb{P})$.

Definition 2.6. ([1]) Let $(\mathbb{T}, \Theta, \mathbb{P})$ be a *PTS*. We provide a function $(\cdot)^\diamond : \mathcal{P}(\mathbb{T}) \to \mathcal{P}(\mathbb{T})$ as $A^\diamond(\mathbb{T}, \Theta, \mathbb{P}) = \{t \in \mathbb{T} : A^c \cup U^c \in \mathbb{P}$ for all $U \in \Theta(t)\}$for any set $A \subseteq \mathbb{T}$. We're going to utilize the symbol $A^\diamond_{\mathbb{P}}$ or $A^\diamond$ to denote $A^\diamond(\mathbb{T}, \Theta, \mathbb{P})$.

Definition 2.7. ([1]) Let $(\mathbb{T}, \Theta, \mathbb{P})$ be a *PTS*. We provide a function $cl^\diamond : \mathcal{P}(\mathbb{T}) \to \mathcal{P}(\mathbb{T})$ as $cl^\diamond(V) = V \cup V^\diamond$, where $V \subseteq \mathbb{T}$.

Definition 2.8. ([1]) Let $(\mathbb{T}, \Theta, \mathbb{P})$ be a *PTS*. The definition of $\Theta^\diamond$ is given as $\Theta^\diamond = \{V \subseteq \mathbb{T} : cl^\diamond(V^c) = V^c\}$. It is a topology on $\mathbb{T}$ induced by topology Θ and primal $\mathbb{P}$ and $\Theta \subseteq \Theta^\diamond$. The elements of $\Theta^\diamond$ are called $\Theta^\diamond$-open and the complement of a $\Theta^\diamond$-open set is called $\Theta^\diamond$-closed. It is clear that if $V^\diamond \subseteq V$, then V is $\Theta^\diamond$-closed more about $\Theta^\diamond$ can be found in ([1, 4, 5]).

Theorem 2.9. ([1]) *Assume that a PTS is* $(\mathbb{T}, \Theta, \mathbb{P})$. *For any two subsets of V and U of* $\mathbb{T}$, *then the following claims are hold:*

(i) *if* $V^c \in \Theta$, *then* $V^\diamond \subseteq V$,
(ii) $\emptyset^\diamond = \emptyset$,
(ii) $cl(V^\diamond) = V^\diamond$,
(iv) $(V^\diamond)^\diamond \subseteq V^\diamond$,
(v) *if* $V \subseteq U$, *then* $V^\diamond \subseteq U^\diamond$,
(vi) $V^\diamond \cup U^\diamond = (V \cup U)^\diamond$,
(vii) $(V \cap U)^\diamond \subseteq V^\diamond \cap U^\diamond$.

Lemma 2.10. ([1, 5]) *Let* $(\mathbb{T}, \Theta, \mathbb{P})$ *be a PTS for any* $V \subseteq \mathbb{T}$. *Then,*

(i) *if* $V^c \notin \mathbb{P}$, *then* $V^\diamond = \emptyset$.
(ii) $V^\diamond \setminus V$, *does not include any nonempty* $\Theta^\diamond$*-open set.*
(iii) *if* $V \notin \mathbb{P}$, *then V is* $\Theta^\diamond$*-open.*

3. A primal kernel sets

This section presents and examines the idea of a primal kernel, which is a topological space's natural generalization of a set's kernel.

Definition 3.1. Let $(\mathbb{T}, \Theta, \mathbb{P})$ be a PTS. For a subset U of $\mathbb{T}$, We provide the following definition for a primal kernel set as $\overline{\kappa}(U)(\mathbb{T}, \Theta) = \{t \in \mathbb{T} : U^c \cup F^c = (U \cap F)^c \in \mathbb{P}$ for every closed set F containing t in $\mathbb{T}\}$. To ensure there is no misunderstanding $\overline{\kappa}(U)(\mathbb{P}, \Theta)$ is briefly denoted by $\overline{\kappa}(U)$ and is called the a primal kernel of U with respect to $\mathbb{P}$ and Θ.

We note that one way to conceptualize the a primal kernel is as an operator that is $\overline{\kappa}(.) : \mathcal{P}(\mathbb{T}) \to \mathcal{P}(\mathbb{T})$ defined by $U \to \overline{\kappa}(U)$. The a primal kernel is not a Kuratowski closure operator, since in general, it dose not satisfy $U \subseteq \overline{\kappa}(U)$ for all $U \subseteq \mathbb{T}$. If it is the case that $U \subseteq \overline{\kappa}(U)$, we say that U is a subset κ-dense in itself.

Theorem 3.2. *Let $(\mathbb{T}, \Theta, \mathbb{P})$ be a PTS, then for all $V \subseteq \mathbb{T}$ the primal kernel $\overline{\kappa}(V) = \{t \in \mathbb{T} : [cl(\{t\}) \cap V]^c \in \mathbb{P}\}$.*

Proof. Let $K = \{t \in \mathbb{T} : [cl(\{t\}) \cap V]^c \in \mathbb{P}\}$ and suppose $t \notin K$, then $[cl(\{t\}) \cap V]^c \notin \mathbb{P}$. Since $cl(\{t\})$ is closed set containing t, then $t \notin \overline{\kappa}(V)$. Conversely, let $t \notin \overline{\kappa}(V)$, then there is closed set F containing t and $[F \cap V]^c \notin \mathbb{P}$. Since $cl(\{t\}) \subseteq F$ we have $[cl(\{t\}) \cap V]^c \notin \mathbb{P}$. Hence, $t \notin K$ and $\overline{\kappa}(V) = \{t \in \mathbb{T} : [cl(\{t\}) \cap V]^c \in \mathbb{P}\}$. □

Corollary 3.3. *Let $(\mathbb{T}, \Theta)$ be a TS, then for all $V \subseteq \mathbb{T}$ the kernel of V is $\kappa(V) = \{t \in \mathbb{T} : cl(\{t\}) \cap V \neq \emptyset\}$.*

Lemma 3.4. *Let $(\mathbb{T}, \Theta)$ be a topological space, $\mathbb{P}$ and $\mathcal{J}$ be primals on $\mathbb{T}$, and let U and V be subsets of $\mathbb{T}$. Then the following characteristics are true:*

1. *If $V \subseteq U$, then $\overline{\kappa}(V) \subseteq \overline{\kappa}(U)$.*
2. *If $\mathbb{P} \subseteq \mathcal{J}$, then $\overline{\kappa}(V)(\mathbb{P}) \supseteq \overline{\kappa}(V)(\mathcal{J})$.*
3. *$\overline{\kappa}(V) = \kappa(\overline{\kappa}(V)) \subseteq \kappa(V)$ (i.e. $\overline{\kappa}(V)$ is a κ-set).*
4. *If $V \subseteq \overline{\kappa}(V)$, then $\overline{\kappa}(V) = \kappa(V)$.*
5. *If $V^c \notin \mathbb{P}$, then $\overline{\kappa}(V) = \emptyset$.*
6. *If $V^c \in \Theta$, then $\overline{\kappa}(V) \subseteq V$.*

Proof. (1) Suppose that $t \notin \overline{\kappa}(U)$. Then there is a closed set F such that $t \in F$ and $(U \cap F)^c \notin \mathbb{P}$. Since $(U \cap F)^c \subseteq (V \cap F)^c$, thus $(V \cap F)^c \notin \mathbb{P}$. Hence $t \notin \overline{\kappa}(V)\}$. Thus $\mathbb{T} \setminus \overline{\kappa}(U) \subseteq \mathbb{T} \setminus \overline{\kappa}(V)$ or $\overline{\kappa}(V) \subseteq \overline{\kappa}(U)$.

(2) Suppose that $t \notin \overline{\kappa}(V)(\mathbb{P})$. Thus, there is closed set F containing t such that $(V \cap F)^c \notin \mathbb{P}$. Since $\mathbb{P} \subseteq \mathcal{J}$, $(V \cap F)^c \notin \mathcal{J}$ and $t \notin \overline{\kappa}(V)(\mathcal{J})$. Hence, $\overline{\kappa}(V)(\mathbb{P}) \supseteq \overline{\kappa}(V)(\mathcal{J})$.

(3) We have $\overline{\kappa}(V) \subseteq \kappa(\overline{\kappa}(V))$ in general. Let $t \in \kappa(\overline{\kappa}(V))$. Then $\overline{\kappa}(V) \cap F \neq \emptyset$ for each closed set F with $t \in F$. Therefore, there is some $s \in \overline{\kappa}(V) \cap F$ and F closed set containing s. Since $s \in \overline{\kappa}(V)$, $(\overline{\kappa}(V) \cap F)^c \in \mathbb{P}$ and hence $t \in \overline{\kappa}(V)$. Hence we get $\kappa(\overline{\kappa}(V)) \subseteq \overline{\kappa}(V)$ and hence $\overline{\kappa}(V) = \kappa(\overline{\kappa}(V))$. Again, let $t \in \kappa(\overline{\kappa}(V)) = \overline{\kappa}(V)$, then $(V \cap F)^c \in \mathbb{P}$ for each closed set F with $t \in F$. This implies $V \cap F \neq \emptyset$ for each closed set F with $t \in F$. Therefore, $t \in \kappa(V)$. This shows that $\overline{\kappa}(V) = \kappa(\overline{\kappa}(V)) \subseteq \kappa(V)$.

(4) For any subset V of $\mathbb{T}$, by (3) we get $\overline{\kappa}(V) = \kappa(\overline{\kappa}(V)) \subseteq \kappa(V)$. Since $V \subseteq \overline{\kappa}(V)$, $\kappa(V) \subseteq \kappa(\overline{\kappa}(V))$ and so $\overline{\kappa}(V) = \kappa(V)$.

(5) Suppose that $t \in \overline{\kappa}(V)$. Thus for any closed set F containing t, $(V \cap F)^c = V^c \cup F^c \in \mathbb{P}$. But since $V^c \notin \mathbb{P}$, $(V \cap F)^c \notin \mathbb{P}$ for some closed set F containing t and $t \notin \overline{\kappa}(V)$. This is a contradiction. Hence $\overline{\kappa}(V) = \emptyset$.

(6) Let $V^c \in \Theta$ and $t \in \overline{\kappa}(V)$. Suppose $t \notin V$. Then $t \in V^c$ and it is closed set containing t. Since $t \in \overline{\kappa}(V)$, then $V^c \cup F^c \in \mathbb{P}$ for all closed set containing t. Therefore, $\mathbb{T} = V \cup V^c = (V^c)^c \cup V^c \in \mathbb{P}$. This is contradiction with $\mathbb{T} \notin \mathbb{P}$. Hence, $\overline{\kappa}(V) \subseteq V$. □

Example 3.5. Let us consider the set $\mathbb{N}$ of natural numbers. In $\mathbb{N}$, define the topological space Θ such that $U \in \Theta$ if and only if $U = \mathbb{N}$ or $3 \notin U$. On $\mathbb{N}$, let $\mathbb{P}$ be defined as $B \in \mathbb{P}$ if and only if $3 \notin B$. Thus, $(\mathbb{N}, \Theta, \mathbb{P})$ is PTS. Let $A \subseteq \mathbb{N}$. Then, there are two cases:

Case 1. If $3 \in A$. Let $n \in \mathbb{N}$ and let V be any closed set containing n. From the definition of Θ, we know that $3 \in V$. Then, we observe hat $3 \notin A^c \cup V^c$ which implies that $A^c \cup V^c \in \mathbb{P}$ and then $n \in \overline{\kappa}(A)$. Hence, $\overline{\kappa}(A) = \mathbb{N}$.

Case 2. If $3 \notin A$. Then, we observe hat $3 \in A^c \cup V^c$ for every closed set V which implies that $A^c \cup V^c \notin \mathbb{P}$. Hence, $\overline{\kappa}(A) = \emptyset$.

$$\therefore \overline{\kappa}(A) = \begin{cases} \mathbb{N}, & \text{if } 3 \in A \\ \emptyset, & \text{if } 3 \notin A \end{cases}$$

Lemma 3.6. *Let $(\mathbb{T}, \Theta, \mathbb{P})$ be a PTS. If F is closed set, then $F \cap \overline{\kappa}(V) = F \cap \overline{\kappa}(F \cap V) \subseteq \overline{\kappa}(F \cap V)$ for any $V \subseteq \mathbb{T}$.*

Proof. Suppose that F is closed set and $t \in F \cap \overline{\kappa}(V)$. Then $t \in F$ and $t \in \overline{\kappa}(V)$. Let U be any closed set containing t. Then $U \cap F$ is closed set containing t and $[U \cap (F \cap V)]^c = [(U \cap F) \cap V]^c \in \mathbb{P}$. This shows that $t \in \overline{\kappa}(F \cap v)$ and hence we obtain $F \cap \overline{\kappa}(V) \subseteq \overline{\kappa}(F \cap V)$. Moreover, $F \cap \overline{\kappa}(V) \subseteq F \cap \overline{\kappa}(F \cap V)$ and by item (1) of Lemma 3.4 $\overline{\kappa}(F \cap V) \subseteq \overline{\kappa}(V)$ and $F \cap \overline{\kappa}(F \cap V) \subseteq F \cap \overline{\kappa}(V)$. Therefore, $F \cap \overline{\kappa}(V) = F \cap \overline{\kappa}(F \cap V)$. □

Theorem 3.7. *Let $V, U \subseteq \mathbb{T}$ and $(\mathbb{T}, \Theta, \mathbb{P})$ be a PTS. The following characteristics are true:*

1. $\overline{\kappa}(\emptyset) = \emptyset$.
2. $\overline{\kappa}(U) \cup \overline{\kappa}(V) = \overline{\kappa}(U \cup V)$.
3. $\overline{\kappa}(\overline{\kappa}(V)) \subseteq \overline{\kappa}(V)$.

Proof. (1) It is obvious to proof.

(2) According to Theorem 3.4 we have $\overline{\kappa}(V \cup U) \supseteq \overline{\kappa}(V) \cup \overline{\kappa}(U)$. Let's illustrate the inclusion in reverse, if $t \notin \overline{\kappa}(V) \cup \overline{\kappa}(U)$. Then, t neither belongs to $\overline{\kappa}(V)$ nor to $\overline{\kappa}(U)$. So, there is two closed sets F and D containing t such that $F^c \cup V^c \notin \mathbb{P}$ and $D^c \cup U^c \notin \mathbb{P}$. Since $\mathbb{P}$ is additive, $(F^c \cup V^c) \cap (D^c \cup U^c) \notin \mathbb{P}$. Additionally, since $\mathbb{P}$ is hereditary and

$$\begin{aligned}(F^c \cup V^c) \cap (D^c \cup U^c) &= [(F^c \cup V^c) \cap D^c] \cup [(F^c \cup V^c) \cap U^c] \\ &= [F^c \cap D^c] \cup [V^c \cap D^c] \cup [F^c \cap U^c] \cup [V^c \cap U^c] \\ &\subseteq [F^c \cap D^c] \cup D^c \cup F^c \cup [V^c \cap U^c] \\ &\subseteq [F \cap D]^c \cup (V \cup U)^c.\end{aligned}$$

Then, $[F \cap D]^c \cup (V \cup U)^c \notin \mathbb{P}$. Since $F \cap D$ is closed set containing t, thus $t \notin \overline{\kappa}(V \cup U)$. Hence, $\overline{\kappa}(V \cup U) \subseteq \overline{\kappa}(V) \cup \overline{\kappa}(U)$. Hence we obtain $\overline{\kappa}(V) \cup \overline{\kappa}(U) = \overline{\kappa}(V \cup U)$.

(3) Let $t \in \overline{\kappa}(\overline{\kappa}(V))$. Then for every closed set F containing t we get $(F \cap \overline{\kappa}(V))^c \in \mathbb{P}$ and hence $F \cap \overline{\kappa}(V) \neq \emptyset$. Let $s \in F \cap \overline{\kappa}(V)$. Then, F is closed set containing s and $s \in \overline{\kappa}(V)$. Hence, we have $(F \cap V)^c \in \mathbb{P}$ and $t \in \overline{\kappa}(V)$. This indicates that $\overline{\kappa}(\overline{\kappa}(V)) \subseteq \overline{\kappa}(V)$. □

Lemma 3.8. *Let $U, V \subseteq \mathbb{T}$ and $(\mathbb{T}, \Theta, \mathbb{P})$ be a PTS. Then $\overline{\kappa}(V) \setminus \overline{\kappa}(U) = \overline{\kappa}(V \setminus U) \setminus \overline{\kappa}(U)$.*

Proof. We get by Theorem 3.7 $\overline{\kappa}(V) = \overline{\kappa}[(V \setminus U) \cup (V \cap U)] = \overline{\kappa}(V \setminus U) \cup \overline{\kappa}(V \cap U) \subseteq \overline{\kappa}(V \setminus U) \cup \overline{\kappa}(U)$. Thus, $\overline{\kappa}(V) \setminus \overline{\kappa}(U) \subseteq \overline{\kappa}(V \setminus U) \setminus \overline{\kappa}(U)$. By Theorem 3.4, $\overline{\kappa}(V \setminus U) \subseteq \overline{\kappa}(V)$ and we obtain $\overline{\kappa}(V \setminus U) \setminus \overline{\kappa}(U) \subseteq \overline{\kappa}(V) \setminus \overline{\kappa}(U)$. Hence, $\overline{\kappa}(V) \setminus \overline{\kappa}(U) = \overline{\kappa}(V \setminus U) \setminus \overline{\kappa}(U)$. □

Corollary 3.9. *Let $(\mathbb{T}, \Theta, \mathbb{P})$ be a PTS and $U, V \subseteq \mathbb{T}$ with $V^c \notin \mathbb{P}$. Then, $\overline{\kappa}(U \cup V) = \overline{\kappa}(U) = \overline{\kappa}(U \setminus V)$.*

Proof. Since $V^c \notin \mathbb{P}$, by Theorem 3.4 $\overline{\kappa}(V) = \emptyset$. By Lemma 3.8, $\overline{\kappa}(U) = \overline{\kappa}(U \setminus V)$ and by Theorem 3.7 $\overline{\kappa}(U \cup V) = \overline{\kappa}(U) \cup \overline{\kappa}(V) = \overline{\kappa}(U)$. □

Theorem 3.10. *Let $(\mathbb{T}, \Theta, \mathbb{P})$ be a PTS, so the subsequent properties are equivalent:*

1. $\Theta \setminus \{\mathbb{T}\} \subseteq \mathbb{P}$;
2. *If $V^c \notin \mathbb{P}$, then $co\kappa(V) = \emptyset$;*
3. *$V \subseteq \overline{\kappa}(V)$ for each closed set V in $\mathbb{T}$;*
4. $\mathbb{T} = \overline{\kappa}(\mathbb{T})$.

Proof. (1) ⇒ (2): Let $V^c \notin \mathbb{P}$ and $\Theta \setminus \{\mathbb{T}\} \subseteq \mathbb{P}$. Assume $t \in co\kappa(V)$. Then, there is a closed set F in $\mathbb{T}$ such that $t \in F \subseteq V$ and $V^c \subseteq F^c$. Since $V^c \notin \mathbb{P}$, then $F^c \notin \mathbb{P}$. This is not the case that $\Theta \setminus \{\mathbb{T}\} \subseteq \mathbb{P}$. Hence, $co\kappa(V) = \emptyset$.

(2)⇒ (3): Let V be closed set and $t \in V$. Suppose $t \notin \overline{\kappa}(V)$ then there is a closed set F containing t such that $V^c \cup F^c \notin \mathbb{P}$ and hence $(V \cap F)^c \notin \mathbb{P}$, thus by item (2) $co\kappa(V \cap F) = \emptyset$. Also $V \cap F$ is closed, then $co\kappa(V \cap F) = V \cap F \neq \emptyset$. This is a logical contradiction. Hence, $t \in \overline{\kappa}(V)$ and $V \subseteq \overline{\kappa}(V)$ for every closed set V.

(3)⇒ (4): Since $\mathbb{T}$ is closed, so we get $\mathbb{T} = \overline{\kappa}(\mathbb{T})$.

(4)⇒ (1): $\mathbb{T} = \overline{\kappa}(\mathbb{T}) = \{t \in \mathbb{T} : F^c \cup \mathbb{T}^c = F^c \in \mathbb{P}$ for each closed set F containing $t\}$. Hence, $\Theta \setminus \{\mathbb{T}\} \subseteq \mathbb{P}$. □

Remark 3.11. Let $(\mathbb{T}, \Theta, \mathbb{P})$ be a PTS. For a proper subset V of $\mathbb{T}$ the following hold:

1. If $\mathbb{P} = \emptyset$, then $\overline{\kappa}(V) = \emptyset$.
2. If $\mathbb{P} = \mathcal{P}(\mathbb{T}) \setminus \mathbb{T}$, then $\overline{\kappa}(V) = \kappa(V)$.

Examples of how the ideas relate to one another are as follows:

Example 3.12. Let $\mathbb{T} = \{3, 2, 1\}$. Define $\Theta = \{\emptyset, \mathbb{T}, \{1\}, \{1, 2\}\}$ and $\mathbb{P} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. If $(\mathbb{T}, \Theta, \mathbb{P})$ be a *PTS* and $V \subseteq \mathbb{T}$, then we have the following table.

V	$\kappa(V)$ kernel	$co\kappa(V)$ co-kernel	$\overline{\kappa}(V)$ primal-kernel
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\{1\}$	$\{1\}$	$\emptyset$	$\emptyset$
$\{2\}$	$\{1, 2\}$	$\emptyset$	$\emptyset$
$\{3\}$	$\mathbb{T}$	$\{3\}$	$\mathbb{T}$
$\{1, 2\}$	$\{1, 2\}$	$\emptyset$	$\emptyset$
$\{1, 3\}$	$\mathbb{T}$	$\{3\}$	$\mathbb{T}$
$\{2, 3\}$	$\mathbb{T}$	$\{2, 3\}$	$\mathbb{T}$

Example 3.13. Let $\mathbb{T} = \{3, 2, 1\}$. Define $\Theta = \{\emptyset, \mathbb{T}, \{1\}, \{1, 2\}\}$ and $\mathbb{P} = \emptyset$. If $(\mathbb{T}, \Theta, \mathbb{P})$ is a *PTS* and $V \subseteq \mathbb{T}$, then we have the following table.

V	$\kappa(V)$ kernel	$co\kappa(V)$ co-kernel	$\overline{\kappa}(V)$ primal-kernel
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\{1\}$	$\{1\}$	$\emptyset$	$\emptyset$
$\{2\}$	$\{1, 2\}$	$\emptyset$	$\emptyset$
$\{3\}$	$\mathbb{T}$	$\{3\}$	$\emptyset$
$\{1, 2\}$	$\{1, 2\}$	$\emptyset$	$\emptyset$
$\{1, 3\}$	$\mathbb{T}$	$\{3\}$	$\emptyset$
$\{2, 3\}$	$\mathbb{T}$	$\{2, 3\}$	$\emptyset$

Example 3.14. Let $\mathbb{T} = \{3, 2, 1\}$. Define $\Theta = \{\emptyset, \mathbb{T}, \{1\}, \{1, 2\}\}$ and $\mathbb{P} = \mathcal{P}(\mathbb{T}) \setminus \{\mathbb{T}\}$. If $(\mathbb{T}, \Theta, \mathbb{P})$ be a *PTS* and $V \subseteq \mathbb{T}$, then we have the following table.

V	$\kappa(V)$ kernel	$co\kappa(V)$ co-kernel	$\overline{\kappa}(V)$ primal-kernel
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\{1\}$	$\{1\}$	$\emptyset$	$\{1\}$
$\{2\}$	$\{1, 2\}$	$\emptyset$	$\{1, 2\}$
$\{3\}$	$\mathbb{T}$	$\{3\}$	$\mathbb{T}$
$\{1, 2\}$	$\{1, 2\}$	$\emptyset$	$\{1, 2\}$
$\{1, 3\}$	$\mathbb{T}$	$\{3\}$	$\mathbb{T}$
$\{2, 3\}$	$\mathbb{T}$	$\{2, 3\}$	$\mathbb{T}$

4. A topology associated with a primal kernel sets

Theorem 4.1. *Let $(\mathbb{T}, \Theta, \mathbb{P})$ be a PTS, then $\overline{V}^{\kappa} = \overline{\kappa}(V) \cup V$ is a Kuratowski operator that is if V, U be subsets of $\mathbb{T}$. Then*

1. $\overline{\emptyset}^{\kappa} = \emptyset$.
2. $V \subseteq \overline{V}^{\kappa}$.
3. $\overline{V \cup U}^{\kappa} = \overline{V}^{\kappa} \cup \overline{U}^{\kappa}$.
4. $\overline{V}^{\kappa} = \overline{\overline{V}^{\kappa}}^{\kappa}$.

Proof. By Theorem 3.7, we obtain

(1) $\overline{\emptyset}^{\kappa} = \overline{\kappa}(\emptyset) \cup \emptyset = \emptyset$.

(2) $V \subseteq V \cup \overline{\kappa}(V) = \overline{V}$.

(3) $\overline{V \cup U}^{\kappa} = \overline{\kappa}(V \cup U) \cup (V \cup U) = \overline{\kappa}(V) \cup \overline{\kappa}(U) \cup (V \cup U) = \overline{V}^{\kappa} \cup \overline{U}^{\kappa}$.

(4) $\overline{\overline{V}^{\kappa}}^{\kappa} = \overline{\overline{\kappa}(V) \cup V}^{\kappa} = \overline{\kappa}(\overline{\kappa}(V) \cup V) \cup (\overline{\kappa}(V) \cup V) = \overline{\kappa}(\overline{\kappa}(V)) \cup \overline{\kappa}(V) \cup (\overline{\kappa}(V) \cup V) = \overline{\kappa}(V) \cup V = \overline{V}^{\kappa}$. □

Lemma 4.2. *Let $(\mathbb{T}, \Theta, \mathbb{P})$ be a PTS and V, U be subsets of $\mathbb{T}$. Then*

1. *If $V \subseteq U$, then $\overline{V}^{\kappa} \subseteq \overline{U}^{\kappa}$.*
2. $\overline{(V \cap U)}^{\kappa} \subseteq \overline{V}^{\kappa} \cap \overline{U}^{\kappa}$.
3. *When F is a closed set, then $F \cap \overline{V}^{\kappa} \subseteq \overline{V \cap F}^{\kappa}$.*

Proof. (1) Since $V \subseteq U$, by Lemma 3.4 we have $\overline{V}^{\kappa} = V \cup \overline{\kappa}(V) \subseteq U \cup \overline{\kappa}(U) = \overline{U}^{\kappa}$.

(2) This is obvious by (1).

(3) Since F is closed set, by Lemma 3.6 we have $F \cap \overline{V}^{\kappa} = F \cap (V \cup \overline{\kappa}(V)) = (F \cap V) \cup (F \cap \overline{\kappa}(V)) \subseteq (F \cap V) \cup \overline{\kappa}(F \cap V) = \overline{F \cap V}^{\kappa}$. □

Corollary 4.3. *Let $(\mathbb{T}, \Theta, \mathbb{P})$ be a PTS and $V \subseteq \mathbb{T}$. If $V \subseteq \overline{\kappa}(V)$, then $\kappa(V) = \overline{V}^{\kappa}$.*

According with Theorem 4.1, if $(\mathbb{T}, \Theta, \mathbb{P})$ is a PTS, we denote by $\overline{\Theta}^{\kappa}$ the topology generated by $\overline{(.)}^{\kappa}$, that is $\overline{\Theta}^{\kappa} = \{V \subseteq \mathbb{T} : \overline{\mathbb{T} \setminus V}^{\kappa} = \mathbb{T} \setminus V\}$. The elements of $\overline{\Theta}^{\kappa}$ are called $\overline{\kappa}$-open and the complement of $\overline{\kappa}$-open is called $\overline{\kappa}$-closed.

Remark 4.4. In Example 3.5 we consider the set $\mathbb{N}$ of natural numbers. In $\mathbb{N}$, define the topological space Θ such that $U \in \Theta$ if and only if $U = \mathbb{N}$ or $3 \notin U$. On $\mathbb{N}$, let $\mathbb{P}$ be defined as $B \in \mathbb{P}$ if and only if $3 \notin B$. Thus, $(\mathbb{N}, \Theta, \mathbb{P})$ is PTS. Let $A \subseteq \mathbb{N}$. Then,

$$\overline{A}^{\kappa} = \begin{cases} \mathbb{N}, & \text{if } 3 \in A \\ A, & \text{if } 3 \notin A \end{cases}$$

Hence, A is $\overline{\kappa}$-closed set if and only if $3 \notin A$.

In general the following topology Θ and $\overline{\Theta}^{\kappa}$ are incomparable as we can see in below example

Example 4.5. Let $\mathbb{T} = \{3, 2, 1\}$ with $\Theta = \{\emptyset, \mathbb{T}, \{1, 3\}\}$ and the primal $\mathbb{P} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 3\}, \{2, 3\}\}$.

V	$\kappa(V)$ kernel	$co\kappa(V)$ co-kernel	$\overline{\kappa}(V)$ primal-kernel	$\overline{V}^{\kappa}$	$V^{\diamond}$	$cl^{\diamond}(V)$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\{1\}$	$\{1,3\}$	$\emptyset$	$\{1,3\}$	$\{1,3\}$	$\{2\}$	$\{1,2\}$
$\{2\}$	$\mathbb{T}$	$\{2\}$	$\mathbb{T}$	$\mathbb{T}$	$\{2\}$	$\{2\}$
$\{3\}$	$\{1,3\}$	$\emptyset$	$\emptyset$	$\{3\}$	$\emptyset$	$\{3\}$
$\{1,2\}$	$\mathbb{T}$	$\{2\}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\{1,3\}$	$\{1,3\}$	$\emptyset$	$\{1,3\}$	$\{1,3\}$	$\mathbb{T}$	$\mathbb{T}$
$\{2,3\}$	$\mathbb{T}$	$\{2\}$	$\mathbb{T}$	$\mathbb{T}$	$\{2\}$	$\{2,3\}$

By above table we obtain $\overline{\Theta}^{\kappa} = \{\emptyset, \mathbb{T}, \{2\}, \{1,2\}\}$ and it clear that Θ and $\overline{\Theta}^{\kappa}$ are incomparable topology. Also $\Theta^{\diamond} = \{\emptyset, \mathbb{T}, \{1\}, \{1,2\}, \{1,3\}\}$

Remark 4.6.

1. Since $\overline{\kappa}(V) = \kappa(\overline{\kappa}(V)) \subseteq \kappa(V)$, then $\overline{V}^{\kappa} \subseteq \kappa(V)$ for all subset V of $\mathbb{T}$. Hence, if V is a κ-set, then V is $\overline{\kappa}$-closed. It follows that each $co\kappa$-set is $\overline{\kappa}$-open.
2. If $\mathbb{P} = \mathcal{P}(\mathbb{T}) \setminus \mathbb{T}$, then $\overline{\Theta}^{\kappa}$ is the collection of all $co\kappa$-set.
3. According to [4], if V is a subset of $\mathbb{T}$ considering $\mathbb{P}$ and Θ, the primal local closure operator of V is $\Pi(V) = \Pi(V)(\mathbb{P}, \Theta) = \{t \in \mathbb{T} : V^c \cup (\overline{F})^c \in \mathbb{P}$ for every $F \in \Theta(t)\}$, where $\Theta(t) = \{F \in \Theta : t \in F\}$. It is clear that $\overline{\kappa}(V) \subseteq \Pi(V) \subseteq cl_{\theta}(V)$. Hence, it follows that if V is a θ-closed set, then V is $\overline{\kappa}$-closed and every θ-open set, then is $\overline{\kappa}$-open.

Proposition 4.7. *A subset V of a PTS $(\mathbb{T}, \Theta, \mathbb{P})$ is $\overline{\kappa}$-closed if and only if $\overline{\kappa}(V) \subseteq V$.*

Proof. Let V be $\overline{\kappa}$-closed, then $V = \overline{V}^{\kappa} = \overline{\kappa}(V) \cup V$. Hence, $\overline{\kappa}(V) \subseteq V$.

Conversely, let $\overline{\kappa}(V) \subseteq V$. Since, $\overline{V}^{\kappa} = \overline{\kappa}(V) \cup V$ and $\overline{\kappa}(V) \cup V \subseteq V$, then $\overline{V}^{\kappa} \subseteq V$ and $\overline{V}^{\kappa} = V$. This show that V is $\overline{\kappa}$-closed. □

Theorem 4.8. *Let $(\mathbb{T}, \Theta, \mathbb{P})$ be a PTS. Then $\beta(\Theta, \mathbb{P}) = \{V \cap P : V$ is a closed set and $P \notin \mathbb{P}\}$ is a basis for $\overline{\Theta}^{\kappa}$.*

Proof. Let $(\mathbb{T}, \Theta, \mathbb{P})$ be a PTS. It is clear that V is $\overline{\kappa}$-closed if and only if $\overline{\kappa}(V) \subseteq V$. Currently, we have $U \in \overline{\Theta}^{\kappa}$ if and only if $\overline{\kappa}(U^c) \subseteq U^c$ if and only if $U \subseteq [\overline{\kappa}(U^c)]^c$. Therefore, $t \in U$ implies that $t \notin \overline{\kappa}(U^c)$. This follows that there is a closed set F such that $t \in F$ and $[F \cap U^c]^c \notin \mathbb{P}$. Now let $B = F \cap [F^c \cup U] \in \beta$ and we have $t \in B \subseteq U$, where F is a closed and $t \in F$ and $F^c \cup U \notin \mathbb{P}$. All that remains to be shown is that β is closed under finite intersection. Let $M, N \in \beta$, then $M = H \cap P_1$ and $N = K \cap P_2$, where H, K are closed sets and $P_1, P_2 \notin \mathbb{P}$. Hence, we obtain

$$M \cap N = (H \cap P_1) \cap (K \cap P_2) = [H \cap K] \cap (P_1 \cap P_2).$$

Since $(P_1 \cap P_2) \notin \mathbb{P}$ and $H \cap K$ is closed set, hence $M \cap N \in \beta$. Hence, the finite intersection in β is closed. Thus $\beta = \{V \cap P : V$ is a closed set and $P \notin \mathbb{P}\}$ is a basis for $\overline{\Theta}^{\kappa}$. □

Example 4.9. Assume that the real numbers with a left ray topology are $\Theta = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$. Let $\mathbb{P}$ be a primal of all subsets A of $\mathbb{R}$ such that $\mathbb{R} \setminus A$ is infinite. Consider the collection $\beta(\Theta, \mathbb{P}) = \{F^n \cap P^n : n \in \mathbb{N}\}$, where $F^n = [\frac{1}{2} - n, \infty)$ which is closed set and $P^n = \mathbb{R} \setminus \{1, 2, ..., n\} \notin \mathbb{P}$. Then $\cup\{F^n \cap P^n : n \in \mathbb{N}\} = \mathbb{R} \setminus \mathbb{N}$ which is not in $\beta(\Theta, \mathbb{P})$. Thus $\beta(\Theta, \mathbb{P})$ is not a topology in general as it is not closed under arbitrary unions.

Lemma 4.10. *Let $(\mathbb{T}, \Theta, \mathbb{P})$ be a PTS and V be subset of $\mathbb{T}$. Then,*

1. *$V \in \overline{\Theta}^{\kappa}$ if and only if for all $t \in V$, a closed set F that contains t exists such that $V \cup F^c \notin \mathbb{P}$.*
2. *if $V \notin \mathbb{P}$, then $V \in \overline{\Theta}^{\kappa}$.*

Proof. (1) Let $V \in \overline{\Theta}^{\kappa}$ if and only if $\overline{V^c}^{\kappa} = \overline{\kappa}(V^c) \cup V^c = V^c$, then $\overline{\kappa}(V^c) \subseteq V^c$ and $V \subseteq [\overline{\kappa}(V^c)]^c$. Hence, for all $t \in V$, we have $t \notin \overline{\kappa}(V^c)$. Thus, for all $t \in V$, there is a closed set F containing t such that $F^c \cup (V^c)^c = F^c \cup V \notin \mathbb{P}$.

(2) Let $V \notin \mathbb{P}$ and $t \in V$. Put $F = \mathbb{T}$. Then F is a closed set containing t. Since $V \notin \mathbb{P}$ and $F^c \cup V = V$, we have $F^c \cup V \notin \mathbb{P}$. By item (1), we have $V \in \overline{\Theta}^{\kappa}$. □

Proposition 4.11. *Let $(\mathbb{T}, \Theta, \mathbb{P})$ be a PTS and $V \subseteq \mathbb{T}$. Then, $\overline{\kappa}(V) \setminus V$ does not contain any nonempty $\overline{\kappa}$-open set.*

Proof. Suppose that $V \subseteq \mathbb{T}$ and U is $\overline{\kappa}$-open set such that $U \subseteq \overline{\kappa}(V) \setminus V$. Then, $U \subseteq \overline{\kappa}(V) \setminus V \subseteq \mathbb{T} \setminus V$, $V \subseteq \mathbb{T} \setminus U$ and $\mathbb{T} \setminus U$ is $\overline{\kappa}$-closed set. By Lemma 3.4 and Proposition 4.7, we have $\overline{\kappa}(V) \subseteq \overline{\kappa}(\mathbb{T} \setminus U) \subseteq \mathbb{T} \setminus U$ and hence $U \subseteq \mathbb{T} \setminus \overline{\kappa}(V)$ since $U \subseteq \overline{\kappa}(V)$ we obtain that $U \subseteq \overline{\kappa}(V) \cap (\mathbb{T} \setminus \overline{\kappa}(V)) = \emptyset$. Thus, $U = \emptyset$ and we get, $\overline{\kappa}(V) \setminus V$ does not contain any nonempty $\overline{\kappa}$-open set. □

Corollary 4.12. *Let $(\mathbb{T},\Theta)$ be a TS and $V \subseteq \mathbb{T}$. Then, $\kappa(V) \setminus V$ does not contain any nonempty coκ-set.*

Theorem 4.13. *Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be primals on a TS $(\mathbb{T},\Theta)$ such that $\mathbb{P}_2 \subseteq \mathbb{P}_1$, then $\overline{\Theta}^{\kappa}(\mathbb{P}_1) \subseteq \overline{\Theta}^{\kappa}(\mathbb{P}_2)$.*

Proof. Let $V \in \overline{\Theta}^{\kappa}(\mathbb{P}_1)$. We will show that $U = \mathbb{T} \setminus V$ is $\overline{\kappa}_{\mathbb{P}_2}$-closed that is $\overline{\kappa}_{\mathbb{P}_2}(U) \subseteq U$. Suppose that $t \notin U$, then $t \notin \overline{\kappa}_{\mathbb{P}_1}(U)$ because U is $\overline{\kappa}_{\mathbb{P}_1}$-closed set. Next, a closed set F exists such that $t \in F$ and $(U \cap F)^c \notin \mathbb{P}_1$. Since $\mathbb{P}_2 \subseteq \mathbb{P}_1$ we get $(U \cap F)^c \notin \mathbb{P}_2$ and hence $t \notin \overline{\kappa}_{\mathbb{P}_2}(U)$ this mean that $\overline{\kappa}_{\mathbb{P}_2}(U) \subseteq U$ and U is $\overline{\kappa}_{\mathbb{P}_2}$-closed set. Hence, $\overline{\Theta}^{\kappa}(\mathbb{P}_1) \subseteq \overline{\Theta}^{\kappa}(\mathbb{P}_2)$. □

Theorem 4.14. *If a PTS $(\mathbb{T},\Theta,\mathbb{P})$ is T_1-space, then for all $t \in \mathbb{T}$, the singleton $\{t\}$ is $\overline{\kappa}$-closed. The converse is true if each singleton is κ-dense in itself.*

Proof. Let t be any point in $\mathbb{T}$. For any $s \in \mathbb{T}$, $t \neq s$ an open set V exists such that $t \in V$ and $s \notin V$. Now, $s \in F = \mathbb{T} \setminus V$ and F is closed and $\{t\} \cap F \subseteq V \cap F = \emptyset$. Hence $(\{t\} \cap F)^c \notin \mathbb{P}$ and so $s \notin \overline{\kappa}(\{t\})$. This show that $\overline{\kappa}(\{t\}) \subseteq \{t\}$ and hence $\{t\}$ is $\overline{\kappa}$-closed by Proposition 4.7. Conversely, let each singleton is $\overline{\kappa}$-closed and κ-dense in itself. Let t be any point of $\mathbb{T}$ and $s \in \mathbb{T} \setminus \{t\}$. Then, $\overline{\kappa}(\{s\}) = \{s\} \subseteq \mathbb{T} \setminus \{t\}$ and hence $t \notin \overline{\kappa}(\{s\})$, then there is a closed set F such that $t \in F$ and $(\{s\} \cap F)^c \notin \mathbb{P}$. We claim that $\{s\} \cap F = \emptyset$ and $s \in F^c \subseteq \mathbb{T} \setminus \{t\} = V$ and F^c is open set. Hence, $\{t\}$ is a closed set and $\mathbb{T} \setminus \{t\}$ is an open set and thus $(\mathbb{T},\Theta,\mathbb{P})$ is a T_1-space. Otherwise we have $\{s\} \cap F \neq \emptyset$, so $s \in F$ and $(\{s\} \cap F)^c \notin \mathbb{P}$. It follows that $s \notin \overline{\kappa}(\{s\})$ which is a contradiction with the fact $\overline{\kappa}(\{s\}) = \{s\}$. □

Corollary 4.15. *If TS $(\mathbb{T},\Theta)$ is T_1-space if and only if for each $t \in \mathbb{T}$, the singleton $\{t\}$ is a κ-set.*

5. g_κ-closed sets in primal spaces

Definition 5.1. A subset V of a PTS $(\mathbb{T},\Theta,\mathbb{P})$ is called g_κ-closed if $cl^\diamond(V) \subseteq U$ whenever $V \subseteq U$ and U is $\overline{\kappa}$-open.

Remark 5.2. In Example 3.5 let $A \subseteq \mathbb{N}$. Then,

$$A^\diamond = \begin{cases} \{3\}, & \text{if } 3 \in A \\ \emptyset, & \text{if } 3 \notin A \end{cases}$$

Hence, $cl^\diamond(A) = A$ for all $A \subseteq \mathbb{N}$, thus A is g_κ-closed for all $A \subseteq \mathbb{N}$.

Remark 5.3. In Example 4.5 we obtained that $\overline{\Theta}^{\kappa} = \{\emptyset, \mathbb{T}, \{2\}, \{1,2\}\}$ and $\Theta^\diamond = \{\emptyset, \mathbb{T}, \{1\}, \{1,2\}, \{1,3\}\}$. Let $U = \{1,2\}$ be $\overline{\kappa}$-open and $V = \{2\}$ it is clearly that $V \subseteq U$ and $cl^\diamond(V) = \mathbb{T} \nsubseteq U$. Hence, V is not g_κ-closed.

Proposition 5.4. *A subset V of a PTS $(\mathbb{T},\Theta,\mathbb{P})$ is g_κ-closed if and only if $V^\diamond \subseteq U$ whenever $V \subseteq U$ and U is $\overline{\kappa}$-open.*

Proof. Let V be g_κ-closed and $V \subseteq U$ where U is $\overline{\kappa}$-open, then $cl^\diamond(V) \subseteq U$ and since $V^\diamond \subseteq cl^\diamond(V)$ for all $V \subseteq \mathbb{T}$. Hence, $V^\diamond \subseteq U$ whenever $V \subseteq U$ and U is $\overline{\kappa}$-open. Conversely, let $V \subseteq U$ and $V^\diamond \subseteq U$ whenever U is $\overline{\kappa}$-open, then $V \cup V^\diamond = cl^\diamond(V) \subseteq U$. Thus, V is g_κ-closed. □

Proposition 5.5. *Let V and U be a subset of a PTS $(\mathbb{T},\Theta,\mathbb{P})$. The listed below items are hold:*

(i) If V is $\Theta^\diamond$-closed, then V is g_κ-closed.
(ii) If V is g_κ-closed and $\overline{\kappa}$-open, then V is $\Theta^\diamond$-closed.
(iii) If V is g_κ-closed and $V \subseteq U \subseteq V^\diamond$, then U is g_κ-closed.

Proof. (i) Let V be $\Theta^\diamond$-closed and $V \subseteq U$ where U is $\overline{\kappa}$-open. Then, $V = cl^\diamond(V) = V^\diamond \cup V$, we get $V^\diamond \subseteq V$ and $V^\diamond \subseteq V \subseteq U$. Therefore, $V^\diamond \subseteq U$ and thus V is g_κ-closed.

(ii) Let V be g_κ-closed and $\overline{\kappa}$-open. Since, $V \subseteq V$ and V is $\overline{\kappa}$-open by Proposition 5.4, $V^\diamond \subseteq V$ and hence V is $\Theta^\diamond$-closed.

(iii) Let V be g_κ-closed, $V \subseteq U \subseteq V^\diamond$, and $U \subseteq W$ where, W is $\overline{\kappa}$-open. Then, $V \subseteq W$ and hence $V^\diamond \subseteq W$ also $U^\diamond \subseteq (V^\diamond)^\diamond \subseteq V^\diamond \subseteq W$. Hence, U is g_κ-closed. □

Theorem 5.6. *A subset V of a PTS $(\mathbb{T}, \Theta, \mathbb{P})$ is g_κ-closed if and only if $V^\diamond \subseteq \kappa_{\overline{\Theta}^\kappa}(V)$ where, $\kappa_{\overline{\Theta}^\kappa}(V)$ is the kernel of V in the topology $\overline{\Theta}^\kappa$.*

Proof. Suppose that V is g_κ-closed, let $V \subseteq U$ where U is $\overline{\kappa}$-open. Then, $cl^\diamond(V) \subseteq U$ and since U is $\overline{\kappa}$-open containing V we get $cl^\diamond(V) \subseteq \kappa_{\overline{\Theta}^\kappa}(V)$.

Conversely, suppose that $cl^\diamond(V) \subseteq \kappa_{\overline{\Theta}^\kappa}(V)$ and U is $\overline{\kappa}$-open containing V. Then, we get that $V \subseteq \kappa_{\overline{\Theta}^\kappa}(V) \subseteq \kappa_{\overline{\Theta}^\kappa}(U) = U$. Hence, $cl^\diamond(V) \subseteq U$ and V is g_κ-closed. □

Theorem 5.7. *If a subset V of a PTS $(\mathbb{T}, \Theta, \mathbb{P})$ is g_κ-closed, then $V^\diamond \setminus V$ does not contain any nonempty $\overline{\kappa}$-closed set.*

Proof. Let V be g_κ-closed set and suppose and F is nonempty $\overline{\kappa}$-closed set such that $F \subseteq V^\diamond \setminus V$. Then, $F \subseteq V^\diamond \setminus V \subseteq \mathbb{T} \setminus V$, $V \subseteq \mathbb{T} \setminus F$ and $\mathbb{T} \setminus F$ is $\overline{\kappa}$-open set. By Lemma 2.9 and Proposition 5.4, we have $V^\diamond \subseteq (\mathbb{T} \setminus F)^\diamond \subseteq \mathbb{T} \setminus F$ and hence $F \subseteq \mathbb{T} \setminus V^\diamond$ since $F \subseteq V^\diamond$ we obtain that $F \subseteq V^\diamond \cap (\mathbb{T} \setminus V^\diamond) = \emptyset$. Thus, $F = \emptyset$ and we get, $V^\diamond \setminus V$ does not contain any non empty $\overline{\kappa}$-closed set. □

Theorem 5.8. *Let V be a g_κ-closed subset of a PTS $(\mathbb{T}, \Theta, \mathbb{P})$. Then, the items below are equivalent:*

1. *V is $\Theta^\diamond$-closed.*
2. *$V^\diamond \setminus V = \emptyset$.*
3. *$V^\diamond \setminus V$ is $\overline{\kappa}$-closed.*

Proof. (1) ⇔ (2) By Definition 2.7 V is $\Theta^\diamond$-closed set if and only if $V^\diamond \subseteq V$, which is equivalently $V^\diamond \setminus V = \emptyset$.

(2) ⇒ (3) It is obvious since $\overline{\Theta}^\kappa$ is a topology.

(3) ⇒ (1) Suppose that $V^\diamond \setminus V$ is $\overline{\kappa}$-closed set. Since V is a g_κ-closed, by Theorem 5.7, we get that $V^\diamond \setminus V = \emptyset$. □

Lemma 5.9. *Let $(\mathbb{T}, \Theta, \mathbb{P})$ be a PTS, for all $t \in \mathbb{T}$, the singleton $\{t\}$ is $\overline{\kappa}$-closed or g_κ-open.*

Proof. Let $\{t\}$ be not $\overline{\kappa}$-closed, then $\mathbb{T} \setminus \{t\}$ is $\overline{\kappa}$-open and the only $\overline{\kappa}$-open set containing $\mathbb{T} \setminus \{t\}$ is $\mathbb{T}$. Since $cl^\diamond(\mathbb{T} \setminus \{t\}) \subseteq \mathbb{T}$, then $\mathbb{T} \setminus \{t\}$ is a g_κ-closed set and hence $\{t\}$ is g_κ-open. □

Definition 5.10. A PTS $(\mathbb{T}, \Theta, \mathbb{P})$ is called $\overline{\kappa}$-$T_{\frac{1}{2}}$ if each g_κ-closed set of $\mathbb{T}$ is $\Theta^\diamond$-closed.

Theorem 5.11. *The PTS $(\mathbb{T}, \Theta, \mathbb{P})$ is $\overline{\kappa}$-$T_{\frac{1}{2}}$ if and only if for each $t \in \mathbb{T}$ the singleton $\{t\}$ is $\overline{\kappa}$-closed or $\Theta^\diamond$-open.*

Proof. Suppose that $(\mathbb{T}, \Theta, \mathbb{P})$ is $\overline{\kappa}$-$T_{\frac{1}{2}}$ space and let $t \in \mathbb{T}$. If $\{t\}$ is not $\overline{\kappa}$-closed, then by Lemma 5.9, $\{t\}$ is g_κ-open and hence $\mathbb{T} \setminus \{t\}$ is g_κ-closed set. Since $(\mathbb{T}, \Theta, \mathbb{P})$ is $\overline{\kappa}$-$T_{\frac{1}{2}}$ we get that $\mathbb{T} \setminus \{t\}$ is $\Theta^\diamond$-closed set. Hence, $\{t\}$ is $\Theta^\diamond$-open. Conversely, let V be a g_κ-closed set and $t \in V^\diamond$. Next, there are two cases:

Case 1: $\{t\}$ is $\overline{\kappa}$-closed set. Then, by Theorem 5.8, $V^\diamond \setminus V$ does not contain any nonempty $\overline{\kappa}$-closed set and hence $t \notin V^\diamond \setminus V$. Since $t \in V^\diamond$ we get that $t \in V$. Thus, $V^\diamond \subseteq V$ and V is $\Theta^\diamond$-closed set and in this case $(\mathbb{T}, \Theta, \mathbb{P})$ is $\overline{\kappa}$-$T_{\frac{1}{2}}$ space.

Case 2: $\{t\}$ is $\Theta^\diamond$-open set. Then, by Lemma 2.10, $V^\diamond \setminus V$ does not contain any nonempty $\Theta^\diamond$-open set and hence $t \notin V^\diamond \setminus V$. Since $t \in V^\diamond$ we conclude that $t \in V$. Thus, $V^\diamond \subseteq V$ and V is $\Theta^\diamond$-closed set and in this case $(\mathbb{T}, \Theta, \mathbb{P})$ is $\overline{\kappa}$-$T_{\frac{1}{2}}$ space. □

Remark 5.12. In Example 3.5 A is $\overline{\kappa}$-closed set if and only if $3 \notin A$ and since $cl^{\diamond}(A) = A$ for all $A \subseteq \mathbb{N}$, thus A is $\Theta^{\diamond}$-open for all $A \subseteq \mathbb{N}$, therefore for each $t \in \mathbb{N}$ the singleton $\{t\}$ is $\overline{\kappa}$-closed or $\Theta^{\diamond}$-open. Hence, $(\mathbb{N}, \Theta, \mathbb{P})$ is $\overline{\kappa}$-$T_{\frac{1}{2}}$ space.

Remark 5.13. In Example 4.5 we obtained that $\overline{\Theta}^{\kappa} = \{\emptyset, \mathbb{T}, \{2\}, \{1,2\}\}$ and $\Theta^{\diamond} = \{\emptyset, \mathbb{T}, \{1\}, \{1,2\}, \{1,3\}\}$. Let $V = \{2\}$ it is clearly that V is neither $\overline{\kappa}$-closed nor $\Theta^{\diamond}$-open set. Hence, $(\mathbb{T}, \Theta, \mathbb{P})$ is not $\overline{\kappa}$-$T_{\frac{1}{2}}$ space.

6. λ_{κ}-closed sets in primal spaces

Definition 6.1. A subset V of a PTS $(\mathbb{T}, \Theta, \mathbb{P})$ is called λ_{κ}-closed if $V = F \cap U$ whenever F is a $\overline{\kappa}$-closed and U is a $\Theta^{\diamond}$-closed set.

Example 6.2. In Example 4.5 when $\mathbb{T} = \{3, 2, 1\}$ with $\Theta = \{\emptyset, \mathbb{T}, \{1,3\}\}$ and the primal $\mathbb{P} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,3\}, \{2,3\}\}$. It is clear that the collection of all $\overline{\kappa}$-closed and $\Theta^{\diamond}$-closed sets are $\{\emptyset, \mathbb{T}, \{1,3\}, \{3\}\}$ and $\{\emptyset, \mathbb{T}, \{2,3\}, \{3\}, \{2\}\}$ respectively. Thus, $\{\emptyset, \mathbb{T}, \{2,3\}, \{3\}, \{2\}, \{1,3\}\}$ is the collection of all λ_{κ}-closed sets.

Remark 6.3. Let $(\mathbb{T}, \Theta, \mathbb{P})$ be a PTS, then

1. If V is a λ-closed set, then V is a λ_{κ}-closed set, because every κ-set is $\overline{\kappa}$-closed and each closed set is $\Theta^{\diamond}$-closed.
2. If $\mathbb{P} = \mathcal{P}(\mathbb{T}) \setminus \mathbb{T}$, then V is a λ_{κ}-closed set if and only if V is a λ-closed set.

Proposition 6.4. *In a PTS $(\mathbb{T}, \Theta, \mathbb{P})$ every $\overline{\kappa}$-closed set is λ_{κ}-closed and each $\Theta^{\diamond}$-closed set is λ_{κ}-closed.*

Proof. (1) Let V be a $\overline{\kappa}$-closed, since $V = \mathbb{T} \cap V$, where, V is a $\overline{\kappa}$-closed and $\mathbb{T}$ is $\Theta^{\diamond}$-closed, then V is a λ_{κ}-closed.

(1) Let V be a $\Theta^{\diamond}$-closed, since $V = \mathbb{T} \cap V$, where, V is a $\Theta^{\diamond}$-closed and $\mathbb{T}$ is $\overline{\kappa}$-closed, then V is a λ_{κ}-closed. □

Lemma 6.5. *Let V be a subset of a PTS $(\mathbb{T}, \Theta, \mathbb{P})$. The following items are equivalent:*

1. *V is a λ_{κ}-closed set.*
2. *$V = F \cap cl^{\diamond}(V)$, where F is a $\overline{\kappa}$-closed set.*
3. *$V = \overline{V}^{\kappa} \cap cl^{\diamond}(V)$.*

Proof. Since $\overline{V}^{\kappa}$ is a $\overline{\kappa}$-closed and $cl^{\diamond}(V)$ is a $\Theta^{\diamond}$-closed set the proof is clear. □

Definition 6.6. The PTS $(\mathbb{T}, \Theta, \mathbb{P})$ is said to be $\overline{\kappa}$-T_0 if for each of distinct points $t, s \in \mathbb{T}$ there is a $\Theta^{\diamond}$-open V containing s but not t or there is $\overline{\kappa}$-closed set F containing t but not s.

Remark 6.7. 1. Every T_0-space is $\overline{\kappa}$-T_0-space, because each open set is $\Theta^{\diamond}$-open and $\overline{\kappa}$-closed.
2. A PTS $(\mathbb{T}, \Theta, \mathbb{P})$ in Example 4.5 is $\overline{\kappa}$-T_0-space but not T_0-space.

Theorem 6.8. *The PTS $(\mathbb{T}, \Theta, \mathbb{P})$ is a $\overline{\kappa}$-T_0 if and only if each $t \in \mathbb{T}$, the singleton $\{t\}$ is a λ_{κ}-closed.*

Proof. For all $t \in \mathbb{T}$ we have $\{t\} \subseteq \overline{\{t\}}^{\kappa} \cap cl^{\diamond}(\{t\})$. Let $t \neq s$, since is a λ_{κ}-closed. is $\overline{\kappa}$-T_0, after which there are two cases:

Case 1: There is a $\Theta^{\diamond}$-open set V containing s but not t. In this case $s \notin cl^{\diamond}(\{t\})$ and hence $s \notin \overline{\{t\}}^{\kappa} \cap cl^{\diamond}(\{t\})$.

Case 2: There is a $\overline{\kappa}$-closed set F containing t but not s. In this case $s \notin \overline{\{t\}}^{\kappa}$ and hence $s \notin \overline{\{t\}}^{\kappa} \cap cl^{\diamond}(\{t\})$.

Hence, we show that $\overline{\{t\}}^{\kappa} \cap cl^{\diamond}(\{t\}) \subseteq \{t\}$. Therefore, $\overline{\{t\}}^{\kappa} \cap cl^{\diamond}(\{t\}) = \{t\}$ and by Lemma 6.5, $\{t\}$ is a λ_{κ}-closed.

Conversely, suppose that a PTS $(\mathbb{T}, \Theta, \mathbb{P})$ is not $\overline{\kappa}$-T_0 space. Then there is two distant points $t, s \in \mathbb{T}$ such that one of the following cases hold:

Case 1: $s \in F$ for each $\overline{\kappa}$-closed set F containing t. Then, we obtain $s \in \overline{\{t\}}^{\kappa}$.

Case 2: $\{t\} \cap V \neq \emptyset$ for each $\Theta^{\diamond}$-open set V containing s. Then, we obtain $s \in cl^{\diamond}(\{t\})$.

Therefore, $s \in \overline{\{t\}}^{\kappa} \cap cl^{\diamond}(\{t\})$. Since $\{t\}$ is a λ_{κ}-closed, then by Lemma 6.5 we have $\{t\} = \overline{\{t\}}^{\kappa} \cap cl^{\diamond}(\{t\})$ and hence $t = s$ it is contradictory therefore, $(\mathbb{T}, \Theta, \mathbb{P})$ is a $\overline{\kappa}$-T_0. □

We now provide a concept that shares some similarities with the concept of R_0 space.

Definition 6.9. The PTS $(\mathbb{T}, \Theta, \mathbb{P})$ is said to be $\overline{\kappa}$-R_0 space if each $\overline{\kappa}$-closed set F and each $t \in F$ we have $cl^{\diamond}(\{t\}) \subseteq F$.

Example 6.10. In Example 3.12 when $\mathbb{T} = \{3, 2, 1\}$ with $\Theta = \{\emptyset, \mathbb{T}, \{1\}, \{1, 2\}\}$ and the primal $\mathbb{P} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. We obtaind the collection of all $\overline{\kappa}$-closed and $\Theta^{\diamond}$-closed sets are $\{\emptyset, \mathbb{T}, \{1, 2\}, \{1\}, \{2\}\}$ and $\{\emptyset, \mathbb{T}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1\}, \{3\}, \{2\}\}$ respectively. It is clear that, each $\overline{\kappa}$-closed set F and each $t \in F$ we have $cl^{\diamond}(\{t\}) = \{t\} \subseteq F$. Hence, Example 3.12 is $\overline{\kappa}$-R_0. But Example 4.5 is not $\overline{\kappa}$-R_0 because $\{1, 3\}$ is $\overline{\kappa}$-closed and $1 \in \{1, 3\}$ but $cl^{\diamond}(\{1\}) = \{1, 2, 3\} \nsubseteq \{1, 3\}$.

Theorem 6.11. *Let a PTS $(\mathbb{T}, \Theta, \mathbb{P})$ be $\overline{\kappa}$-R_0 space. A singleton $\{t\}$ is λ_κ-closed if and only if $\{t\}$ is $\Theta^{\diamond}$-closed.*

Proof. Let $\{t\}$ be a λ_κ-closed set. By lemma 6.5 we have $\{t\} = \overline{\{t\}}^{\kappa} \cap cl^{\diamond}(\{t\})$. Now for each $\overline{\kappa}$-closed set F containing $\{t\}$, $cl^{\diamond}(\{t\}) \subseteq F$. Thus, $cl^{\diamond}(\{t\}) \subseteq \overline{\{t\}}^{\kappa}$. Hence, we obtain that $\{t\} = \overline{\{t\}}^{\kappa} \cap cl^{\diamond}(\{t\}) = cl^{\diamond}(\{t\})$. Therefore, $\{t\}$ is a $\Theta^{\diamond}$-closed set.

Conversely, suppose that $\{t\}$ is a $\Theta^{\diamond}$-closed set. Then, $\{t\} = cl^{\diamond}(\{t\})$, and we obtain that $\{t\} \subseteq \overline{\{t\}}^{\kappa} \cap cl^{\diamond}(\{t\}) = \overline{\{t\}}^{\kappa} \cap \{t\} = \{t\}$. Hence, $\{t\} = \overline{\{t\}}^{\kappa} \cap cl^{\diamond}(\{t\})$ and by lemma 6.5, a singleton $\{t\}$ is λ_κ-closed set. □

Corollary 6.12. *Let a PTS $(\mathbb{T}, \Theta, \mathbb{P})$ be $\overline{\kappa}$-R_0 space. Then, $(\mathbb{T}, \Theta, \mathbb{P})$ a $\overline{\kappa}$-T_0 space if and only $(\mathbb{T}, \Theta^{\diamond})$ is a T_1 space.*

Example 6.13. In Example 4.5 for each distinct points $t, s \in \mathbb{T}$ there is a $\Theta^{\diamond}$-open V containing s but not t or there is $\overline{\kappa}$-closed set F containing t but not s. Hence, Example 4.4 is $\overline{\kappa}$-T_0. But $(\mathbb{T}, \Theta^{\diamond})$ it is not a T_1 space because $\{1\}$ is not $\Theta^{\diamond}$-closed. Thus, the condition in Corollary 6.12 is a necessary.

Acknowledgement

The authors wish to thank the referee for useful comments and suggestions.

References

[1] S. Acharjee, M. Özkoç, F.Y. Issaka, *Primal topological spaces*, Bol. Soc. Paran. Mat. (3s.) v. **43** (2025), 1–9.
[2] A. Al-Omari, O. Alghamdi, *Regularity and normality on primal spaces*, AIMS Mathematics **9** (3) (2024), 7662-7672, doi: 10.3934/math.2024372
[3] A. Al-Omari, M. H. Alqahtani, *Some operators in soft primal spaces*, AIMS Mathematics **9** (5) (2024), 10756–10774.
[4] A. Al-Omari, M. H. Alqahtani, *Primal structure with vclosure operators and their applications* Mathematics **11** (2023), 4946.
[5] A. Al-Omari, S. Acharjee, M. Özkoç, *A new operator of primal topological spaces*, Mathematica **65** (88) (2023), 175–183.
[6] A. Al-Omari, M. Özkoç, S. Acharjee, *Primal-proximity spaces*, Mathematica **66** (89) (2024), 151–167.
[7] A. Al-Omari, T. Noiri, *On $\Psi_{\mathcal{G}}$-sets in grill topological spaces*, Filomat **25** (2011), 187–196.
[8] T. M. Al-shami, Z. A. Ameen, R. Abu-Gdairi, A. Mhemdi, *On Primal doft topology*, Mathematics **11** (2023), 2329.
[9] Z. Ameen, S. Al Ghour, *Cluster soft sets and cluster soft topologies*, Comp. Appl. Math. **42** (2023), 337.
[10] Z. A. Ameen, R. A. Mohammed, T. M. Al-shami, B. A. Asaad, *Novel fuzzy topologies from old through fuzzy primals*, J. Intell. Fuzzy Syst. **48** (2024), 1–10.
[11] F. G. Arenas, J. Dontchev, M. Ganster, *On λ-sets and the dual of generalized continuity*, Quest. Answers Gen. Topology **15** (1997), 3–13.
[12] K. K. Azad, *Fuzzy grills and a characterization of fuzzy proximity*, J. Math. Anal. Appl. **79** (1981), 13–17.
[13] G. Choquet, *Sur les notions de filter et grille*, C. R. Acad. Sci. **224**, (1947), 171–173.
[14] A. S. Davis. *Indexed systems of neighborhoods for general topological spaces*, Amer. Math. Soc. **68** (1961), 886–893.
[15] D. Janković, T. R. Hamlett, *New topologies from old via ideals*, Am. Math. Mon.**97** (1990), 295–310.
[16] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh, A. M. Abd El-latif, *Soft ideal theory,soft local function and generated soft topological spaces*, Appl. Math. Inf. Sci. **8** (2014),, 1595–1603.
[17] k. Kuratowski, *Topologie I* , Warszawa, Academic Press, 1966.
[18] N. Levine, *Generalized closed sets in topology*, Rend. Circ. Mat.Palermo **19** (1970), 89–96.
[19] H. Maki, *Generalized Λ-sets and the associated closure operator*, The Special issue in Commemoration of Prof. Kazusada IKEDA's Retirement **1** (1986), 139–146.
[20] D. Sarkar, *Fuzzy ideal theory fuzzy local Ffunction and generated fuzzy topology*, Fuzzy Sets Syst. **87** (1997), 117–123.
[21] R. Vaidyanathaswamy, *The localisation theory in set topology*, P. Indian AS-Math. Sci. **20** (1944), 51–61.
[22] R. Vaidyanathaswamy, *Set Theory*, Chelsea Publishing Company, Second Edition, 1960.