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Isotropic space form Riemannian submersions

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Abstract. We introduce the notion of isotropic space form submersions between Riemannian manifolds in this paper. We begin with a concrete example to demonstrate this new concept. We then characterize isotropic space form submersions in terms of O'Neill's tensor field, $\tilde{\mathbf{T}}$, and explore some relationships between the sectional curvatures of the base manifold and the total manifold. Particularly, considering an isotropic lift $\tilde{\mathbf{M}}^{\ell}$ (where $\ell \geq 3$) into a space form $\tilde{\mathbf{N}}^{n+p}(\ddot{c})$ with constant $\ddot{c}$ sectional curvature, We demonstrate that the $\tilde{\mathbf{T}}$-fundamental tensor of $\tilde{\mathbf{N}}^{\ell+p}$ with respect to $\tilde{\mathbf{M}}^{\ell}$ is parallel if the mean curvature vector of $\tilde{\mathbf{M}}^{\ell}$ is parallel and the sectional curvature $\mathbb{K}$ of $\tilde{\mathbf{N}}^{\ell+p}$ satisfies a given inequality. Accordingly, $\tilde{\mathbf{N}}^{\ell+p}$ is a space form with lift.

1. Introduction

Let $\tilde{\mathbf{M}}$ be a submanifold of the Riemannian manifold $(\tilde{\mathbf{N}}, \tilde{g})$. If for a point $\tilde{q} \in \tilde{\mathbf{M}}$ and for any tangent vector $\tilde{Y}$ at $\tilde{q}$, the condition

$$g(h(\tilde{Y}, \tilde{Y}), h(\tilde{Y}, \tilde{Y})) = \tilde{\lambda}(\tilde{q}) g(\tilde{Y}, \tilde{Y})^2 \tag{1}$$

is satisfied, then the manifold $\tilde{\mathbf{M}}$ is said to be isotropic at the point $\tilde{q}$. Here, $\tilde{\lambda}$ is a function on $(\tilde{\mathbf{M}}, g)$, and h denotes the second fundamental form of the immersion. O'Neill first proposed the idea of isotropic submanifolds in [23]. Although all umbilical submanifolds are known to be isotropic, the opposite is not always true. A number of isotropic immersion-related topics have been examined in publications like [4, 19, 20].

The idea of Riemannian submersions was independently put forward by O'Neill [22] and Gray [14]. They were seen as the opposite of isometric immersions in semi-Riemannian, Lorentzian, almost Hermitian, and contact-complex submersions, among other contexts. Isometric immersions have been the main focus of differential geometry research on smooth maps, resulting in the production of multiple volumes and monographs on the subject [7–9]. Riemannian submersions have been the focus of a number of specialized works, despite being less well studied [13, 17, 18, 26]. Numerous studies on Riemannian submersions have been published recently, and many different kinds of Riemannian submersions titled semi-Riemannian submersions, Lorentzian submersions [13], almost Hermitian submersions [29], contact-complex submersions

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[17], and quaternionic submersions [18] have been introduced under multiple names. Riemannian submersions have been used practically in different areas of study in addition to their theoretical advancement. Notably, they have been utilized in robotics for forward kinematics [2] and mathematical physics, namely in the Kaluza-Klein model and superstring theories [5, 13].

Sahin [25] recently introduced the concept of anti-invariant Riemannian submersions, which are Riemannian submersions defined on almost Hermitian manifolds, where the vertical distribution is invariant under the almost complex structure in an anti-invariant manner. This concept has been extended and explored in numerous other works, including [1, 3, 6, 10, 12, 15, 16, 21, 24, 26–28, 30].

The idea of isotropic space form submersions is put forward in this paper together with an analysis of the connection between the base and total manifolds in this context. The paper is organized as follows: We give an overview of the fundamental ideas needed to develop the subject in Section 2. The definition of isotropic space form submersions and the necessary and sufficient requirements for these submersions to possess qualities like being minimum, entirely geodesic, and totally umbilical are established in Section 3. We analyze the connection between the base manifold's and the total manifold's curvatures. O'Neill's tensor field $\tilde{\mathbf{T}}$ is applied to analyze this relationship, and we additionally investigate the connections that exist among the sectional curvatures of the two manifolds. In particular, for a space form $\tilde{\mathbf{N}}^{\ell+p}$ with constant sectional curvature $\ddot{c}$ and an isotropic lift $\tilde{\mathbf{M}}^{\ell}$ (with $\ell \geq 3$), we demonstrate that the fundamental tensor $\tilde{\mathbf{T}}$ of $\tilde{\mathbf{N}}^{\ell+p}$ in $\tilde{\mathbf{M}}^{\ell}$ is parallel if the mean curvature vector of $\tilde{\mathbf{M}}^{\ell}$ is parallel and the sectional curvature $\breve{K}$ of $\tilde{\mathbf{N}}^{\ell}$ satisfies particular criteria. In turn, a space form is formed by the lift in $\tilde{\mathbf{N}}^{\ell+p}$.

2. Preliminaries

Let $\tilde{\mathbf{M}}$ and $\tilde{\mathbf{N}}$ be Riemannian manifolds of dimensions m and n, respectively. A smooth map $\tilde{\theta} : \tilde{\mathbf{M}} \longrightarrow \tilde{\mathbf{N}}$ is called a Riemannian submersions if it fulfills the following criteria.

(S1) The map $\tilde{\theta}$ is required to have max rank.

(S2) The lengths of horizontal vectors are preserved for every point $\tilde{q} \in \tilde{\mathbf{M}}$ by the differential $\tilde{\theta}_*$.

The fiber $\tilde{\theta}_x = \tilde{\theta}^{-1}(x)$ creates a submanifold of $(\tilde{\mathbf{M}}, g)$ with dimension $s = (m - n)-$ for any point $x \in \tilde{\mathbf{N}}$. The kernel of the differential map $\tilde{\theta}_{*\tilde{q}}$ denoted by $\tilde{\mathbf{M}}$ is defined by $\dot{\vartheta}_{\tilde{q}} = Ker\tilde{\theta}_{*\tilde{q}}$, defines an integrable distribution on $\tilde{\mathbf{M}}$. This distribution is referred to as vertical distribution of submersion $\tilde{\theta}$.

We observe that for any point $\tilde{q} \in \tilde{\mathbf{M}}$, the tangent space of the submanifold $\tilde{\theta}^{-1}$ is identical. The distribution $\dot{\varkappa}_{\tilde{q}} = \left(\dot{\vartheta}_{\tilde{q}}\right)^{\perp}$, which is orthogonal and complementary to the vertical distribution, is called the horizontal distribution of the submersion. Thus for every $\tilde{q} \in \tilde{\mathbf{M}}$,

$$\tilde{T}_{\tilde{q}}\tilde{\mathbf{M}} = \dot{\vartheta}_{\tilde{q}} \oplus \dot{\varkappa}_{\tilde{q}} = \dot{\vartheta}_{\tilde{q}} \oplus \left(\dot{\vartheta}_{\tilde{q}}\right)^{\perp}.$$

O'Neill's tensors $\tilde{\mathbf{T}}$ and $\tilde{\mathbf{A}}$ provide a description of the geometry of a Riemannian submersion. These tensors are defined for vector fields $\tilde{\mathbb{E}}$ and $\tilde{\mathbb{F}}$ on the manifold $\tilde{\mathbf{M}}$ as follows:

$$\tilde{\mathbf{A}}_{\tilde{\mathbb{E}}}\tilde{\mathbb{F}} = \dot{\varkappa}\nabla_{\dot{\varkappa}\tilde{\mathbb{E}}}\dot{\vartheta}\tilde{\mathbb{F}} + \dot{\vartheta}\nabla_{\dot{\varkappa}\tilde{\mathbb{E}}}\dot{\varkappa}\tilde{\mathbb{F}}, \ \tilde{\mathbf{T}}_{\tilde{\mathbb{E}}}\tilde{\mathbb{F}} = \dot{\varkappa}\nabla_{\dot{\vartheta}\tilde{\mathbb{E}}}\dot{\vartheta}\tilde{\mathbb{F}} + \dot{\vartheta}\nabla_{\dot{\vartheta}\tilde{\mathbb{E}}}\dot{\varkappa}\tilde{\mathbb{F}} \tag{2}$$

where ∇ is the Levi-Civita connection of $\tilde{\mathbf{M}}$. It is clear that a Riemannian submersion $\tilde{\theta} : (\tilde{\mathbf{M}}^m, g) \to (\tilde{\mathbf{N}}^n, \tilde{g})$ has totally geodesic fibres if and only if $\tilde{\mathbf{T}}$ vanishes everywhere. It is also straightforward to observe that $\tilde{\mathbf{T}}$ is vertical, $\tilde{\mathbf{T}}_{\tilde{\mathbb{E}}} = \tilde{\mathbf{T}}_{\dot{\vartheta}\tilde{\mathbb{E}}}$, and $\tilde{\mathbf{A}}$ is horizontal, $\tilde{\mathbf{A}}_{\tilde{\mathbb{E}}} = \tilde{\mathbf{A}}_{\dot{\varkappa}\tilde{\mathbb{E}}}$. Additionally, we note that the tensor field $\tilde{\mathbf{T}}$ satisfies

$$\tilde{\mathbf{T}}_{\tilde{U}}\tilde{W} = \tilde{T}_{\tilde{W}}\tilde{U} \qquad \forall \tilde{U}, \tilde{W} \in \Gamma(\ker \tilde{\theta}_*),$$

from here we have

$$\nabla_{\tilde{V}}\tilde{W} = \tilde{\mathbf{T}}_{\tilde{V}}\tilde{W} + \hat{\nabla}_{\tilde{V}}\tilde{W} \qquad \forall \tilde{V}, \tilde{W} \in \Gamma(\ker \tilde{\theta}_*),$$

where $\hat{\nabla}_{\tilde{V}}\tilde{W} = \dot{\vartheta}\nabla_{\tilde{V}}\tilde{W}$.

Now, let's go over some of the theorems and lemmas that will be referenced throughout this paper.

Theorem 2.1. *[22] Let $(\tilde{\mathbf{M}}^m, g)$ and $(\tilde{\mathbf{N}}^n, \tilde{g})$ be Riemannian manifolds with $\tilde{\theta} : (\tilde{\mathbf{M}}^m, g) \to (\tilde{\mathbf{N}}^n, \tilde{g})$ denote a Riemannian submersion. Assuming that $\alpha : I \to \tilde{\mathbf{M}}$ is a regular curve, let $\mathbb{E}_1(t)$ and $\tilde{W}(t)$ stand for the horizontal and vertical components of its tangent vector field, respectively. On $\tilde{\mathbf{M}}$, α is a geodesic if and only if*

$$(\hat{\nabla}_{\dot{\alpha}}\mathbb{E}_1 + \tilde{\mathbf{T}}_{\tilde{W}}\mathbb{E}_1)(t) = 0$$

and

$$(\hat{\nabla}_{\mathbb{E}_1}\mathbb{E}_1 + 2\tilde{\mathbf{A}}_{\mathbb{E}_1}\tilde{W} + \tilde{\mathbf{T}}_{\tilde{W}}\tilde{W})(t) = 0,$$

where $\mathbb{E}_1 \in \chi(\tilde{\mathbf{M}})$ and $\hat{\nabla}$ is Schouten connection.

Lemma 2.2. *[22] Let $\tilde{\theta} : (\tilde{\mathbf{M}}^m, g) \to (\tilde{\mathbf{N}}^n, \tilde{g})$ be a Riemannian submersion betweeen $(\tilde{\mathbf{M}}^m, g)$ and $(\tilde{\mathbf{N}}^n, \tilde{g})$ Riemannian manifolds. For*

$$\begin{aligned}(\nabla_{\tilde{V}}\tilde{\mathbf{A}})_{\tilde{W}} &= -\tilde{\mathbf{A}}_{\tilde{\mathbf{T}}_{\tilde{V}}\tilde{W}}, \qquad (\nabla_{\tilde{X}}\tilde{\mathbf{T}})_{\tilde{Y}} = -\tilde{\mathbf{T}}_{\tilde{\mathbf{A}}_{\tilde{X}}\tilde{Y}},\\ (\nabla_{\tilde{X}}\tilde{\mathbf{A}})_{\tilde{W}} &= -\tilde{\mathbf{A}}_{\tilde{\mathbf{A}}_{\tilde{X}}\tilde{W}} \qquad (\nabla_{\tilde{V}}\tilde{\mathbf{T}})_{\tilde{Y}} = -\tilde{\mathbf{T}}_{\tilde{\mathbf{T}}_{\tilde{V}}\tilde{Y}},\end{aligned}$$

where $\tilde{X}, \tilde{Y} \in \chi^v(\tilde{\mathbf{M}})$ and $\tilde{W}, \tilde{V} \in \chi^h(\tilde{\mathbf{M}})$

Theorem 2.3. *[22] Let $\tilde{\theta} : (\tilde{\mathbf{M}}^m, g) \to (\tilde{\mathbf{N}}^n, \tilde{g})$ be a Riemannian submersion between Riemann manifolds $(\tilde{\mathbf{M}}^m, g)$ and $(\tilde{\mathbf{N}}^n, \tilde{g})$. Sectional curvatures $\mathbb{K}$, $\hat{\mathbb{K}}$, and $\dot{\mathbb{K}}$, which represent the whole space, base space, and fibers, respectively, are thus represented.*

$$\begin{aligned}\mathbb{K}(\tilde{U}, \tilde{V}) &= \dot{\mathbb{K}}(\tilde{U}, \tilde{V}) + \left\|\tilde{\mathbf{T}}_{\tilde{U}}\tilde{V}\right\|^2 - g(\tilde{\mathbf{T}}_{\tilde{U}}\tilde{U}, \tilde{\mathbf{T}}_{\tilde{V}}\tilde{V}),\\ \mathbb{K}(\tilde{X}, \tilde{Y}) &= \hat{\mathbb{K}}(\tilde{X}, \tilde{Y}) \circ \tilde{\theta} - 3\left\|\tilde{\mathbf{A}}_{\tilde{X}}\tilde{Y}\right\|^2,\\ \mathbb{K}(\tilde{X}, \tilde{V}) &= g((\nabla_{\tilde{X}}\tilde{\mathbf{T}})_{\tilde{V}}\tilde{V}, \tilde{X}) + \left\|\tilde{\mathbf{T}}_{\tilde{V}}\tilde{X}\right\|^2 - \left\|\tilde{\mathbf{A}}_{\tilde{X}}\tilde{V}\right\|^2,\end{aligned}$$

for $\tilde{X}, \tilde{Y} \in \chi^h(\tilde{\mathbf{M}})$ and $\tilde{U}, \tilde{V} \in \chi^v(\tilde{\mathbf{M}})$.

It is important to remember that a Riemannian submersion $\tilde{\theta}$ is said to have totally umbilical fibers if the following criteria is met:

$$\tilde{\mathbf{T}}_{\tilde{V}}\tilde{W} = g(\tilde{V}, \tilde{W})H,$$

where $\Gamma(\ker \tilde{\theta}_*)$ contains $\tilde{V}, \tilde{W}$ and H stands for the fibers' mean curvature vector field.

Definition 2.4. *[23] Let $(\tilde{\mathbf{M}}, g)$ be a submanifold embedded in a Riemannian manifold $(\tilde{\mathbf{N}}, \tilde{g})$. For a point $\tilde{q}$ on $\tilde{\mathbf{M}}$ and any tangent vector $\tilde{Y}$ at $\tilde{q}$, if the equation (1.1) is satisfied, then $\tilde{\mathbf{M}}$ is said to be isotropic at the point $\tilde{q}$. Here, $\tilde{\lambda}$ is a function on $(\tilde{\mathbf{M}}, g)$, and h denotes the second fundamental form of the immersion.*

Definition 2.5. *[11] $\tilde{\theta} : \tilde{\mathbf{M}} \longrightarrow \tilde{\mathbf{N}}$ is known to be a Riemannian submersion. If the following holds for $u_1 \in \chi^v(\tilde{\mathbf{M}})$*

$$g(\tilde{\mathbf{T}}_{u_1}u_1, \tilde{\mathbf{T}}_{u_1}u_1) = \tilde{\lambda}g(u_1, u_1)^2,$$

$\tilde{\theta}$ is called $\tilde{\lambda}$-isotropic for every $\tilde{q} \in \tilde{\mathbf{M}}$. For each $\tilde{q} \in \tilde{\mathbf{M}}$, if $\tilde{\lambda}$ is constant, then $\tilde{\theta}$ is known as $\tilde{\lambda}-$ constant isotropic.

Theorem 2.6. *[11] Let $\tilde{\theta} : (\tilde{\mathbf{M}}, g) \to (\tilde{\mathbf{N}}, \tilde{g})$ be a Riemannian submersion betweeen Riemannian manifolds $\tilde{\mathbf{M}}$ and $\tilde{\mathbf{N}}$. If $\tilde{\theta}$ is $\tilde{\lambda}$-isotropic, then for any orthogonal vectors $\tilde{Y}, \tilde{U} \in \chi^v(\tilde{\mathbf{M}})$, we have*

$$g(\tilde{\mathbf{T}}_{\tilde{Y}}\tilde{Y}, \tilde{\mathbf{T}}_{\tilde{Y}}\tilde{U}) = 0.$$

We now give an example of isotropic submersions.

Example 2.7. *[11] Let (R^4, g) and $(R^2, \tilde{g})$ be Riemannian manifolds and $g = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$ and $\tilde{g} = \frac{1-\lambda}{\lambda}(dx_1^2 + dx_2^2)$, $0 < \lambda < 1$. Let's look at the map below.*

$$\begin{aligned} \tilde{\phi} : R^4 &\rightarrow R^2 \\ (x_1, x_2, x_3, x_4) &\rightarrow (\sqrt{1 - x_1^2 - x_2^2}, \sqrt{1 - x_3^2 - x_4^2}), \end{aligned}$$

such that $x_1^2 + x_2^2 = \lambda < 1$, $x_3^2 + x_4^2 = \lambda < 1$. Given that rank$\tilde{\phi}_* = \dim R^2$, $\tilde{\phi}$ is a submersion. Taking the above condition into account, we obtain

$$\ker \tilde{\phi}_* = \dot{\vartheta} = span\,\{\tilde{v} = (x_2, -x_1, 0, 0), \tilde{u} = (0, 0, x_4, -x_3)\}$$

$$\ker \tilde{\phi}_*^{\perp} = \dot{\varkappa} = span\left\{\tilde{X} = (-x_1, -x_2, 0, 0), \tilde{Y} = (0, 0, -x_3, -x_4)\right\}.$$

Given that we have

$$\begin{aligned} g(\tilde{\phi}_*\tilde{X}, \tilde{\phi}_*\tilde{X}) &= \tilde{g}(\tilde{X}, \tilde{X}), \\ g(\tilde{\phi}_*\tilde{Y}, \tilde{\phi}_*\tilde{Y}) &= \tilde{g}(\tilde{Y}, \tilde{Y}). \end{aligned}$$

g is a submersion of Riemannians. In addition, we discover that

$$\nabla_{\tilde{v}}\tilde{v} = \tilde{X} \in \ker \tilde{\phi}_*^{\perp},\ \nabla_{\tilde{u}}\tilde{u} = \tilde{Y} \in \ker \tilde{\phi}_*^{\perp}.$$

Thus, we demonstrate that $\tilde{\phi}$ is an isotropic Riemannian submersion.

Lemma 2.8. *[11]Let $\tilde{\theta} : (\tilde{\mathbf{M}}^m, g) \rightarrow (\tilde{\mathbf{N}}^n, \tilde{g})$ be a $\tilde{\lambda}$- isotropic Riemannian submersion. Then we have*

1. $$g(\tilde{\mathbf{T}}_{u_1}u_1, \tilde{\mathbf{T}}_{u_2}u_2) + 2g(\tilde{\mathbf{T}}_{u_1}u_2, \tilde{T}_{u_1}u_2) = \tilde{\lambda},\ for\ \|u_1\| = \|u_2\| = 1,$$

2. $$g(\tilde{\mathbf{T}}_{u_1}u_1, \tilde{\mathbf{T}}_{u_3}u_4) + 2g(\tilde{\mathbf{T}}_{u_1}u_3, \tilde{\mathbf{T}}_{u_1}u_4) = 0,$$

3. $$g(\tilde{\mathbf{T}}_{u_1}u_2, \tilde{\mathbf{T}}_{u_3}u_4) + (\tilde{\mathbf{T}}_{u_1}u_3, \tilde{\mathbf{T}}_{u_2}u_4) + g(\tilde{\mathbf{T}}_{u_1}u_4, \tilde{\mathbf{T}}_{u_2}u_3) = 0,$$

for all orthogonal $u_1, u_2, u_3, u_4 \in \chi^v(\tilde{\mathbf{M}})$.

Lemma 2.9. *[11]Let $\tilde{\theta} : (\tilde{\mathbf{M}}^m, g) \rightarrow (\tilde{\mathbf{N}}^n, \tilde{g})$ be a $\tilde{\lambda}$ isotropic Riemannian submersion. We have the following for orthonormal $u_1, v_1 \in \chi^v(\tilde{\mathbf{M}})$*

$$\mathbb{K}(u_1, v_1) = \hat{\mathbb{K}}(u_1, v_1) + 3\left\|\tilde{\mathbf{T}}_{u_1}v_1\right\|^2 - \tilde{\lambda}, \tag{3}$$

$$2\mathbb{K}(u_1, v_1) = 2\hat{\mathbb{K}}(u_1, v_1) - 3g(\tilde{\mathbf{T}}_{u_1}u_1, \tilde{\mathbf{T}}_{v_1}v_1) + \tilde{\lambda}. \tag{4}$$

Using the previously mentioned Lemma 2.9, we arrive at the following conclusion.

Proposition 2.10. *[11]Let $\tilde{\theta} : (\tilde{\mathbf{M}}^m, g) \rightarrow (\tilde{\mathbf{N}}^n, \tilde{g})$ be a $\tilde{\lambda}$- isotropic Riemannian submersion. The following expressions are equivalent for orthonormal $u_1, v_1 \in \chi^v(\tilde{\mathbf{M}})$ the following expressions are equivalent*

1. $\mathbb{K}(u_1, v_1) = \hat{\mathbb{K}}(u_1, v_1) - \tilde{\lambda}$,

2. $\tilde{\mathbf{T}}_{u_1}v_1 = 0$.

Applying Proposition 2.1, we obtain the following result.

Theorem 2.11. *[11]Let $\tilde{\theta} : (\tilde{\mathbf{M}}^m, g) \to (\tilde{\mathbf{N}}^n, \tilde{g})$ be a $\tilde{\lambda}$- isotropic Riemannian submersion between $\tilde{\mathbf{M}}$ and $\tilde{\mathbf{N}}$. The relation $\mathbb{K}(\tilde{\mathcal{P}}) = \hat{\mathbb{K}}(\tilde{\mathcal{P}}) - \tilde{\lambda}$ is satisfied for any vertical plane $\tilde{\mathcal{P}}$ spanned by the vectors u_1 and v_1 if and only if $\tilde{\theta}$ is a Riemannian submersion with minimal fibers.*

The results of applying Proposition 2.10 are as follows.

Corollary 2.12. *[11]Consider the Riemannian submersion $\tilde{\theta} : (\tilde{\mathbf{M}}^m, g) \to (\tilde{\mathbf{N}}^n, \tilde{g})$ which is $\tilde{\lambda}-$ isotropic. Denote by $\{u_\ell\}_{1\leq \ell \leq r}$ a local orthonormal frame for the vertical distribution of $\tilde{\mathbf{M}}^m$. Let $\mathbb{N}^*$ be a horizontal vector field on $(\tilde{\mathbf{M}}^m, g)$, where*

$$\|\mathbb{N}^*\|^2 = s^2\tilde{\lambda}.$$

Corollary 2.13. *[11]Let $\tilde{\theta} : (\tilde{\mathbf{M}}^m, g) \to (\tilde{\mathbf{N}}^n, \tilde{g})$ be a $\tilde{\lambda}-$ isotropic Riemannian submersion. When considering a geodesic curve $\alpha : I \to \tilde{\mathbf{M}}^m$, the curve $\gamma = \tilde{\theta} \circ \alpha$ is geodesic if and only if the condition $2\tilde{\mathbf{A}}_{\mathbb{E}_1}\tilde{W} + \tilde{\mathbf{T}}_{\tilde{W}}\tilde{W} = 0$ is fulfilled.*

$$\left\|\tilde{\mathbf{A}}_{\mathbb{E}_1}\tilde{W}\right\|^2 = \frac{\tilde{\lambda}}{4}\left\|\tilde{W}\right\|^2,$$

where the horizontal component of curve α is $\mathbb{E}_1(t)$, and the vertical component is $\tilde{W}(t)$.

Let $\tilde{\theta} : (\tilde{\mathbf{M}}^m, g) \to (\tilde{\mathbf{N}}^n, \tilde{g})$ be a $\tilde{\lambda}-$ isotropic Riemannian submersion, then we have

$$g(\tilde{\mathbf{T}}_{u_1}u_1, \tilde{\mathbf{T}}_{u_1}u_1) = \tilde{\lambda}.$$

As a result, we obtain

$$g(\nabla_{u_1}\tilde{\mathbf{T}}_{u_1}u_1, \tilde{\mathbf{T}}_{u_1}u_1) = 0 \Rightarrow g((\nabla_{u_1}\tilde{\mathbf{T}}_1)_{u_1}u_1, \tilde{\mathbf{T}}_{u_1}u_1) + 2g(\tilde{\mathbf{T}}_{u_1}\nabla_{u_1}u_1, \tilde{\mathbf{T}}_{u_1}u_1) = 0.$$

If $(\nabla_{u_1}\tilde{\mathbf{T}})_{u_1}u_1 = 0$, then $g(\tilde{\mathbf{T}}_{u_1}\nabla_{u_1}u_1, \tilde{\mathbf{T}}_{u_1}u_1) = 0$. If $\nabla_{u_1}u_1 \in \chi^h(\tilde{\mathbf{M}}^m)$, then we can conclude that $\nabla_{u_1}u_1 \perp u_1 \in \dot{\vartheta}_{\tilde{q}}$. If $\nabla_{u_1}u_1 \in \chi^h(\tilde{\mathbf{M}}^m)$, then we find $\tilde{\mathbf{T}}_{u_1}u_1 = 0$. Hence we conclude that $\tilde{\mathbf{M}}$ is totally geodesic when $\tilde{\mathbf{T}} = 0$. The opposite is easy to see. As a consequence, the following theorem emerges.

Theorem 2.14. *[11]Let $\tilde{\theta} : (\tilde{\mathbf{M}}^m, g) \to (\tilde{\mathbf{N}}^n, \tilde{g})$ be a constant $\tilde{\lambda}$- isotropic Riemannian submersion. For any $u_1 \in \dot{\vartheta}_{\tilde{q}}$, if $(\nabla_{u_1}\tilde{\mathbf{T}})_{u_1}u_1 = 0$, then one of the following is true:*

1. $\nabla_{u_1}u_1 \perp u_1$ *in* $\chi^v(\tilde{\mathbf{M}}^m)$.
2. *With totally geodesic fibers, $\tilde{\theta}$ is a Riemannian submersion.*

3. Isotropic Riemannian Space Form

In this section, we define isotropic space form submersions between Riemannian manifolds and explore their properties. We give a characterisation and investigate the relationships between the sectional curvatures of the base space and the total space using the O'Neill's tensor field $\tilde{\mathbf{T}}$. For an isotropic lift $\tilde{\mathbf{M}}^\ell(\ell \geq 3)$ over a space form $\tilde{\mathbf{N}}^{\ell+p}(\ddot{c})$ having constant sectional curvature $\ddot{c}$, We show that if the sectional curvature $\tilde{\mathbb{K}}$ of $\tilde{\mathbf{N}}^\ell$ meets a particular inequality and the mean curvature vector of $\tilde{\mathbf{M}}^\ell$ is parallel, The fundamental tensor $\tilde{\mathbf{T}}$ of $\tilde{\mathbf{N}}^{\ell+p}$ in $\tilde{\mathbf{M}}^\ell$ is parallel, therefore. This suggests that a space form is formed by the lift in $\tilde{\mathbf{N}}^{\ell+p}$. Furthermore, in isotropic Riemannian submersions, some relations are known to hold.

$$
\begin{aligned}
< \tilde{\mathbf{T}}_{u_1}u_1, \tilde{\mathbf{T}}_{u_1}u_1 > &= \tilde{\lambda}^2 < u_1, u_1 > \qquad (5)\\
< \tilde{\mathbf{T}}_{u_1}u_1, \tilde{\mathbf{T}}_{u_1}u_2 > &= 0\\
\begin{pmatrix} < \tilde{\mathbf{T}}_{u_1}u_2, \tilde{\mathbf{T}}_{u_3}u_4 > \\ + < \tilde{\mathbf{T}}_{u_1}u_3, \tilde{\mathbf{T}}_{u_2}u_4 > \\ + < \tilde{\mathbf{T}}_{u_1}u_4, \tilde{\mathbf{T}}_{u_2}u_3 > \end{pmatrix} &= \tilde{\lambda}^2 \begin{pmatrix} < u_1, u_2 >< u_3, u_4 > \\ + < u_1, u_3 >< u_2, u_4 > \\ + < u_1, u_4 >< u_2, u_3 > \end{pmatrix} \qquad (6)
\end{aligned}
$$

for any $u_1, u_2, u_3, u_4 \in \chi^v(\tilde{\mathbf{M}}(\dot{c}))$. Let $\tilde{\mathbf{M}}(\dot{c})$ and $\tilde{\mathbf{N}}(\ddot{c})$ are Riemannian manifolds and

$$\tilde{\theta} : (\tilde{\mathbf{M}}(\dot{c}), g) \longrightarrow (\tilde{\mathbf{N}}(\ddot{c}), \tilde{g})$$

is a Riemannian submersion and $\tilde{R}, \dot{\tilde{R}}$ are curvature tensors of $\tilde{\mathbf{M}}$ and leaf $(\tilde{\theta}^{-1}(p), \tilde{g}_p)$, respectively. The equations are

$$
\begin{aligned}
< \tilde{R}(u_1, v_1)\tilde{w}, f_1 > &= < \dot{\tilde{R}}(u_1, v_1)\tilde{w}, f_1 > + < \tilde{\mathbf{T}}_{u_1}\tilde{w}, \tilde{\mathbf{T}}_{v_1}f_1 > - < \tilde{\mathbf{T}}_{v_1}\tilde{w}, \tilde{\mathbf{T}}_{u_1}f_1 > \qquad (7)\\
< \tilde{R}(u_1, v_1)\tilde{w}, \tilde{x} > &= < (\nabla_{u_1}\tilde{\mathbf{T}})_{v_1}\tilde{w}, \tilde{x} > - < (\nabla_{v_1}\tilde{\mathbf{T}})_{u_1}\tilde{w}, \tilde{x} > \qquad (8)
\end{aligned}
$$

Gauss and Codazzi equations for submersions, respectively, for $u_1, v_1, \tilde{w}, f_1 \in \chi^v(\tilde{\mathbf{M}})$ and $\tilde{x} \in \chi^h(\tilde{\mathbf{M}})$. If both spaces with constant curvature, then we obtain

$$
\begin{aligned}
\langle \tilde{\mathbf{T}}(u_1, u_2), \tilde{\mathbf{T}}(u_3, u_4)\rangle - \langle \tilde{\mathbf{T}}(u_3, u_2), \tilde{\mathbf{T}}(u_1, u_4)\rangle &= (\dot{c} - \ddot{c})\{\langle u_1, u_2\rangle\langle u_3, u_4\rangle - \langle u_3, u_2\rangle\langle u_1, u_4\rangle\}\\
(\nabla_{u_1}\tilde{\mathbf{T}})(u_2, u_3) &= (\nabla_{u_2}\tilde{\mathbf{T}})(u_1, u_3).
\end{aligned}
$$

Now let's prepare a lemma. Let's define

$$\Delta_{u_1u_2} = \left\langle \tilde{\mathbf{T}}_{u_1}u_1, \tilde{\mathbf{T}}_{u_2}u_2 \right\rangle - \left\| \tilde{\mathbf{T}}_{u_1}u_2 \right\|^2 \qquad (9)$$

for orthonormal $u_1, u_2 \in \chi^{\mathfrak{F}}(\tilde{\mathbf{M}}(\dot{c}))$. Thus, from the result

$$\left\langle \tilde{\mathbf{T}}_{u_1}u_1, \tilde{\mathbf{T}}_{u_2}u_2 \right\rangle + 2\left\langle \tilde{\mathbf{T}}_{u_1}u_2, \tilde{\mathbf{T}}_{u_1}u_2 \right\rangle = \tilde{\lambda}^2, \qquad \|u_1\| = \|u_2\| = 1 \qquad (10)$$

for isotropic Riemannian submersions, we obtain the following lemma.

Lemma 3.1. *If $\tilde{\mathbf{T}}$ is $\tilde{\lambda}$- isotropic*

$$
\begin{aligned}
\Delta_{u_1u_2} + 3\left\| \tilde{\mathbf{T}}_{u_1}u_2 \right\|^2 &= \tilde{\lambda}^2\\
2\Delta_{u_1u_2} + \tilde{\lambda}^2 &= 3\left\langle \tilde{\mathbf{T}}_{u_1}u_1, \tilde{\mathbf{T}}_{u_2}u_2 \right\rangle
\end{aligned}
$$

for $u_1, u_2 \in \chi^{\mathfrak{F}}(\tilde{\mathbf{M}}(\dot{c}))$.

Proof. If we subtitude $\left\| \tilde{\mathbf{T}}_{u_1}u_2 \right\|^2$ into equation (10) and simplify, we obtain,

$$\left\langle \tilde{\mathbf{T}}_{u_1}u_1, \tilde{\mathbf{T}}_{u_2}u_2 \right\rangle + 2\left\langle \tilde{\mathbf{T}}_{u_1}u_2, \tilde{\mathbf{T}}_{u_1}u_2 \right\rangle + \left\| \tilde{\mathbf{T}}_{u_1}u_2 \right\|^2 - \left\| \tilde{\mathbf{T}}_{u_1}u_2 \right\|^2 = \tilde{\lambda}^2,$$

from this, we derive

$$\Delta_{u_1u_2} + 3\left\| \tilde{\mathbf{T}}_{u_1}u_2 \right\|^2 = \tilde{\lambda}^2,$$

which proves first equation. Similarly, if equation (9) is taken into account in the expression (10), we obtain

$$3\left\langle \tilde{\mathbf{T}}_{u_1}u_1, \tilde{\mathbf{T}}_{u_2}u_2 \right\rangle = 2\Delta_{u_1u_2} + \tilde{\lambda}^2,$$

which gives us the proof of the second equation of the above lemma.

□

Corollary 3.2. *If $\tilde{\mathbf{T}}$ is $\tilde{\lambda}$- isotropic, the followings are equivalent.*

1 $\Delta_{u_1 u_2} = -2\tilde{\lambda}^2$

2 $\left\|\tilde{\mathbf{T}}_{u_1} u_2\right\| = \tilde{\lambda}$

3 $\tilde{\mathbf{T}}_{u_1} u_1 + \tilde{\mathbf{T}}_{u_2} u_2 = 0$

Proof. (1) ⇒ (2) is evident from Lemma 3.1 Also

$$\left\langle \tilde{\mathbf{T}}_{u_1} u_1 + \tilde{\mathbf{T}}_{u_2} u_2, \tilde{\mathbf{T}}_{u_1} u_1 + \tilde{\mathbf{T}}_{u_2} u_2 \right\rangle = \left\|\tilde{\mathbf{T}}_{u_1} u_1\right\|^2 + 2\left\langle \tilde{\mathbf{T}}_{u_1} u_1, \tilde{\mathbf{T}}_{u_2} u_2 \right\rangle + \left\|\tilde{\mathbf{T}}_{u_2} u_2\right\|^2 \tag{11}$$

here, from the equation (11), we obtain

$$\left\langle \tilde{\mathbf{T}}_{u_1} u_1, \tilde{\mathbf{T}}_{u_2} u_2 \right\rangle = \tilde{\lambda}^2.$$

Also, from (11), we have

$$\left\|\tilde{\mathbf{T}}_{u_1} u_1 + \tilde{\mathbf{T}}_{u_2} u_2\right\|^2 = 0 \Longrightarrow \tilde{\mathbf{T}}_{u_1} u_1 + \tilde{\mathbf{T}}_{u_2} u_2 = 0.$$

□

From this result and Lemma 3.1 we can give the following theorem.

Theorem 3.3. *If $\tilde{\mathbf{T}}$ is $\tilde{\lambda}$- isotropic, then the discriminant of $\tilde{\mathbf{T}}$ satisfies the condition $-2\tilde{\lambda}^2 \leq \Delta \leq \tilde{\lambda}^2$. Moreover, if II is the plane at $(\tilde{\mathbf{M}}(\dot{c}))$, $\Delta(II) = \tilde{\lambda}^2$ iff $\tilde{\mathbf{T}}$ is umbilical in the II plane, $\Delta(II) = -2\tilde{\lambda}^2$ iff $\tilde{\mathbf{T}}$ is minimal in the II plane.*

Let $\{\dot{e}_1, \dot{e}_2, ..., \dot{e}_p\}$ be a orthonormal basis for $\chi^h(\tilde{\mathbf{M}}^n(\dot{c}))$. Also, Let A_{ij} be any $\frac{p(p+1)}{2}$ vectors in $\chi^h(\tilde{\mathbf{M}}^n(\dot{c}))$ for $1 \leq i \leq j \leq n$. Then,

- There is only one symmetric bilinear function from $\chi^v(\tilde{\mathbf{M}}^n(\dot{c})) \times \chi^v(\tilde{\mathbf{M}}^n(\dot{c}))$ to $\chi^h(\tilde{\mathbf{N}}^{\ell-k}(\ddot{c}))$ such that $\tilde{\mathbf{T}}_{e_i} e_j = A_{ij}$
- If A_{ij} satisfies conditions

$$\begin{aligned} \langle \tilde{\mathbf{T}}_{e_i} e_i, \tilde{\mathbf{T}}_{e_j} e_j \rangle &= 0 & (12)\\ \langle \tilde{\mathbf{T}}_{u_1} u_1, \tilde{\mathbf{T}}_{v_1} v_1 \rangle + 2\|\tilde{\mathbf{T}}_{u_1} v_1\| &= \tilde{\lambda}^2, \|u_1\|\|_{\|} v_1\| = 1 & (13)\\ \langle \tilde{\mathbf{T}}_{u_1} u_1, \tilde{\mathbf{T}}_{u_2} u_3 \rangle + 2\langle \tilde{\mathbf{T}}_{u_1} u_2, \tilde{\mathbf{T}}_{u_1} u_3 \rangle &= 0 & (14)\\ \langle \tilde{\mathbf{T}}_{u_1} u_2, \tilde{\mathbf{T}}_{u_3} u_4 \rangle + \langle \tilde{\mathbf{T}}_{u_1} u_3, \tilde{\mathbf{T}}_{u_2} u_4 \rangle \langle \tilde{\mathbf{T}}_{u_1} u_4, \tilde{\mathbf{T}}_{u_2} u_3 \rangle &= 0 & (15) \end{aligned}$$

  $\tilde{\mathbf{T}}$ is $\tilde{\lambda}$ isotropic.
- The discriminant Δ of $\tilde{\mathbf{T}}$ is constant if and only if Δ is constant on all planes spanned by e_i, e_j, and

  $$\langle A_{ij}, A_{kl} \rangle = \langle A_{il}, A_{kj} \rangle \text{ and } \langle A_{ii}, A_{jk} \rangle = \langle A_{ij}, A_{ik} \rangle$$

  for all different i, j, k, l.

From now on, we accept that $\tilde{\mathbf{T}}$ is $\tilde{\lambda}$-isotropic and Δ is constant. Let the first normal fiber ß of $\tilde{\mathbf{T}}$ be a subfiber of $\chi^h(\tilde{\mathbf{M}}^n(\dot{c}))$ spanned by $\tilde{\mathbf{T}}_{u_1} v_1$ vectors for any $u_1, v_1 \in \chi(\tilde{\mathbf{M}}^n(\dot{c}))$. Now we have to show that the dimension of the first normal fiber is $-2\tilde{\lambda}^2 \leq \Delta \leq \tilde{\lambda}^2$. Since $\tilde{\mathbf{T}}$ is minimal in general, we have the expression $-2\tilde{\lambda}^2 \leq \Delta \leq \tilde{\lambda}^2$ for every frame $e_1, ..., e_p$. Also, $\tilde{\mathbf{T}}$ being umbilical ensures that $\chi^h(\tilde{\mathbf{M}}^n(\dot{c}))$ has the same value for each unit vector in $\tilde{\mathbf{T}}_{u_1} u_1$.

Theorem 3.4. *Let $\tilde{\mathbf{T}}$ be $\tilde{\lambda}$- isotropic for $\tilde{\lambda} > 0$ and discriminant be Δ. Then*

$$-h_p\tilde{\lambda}^2 \leq \Delta \leq \tilde{\lambda}^2$$

moreover, if ß is the first normal fiber of $\tilde{\mathbf{T}}$,

- *1- $\Delta = \tilde{\lambda}^2 \Longleftrightarrow \tilde{\mathbf{T}}$ is umbilical $\Leftrightarrow \dim$ ß $= 1$*
- *2- $\Delta = -h_p\tilde{\lambda}^2 \Longleftrightarrow \tilde{\mathbf{T}}$ is minimal $\Leftrightarrow \dim$ ß $= m_p - 1$*
- *3- $-h_p\tilde{\lambda}^2 < \Delta < \tilde{\lambda}^2 \Leftrightarrow \dim$ ß $= m_p$,*

where $m_p = \frac{p(p+1)}{2}$ and $h_p = \frac{p+2}{2(p-1)}$.

Proof. From the fundamental effect of the constancy of Δ on $\tilde{\mathbf{T}}$, we have

$$\Delta_{u_1u_2} = \left\langle \tilde{\mathbf{T}}_{u_1}u_1, \tilde{\mathbf{T}}_{u_2}u_2 \right\rangle - \left\langle \tilde{\mathbf{T}}_{u_1}u_2, \tilde{\mathbf{T}}_{u_1}u_2 \right\rangle.$$

From the definition, the permutations of $\left\langle \tilde{\mathbf{T}}_{u_1}u_2, \tilde{\mathbf{T}}_{u_3}u_4 \right\rangle$ and $\left\langle \tilde{\mathbf{T}}_{u_1}u_1, \tilde{\mathbf{T}}_{u_3}u_4 \right\rangle$ with respect to u_1, u_2, u_3, u_4 do not change for orthogonal $u_1, u_2, u_3, u_4 \in \chi^v(\tilde{\mathbf{M}}(\dot{c}))$. From the equation (13)-(15), $\left\langle \tilde{\mathbf{T}}_{u_1}u_2, \tilde{\mathbf{T}}_{u_3}u_4 \right\rangle$, $\left\langle \tilde{\mathbf{T}}_{u_1}u_1, \tilde{\mathbf{T}}_{u_3}u_4 \right\rangle$, and $\left\langle \tilde{\mathbf{T}}_{u_1}u_3, \tilde{T}_{u_1}u_4 \right\rangle$ are zero when u_1, u_2, u_3, u_4 are orthogonal. Let the orthonormal basis for $\chi^h(\tilde{\mathbf{M}}^n(\dot{c}))$ be $\dot{e_1}, \dot{e_2}, ..., \dot{e_p}$ and $z_i = \tilde{\mathbf{T}}_{e_i}e_i, 1 \leq i \leq p$. Now let $\frac{p(p-1)}{2}$ number $\tilde{\mathbf{T}}_{e_1}e_j, (i < j)$ be perpendicular and orthogonal to the lower fiber Z, each of which is stretched by $z_1, z_2, ..., z_p$. Claim 1 is clear. Now let's prove the second claim. We exclude $\Delta = \tilde{\lambda}^2$ in case of being minimal. Then, if $\tilde{\mathbf{T}}$, $\tilde{\lambda}$ is isotropic, then for the u_1, u_2 orthonormal vectors, the $\tilde{\mathbf{T}}_{e_i}e_j$, $i < j$ all have nonzero length. Hence dimß $= m_p - 1$. Also, Lemma 3.1 shows that all $\langle z_i, z_j \rangle$, $(i \neq j)$ are equal. Then we have $\langle z_i, z_j \rangle = \tilde{\lambda}^2 \cos\theta$. According to Euclidean geometry, $\cos\theta \leq \frac{-1}{p-1}$ and equivalence are valid when vectors $z_1, z_2, ..., z_p$ are linearly dependent. The reverse is obvious. If $\cos\theta = \frac{-1}{p-1}$, we find $\Delta = \frac{-(p+1)\tilde{\lambda}^2}{2(p-1)}$, If $h_p = \frac{p+2}{2(p-1)}$, found $\Delta = -h_p\tilde{\lambda}^2$ and $dim Z = n-1, dim$ß $= m_p - 1$. From $z_1 + z_2 + ... + z_p = 0$, $\tilde{\mathbf{T}}$ is minimal. Similarly, $\Delta > -\tilde{\lambda}^2 h_p$, $\dim Z = n$, boyß $= m_p$. But with lemma3.1 we always have $\Delta^2 \leq \tilde{\lambda}$, so excluding case $\Delta^2 = \tilde{\lambda}$, the proof is complete. □

This theorem shows us that in the case of fixed Δ large co-dimension is required if the isotropic submersion is not umbilical.

Theorem 3.5. *Let the transformation $\dot{\phi}$ be an isotropic Riemannian submersion from an $\ell(\geq 2)$ spaceform $\tilde{\mathbf{M}}^n(\dot{c})$ with dimension $(k < \ell)$ to a with dimension $(k < \ell)$ space form $(\tilde{\mathbf{N}}(\ddot{c}))$. Suppose it is $k \leq \frac{1}{2}\ell(\ell+1) - 1$. Then $\dot{\phi}$ is parallel submersion, furthermore $\dot{\phi}$ is locally equivalent to one of the followings.*

1- $\dot{\pi} : \tilde{\mathbf{M}}^\ell(\dot{c}) \rightarrow \tilde{\mathbf{N}}^{\ell-k}(\ddot{c})$, $\dot{c} > \ddot{c}$ and $k \leq \frac{1}{2}\ell(\ell+1) - 1$ is a totally umbilical submersion.

2- $\dot{\pi} : \tilde{\mathbf{M}}^\ell(\dot{c}) = S^\ell(\dot{c}) \rightarrow \tilde{\mathbf{N}}^{\ell-k}(\ddot{c}) = S^{\ell-k}(\ddot{c})$, $\ddot{c} = \frac{2(\ell+2)\dot{c}}{\ell}$ and $k = \frac{\ell(\ell+1)}{2} - 1$ is a second standard minimal submersion.

Proof. From our hypothesis, we have (5) and (6) for any $u_1, u_2, u_3, u_4 \in \chi^v(\tilde{\mathbf{M}}(\dot{c}))$. If $\tilde{\mathbf{M}}^\ell$ and $\tilde{\mathbf{N}}^{\ell-k}$ are space forms, from the Gauss and Codazzi equations, we obtain

$$\left\langle \tilde{\mathbf{T}}_{u_1}u_3, \tilde{\mathbf{T}}_{u_2}u_4 \right\rangle - \left\langle \tilde{\mathbf{T}}_{u_2}u_3, \tilde{\mathbf{T}}_{u_1}u_4 \right\rangle = (\dot{c} - \ddot{c})\{\langle u_2, u_3 \rangle \langle u_1, u_4 \rangle - \langle u_1, u_3 \rangle \langle u_2, u_4 \rangle\} \tag{16}$$

also we have

$$\begin{pmatrix} \left\langle \tilde{\mathbf{T}}_{u_1}u_2, \tilde{\mathbf{T}}_{u_3}u_4 \right\rangle \\ + \left\langle \tilde{\mathbf{T}}_{u_1}u_3, \tilde{\mathbf{T}}_{u_2}u_4 \right\rangle + \left\langle \tilde{\mathbf{T}}_{u_1}u_4, \tilde{\mathbf{T}}_{u_2}u_3 \right\rangle \end{pmatrix} = \tilde{\lambda}^2 \begin{pmatrix} \langle u_1, u_2 \rangle \langle u_3, u_4 \rangle \\ + \langle u_1, u_3 \rangle \langle u_2, u_4 \rangle + \langle u_1, u_4 \rangle \langle u_2, u_3 \rangle \end{pmatrix}. \tag{17}$$

from the (16) and (17), we find

$$\begin{pmatrix} 2\left\langle \tilde{\mathbf{T}}_{u_1}u_3, \tilde{\mathbf{T}}_{u_2}u_4 \right\rangle \\ +\left\langle \tilde{\mathbf{T}}_{u_1}u_2, \tilde{\mathbf{T}}_{u_3}u_4 \right\rangle \end{pmatrix} = \begin{array}{l} \tilde{\lambda}^2 \begin{pmatrix} \langle u_1,u_2\rangle\langle u_3,u_4\rangle \\ +\langle u_1,u_3\rangle\langle u_2,u_4\rangle + \langle u_1,u_4\rangle\langle u_2,u_3\rangle \end{pmatrix} \\ +(\dot{c}-\ddot{c})\{\langle u_2,u_3\rangle\langle u_1,u_4\rangle - \langle u_1,u_3\rangle\langle u_2,u_4\rangle\}. \end{array} \tag{18}$$

Therefore from (16) and (18), we have

$$\begin{array}{rcl} 3\left\langle \tilde{\mathbf{T}}_{u_1}u_3, \tilde{\mathbf{T}}_{u_2}u_4 \right\rangle &=& \tilde{\lambda}^2 \begin{pmatrix} \langle u_1,u_2\rangle\langle u_3,u_4\rangle + \langle u_1,u_3\rangle\langle u_2,u_4\rangle \\ +\langle u_1,u_4\rangle\langle u_2,u_3\rangle \end{pmatrix} \\ && +(\dot{c}-\ddot{c})\{\langle u_2,u_3\rangle\langle u_1,u_4\rangle - 2\langle u_1,u_3\rangle\langle u_2,u_4\rangle \\ && +\langle u_1,u_2\rangle\langle u_3,u_4\rangle\}. \end{array} \tag{19}$$

If the total geodesic submersion of $\ddot{\pi}$ is considered first, the first case of the theorem is realized. Let us now consider the case where $\ddot{\pi}$ is not a total geodesic submersion. Since $\tilde{\lambda}$ is a constant function in $\tilde{\mathbf{M}}^{\ell}(\dot{c})$, x_0 has a U neighborhood such that $\tilde{\lambda} > 0$ on U. (From now on we will work on the open subset U). From Theorem 3.2, from the assumption of our theorem and from the fact that $\tilde{\lambda}$ is constant, we see that $\tilde{\lambda}$ is constant on U. So it is $\tilde{\lambda}^2 = (\dot{c}-\ddot{c})$ or $\tilde{\lambda}^2 = \frac{2(p-1)(\ddot{c}-\dot{c})}{2(\ell+2)}$. If $\tilde{\lambda}^2 = (\dot{c}-\ddot{c})$, the first case of our theorem from equation (19) is realized. If $\tilde{\lambda}^2 = \frac{2(p-1)(\ddot{c}-\dot{c})}{2(\ell+2)}$ then it is $\dim ß = \frac{p(p+1)}{2} - 1$ from Theorem3.2 and since $\tilde{\lambda}$ is constant, if we derive the equation (19), then

$$\left\langle (\nabla_v \tilde{\mathbf{T}})(u_1,u_3), \tilde{\mathbf{T}}(u_2,u_4) \right\rangle + \left\langle (\nabla_v \tilde{\mathbf{T}})(u_2,u_4), \tilde{\mathbf{T}}(u_1,u_3) \right\rangle = 0$$

according to $v \in \tilde{\mathbf{M}}^{\ell}(\dot{c})$. From here, we have

$$\left\langle (\nabla_v \tilde{\mathbf{T}})(u_1,u_3), \tilde{\mathbf{T}}(u_2,u_4) \right\rangle = -\left\langle (\nabla_v \tilde{\mathbf{T}})(u_2,u_4), \tilde{\mathbf{T}}(u_1,u_3) \right\rangle. \tag{20}$$

Considering the codazzi equation repeatedly in equation (20), we find

$$\begin{array}{rcl} \left\langle (\nabla_v \tilde{\mathbf{T}})(u_1,u_3), \tilde{\mathbf{T}}(u_2,u_4) \right\rangle &=& -\left\langle (\nabla_{u_2} \tilde{\mathbf{T}})(v,u_4), \tilde{\mathbf{T}}(u_1,u_3) \right\rangle, \\ &=& -\left\langle (\nabla_{u_3} \tilde{\mathbf{T}})(v,u_4), \tilde{\mathbf{T}}(u_1,u_2) \right\rangle, \\ &=& \left\langle (\nabla_{u_4} \tilde{\mathbf{T}})(u_1,u_2), \tilde{\mathbf{T}}(v,u_3) \right\rangle, \\ &=& -\left\langle (\nabla_{u_1} \tilde{\mathbf{T}})(v,u_3), \tilde{\mathbf{T}}(u_4,u_2) \right\rangle, \\ &=& -\left\langle (\nabla_v \tilde{\mathbf{T}})(u_1,u_3), \tilde{\mathbf{T}}(u_4,u_2) \right\rangle. \end{array}$$

From here, we have

$$\left\langle (\nabla_v \tilde{\mathbf{T}})(u_1,u_3), \tilde{\mathbf{T}}(u_2,u_4) \right\rangle = 0.$$

This shows that $\dim ß = co\dim \tilde{\mathbf{M}}^{\ell}(\dot{c})$ and $\ddot{\pi}$ are parallel on U. Thus, we show the second case of the theorem. □

Lemma 3.6. *Let $\tilde{\theta} : \tilde{\mathbf{M}}^{\ell}(\dot{c}) \to \tilde{\mathbf{N}}^{\ell-k}(\ddot{c})$, $\ell \geq 2$, $\ell > \tilde{q}$ be an isotropic Riemannian submersion. For any $u_1, v_1 \in \tilde{\mathbf{M}}^{\ell}(\dot{c})$, we have*

$$\left\langle (\nabla_{u_1} \tilde{\mathbf{T}})(u_1,u_1), \tilde{\mathbf{T}}(u_1,v_1) \right\rangle = \begin{array}{c} d\tilde{\lambda}^2(u_1) < u_1,u_1 >< u_1,v_1 > \\ -\frac{1}{2}d\tilde{\lambda}^2(v_1) < u_1,u_1 >< u_1,u_1 > . \end{array}$$

Proof. Let $u_1, v_1 \in \tilde{\mathbf{M}}^{\ell}(\dot{c})$ be arbitrary vectors. Let $\gamma : (-\varepsilon,\varepsilon) \to \tilde{\theta}^{-1}(\tilde{q})$, where $\tilde{\theta}(\tilde{q}) = u_1$, be a curve. Such that $\gamma'(0) = v_1$ and define a parallel vector field $u(t)$ along γ with $\gamma(0) = u_1$. Then, for any $t \in (-\varepsilon,\varepsilon)$, we have

$$\left\langle \tilde{\mathbf{T}}(u_1(t),u_1(t)), \tilde{\mathbf{T}}(u_1(t),u_1(t)) \right\rangle = \tilde{\lambda}^2 \langle u_1(t),u_1(t) \rangle$$

and if this expression is derived at $t = 0$, for $u_1, v_1 \in \chi^{\tilde{\delta}}(\tilde{\mathbf{M}}(\dot{c}))$, we obtain

$$2\left\langle (\nabla_{v_1}\tilde{\mathbf{T}})(u_1,u_1), \tilde{\mathbf{T}}(u_1,u_1)\right\rangle = d\tilde{\lambda}^2(v_1)\langle u_1,u_1\rangle^2 \tag{21}$$

If we set $u_1 = v_1$ in equation (21), we get the equation below,

$$2\left\langle (\nabla_{u_1}\tilde{\mathbf{T}})(u_1,u_1), \tilde{\mathbf{T}}(u_1,u_1)\right\rangle = d\tilde{\lambda}^2(u_1)\langle u_1,u_1\rangle^2 .$$

Using the Codazzi equation in (21) and considering the symmetry of $\nabla\tilde{\mathbf{T}}$, we obtain

$$\left\langle (\nabla_{u_1}\tilde{\mathbf{T}})(u_1,u_1), \tilde{\mathbf{T}}(u_1,v_1)\right\rangle = d\tilde{\lambda}^2(u_1) < u_1,u_1 >< u_1,v_1 > -\frac{1}{2}d\tilde{\lambda}^2(v_1) < u_1,u_1 >^2 . \tag{22}$$

□

Proposition 3.7. *Let $\tilde{\pi} : \tilde{\mathbf{M}}^{\ell}(\dot{c}) \to \tilde{\mathbf{N}}^{\ell-k}(\ddot{c})$, $n \geq 2, \ell - 1 \geq p$ be the isotropic Riemannian submersion. Then $\tilde{\lambda}$ is a constant.*

Proof. We need to show that $\tilde{\lambda}(p) = 0$ at any point in $\tilde{\mathbf{M}}^{\ell}$. First, let's work on $\tilde{\lambda}(p) = 0$, $p \in \tilde{\mathbf{M}}$. If we take $u_1 = v_1$ in equation (22), we get

$$\left\langle \nabla_{u_1}\tilde{\mathbf{T}}(u_1,u_1), \tilde{\mathbf{T}}(u_1,u_1)\right\rangle = \frac{1}{2}d\tilde{\lambda}^2(u_1)\langle u_1,u_1\rangle^2 = 0$$

for any $u_1 \in \chi^{v}(\tilde{\mathbf{M}}(\dot{c}))$. Thus, $d\tilde{\lambda}^2 = 0$ for $p \in \tilde{\mathbf{M}}$. Now let's work on a $p \in \tilde{\mathbf{M}}$ point where $\tilde{\lambda}(p) > 0$. Let $\chi^{h}(\tilde{\mathbf{M}}(\dot{c})) \leq \chi^{v}(\tilde{\mathbf{M}}(\dot{c})) - 1$ be $\chi^{h}(\tilde{\mathbf{M}}(\dot{c}))$ being the horizontal fiber at point p. for any $v_1 \in \chi^{v}(\tilde{\mathbf{M}}(\dot{c}))$ there is a nonzero $u_1 \in \chi^{v}(\tilde{\mathbf{M}}(\dot{c}))$ such that $\tilde{\mathbf{T}}(u_1,v_1) = 0$. Then, from the equation (17) becomes $\langle u_1,v_1\rangle = 0$. Thus, from equation (22), it is seen that for any $v_1 \in \chi^{v}(\tilde{\mathbf{M}}(\dot{c}))$, $\tilde{\lambda}$ is constant since $d\tilde{\lambda}^2 = 0$. □

Using the fundamental tensor $\tilde{\mathbf{A}}$, we will build the notion of an isotropic Riemannian submersion, which we previously described based on the fundamental tensor $\tilde{\mathbf{T}}$, and derive the corresponding characterizations in this section.

Definition 3.8. *Let $\tilde{\theta} : (\tilde{\mathbf{M}}, g) \to (\tilde{\mathbf{N}}, \tilde{g})$ be a Riemannian submersion. If, for each $\tilde{q} \in \tilde{\mathbf{M}}$, the following condition is satisfied for $u, v \in \Gamma(ker, \tilde{\theta}_*)^{\perp}$, then $\tilde{\theta}$ is called $\tilde{\lambda}$-horizontally isotropic.*

$$g(\tilde{\mathbf{A}}_u v, \tilde{\mathbf{A}}_u v) = \tilde{\lambda} g(u - v, u + v)^2$$

If $\tilde{\lambda}$ is constant for $\tilde{q} \in \tilde{\mathbf{M}}$, $\tilde{\lambda}$ is constant horizontally isotropic.
If $\tilde{\mathbf{A}}$ is horizontally isotropic,

$$g(v\nabla_{\tilde{U}}\tilde{V}, v\nabla_{\tilde{U}}\tilde{V}) = \tilde{\lambda} g(\tilde{U} - \tilde{V}, \tilde{U} + \tilde{V})^2$$

Let us introduce a 4-linear function on the Riemannian manifold $\tilde{\mathbf{M}}$:

$$\tilde{\varphi} : \chi^{h}(\tilde{\mathbf{M}}) \times \chi^{h}(\tilde{\mathbf{M}}) \times \chi^{h}(\tilde{\mathbf{M}}) \times \chi^{h}(\tilde{\mathbf{M}}) \longrightarrow C^{\infty}(\tilde{\mathbf{M}})$$

given by

$$\tilde{\varphi}(\tilde{X}, \tilde{Z}, \tilde{U}, \tilde{V}) = g(\tilde{\mathbf{A}}_{\tilde{X}}\tilde{Z}, \tilde{\mathbf{A}}_{\tilde{X}}\tilde{Z}) - \tilde{\lambda} g(\tilde{X} - \tilde{Y}, \tilde{X} + \tilde{Y}) g(\tilde{U} - \tilde{V}, \tilde{U} + \tilde{V})$$

The function $\tilde{\varphi}$ satisfies the following properties:
1)$\tilde{\varphi}(\tilde{X}, \tilde{Z}, \tilde{U}, \tilde{V}) = \tilde{\varphi}(\tilde{U}, \tilde{V}, \tilde{X}, \tilde{Z})$
2)$\tilde{\varphi}(\tilde{X}, \tilde{Z}, \tilde{U}, \tilde{V}) = \tilde{\varphi}(\tilde{Z}, \tilde{X}, \tilde{V}, \tilde{U})$

for $\tilde{U} \in \chi^h(\tilde{\mathbf{M}})$.

If the distribution $\tilde{F}$ *is horizontally* $\tilde{\lambda}$*- isotropic, then for every* $\tilde{U} \in \chi^h(\tilde{M})$*, we have*

$$\tilde{\phi}(\tilde{U}) = \tilde{\varphi}(\tilde{U}, \tilde{U}, \tilde{U}, \tilde{U}) = 0.$$

From here, for all $\tilde{X}, \tilde{Z} \in \chi^h(\tilde{\mathbf{M}})$*, we obtain*

$$\tilde{\phi}(\tilde{X} + \tilde{Z}) + \tilde{\phi}(\tilde{X} - \tilde{Z}) = 0,$$

$$\tilde{\phi}(\tilde{X} + \tilde{Z}) - \tilde{\phi}(\tilde{X} - \tilde{Z}) = 0.$$

Let's find the expressions for $\tilde{\phi}(\tilde{X} + \tilde{Z})$ *and* $\tilde{\phi}(\tilde{X} - \tilde{Z})$*. After performing the necessary operations and simplifications, we obtain*

$$\tilde{\phi}(\tilde{X} + \tilde{Z}) = \tilde{\varphi}(\tilde{X}, \tilde{Z}, \tilde{Z}, \tilde{X}) + \tilde{\varphi}(\tilde{Z}, \tilde{X}, \tilde{X}, \tilde{Z}) = 0.$$

This result follows from the property of the function $\tilde{\varphi}$ *given above.*

$$2\tilde{\varphi}(\tilde{X}, \tilde{Z}, \tilde{Z}, \tilde{X}) = 0 \Rightarrow \tilde{\varphi}(\tilde{X}, \tilde{Z}, \tilde{Z}, \tilde{X}) = 0.$$

Likewise, if performing the necessary operations and simplifications, we obtain

$$\tilde{\phi}(\tilde{X} - \tilde{Z}) = \tilde{\varphi}(\tilde{X}, \tilde{Z}, \tilde{Z}, \tilde{X}) = 0,$$

and from here,

$$\tilde{\phi}(\tilde{X} + \tilde{Z}) + \tilde{\phi}(\tilde{X} - \tilde{Z}) = 2\tilde{\varphi}(\tilde{X}, \tilde{Z}, \tilde{Z}, \tilde{X}) + 2\tilde{\varphi}(\tilde{X}, \tilde{Z}, \tilde{Z}, \tilde{X}) = 0,$$

$$\tilde{\varphi}(\tilde{X}, \tilde{Z}, \tilde{Z}, \tilde{X}) = 0 \tag{23}$$

and

$$\tilde{\phi}(\tilde{X} + \tilde{Z}) - \tilde{\phi}(\tilde{X} - \tilde{Z}) = 2\tilde{\varphi}(\tilde{X}, \tilde{Z}, \tilde{Z}, \tilde{X}) - 2\tilde{\varphi}(\tilde{X}, \tilde{Z}, \tilde{Z}, \tilde{X}) = 0. \tag{24}$$

For (22) *equation,*

$$\begin{aligned}\tilde{\varphi}(\tilde{X}, \tilde{Z}, \tilde{Z}, \tilde{X}) =& g(\tilde{\mathbf{A}}_{\tilde{X}}\tilde{Z}, \tilde{\mathbf{A}}_{\tilde{Z}}\tilde{X}) - \tilde{\lambda} g(\tilde{X} - \tilde{Z}, \tilde{X} + \tilde{Z}) g(\tilde{Z} - \tilde{X}, \tilde{Z} + \tilde{X}) = 0\\ & g(\tilde{\mathbf{A}}_{\tilde{X}}\tilde{Z}, \tilde{\mathbf{A}}_{\tilde{Z}}\tilde{X}) = \tilde{\lambda} g(\tilde{X} - \tilde{Z}, \tilde{X} + \tilde{Z})^2\\ & g(\tilde{\mathbf{A}}_{\tilde{X}}\tilde{Z}, \tilde{\mathbf{A}}_{\tilde{Z}}\tilde{X}) = -g(\tilde{\mathbf{A}}_{\tilde{X}}\tilde{Z}, \tilde{\mathbf{A}}_{\tilde{X}}\tilde{Z})\end{aligned}$$

Specifically, if we take $\tilde{X} \perp \tilde{Z}$ *and* $\|\tilde{X}\| = \|\tilde{Z}\|$*, then we have*

$$\begin{aligned}g(\tilde{\mathbf{A}}_{\tilde{X}}\tilde{Z}, \tilde{\mathbf{A}}_{\tilde{X}}\tilde{Z}) &= -\tilde{\lambda} g(\tilde{X} - \tilde{Z}, \tilde{X} + \tilde{Z})^2\\ &= -\tilde{\lambda}\left[g(\tilde{X}, \tilde{X}) - g(\tilde{Z}, \tilde{Z})\right]^2\\ &= -\tilde{\lambda}\left[\|\tilde{X}\|^2 - \|\tilde{Z}\|^2\right]\\ &= 0.\end{aligned}$$

Here,

$$\begin{aligned}\tilde{g}(\tilde{\mathbf{A}}_{\tilde{X}}\tilde{Z}, \tilde{\mathbf{A}}_{\tilde{X}}\tilde{Z}) = 0 &\Leftrightarrow \|\tilde{\mathbf{A}}_{\tilde{X}}\tilde{Z}\| = 0\\ &\Leftrightarrow \tilde{\mathbf{A}}_{\tilde{X}}\tilde{Z} = 0.\end{aligned}$$

Also,

$$\tilde{\mathbf{A}}_{\tilde{X}}\tilde{X} = 0 \Leftrightarrow \tilde{\mathbf{A}} = 0.$$

Based on this result, we can state the following theorem.

Theorem 3.9. *Let $\tilde{\theta} : \tilde{\mathbf{M}}^m \rightarrow \tilde{\mathbf{N}}^n$ be a $\tilde{\lambda}$-horizontally isotropic Riemannian submersion. For any orthogonal $\tilde{U}_1, \tilde{U}_2 \in \chi^h(\tilde{\mathbf{M}})$, we have $\tilde{\mathbf{A}} = 0$, which corresponds to the integrability of the horizontal distribution.*

Theorem 3.10. *Let $\tilde{\theta} : \tilde{\mathbf{M}}^m \rightarrow \tilde{\mathbf{N}}^n$ be a horizontally isotropic Riemannian submersion. Let $\mathbb{K}$ and $\hat{\mathbb{K}}$ denote the sectional curvatures of the manifolds $\tilde{\mathbf{M}}$ and $\tilde{\mathbf{N}}$, respectively. For orthonormal horizontal vector fields $\tilde{X}_1$ and $\tilde{Y}_1$, the following equality holds between these curvatures:*

$$\mathbb{K}(\tilde{X}_1, \tilde{Y}_1) = \hat{\mathbb{K}}(\tilde{X}_1, \tilde{Y}_1) \circ \tilde{\theta}.$$

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