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Quasi-pseudometric modular spaces as $\mathcal{Q}$-categories

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Abstract. We prove that the category of quasi-pseudometric modular spaces whose morphisms are the nonexpansive mappings is isomorphic to a quantale enriched category. To achieve this, we construct an appropriate quantale of isotone functions. We also show that, by means of this isomorphism, the topology associated with a quasi-pseudometric modular coincides with that generated by its corresponding quantale enriched category.

Furthermore, we demonstrate that the class of quasi-pseudometrizable topological spaces coincides with the topological spaces whose topology is induced by a quasi-pseudometric modular.

1. Introduction

Nakano introduced the concept of modular [24] to obtain a more detailed theory of Dedekind complete Riesz spaces and it was further extended to Riesz spaces and vector spaces. A modular on a vector space is a nonnegative real-valued function, symmetric, convex, left-continuous, and non-identically null in each half-line. Its importance comes from the fact that you can construct a normed vector subspace from a modular, with the so-called Luxemburg norm [17]. Moreover, modular spaces extend the Lebesgue, Riesz, and Orlicz spaces.

Recently, motivated by problems from multivalued analysis, Chistyakov [2, 3] introduced a general theory of modulars in arbitrary sets (removing the requirement of an algebraic structure in the underlying set) under the name of metric modular space. Roughly speaking, a metric modular space is a nonempty set endowed with a parameterized family $\{w_t\}_{t>0}$ of two-variable functions valued at $[0, +\infty]$ satisfying certain axioms that are consistent with the classical theory of modulars (see Definition 2.1). The monograph [4] written by Chistyakov is a comprehensive study of the metric and topological properties of metric modular spaces. In particular, he introduced two different topologies in a metric modular space: the so-called metric topology and modular topology. The modular topology turns out to be the topologization [4, Theorem

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4.3.5] of a non-topological convergence called *modular convergence* [4, Definition 4.2.1], that extends the modular convergence defined by Musielak and Orlicz [23].

There also exists an asymmetric version of metric modular spaces, named quasi-pseudometric modular spaces, introduced by Sebogodi [27] in 2019, for which there is also a parallel theory to a certain extent.

The purpose of this paper is to keep on exploring the theory of quasi-pseudometric modular spaces. Specifically, our objective is twofold. First, we aim to contribute to the basic theory of quasi-pseudometric modular spaces. This will be addressed in Section 2. After recalling the basic definitions, we introduce a quasi-uniformity (Proposition 2.4) on every quasi-pseudometric modular space, having as entourages the modular entourages considered by Chistyakov [4, Section 4.1.2]. The topology generated by this quasi-uniformity is the quasi-pseudometric topology of the quasi-pseudometric modular. Moreover, we will show that the topology of a quasi-pseudometric space is also induced by a quasi-pseudometric modular (Theorem 2.10). In addition, we analyze some concepts of functions between quasi-pseudometric modular spaces that can be considered as morphisms for the category of quasi-pseudometric modular spaces, which will be necessary for the second aim of the paper that we next discuss.

The study of the metrizability of a topological space has been one of the main research areas of general topology. Since not every topological space is metrizable, some authors have taken a different approach to this problem, searching for a more general concept of metric in such a way that every topology comes from a generalized metric. Quasi-pseudometrics (metrics that do not satisfy neither the symmetry axiom nor the non-degeneracy axiom) are probably the first generalized metrics but there are still topologies that are not quasi-pseudometrizable [12].

In 1978, Trillas and Alsina [28] replaced the codomain of non-negative reals of a classic metric with an ordered algebraic structure. Kopperman tackled a similar approach [18] in 1988, introducing the so-called continuity spaces by considering a value semigroup as the codomain of the metric. This afforded him to prove that every topological space is a continuity space. Later on, Flagg [10, 11] modified Kopperman's approach by evaluating a metric in a value quantale (see Definition 3.20 and sections 3 and 4) which provides important advantages with respect to the original continuity spaces (see [6]). Furthermore, Flagg noticed that, in the same way that quasi-pseudometric spaces are enriched categories as first noticed by Lawvere [20], continuity spaces are just enriched categories over a value quantale. Roughly speaking, an enriched category is a generalization of the concept of a category where the set of morphisms are objects of a monoidal category. Therefore, the continuity spaces are $\mathcal{Q}$-categories [15, Section III.1.3] where $\mathcal{Q}$ is a value quantale.

The second goal of this paper is to demonstrate that the category of quasi-pseudometric modular spaces is isomorphic to a $\mathcal{Q}$-category for a concrete value quantale $\mathcal{Q}$. We will show this in Section 5 where we prove that the family ∇ of all isotone functions between $(0, +\infty)$ and $[0, +\infty]^{op}$ can be endowed with a specific order and operation that makes it a value quantale (Proposition 5.5). Then, we provide an isomorphism between the category of quasi-pseudometric modular spaces and the Δ-category (Theorem 5.6). Furthermore, we show that this isomorphism also preserves the topologies of the objects (Theorem 5.9). These results establish the enriched category theory as a frame for studying quasi-pseudometric modular spaces that could allow for analyzing their relationship with other topological structures.

2. Quasi-pseudometric modular spaces

We start by recalling the definition of a quasi-pseudometric modular [27], the asymmetric version of the metric modular introduced by Chistyakov [2, 4].

Definition 2.1. ([4, 27]) Let X be a nonempty set. A function $w : (0, +\infty) \times X \times X \to [0, +\infty]$ is a **quasi-pseudometric modular** on X if for every $x, y, z \in X$ and all $t, s > 0$ it verifies:

(M1) $w(t, x, x) = 0$ for all $t > 0$;
(M2) $w(t + s, x, y) \leq w(t, x, z) + w(s, z, y)$.

If, in addition, w satisfies

(M3) $w(t, x, y) = w(t, y, x) = 0$ for all $t > 0$ if and only if $x = y$

then w is called a **quasi-metric modular**.
If a quasi-(pseudo)metric modular w verifies

(M4) $w(t, x, y) = w(t, y, x)$ for all $x, y \in X$ and all $t > 0$

then w is said to be a (pseudo)metric modular on X.

The pair (X, w) is known as a **(quasi)-(pseudo)metric modular space**.

Moreover, a (quasi)-(pseudo)metric modular w on X is said to be **left-continuous** if $w(\cdot, x, y) : (0, +\infty) \to [0, +\infty]$ is left-continuous for every $x, y \in X$. In this case we say that (X, w) is a **left-continuous (quasi)-(pseudo)metric modular space**.

Example 2.2. ([4]) Given a (quasi)-(pseudo)metric space (X, d) and a nonincreasing function $g : (0, +\infty) \to [0, +\infty]$ non-identically zero, then $w_g : (0, +\infty) \times X \times X \to [0, +\infty]$ defined as

$$w_g(t, x, y) = g(t) \cdot d(x, y)$$

for all $x, y \in X$ and all $t > 0$, is a (quasi)-(pseudo)metric modular on X.

If $g(t) = \frac{1}{t}$ for all $t > 0$ then w_g will be called the **standard (quasi)-(pseudo) metric modular induced by** d and will be denoted by w_d, that is,

$$w_d(t, x, y) = \frac{d(x, y)}{t}$$

for all $x, y \in X$ and all $t > 0$.

One of the most important properties of a quasi-pseudometric modular, which can be deduced from (M2), the triangular inequality, is the following:

Proposition 2.3. ([27, Lemma 3.1.1]) *Let (X, w) be a quasi-pseudometric modular space. Then the function $w(\_, x, y) : (0, +\infty) \to [0, +\infty]$ is non-increasing for all $x, y \in X$.*

In [4], Chistyakov considered two different topologies in a metric modular space that were later studied in the realm of quasi-pseudometric modular spaces in [27]: the metric topology and the modular topology. We provide here a new approach to the introduction of the metric topology by defining a quasi-uniformity from a quasi-pseudometric modular.

Proposition 2.4. *Let (X, w) be a quasi-pseudometric modular space. Given $t, \varepsilon > 0$, define*

$$W^w_{t,\varepsilon} := \{(x, y) \in X^2 : w(t, x, y) < \varepsilon\}$$

(we will omit the superscript w if no confusion arises).

1. *The family $\mathcal{B} = \{W^w_{t,\varepsilon} : t, \varepsilon > 0\}$ is a base for a quasi-uniformity $\mathcal{W}_w$ on X. The elements $W^w_{t,\varepsilon}$ will be called **modular entourages**.*
2. *$\left\{W^w_{\frac{1}{n},\frac{1}{n}} : n \in \mathbb{N}\right\}$ is a countable base for $\mathcal{W}_w$.*
3. *If w is a pseudometric modular, then $\mathcal{W}_w$ is a uniformity on X.*

Proof. We prove (1).

By (M1), it is obvious that $\{(x, x) : x \in X\} \subseteq W_{t,\varepsilon}$ for all $t, \varepsilon > 0$.

Let us see that $\mathcal{B}$ is a filter base. Given $t_1, t_2, \varepsilon_1, \varepsilon_2 > 0$, we claim that $W_{t_1 \wedge t_2, \varepsilon_1 \wedge \varepsilon_2} \subseteq W_{t_1,\varepsilon_1} \cap W_{t_2,\varepsilon_2}$. In fact if $(x, y) \in W_{t_1 \wedge t_2, \varepsilon_1 \wedge \varepsilon_2}$ then $w(t_1 \wedge t_2, x, y) < \varepsilon_1 \wedge \varepsilon_2$. Since $w(\_, x, y)$ is non-increasing then $\max\{w(t_1, x, y), w(t_2, x, y)\} \leq w(t_1 \wedge t_2, x, y) < \varepsilon_1 \wedge \varepsilon_2$, that is, $(x, y) \in W_{t_1,\varepsilon_1} \cap W_{t_2,\varepsilon_2}$.

Now, by (M1), it is obvious that $\{(x, x) : x \in X\} \subseteq W_{t,\varepsilon}$ for all $t, \varepsilon > 0$.

Let $t, \varepsilon > 0$. Let us prove that $W_{\frac{t}{2},\frac{\varepsilon}{2}} \circ W_{\frac{t}{2},\frac{\varepsilon}{2}} \subseteq W_{t,\varepsilon}$. If $(x, y) \in W_{\frac{t}{2},\frac{\varepsilon}{2}} \circ W_{\frac{t}{2},\frac{\varepsilon}{2}}$, then there exists some $z \in X$ such that $(x, z), (z, y) \in W_{\frac{t}{2},\frac{\varepsilon}{2}}$, that is,

$$\max\left\{w\left(\frac{t}{2}, x, z\right), w\left(\frac{t}{2}, z, x\right)\right\} < \frac{\varepsilon}{2}.$$

By (M2) we have that

$$w(t, x, y) = w\left(\frac{t}{2} + \frac{t}{2}, x, y\right) \leq w\left(\frac{t}{2}, x, z\right) + w\left(\frac{t}{2}, z, x\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, $(x, y) \in W_{t,\varepsilon}$ and $\mathcal{B}$ is a base of a quasi-uniformity on X.

We next prove (2). Given some arbitrary $t, \varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \min\{\varepsilon, t\}$. We claim that $W_{\frac{1}{n_0},\frac{1}{n_0}} \subseteq W_{t,\varepsilon}$.

Take some $(x, y) \in W_{\frac{1}{n_0},\frac{1}{n_0}}$. Then $w\left(\frac{1}{n_0}, x, y\right) < \frac{1}{n_0}$. Hence, since $w(\_, x, y)$ is non-increasing

$$w(t, x, y) \leq w\left(\frac{1}{n_0}, x, y\right) < \frac{1}{n_0} < \varepsilon,$$

so $(x, y) \in W_{t,\varepsilon}$.

Finally, to see (3), it is obvious that the modular entourages are symmetric in case that w is a pseudometric modular. Thus, they form a base for a uniformity on X. □

Remark 2.5. It is important to mention that a version of this result for metric modular spaces appears concurrently in [22, Theorem 2].

Definition 2.6. Let (X, w) be a quasi-pseudometric modular space. The topology $\mathcal{T}(\mathcal{W}_w)$ generated by the quasi-uniformity $\mathcal{W}_w$ on X will be called the **topology associated to the quasi-pseudometric modular** w. For simplicity, it will be also denoted by $\mathcal{T}(w)$.

Then $\mathcal{T}(w)$ has as neighborhood base at $x \in X$ the family $\{W_{t,\varepsilon}(x) : t, \varepsilon > 0\}$ where

$$W_{t,\varepsilon}(x) = \{y \in X : w(t, x, y) < \varepsilon\}.$$

Example 2.7. Let (X, d) be a quasi-pseudometric space. Consider the standard quasi-pseudometric modular w_d on X induced by d (see Example 2.2) given by

$$w(t, x, y) = \frac{d(x, y)}{t}$$

for all $x, y \in X$ and all $t > 0$. Then $\mathcal{T}(w) = \mathcal{T}(d)$. Let us check this.

Notice first that for all $t, \varepsilon > 0$ and $x \in X$,

$$W_{t,\varepsilon}(x) = \left\{y \in X : w(t, x, y) = \frac{d(x, y)}{t} < \varepsilon\right\} = \{y \in X : d(x, y) < t\varepsilon\} = B(x, t\varepsilon).$$

Hence, the neighborhood base at any $x \in X$ in $\mathcal{T}(w)$ coincides with all the open balls in $\mathcal{T}(d)$, so they generate the same topology.

Remark 2.8. Given a pseudometric modular space (X, w), Chistyakov [4, Theorem 2.2.1] (see also [2, Theorem 2.6]) proved that the function $d_w(x, y) : X \times X \to [0, +\infty)$ given by

$$d_w(x, y) = \inf\{t > 0 : w(t, x, y) \leq t\}$$

for all $x, y \in X$, is an extended pseudometric on X (i.e., a pseudometric that it is allowed to take the value $+\infty$). In case that (X, w) is a quasi-pseudometric modular space, then d_w is an extended quasi-pseudometric on X [27, Theorem 3.1.2].

Hence we can consider the open ball topology $\mathcal{T}(d_w)$ generated by the extended quasi-pseudometric d_w on X. For pseudometric modulars, Chistyakov [4, Section 4.1.2] studied this topology, that he called *metric topology*. The corresponding study in which w is a quasi-pseudometric modular was performed in [27].

Moreover, in a quasi-pseudometric modular space (X, w), we have that $\mathcal{T}(d_w) = \mathcal{T}(w)$ on X, that is, the topology generated by the quasi-uniformity $\mathcal{W}_w$ is equal to the topology associated with the extended quasi-pseudometric d_w. To see this, it suffices to observe that the quasi-uniformity $\mathcal{U}_{d_w}$ is in fact $\mathcal{W}_w$ since

$$\{(x, y) \in X \times X : d_w(x, y) < \min\{t, \varepsilon\}\} \subseteq W_{t,\varepsilon},$$

$$W_{\varepsilon,\varepsilon} \subseteq \{(x, y) \in X \times X : d_w(x, y) \leq \varepsilon\}.$$

Therefore, the topology associated with a quasi-pseudometric modular is generated by a quasi-pseudometric, that is, it is quasi-pseudometrizable. But the converse is also true as we next show.

Definition 2.9. A topological space $(X, \mathcal{T})$ is said to be **quasi-pseudomodulable** if $\mathcal{T} = \mathcal{T}(w)$ for some quasi-pseudometric modular w on X.

Theorem 2.10. *A topological space is quasi-pseudomodulable if and only if is quasi-pseudometrizable.*

Proof. Let $(X, \mathcal{T})$ be a quasi-pseudometrizable topological space. Then there exists some quasi-pseudometric d on X such that $\mathcal{T} = \mathcal{T}(d)$. By Example 2.7, $\mathcal{T}(d) = \mathcal{T}(w_d)$, where w_d is the standard quasi-pseudometric modular associated with d. Hence $\mathcal{T}$ is quasi-pseudomodulable.

Conversely, suppose that there exists a quasi-pseudometric modular w on X such that $\mathcal{T} = \mathcal{T}(w)$. Since $\mathcal{T}(w)$ is induced by a quasi-uniformity $\mathcal{W}_w$ with a countable base, then it is quasi-pseudometrizable [12]. □

We observe that, in general, the set $W_{t,\varepsilon}(x)$ is not open in $\mathcal{T}(w)$ even for metric modulars, as the next example shows.

Example 2.11. Let $X = \{x, y\} \cup \{z_n\}_{n\in\mathbb{N}}$ and define $w : (0, +\infty) \times X \times X \to [0, +\infty]$ as

$$w(t, a, a) = 0, \ \forall\, a \in X, \ \forall\, t > 0.$$

$$w(t, x, y) = \begin{cases} 1 & \text{if } 0 \leq t < 1, \\ 0 & \text{if } t \geq 1, \end{cases}$$

$$w(t, x, z_n) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1, \\ 0 & \text{if } t > 1, \end{cases}$$

$$w(t, y, z_n) = \begin{cases} \frac{1}{n} & \text{if } 0 \leq t < 1, \\ 0 & \text{if } t \geq 1, \end{cases}$$

$$w(t, z_n, z_m) = \begin{cases} \frac{1}{\min\{n,m\}} & \text{if } 0 \leq t < 1, \\ 0 & \text{if } t \geq 1. \end{cases}$$

It is straightforward to check that (X, w) is a metric modular space.

Let us see that $W_{1,\frac{1}{2}}(x)$ is not open in $\mathcal{T}(w)$. Since $w(1, x, y) = 0$ then $y \in W_{1,\frac{1}{2}}(x)$. We show that for all $t, \varepsilon > 0$, $W_{t,\varepsilon}(y) \not\subseteq W_{1,\frac{1}{2}}(x)$ which shows that $W_{1,\frac{1}{2}}(x)$ is not open.

Given any $t, \varepsilon > 0$, there exists some $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \varepsilon$. Thus,

$$w(t, y, z_{n_0}) = \left\{ \begin{array}{ll} \frac{1}{n_0} & \text{if } 0 \leq t < 1 \\ 0 & \text{if } t \geq 1 \end{array} \right\} < \varepsilon.$$

Hence, $z_{n_0} \in W_{t,\varepsilon}(y)$. Nevertheless,

$$w(1, x, z_{n_0}) = 1 > \frac{1}{2},$$

which implies that $z_{n_0} \notin W_{1,\frac{1}{2}}(x)$. In conclusion, $W_{t,\varepsilon}(y) \nsubseteq W_{1,\frac{1}{2}}(x)$ for all $t, \varepsilon > 0$.

Observe that in the previous example, $w(\_, x, y)$ is not left-continuous. This fact is not casual as it is inferred from the next result.

Proposition 2.12. *Let (X, w) be a left-continuous quasi-pseudometric modular space. Then $W_{t,\varepsilon}(x)$ is open for all $t, \varepsilon > 0$ and for all $x \in X$.*

Proof. Let $x \in X$, $t, \varepsilon > 0$, and $y \in W_{t,\varepsilon}(x)$. Define $\eta := \varepsilon - w(t, x, y) > 0$ and $t_n := t - \frac{t}{2n}$, so $(t_n)_{n\in\mathbb{N}}$ converges to t. Since $w(\_, x, y)$ is left-continuous then $(w(t_n, x, y))_{n\in\mathbb{N}}$ converges to $w(t, x, y)$. Hence there exists $n_0 \in \mathbb{N}$ such that $|w(t_n, x, y) - w(t, x, y)| < \eta$ for all $n \geq n_0$. In particular

$$w(t_{n_0}, x, y) < \eta + w(t, x, y) = \varepsilon.$$

Let us define $\delta := \varepsilon - w(t_{n_0}, x, y)$. We claim that $W_{t-t_{n_0},\delta}(y) \subseteq W_{t,\varepsilon}(x)$. Take $z \in W_{t-t_{n_0},\delta}(y)$. Then $w(t - t_{n_0}, y, z) < \delta$ so

$$w(t, x, z) = w(t - t_{n_0} + t_{n_0}, x, z) \leq w(t_{n_0}, x, y) + w(t - t_{n_0}, y, z) < w(t_{n_0}, x, y) + \delta = \varepsilon.$$

Hence, $W_{t-t_{n_0},\delta}(y) \subseteq W_{t,\varepsilon}(x)$. □

Next, we study which morphisms can be considered between quasi-pseudometric modular spaces to obtain an appropriate category.

We first recall the following concept introduced in [21, 25] for pseudometric modular spaces (see also [8]).

Definition 2.13. ([21, 25]) A function $f : (X, w_1) \to (Y, w_2)$ between two quasi-pseudometric modular spaces is said to be **Lipschitz** if there exists $k > 0$ such that

$$w_2(k \cdot t, f(x), f(y)) \leq w_1(t, x, y)$$

for every $x, y \in X$ and every $t > 0$.

If $k = 1$ then f is called **nonexpansive**.

Remark 2.14. If the above condition is only satisfied when the parameter t belongs to an interval $(0, t_0]$, then f is called *modular Lipschitzian* [5].

We next introduce a new notion.

Definition 2.15. A function $f : (X, w_1) \to (Y, w_2)$ between two quasi-pseudometric modular spaces is said to be **strongly uniformly continuous** if given $t > 0$ there exists $s > 0$ such that

$$w_2(t, f(x), f(y)) \leq w_1(s, x, y)$$

for every $x, y \in X$.

Proposition 2.16. *Let $(X, w_1), (Y, w_2)$ be two quasi-pseudometric modular spaces. Each statement implies its successor:*

(1) *$f : (X, w_1) \to (Y, w_2)$ is Lipschitz;*
(2) *$f : (X, w_1) \to (Y, w_2)$ is strongly uniformly continuous;*
(3) *$f : (X, \mathcal{W}_{w_1}) \to (Y, \mathcal{W}_{w_2})$ is uniformly continuous.*

Proof. (1) ⇒ (2) By assumption, there exists $k > 0$ such that

$$w_2\left(k \cdot \frac{t}{k}, f(x), f(y)\right) = w_2(t, f(x), f(y)) \leq w_1\left(\frac{t}{k}, x, y\right)$$

for every $x, y \in X$ and every $t > 0$. Hence, f is strongly uniformly continuous.

(2) ⇒ (3) Suppose that $f : (X, w_1) \to (Y, w_2)$ is strongly uniformly continuous. Let $V \in \mathcal{W}_{w_2}$. Then we can find $t, \varepsilon > 0$ such that $W^{w_2}_{t,\varepsilon} \subseteq V$. By assumption, there exists $s > 0$ such that

$$w_2(t, f(x), f(y)) \leq w_1(s, x, y)$$

for every $x, y \in X$. Hence if $(x, y) \in W^{w_1}_{s,\varepsilon}$ then $(f(x), f(y)) \in W^{w_2}_{t,\varepsilon}$ so $f : (X, \mathcal{W}_{w_1}) \to (Y, \mathcal{W}_{w_2})$ is uniformly continuous. □

Notice that for standard quasi-pseudometric modulars, Lipschitz functions are equal to strongly uniformly continuous functions.

Proposition 2.17. *Let $(X, d), (Y, q)$ be two quasi-pseudometric spaces. The following statements are equivalent:*

(1) *$f : (X, d) \to (Y, q)$ is Lipschitz;*
(2) *$f : (X, w_d) \to (Y, w_q)$ is Lipschitz;*
(3) *$f : (X, w_d) \to (Y, w_q)$ is strongly uniformly continuous.*

Proof. (1) ⇒ (2) Since f is Lipschitz there exists $k > 0$ such that $q(f(x), f(y)) \leq k \cdot d(x, y)$ for all $x, y \in X$. Hence, for any $t > 0$

$$w_q(k \cdot t, f(x), f(y)) = \frac{q(f(x), f(y))}{k \cdot t} \leq \frac{d(x, y)}{t} = w_d(t, x, y)$$

which proves the statement.

(2) ⇒ (3) This follows from the previous Proposition.

(3) ⇒ (1). For $t = 1$ we can find $s > 0$ such that

$$w_q(1, f(x), f(y)) = q(f(x), f(y)) \leq w_d(s, x, y) = \frac{d(x, y)}{s}$$

for every $x, y \in X$. Hence $f : (X, d) \to (Y, q)$ is Lipschitz with constant $\frac{1}{s}$. □

We denote by QPMod the category whose objects are the quasi-pseudometric modular spaces and whose morphisms are the strongly uniformly continuous maps. When we consider the nonexpansive maps as morphisms, we denote this category by QPMod_n. Then QPMod_n is a subcategory of QPMod.

Moreover, we denote by LQPMod (resp. LQPMod_n) the full subcategory of QPMod (resp. QPMod_n) whose objects are the left-continuous quasi-pseudometric modular spaces. It turns out that LQPMod is a reflective subcategory of QPMod.

Proposition 2.18. LQPMod *is a reflective full subcategory of* QPMod *whose reflector is the functor $\mathcal{L}$: QPMod → LQPMod given by $\mathcal{L}((X, w)) = (X, \widetilde{w})$ and leaving morphisms unchanged, where $\widetilde{w}$ is the left regularization of w defined as*

$$\widetilde{w}(t, x, y) = \bigwedge_{0<s<t} w(s, x, y),$$

for every $x, y \in X$ and every $t > 0$ (see [4, Definition 1.2.4]).

Proof. Following [4, Proposition 1.2.5] we have that $\widetilde{w}$ is a left-continuous quasi-pseudometric modular on X.

Moreover, let $f : (X, w_1) \to (Y, w_2)$ be a strongly uniformly continuous mapping. Given $t > 0$ and $0 < r < t$ there exists $s > 0$ such that

$$w_2(r, f(x), f(y)) \leq w_1(s, x, y)$$

for all $x, y \in X$. Therefore,

$$\widetilde{w_2}(t, f(x), f(y)) = \bigwedge_{0<t'<t} w_2(t', f(x), f(y)) \leq w_2(r, f(x), f(y))$$
$$\leq w_1(s, x, y) \leq \widetilde{w_1}(s, x, y) = \bigwedge_{0<s'<s} w_1(s', x, y)$$

so $f : (X, \widetilde{w_1}) \to (Y, \widetilde{w_2})$ is strongly uniformly continuous. Hence $\mathcal{L}$ is a functor.

We next check that $\mathcal{L}$ is the left adjoint of the inclusion functor $\mathcal{I} : \mathsf{LQPMod} \to \mathsf{QPMod}$. Let $(X, \omega_1) \in \mathsf{QPMod}$ and $(Y, w_2) \in \mathsf{LQPMod}$. Suppose that $f : (X, w_1) \to (Y, w_2)$ is strongly uniformly continuous. Given $t > 0$ there exists $s > 0$ such that $w_2(t, f(x), f(y)) \leq w_1(s, x, y)$ for all $x, y \in X$. Since $w_1(s, x, y) \leq \widetilde{w_1}(s, x, y)$ then $f : (X, \widetilde{w_1}) \to (Y, w_2)$ is also strongly uniformly continuous.

Now, let $g : (X, \widetilde{w_1}) \to (Y, w_2)$ be strongly uniformly continuous. Given $t > 0$ we can find $s > 0$ such that

$$w_2(t, f(x), f(y)) \leq \widetilde{w_1}(s, x, y) = \bigwedge_{0<r<s} w_1(r, x, y).$$

Hence $g : (X, w_1) \to (Y, w_2)$ is strongly uniformly continuous. □

Remark 2.19. We observe that given $x, y \in X$, then $\widetilde{w}(\_, x, y)$ is the upper semicontinuous regularization or upper envelope of $w(\_, x, y)$, since this function is non-increasing (see [1, Chapter 1.3]).

Remark 2.20. Observe that the above proof does not work using the categories LQPMod_n and QPMod_n, although the same mapping between these two categories is still a functor. For example, let X be a set with at least two different points and consider the modular metric x on X given by

$$w(t, x, y) = \begin{cases} 0 & \text{if } x = y, t > 0, \\ 1 & \text{if } x \neq y, 0 < t < 1, \\ 0 & \text{if } x \neq y, t \geq 1, \end{cases}$$

for all $x, y \in X$, $t > 0$. It is clear that its left regularization is

$$\widetilde{w}(t, x, y) = \begin{cases} 0 & \text{if } x = y, t > 0, \\ 1 & \text{if } x \neq y, 0 < t \leq 1, \\ 0 & \text{if } x \neq y, t > 1, \end{cases}$$

for all $x, y \in X$, $t > 0$.

The identity map $i : X \to X$ is nonexpansive when X is endowed with the metric modular w. However $i : (X, \widetilde{w}) \to (X, w)$ is not nonexpansive since

$$\widetilde{w}(1, x, y) = 1 \nleq w(1, x, y) = 0$$

where x, y are two distinct points of X.

3. Lattices and quantales

The second goal of this paper is to establish an equivalence between the category of quasi-pseudometric modular spaces and a category enriched over a quantale (see Section 5). Thus we need some preliminary concepts about order theory that will be useful later. Our main references for this section are [9, 13, 15].

Recall that a *partial order* $\leq$ on a nonempty set X is a reflexive, antisymmetric, and transitive relation on X. In this case, the pair $(X, \leq)$ is a *partially ordered set* (a *poset* for short). The opposite relation $\leq^{\text{op}}$ given by

$$x \leq^{\text{op}} y \text{ if and only if } y \leq x$$

for all $x, y \in X$, is also a partial order on X. If no confusion arises, we will write X^{op} as short for $(X, \leq^{op})$.

A function $f : (X, \leq_1) \to (Y, \leq_2)$ between partially ordered sets is called *isotone* if

$$x \leq_1 y \text{ implies } f(x) \leq_2 f(y)$$

for all $x, y \in X$. The category of partially ordered sets with isotone maps as morphisms will be denoted by POSet.

Furthermore, a poset $(L, \leq)$ where every finite subset has an infimum and supremum is a *lattice*. If every subset has an infimum and supremum, then it is a *complete lattice*. If $A \subseteq L$ then $\bigvee A$, $\bigwedge A$ will denote the supremum and the infimum of A respectively. If we want to emphasize the partial order that is used to compute the supremum or the infimum, we will write $\overset{\leq}{\bigvee}, \overset{\leq}{\bigwedge}$.

Definition 3.1. Let $(L, \leq)$ be a complete lattice. Given $a, b \in L$, then a is **well-below** b ($a \lhd b$) if

$$\text{for all } S \subseteq L \text{ such that } b \leq \bigvee S, \text{ there exists } s_0 \in S \text{ such that } a \leq s_0.$$

Proposition 3.2 (Properties of the well-below order). *Let $(L, \leq)$ be a complete lattice.*

1. $x \lhd y \Rightarrow x \leq y$.
2. $x \lhd y \leq z$ *or* $x \leq y \lhd z$ *implies* $x \lhd z$.
3. $\perp \lhd x$ *if and only if* $x \neq \perp$.

Example 3.3. In the complete lattice $([0, 1], \leq)$, we have that $x \lhd y$ if and only if $x < y$. Let us check this. By Proposition 3.2, $x \lhd y$ implies $x \leq y$. Moreover $x \neq y$. Otherwise taking $S = \{s \in [0, 1] : s < y\}$, we have that $y = \bigvee S$, although $s < x = y$ for all $s \in S$ which contradicts $x \lhd y$.

Suppose now that $x < y$. Let $S \subseteq [0, 1]$ such that $y \leq \bigvee S$. Then for all $\varepsilon > 0$, there exists some $s_0 \in S$ such that $y - \varepsilon \leq s_0$. Taking $\varepsilon = y - x > 0$ we are done.

Example 3.4. Let us see that in $(\mathcal{P}(X), \subseteq)$, if $A, B \neq \emptyset$, then $A \lhd B$ if and only if $A = \{b\}$ for some $b \in B$. Suppose that $A \lhd B$, then taking $\mathcal{S} = \{\{b\}\}_{b \in B}$ we have that $B = \bigvee \mathcal{S} = \bigcup \mathcal{S}$ and thus, there exists some $b_0 \in B$ such that $A \subseteq \{b_0\}$, so $A = \{b_0\}$.

Conversely, suppose that $A = \{b\}$ for some $b \in B$. Let $\mathcal{S} \subseteq \mathcal{P}(X)$ such that $B \subseteq \bigvee \mathcal{S}$. Since $b \in B \subseteq \bigvee \mathcal{S}$, then there is some $S_0 \in \mathcal{S}$ such that $b \in S_0$, which means that $A = \{b\} \subseteq S_0$. Hence $A \lhd B$.

Observe that $\emptyset \lhd A$ for all $A \in \mathcal{P}(X)$.

The proof of the following result is trivial so it is omitted.

Lemma 3.5. *Suppose that $(L, \leq)$ is a complete lattice. Then $\perp \lhd \top$ if and only if L is not trivial.*

Definition 3.6. ([13, Definition I-2.8.]) A complete lattice $(L, \leq)$ is said to be **completely distributive** if given $\{a_{ij} : i \in I,\ j \in K(i)\} \subseteq L$ then

$$\bigwedge_{i \in I} \bigvee_{j \in K(i)} a_{ij} = \bigvee_{f \in M} \bigwedge_{i \in I} a_{i, f(i)},$$

where $M = \prod_{i \in I} K(i)$.

Theorem 3.7. ([26]) *A complete lattice $(L, \leq)$ is completely distributive if and only if $\forall b \in L$,*

$$b = \bigvee \{a \in L : a \lhd b\}.$$

Example 3.8. The complete lattice $([0, 1], \leq)$ is completely distributive. By Example 3.3, $b = \bigvee \{a \in [0, 1] : a < b\} = \bigvee \{a \in [0, 1] : a \lhd b\}$.

Example 3.9. The complete lattice $(\mathcal{P}(X), \subseteq)$ is completely distributive. By Example 3.4, $B = \bigcup_{b\in B}\{b\} = \bigcup\{A \subseteq X : A \lhd B\}$ for every nonempty set B. On the other hand, by Proposition 3.2 (3), $\{A \subseteq X : A \lhd \varnothing\} = \varnothing$ and this concludes our claim.

Remark 3.10. Notice that if $\{(L_\lambda, \leq_\lambda) : \lambda \in \Lambda\}$ is an arbitrary family of completely distributive lattices then its Cartesian product $(\prod_{\lambda\in\Lambda} L_\lambda, \preceq)$ endowed with the componentwise partial order $\preceq$ is also completely distributive. This is clear since given $\{a_{ij} : i \in I,\ j \in K(i)\} \subseteq \prod_{\lambda\in\Lambda} L_\lambda$, then for every $\lambda \in \Lambda$

$$\bigwedge_{i\in I} \bigvee_{j\in K(i)} a_{ij}(\lambda) = \bigvee_{f\in M} \bigwedge_{i\in I} a_{i,f(i)}(\lambda),$$

where $M = \prod_{i\in I} K(i)$, since $(L_\lambda, \leq_\lambda)$ is completely distributive. As the supremum and infimum on $\prod_{\lambda\in\Lambda} L_\lambda$ is computed componentwisely then

$$\bigwedge_{i\in I} \bigvee_{k\in K(i)} a_{ij} = \bigvee_{f\in M} \bigwedge_{i\in I} a_{i,f(i)},$$

so $(\prod_{\lambda\in\Lambda} L_\lambda, \preceq)$ is completely distributive.

Definition 3.11. ([10, 11]) A **value distributive lattice** is a completely distributive lattice $(L, \leq)$ such that

(VDL1) $\perp \lhd \top$;
(VDL2) $a, b \lhd \top \Rightarrow a \vee b \lhd \top$.

Remark 3.12. Notice that:

- (VDL1) just means that L is not trivial by Lemma 3.5.
- (VDL2) just means that $\{a : a \lhd \top\}$ is directed.

Example 3.13. Let **2** be the two element set $\{0, 1\}$ endowed with the usual order $\leq$. Then $(\mathbf{2}, \leq)$ is a value distributive lattice.

Example 3.14. $([0, +\infty], \leq)$ (where $+$ represents the usual sum on the real numbers extended to $+\infty$ as usual) is a value distributive lattice.

We next introduce a crucial notion in our work: a quantale. This structure is a combination of order and a binary operation with some compatibility between them.

Definition 3.15. ([9]) A **quantale** is a triple $(\mathcal{Q}, \leq, *)$ where $(\mathcal{Q}, \leq)$ is a complete lattice and $*$ is a binary operation on $\mathcal{Q}$ such that

(q1) $(\mathcal{Q}, *)$ is a semigroup.
(q2) $a * (\bigvee_{i\in I} b_i) = \bigvee_{i\in I}(a * b_i)$.
(q3) $(\bigvee_{i\in I} b_i) * a = \bigvee_{i\in I}(b_i * a)$.

where $\{b_i : i \in I\} \subseteq \mathcal{Q}$ and $a \in \mathcal{Q}$.

A quantale $(\mathcal{Q}, \leq, *)$ is said to be:

- **commutative** if $*$ is commutative;
- **unital** if $(\mathcal{Q}, *)$ is a monoid with unit $1_\mathcal{Q}$;
- **integral** if it is unital and the unit is the top element of $(\mathcal{Q}, \leq)$, that is, $1_\mathcal{Q} = \top_\mathcal{Q}$. If no confusion arises we will simply write $\top$ instead of $\top_\mathcal{Q}$.

In the remainder of the paper, we will refer to commutative integral quantales as CI-quantales. Moreover, if no confusion arises, we will denote a quantale $(\mathcal{Q}, \leq, *)$ only by its underlying set $\mathcal{Q}$.

Notice that in an integral quantale $(\mathcal{Q}, \leq, *)$ we have that $u * v \leq u \wedge v$ for all $u, v \in \mathcal{Q}$. In fact, $u = u * (v \vee \top) = (u * \top) \vee (u * v) = u \vee (u * v)$ so $u * v \leq u$. In a similar way, $u * v \leq v$.

Example 3.16. $(\mathbf{2}, \leq, \wedge)$ is a CI-quantale.

Example 3.17. $([0, +\infty], \leq, +)$ is a commutative unital quantale, but it is not integral since its unit is $0 \neq \top = +\infty$.

On the other hand, $\mathsf{P}_+ = ([0, +\infty], \leq^{\mathrm{op}}, +)$ is a CI-quantale. This quantale is sometimes called the Lawvere quantale [7] (see also [15, Example II.1.10.1.(3)]).

Example 3.18. Let X be a nonempty set and $(\mathcal{Q}, \leq, *)$ be a quantale. Then we can endow the set $\mathcal{Q}^X$ of all maps $f : X \to \mathcal{Q}$ with the pointwise order that for simplicity we also denote by $\leq$. Then $(\mathcal{Q}^X, \leq)$ is also a complete lattice (see for example [9, Example 2.1.9]). Notice that meet and joins in $\mathcal{Q}^X$ are computed pointwisely.

Moreover, defining a binary operation on $\mathcal{Q}^X$ pointwisely by means of $*$, that we again denote by $*$, turns $(\mathcal{Q}^X, \leq, *)$ into a quantale.

Furthermore, if X is not only a set but also a partially ordered set, then the family $\mathcal{I}(\mathcal{Q}^X)$ of all the isotone maps between X and $\mathcal{Q}$ is a sublattice of $\mathcal{Q}^X$ which is also a quantale.

Example 3.19. A complete lattice $(X, \leq)$ such that $(X, \leq, \wedge)$ is a quantale, is called a **complete Heyting algebra** or a **frame** [13].

In particular, a topology $\mathcal{T}$ on a nonempty set X has a quantale structure $(\mathcal{T}, \subseteq, \cap)$.

The following concept was introduced in [10] to obtain a generalization of the notion of a metric, as it replicates the essential properties of $[0, +\infty]$, the codomain of an extended metric.

Definition 3.20. ([6, 10, 11]) A **value quantale** is a quantale $(\mathcal{Q}, \leq, *)$ such that $(\mathcal{Q}, \leq)$ is a value distributive lattice.

Example 3.21. $(\mathbf{2}, \leq, \wedge)$ is a value quantale.

Example 3.22. $([0, 1], \leq, \cdot)$ *is a value quantale. It is obvious that it is a quantale. Notice that* $\lhd$ *is precisely* $<$, *so it immediately follows that it is a value quantale.*

Example 3.23. $(\mathcal{P}(X), \subseteq, \bigcap)$ is a quantale but not a value quantale if $|X| > 1$. Let $x, y \in X$ be two different points. By Example 3.4, $\{x\} \lhd X$ and $\{y\} \lhd X$. Nevertheless, $\{x\} \vee \{y\} = \{x, y\} \ntriangleleft X$ again by Example 3.4, so (VDL2) does not hold.

4. $\mathcal{Q}$-categories

As we have mentioned in the introduction, one of our main goals is to present quasi-pseudometric modular spaces as a particular example of a $\mathcal{Q}$-category, that is, an enriched category over a commutative unital quantale (see [15, 16]). This will be developed in the next section but we first present a summary of the notions that will be needed.

Definition 4.1. ([15, Section III.1.3], c.f. [11, Definition 3.1]) Let $(\mathcal{Q}, \leq, *)$ be a commutative unital quantale. A $\mathcal{Q}$**-category** is a pair (X, q) where X is a nonempty set and $q : X \times X \to \mathcal{Q}$ is a map such that:

(QC1) $\top \leq q(x, x)$,
(QC2) $q(x, z) * q(z, y) \leq q(x, y)$,

for all $x, y, z \in X$.
A $\mathcal{Q}$**-functor** is a map $f : (X, a) \to (Y, b)$ between $\mathcal{Q}$-categories such that

$$a(x, y) \leq b(f(x), f(y))$$

for every $x, y \in X$.
$\mathcal{Q}$-categories and $\mathcal{Q}$-functors form a category denoted by $\mathcal{Q}$-Cat.

Definition 4.2. A $\mathcal{Q}$-category (X, q) is said to be:

- *separated* if given $x, y \in X$, whenever $\top \leq q(x, y)$ and $\top \leq q(y, x)$ then $x = y$.
- *symmetric* if $q(x, y) = q(y, x)$ for all $x, y \in X$.

We next provide several well-known examples of $\mathcal{Q}$-categories [10, 15].

Example 4.3. ([15, Example III.1.3.1.(1)]) **2**-categories and preordered sets are equivalent concepts. If (X, a) is a **2**-category, the binary relation $\leq_a$ on X given by

$$x \leq_a y \Leftrightarrow a(x, y) = 1$$

is a preorder on X. A similar argument allows to convert a preordered set $(X, \leq)$ into a **2**-category.

Furthermore, a **2**-functor between two **2**-categories (X, a) and (Y, b) is an isotone function between the preordered sets $(X, \leq_a)$ and $(Y, \leq_b)$. So **2**-Cat is isomorphic to the category of preordered sets and isotone maps.

Example 4.4. ([15, Example III.1.3.1.(2)]) P_+-categories (see Example 3.17) are extended quasi-pseudometric spaces.

If (X, a) is a P_+-category then $a : X \times X \to [0, +\infty]$ is a map verifying

$$0 \geq a(x, x) \qquad \text{and} \qquad a(x, z) + a(z, y) \geq a(x, y)$$

for all $x, y, z \in X$, so (X, a) is an **extended quasi-pseudometric space** [19] or an hemi-metric space [14].

Moreover, a P_+-functor between two P_+-categories (X, a), (Y, b) is map $f : (X, a) \to (Y, b)$ verifying

$$a(x, y) \geq b(f(x), f(y))$$

for all $x, y \in X$, that is, a nonexpansive mapping between the extended quasi-pseudometric spaces (X, a), (Y, b).

Thus, P_+-Cat is exactly the category EQPMet of extended quasi-pseudometric spaces.

In [10] (see also [6]), Flagg introduced a topology in a continuity space, that is, a $\mathcal{Q}$-category where $\mathcal{Q}$ is a value quantale. This topology was inspired by the classic open ball topology of a metric space, and the topology of the original continuity spaces of Kopperman [18], where metrics are valued in what he called a *value semigroup* (see [6]). This topology is important since it allows to prove that every topology comes from a $\mathcal{Q}$-category for a certain value quantale $\mathcal{Q}$ ([6, 10]). We recall the definition of this topology.

Definition 4.5. ([10, 11]) Let (X, a) be a $\mathcal{Q}$-category, with $\mathcal{Q}$ being a value quantale. Given $x \in X$ and $r \in \mathcal{Q}$ such that $r \lhd \top$, the **open ball** centered in x with radius r is defined as

$$B(x, r) := \{y \in X : r \lhd a(x, y)\}.$$

Proposition 4.6. ([10, 11]) *Let (X, a) be a $\mathcal{Q}$-category, with $\mathcal{Q}$ being a value quantale. Then $\{B(x, r) : r \lhd \top,\ x \in X\}$ is a base for a topology $\mathcal{T}(a)$ on X.*

5. Quasi-pseudometric modular spaces as Ω-categories

In this section, we address the main goal of the paper: to establish an equivalence between quasi-pseudometric modular spaces and certain Ω-categories. To achieve this, we need to consider a particular quantale that we define using the next few results.

Lemma 5.1. *Let us consider the set*

$$\nabla := \left\{ f : (0, +\infty) \to [0, +\infty]^{\mathrm{op}} \text{ such that } f \text{ is isotone} \right\}$$

endowed with the pointwise order induced by the order $\leq^{\mathrm{op}}$ *on the codomain* $[0, +\infty]$, *that is,*

$$f \leq^{\mathrm{op}} g \Leftrightarrow f(t) \leq^{\mathrm{op}} g(t), \text{ for all } t \in (0, \infty).$$

Then $(\nabla, \leq^{\mathrm{op}})$ *is a completely distributive lattice where the top and bottom elements are the constant 0 function denoted by* $\mathbf{0}$, *and the constant* ∞ *function denoted by* $\boldsymbol{\infty}$, *respectively.*

Proof. It is straightforward to verify that $(\nabla, \leq^{\mathrm{op}})$ is a partially ordered set. Moreover, it is easy to check that the supremum and infimum in ∇ are computed pointwisely, that is, if $F \subseteq \nabla$ then

$$\left(\bigvee^{\leq^{\mathrm{op}}} F \right)(t) = \bigvee^{\leq^{\mathrm{op}}} \{ f(t) : f \in F \}$$
$$\left(\bigwedge^{\leq^{\mathrm{op}}} F \right)(t) = \bigwedge^{\leq^{\mathrm{op}}} \{ f(t) : f \in F \}$$

for all $t > 0$. Hence $(\nabla, \leq^{\mathrm{op}})$ is a complete lattice.

Moreover, $([0, +\infty]^{(0,+\infty)}, \leq^{\mathrm{op}})$ is completely distributive since it is the Cartesian product of completely distributive lattices (see Remark 3.10.) Since $(\nabla, \leq^{\mathrm{op}})$ is a sublattice of $([0, +\infty]^{(0,+\infty)}, \leq^{\mathrm{op}})$ then it is completely distributive. □

Proposition 5.2. *Let*

$$\nabla_L = \left\{ f : (0, +\infty) \to [0, +\infty]^{\mathrm{op}} \text{ such that } f \text{ is isotone and } f(t) = \bigvee_{0<s<t}^{\leq^{\mathrm{op}}} f(s) \right\}.$$

Then $(\nabla_L, \leq^{\mathrm{op}})$ *is a complete sublattice of* $(\nabla, \leq^{\mathrm{op}})$.

Proof. We only prove that the supremum of a family $\mathcal{F} \subseteq \nabla_L$ belongs to ∇_L and that this supremum is computed pointwisely. In this way, let us define $F : (0, +\infty) \to [0, +\infty]^{\mathrm{op}}$ as $F(t) = \bigvee^{\leq^{\mathrm{op}}} \{ f(t) : f \in \mathcal{F} \}$ for all $t \in (0, +\infty)$.

It is obvious that F is isotone. Moreover, let $t \in (0, +\infty)$. Since F is isotone then $\bigvee_{0<s<t}^{\leq^{\mathrm{op}}} F(s) \leq^{\mathrm{op}} F(t)$. On the other hand, for all $f \in \mathcal{F}$ we have that $f(t) = \bigvee_{0<s<t}^{\leq^{\mathrm{op}}} f(s)$ so

$$F(t) = \bigvee^{\leq^{\mathrm{op}}} \{ f(t) : f \in \mathcal{F} \} = \bigvee^{\leq^{\mathrm{op}}} \Big\{ \bigvee_{0<s<t}^{\leq^{\mathrm{op}}} f(s) : f \in \mathcal{F} \Big\} \leq \bigvee_{0<s<t}^{\leq^{\mathrm{op}}} \Big\{ \bigvee^{\leq^{\mathrm{op}}} f(s) : f \in \mathcal{F} \Big\}$$
$$= \bigvee_{0<s<t}^{\leq^{\mathrm{op}}} F(s).$$

Therefore, $F \in \nabla_L$. □

To prove that $(\nabla, \leq^{\mathrm{op}})$ is a value distributive lattice, the following lemma will be useful.

Lemma 5.3. *Let* $\boldsymbol{\infty} \neq f \in \nabla$. *Then* $f \lhd^{\mathrm{op}} \mathbf{0}$ *if and only if*

(1) *there exists* $s \in (0, +\infty)$ *such that* $f(t) = \infty$ *for every* $t \in (0, s)$ *and*

(2) $\bigvee^{\leq^{\mathrm{op}}} \{f(t) : t \in (0, +\infty)\} <^{\mathrm{op}} 0$.

Proof. Suppose that $f \lhd^{\mathrm{op}} \mathbf{0}$.

We first prove (1). Suppose that $f(t) \neq \infty$ for every $t \in (0, +\infty)$. For each $n \in \mathbb{N}$, let $g_n : (0, +\infty) \to [0, +\infty]$ defined as

$$g_n(t) = \begin{cases} f(t) + 1 & \text{if } 0 < t < \frac{1}{n}, \\ 0 & \text{if } \frac{1}{n} \leq t. \end{cases}$$

It is obvious that $g_n \in \nabla$ for every $n \in \mathbb{N}$ and $\bigvee^{\leq^{\mathrm{op}}} g_n = \mathbf{0}$. However, $f \not\leq^{\mathrm{op}} g_n$ for every $n \in \mathbb{N}$, which contradicts $f \lhd^{\mathrm{op}} \mathbf{0}$.

Consequently, there exists $t_0 \in (0, +\infty)$ such that $f(t_0) = +\infty$. Since f is isotone then $f(t) = +\infty$ for very $t \leq t_0$. Define $s := \bigvee\{t \in (0, +\infty) : f(t) = \infty\} < \infty$ since $f \neq \boldsymbol{\infty}$. Therefore $f(t) = \infty$ for every $t \in (0, s)$.

We next prove (2). Suppose that $\bigvee^{\leq^{\mathrm{op}}} \{f(t) : t \in (0, +\infty)\} = 0$. For every $n \in \mathbb{N}$, let $h_n : (0, +\infty) \to [0, +\infty]$ defined as

$$h_n(t) = \frac{1}{n} \text{ for all } t \in (0, +\infty).$$

Obviously, $h_n \in \nabla$ for every $n \in \mathbb{N}$ and $\bigvee^{\leq^{\mathrm{op}}}_{n \in \mathbb{N}} h_n = \mathbf{0}$. However, given $n \in \mathbb{N}$ we can find $t_n \in (0, +\infty)$ such that $h_n(t_n) = \frac{1}{n} <^{\mathrm{op}} f(t_n) \leq^{\mathrm{op}} 0$. Hence $f \not\leq^{\mathrm{op}} h_n$ for all $n \in \mathbb{N}$, which contradicts $f \lhd^{\mathrm{op}} \mathbf{0}$.

Conversely, let $s \in (0, +\infty)$ such that $f(t) = \infty$ for very $t \in (0, s)$ and let $a = \bigvee^{\leq^{\mathrm{op}}} \{f(t) : t \in (0, +\infty)\} <^{\mathrm{op}} 0$. Let us prove that $f \lhd^{\mathrm{op}} \mathbf{0}$. Let $\mathcal{F} \subseteq \nabla$ such that $\bigvee^{\leq^{\mathrm{op}}} \mathcal{F} = \mathbf{0}$. Then we can find $g \in \mathcal{F}$ such that $a \leq^{\mathrm{op}} g(s)$. Now, if $t < s$, then by hypothesis (1), $f(t) = \infty \leq^{\mathrm{op}} g(t)$. On the other hand, if $t \geq s$, then by hypothesis (2) and the fact that g is isotone, $f(t) \leq^{\mathrm{op}} a \leq^{\mathrm{op}} g(s) \leq^{\mathrm{op}} g(t)$. □

Corollary 5.4. $(\nabla, \leq^{\mathrm{op}})$ *is a value distributive lattice.*

Proof. We already know by Lemma 5.1 that $(\nabla, \leq^{\mathrm{op}})$ is completely distributive.

Moreover, it is obvious that $\boldsymbol{\infty} \lhd^{\mathrm{op}} \mathbf{0}$.

Consider $f, g \in \nabla$ such that $f \lhd^{\mathrm{op}} \mathbf{0}$ and $g \lhd^{\mathrm{op}} \mathbf{0}$. By the previous lemma, we can find $s_f, s_g \in (0, +\infty)$ such that $f(t) = \infty$ for every $t \in (0, s_f)$ and $g(t) = \infty$ for every $t \in (0, s_g)$. Hence $(f \vee^{\leq^{\mathrm{op}}} g)(t) = \infty$ for every $t \in (0, s_f \wedge s_g)$ so $f \vee^{\leq^{\mathrm{op}}} g$ verifies condition (1) of the above lemma.

Furthermore, $\bigvee^{\leq^{\mathrm{op}}} \{f(t) : t \in (0, +\infty)\} <^{\mathrm{op}} 0$ and $\bigvee^{\leq^{\mathrm{op}}} \{g(t) : t \in (0, +\infty)\} <^{\mathrm{op}} 0$ which obviously implies $\bigvee^{\leq^{\mathrm{op}}} \{(f \vee^{\leq^{\mathrm{op}}} g)(t) : t \in (0, +\infty)\} <^{\mathrm{op}} 0$. By the preceding lemma, $f \vee^{\leq^{\mathrm{op}}} g \lhd^{\mathrm{op}} \mathbf{0}$. Consequently, $(\nabla, \leq^{\mathrm{op}})$ is a value distributive lattice. □

Proposition 5.5. *Consider the binary operation* $\oplus : \nabla \times \nabla \to \nabla$ *given by*

$$(f \oplus g)(t) := \bigvee_{r+s \leq t}^{\leq^{\mathrm{op}}} (f(r) + g(s)) = \bigvee_{r+s=t}^{\leq^{\mathrm{op}}} (f(r) + g(s))$$

for all $t > 0$. *Then* $(\nabla, \leq^{\mathrm{op}}, \oplus)$ *and* $(\nabla_L, \leq^{\mathrm{op}}, \oplus)$ *are CI-value quantales.*

Proof. It is straightforward to check that $f \oplus g \in \nabla$ for every $f, g \in \nabla$.

Given $t > 0$, we prove that

$$\bigvee_{r+s=t}^{\leq^{\mathrm{op}}} f(r) + g(s) = \bigvee_{r+s \leq t}^{\leq^{\mathrm{op}}} f(r) + g(s).$$

Notice that $\{f(r) + g(s) : r + s = t\} \subseteq \{f(r) + g(s) : r + s \leq t\}$, so

$$\bigvee_{r+s=t}^{\leq^{\mathrm{op}}} (f(r) + g(s)) \leq \bigvee_{r+s \leq t}^{\leq^{\mathrm{op}}} (f(r) + g(s)) = (f \oplus g)(t).$$

On the other hand, if $r + s \leq t$, then $s \leq t - r$. By isotonicity of g, $g(s) \leq^{\mathrm{op}} g(t - r)$ so

$$f(r) + g(s) \leq^{\mathrm{op}} f(r) + g(t-r) \leq^{\mathrm{op}} \bigvee_{r\leq t}^{\leq^{\mathrm{op}}} (f(r) + g(t-r)) = \bigvee_{r+s=t}^{\leq^{\mathrm{op}}} (f(r) + g(s)).$$

Hence,

$$(f \oplus g)(t) = \bigvee_{r+s\leq t}^{\leq^{\mathrm{op}}} (f(r) + g(s)) \leq^{\mathrm{op}} \bigvee_{r+s=t}^{\leq^{\mathrm{op}}} (f(r) + g(s)),$$

proving the desired equality.

We next check that $(\nabla, \oplus)$ is a commutative monoid.

First, commutativity is clear from the commutativity of the sum. Furthermore, given $f \in \nabla$,

$$(f \oplus \mathbf{0})(t) = \bigvee_{r+s=t}^{\leq^{\mathrm{op}}} f(r) + \mathbf{0}(s) = \bigvee_{r+s=t}^{\leq^{\mathrm{op}}} f(r) = f(t),$$

where the last inequality holds since f is isotone. So $\mathbf{0}$ is the neutral element for $\oplus$.

Finally, to prove the associativity property, one has that

$$\begin{aligned}((f \oplus g) \oplus h)(t) &= \bigvee_{r+s=t}^{\leq^{\mathrm{op}}} (f \oplus g)(r) + h(s) = \bigvee_{r+s=t}^{\leq^{\mathrm{op}}} \left(\bigvee_{u+v=r} f(u) + g(v) \right) + h(s) = \\ &= \bigvee_{\substack{r+s=t \\ u+v=r}}^{\leq^{\mathrm{op}}} f(u) + g(v) + h(s) = \bigvee_{u+v+s=t}^{\leq^{\mathrm{op}}} f(u) + g(v) + h(s),\end{aligned}$$

where the last equalities hold by the continuity of the sum. Since the final expression does not depend on the order of the elements, we can assure that $\oplus$ is associative.

To prove the distributivity property of $\oplus$ with respect to suprema, we only need to show it for one side because of the commutativity of the operation. For any $\{g_i\}_{i\in I} \subseteq \nabla$,

$$\begin{aligned}\left(f \oplus \bigvee_{i\in I}^{\leq^{\mathrm{op}}} g_i \right)(t) &= \bigvee_{r+s=t}^{\leq^{\mathrm{op}}} \left(f(r) + \bigvee_{i\in I} g_i(s) \right) = \bigvee_{r+s=t}^{\leq^{\mathrm{op}}} \bigvee_{i\in I}^{\leq^{\mathrm{op}}} f(r) + g_i(s) \\ &= \bigvee_{i\in I}^{\leq^{\mathrm{op}}} \bigvee_{r+s=t}^{\leq^{\mathrm{op}}} f(r) + g_i(s) = \left(\bigvee_{i\in I} (f \oplus g_i) \right)(t).\end{aligned}$$

Finally, since $f \leq^{\mathrm{op}} \mathbf{0}$ for all $f \in \nabla$ then $\mathbf{0} = \top$ so the quantale $(\nabla, \leq^{\mathrm{op}}, \oplus)$ is integral.

Additionally, a routine check shows that $f \oplus g \in \nabla_L$ for every $f, g \in \nabla_L$. Since $(\nabla_L, \leq^{\mathrm{op}})$ is a sublattice of $(\nabla, \leq^{\mathrm{op}})$ with the same top and bottom, then $(\nabla_L, \leq^{\mathrm{op}}, \oplus)$ is also a CI-quantale. □

We arrive at the main result of the paper that proves that the category of quasi-pseudometric modular spaces with nonexpansive maps is isomorphic to the category of ∇-categories.

Theorem 5.6. ∇-Cat *is isomorphic to* QPMod_n.

Proof. Let us define $\mathcal{E}_\nabla : \nabla\text{-}\mathsf{Cat} \to \mathsf{QPMod}_n$ leaving morphisms unchanged and $\mathcal{E}_\nabla((X, a)) = (X, w_a)$ for every ∇-category (X, a), where $w_a(t, x, y) = a(x, y)(t)$ for all $x, y \in X$, $t > 0$. It is clear that $\mathcal{E}_\nabla$ is a functor.

On the other hand, consider $\mathcal{E}_{\mathsf{Mod}} : \mathsf{QPMod}_n \to \nabla\text{-}\mathsf{Cat}$ leaving morphisms unchanged and $\mathcal{E}_{\mathsf{Mod}}((X, w)) = (X, a_w)$ for every quasi-pseudometric modular space (X, w), where $a_w(x, y)(t) = w(t, x, y)$ for all $x, y \in X$, $t > 0$. It is straightforward to check that (X, a_w) is a ∇-category (notice that, due to Proposition 2.3, $a_w(x, y)$ is isotone for every $x, y \in X$).

To check (QC1), notice that for any $x \in X$, $a_w(x,x)(t) = w(t,x,x) = 0$ by (M1). Hence, $a_w(x,x) = \mathbf{0}$, where $\mathbf{0}$ is the unit element of ∇.

We next prove (QC2). Take any $x, y, z \in X$. Then

$$\begin{aligned}(a_w(x,z) \oplus a_w(z,y))(t) &= \bigvee_{r+s=t}^{\leq^{\mathrm{op}}} (a_w(x,z)(r) + a_w(z,y)(s)) \\ &= \bigvee_{r+s=t}^{\leq^{\mathrm{op}}} (w(r,x,z) + w(s,z,y)) \leq^{\mathrm{op}} \\ &\leq^{\mathrm{op}} \bigvee_{r+s=t}^{\leq^{\mathrm{op}}} w(r+s,x,y) = w(t,x,y) = a_w(x,y)(t),\end{aligned}$$

for all $t > 0$. Consequently, $a_w(x,z) \oplus a_w(z,y) \leq^{\mathrm{op}} a_w(x,y)$.

Moreover, it is obvious that a nonexpansive function between quasi-pseudometric modular spaces is a ∇-functor between their corresponding ∇-categories.

Finally, it easily follows that $\mathcal{E}_{\mathsf{Mod}} \circ \mathcal{E}_{\nabla} = \mathcal{I}_{\nabla\text{-}\mathsf{Cat}}$ and $\mathcal{E}_{\nabla} \circ \mathcal{E}_{\mathsf{Mod}} = \mathcal{I}_{\mathsf{QPMod}_n}$. □

If we change the quantale ∇ by ∇_L, we obtain the category of left-continuous quasi-pseudometric modular spaces.

Theorem 5.7. ∇_L-Cat *is isomorphic to* LQPMod_n.

Proof. Let us define $\mathcal{E}_{\nabla_L}$ as the restriction of the functor $\mathcal{E}_{\nabla}$ to the category ∇_L-Cat. Notice that in this case $\mathcal{E}_{\nabla_L}((X,a)) = (X, w_a)$ is a left-continuous quasi-pseudometric modular space. In fact,

$$w_a(t,x,y) = a(x,y)(t) = \bigvee_{0<s<t}^{\leq^{\mathrm{op}}} a(x,y)(s) = w_a(s,x,y)$$

for every $x, y \in X$ and $t > 0$. Consequently, $\mathcal{E}_{\nabla_L} : \nabla_L\text{-}\mathsf{Cat} \to \mathsf{LQPMod}_n$ is well-defined.

The rest of the proof is similar to the previous one. □

Theorem 5.8. *The following diagram commutes:*

$$\begin{array}{ccc} \nabla\text{-}\mathsf{Cat} & \xrightarrow{\quad \mathcal{E}_{\nabla} \quad} & \mathsf{QPMod}_n \\ \downarrow \mathcal{U} & & \downarrow \mathcal{L} \\ \nabla_L\text{-}\mathsf{Cat} & \xrightarrow{\quad \mathcal{E}_{\nabla_L} \quad} & \mathsf{LQPMod}_n \end{array}$$

where $\mathcal{U} : \nabla\text{-}\mathsf{Cat} \to \nabla_L\text{-}\mathsf{Cat}$ *is the functor given by* $\mathcal{U}((X,a)) = (X, \widetilde{a})$ *and leaving morphisms unchanged, where* $\widetilde{a}$ *is given by*

$$\widetilde{a}(x,y)(t) = \bigwedge_{0<s<t} a(x,y)(s).$$

Proof. It is straightforward to check that $\mathcal{U}$ is a functor. By Remark 2.20, $\mathcal{L}$ is a functor.

Moreover,

$$w_{\widetilde{a}}(t,x,y) = \widetilde{a}(x,y)(t) = \bigwedge_{0<s<t} a(x,y)(s) = \bigwedge_{0<s<t} w_a(s,x,y) = \widetilde{w_a}(t,x,y)$$

for all $x, y \in X$ and all $t > 0$. Hence

$$(\mathcal{E}_{\nabla_L} \circ \mathcal{U})(X, a) = (X, w_{\widetilde{a}}) = (X, \widetilde{w_a}) = (\mathcal{L} \circ \mathcal{E}_{\nabla})(X, a)$$

which proves the commutativity of the diagram. □

We finish the paper by showing that the isomorphism between the categories ∇-Cat and QPMod_n also behaves well with respect to topology.

Theorem 5.9. *Let (X, w) be a quasi-pseudometric modular space. Then $\mathcal{T}(w) = \mathcal{T}(a_w)$.*

Proof. Let $G \in \mathcal{T}(w)$. Then given $x \in G$ there exists $t, \varepsilon > 0$ such that $x \in W_{t,\varepsilon}(x) \subseteq G$. Define $f_{t,\varepsilon} : (0, +\infty) \to [0, +\infty]$ by

$$f_{t,\varepsilon}(s) = \begin{cases} +\infty & \text{if } 0 < s < t \\ \varepsilon & \text{if } t \leq s \end{cases}.$$

By Lemma 5.3, $f_{t,\varepsilon} \lhd^{\mathrm{op}} \mathbf{0}$. We next show that $x \in B_{a_w}(x, f_{t,\varepsilon}) \subseteq W_{t,\varepsilon}(x) \subseteq G$. If $y \in B_{a_w}(x, f_{t,\varepsilon})$ then $f_{t,\varepsilon} \lhd^{\mathrm{op}} a_w(x, y)$. In particular, $f_{t,\varepsilon}(t) = \varepsilon <^{\mathrm{op}} a_w(x, y)(t) = w(t, x, y)$ so $y \in W_{t,\varepsilon}(x)$.

Conversely, let $O \in \mathcal{T}(a_w)$ and $x \in O$. Then we can find $g \in \nabla$ with $g \lhd^{\mathrm{op}} \mathbf{0}$ such that $B_{a_w}(x, g) \subseteq G$. By Lemma 5.3 we know that $\varepsilon := \bigvee^{\leq^{\mathrm{op}}}\{g(t) : t \in (0, +\infty)\} <^{\mathrm{op}} 0$ and there exists $s \in (0, +\infty)$ such that $g(t) = \infty$ for every $t \in (0, s)$. Let $0 < t_0 < s$. We assert that $x \in W_{t_0,\varepsilon}(x) \subseteq B_{a_w}(x, g) \subseteq G$. In fact, let $y \in W_{t_0,\varepsilon}(x)$, that is, $w(t_0, x, y) < \varepsilon$. Then $g \leq^{\mathrm{op}} f_{t_0,\varepsilon} \lhd^{\mathrm{op}} w(\_, x, y) = a_w(x, y)$ so $y \in B_{a_w}(x, f)$. □

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