Filomat 39:19 (2025), 6723–6742
https://doi.org/10.2298/FIL2519723M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the well-posedness of mild solutions to certain nonlinear wave equations

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Abstract. We are devoted to the study of the Cauchy problem for Love equation.We establish the existence and uniqueness of the solution under different conditions imposed on the nonlinear source function.We analyze the continuous dependence of the solution with respect to the parameters involved in the equation. Additionally, in this paper, we also shown that the solution to the Love equation converges to the solution of the wave equation.

1. Introduction

Let Ω be a simply connected and bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Let T be a positive real number. In this paper, we study the initial value problem of the Love equation

$$\begin{cases} u_{tt}(x,t) + (-\Delta)^q u(x,t) - \mu\Delta u_{tt}(x,t) = Z(u(x,t)), & \text{in} \quad \Omega \times (0,T], \\ u|_{\partial\Omega} = 0, & \text{in} \quad \Omega, \end{cases} \tag{1}$$

with the initial conditions

$$u(x,0) = \varphi(x), \;\; u_t(x,0) = \psi(x) \quad \text{in} \quad \Omega, \tag{2}$$

where Z is the source function, representing the effect of an external force, and u describes the distribution at time t and space x. Here φ and ψ are the initial conditions which are defined later. Let us describe the main equation in (1) which is called the Love equation.
In mathematical physics, partial differential equations that govern wave propagation and elastic behavior form a fundamental framework for analyzing and modeling complex physical phenomena. A substantial

2020 *Mathematics Subject Classification*. Primary 35L70; Secondary 35A01, 35D30.
Keywords. Well posedness, mild solution,Love equation,existence and uniqueness, wave equation, fixed point method.
Received: 07 May 2025; Revised: 03 June 2025; Accepted: 02 July 2025
Communicated by Paola Bonacini
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body of research on models associated with this class of equations is documented in [6], [7],[1], [11]. Among these, the Love equation represents a prototypical model in the theory of linear elasticity, particularly well-suited for describing the motion of isotropic elastic materials. This equation is frequently employed to investigate shear wave propagation in elastic media, with wide-ranging applications in geophysics and solid mechanics. The classical Love equation originates from the study of elastic wave propagation in layered media, with particular emphasis on the behavior of surface waves that are now referred to as Love waves. This model was first introduced by A. E. H. Love in 1911. In recent decades, considerable attention has been devoted to extending the classical formulation to incorporate more realistic physical effects such as nonlinearity, dispersion, and various forms of dissipation and scattering.

In order to fully comprehend the significance of the model under study, we provide a detailed interpretation of the physical meaning of the terms involved in the problem (1)-(2).

From a physical perspective, the inclusion of the fractional Laplacian accounts for nonlocal interactions within the material. Unlike the standard Laplacian, which models purely local diffusion or elasticity, the operator $(-\Delta)^q$ with $q \in (0,1)$ introduces long-range spatial dependence, which is crucial for accurately modeling heterogeneous, fractal-like, or memory-influenced media. This is particularly relevant in geophysics, where Earth's subsurface structures often display irregular, layered properties that cannot be fully described by classical integer-order models. The parameter q in $(-\Delta)^q$ controls the degree of nonlocality and thus significantly affects the dispersion and regularity properties of the solution. Smaller values of q indicate stronger nonlocal effects and can lead to slower decay or different wavefront propagation characteristics compared to the standard wave equation.

In parallel, the damping term $\mu\Delta u_{tt}$ models internal friction or viscoelastic resistance. This term arises in many physical contexts involving energy dissipation, such as in polymers, biological tissues, or stratified rock formations. The coefficient $\mu > 0$ determines the intensity of the damping effect. Mathematically, this introduces a pseudo-parabolic component to the equation, enriching the model's structure by balancing the dispersive effects of the fractional operator with dissipative behavior.

The nonlinearity of the function $Z(u(x,t))$ plays a pivotal role in determining the behavior of the solutions, and different forms of nonlinearity (e.g., power-type, exponential) will lead to varying outcomes.

The initial data $\varphi(x)$ and $\psi(x)$ prescribe the initial displacement and velocity fields, respectively, thereby establishing the foundational state governing the subsequent motion and interaction of material points. From these initial conditions, the entire temporal evolution of the system is uniquely determined via the governing equations, elucidating the spatiotemporal propagation of waves or stresses throughout the material domain.

From a mathematical viewpoint, this problem presents a rich interplay between hyperbolic, parabolic, and nonlocal phenomena. The combined effect of nonlinearity $Z(u)$, nonlocal dispersion $(-\Delta)^q u$, and viscous damping $\mu\Delta u_{tt}$ makes the analysis of existence, uniqueness, regularity, and qualitative behavior of solutions both challenging and meaningful. In particular, understanding how the parameters q and μ influence the behavior of solutions, including aspects such as smoothing effects, energy decay, and the possibility of blowup, is essential for both theoretical analysis and practical modeling. Moreover, the imposed Dirichlet boundary condition $u|_{\partial\Omega} = 0$ reflects a physically constrained system, such as a fixed boundary in an elastic medium. The well-posedness and stability of this boundary-initial value problem thus become important for ensuring that the model behaves consistently with physical expectations.

As previously analyzed, the Love equation, or wave equations of Love-type, possesses a wide range of practical applications. In recent years, substantial research has been conducted by various mathematicians on this model. The following section is devoted to presenting several notable results from prominent works related to this model.

In [8], Ngoc, Triet, Duy and Long investigated an initial-boundary value problem for a nonlinear Love-type equation of the form

$$\begin{cases} w_{tt}(x,t) - w_{xx}(x,t) - w_{xxtt} = H(x,t,w,w_t), & \text{in} \quad 0 < x < 1,\ 0 < t < T \\ w|_{\partial\Omega} = 0, & \text{in} \quad \Omega, \\ w(x,0) = \widetilde{w}_0(x), \quad w_t(x,0) = \widetilde{w}_1(x), & 0 < x < 1. \end{cases} \tag{3}$$

Here some function $\widetilde{w}_0, \widetilde{w}_1, H$ are are specified later. To establish the local existence of solutions to Problem (3), the Faedo-Galerkin method was employed.
In [9] Ngoc, Duy and Long considered a nonlinear Love-type equation supplemented with initial conditions and mixed nonhomogeneous boundary conditions, given by the following problem

$$w_{tt} - w_{xx} - \varepsilon w_{xxtt} + \kappa |w_t|^{k-2} w_t + K|w|^{h-2} w \\ = H(x,t), \quad x \in \Omega = (0,1), \quad 0 < t < T,$$

$$\delta w_{xtt}(0,t) + w_x(0,t) = mu(0,t) + v(t), \tag{4}$$

$$w(1,t) = 0, \tag{5}$$

$$w(x,0) = \tilde{w}_0(x), w_t(x,0) = \tilde{u}_1(x), \tag{6}$$

where $k > 1, h > 1, \delta > 0, \kappa > 0, K > 0, m \geq 0$ are constants and $\tilde{w}_0, \tilde{w}_1, H, v$ are assumed to be given and to satisfy certain regularity and compatibility conditions, which will be specified in the subsequent sections. The existence of solutions was obtained by means of the Faedo-Galerkin method, together with compactness techniques and monotonicity arguments. The authors established the existence, uniqueness, regularity, and asymptotic behavior of the weak solution.
In [4] Nam, Nghia, Phuong study the initial value problem

$$\begin{cases} w_{tt}(x,t) + (-\Delta)^s w(x,t) - m\Delta w_{tt}(x,t) = G(x,t), & \text{in} \quad \Omega \times (0,T], \\ w|_{\partial\Omega} = 0, & \text{in} \quad \Omega, \end{cases} \tag{7}$$

with the initial conditions

$$w(x,0) = f(x), \;\; w_t(x,0) = g(x) \quad \text{in} \quad \Omega. \tag{8}$$

The authors are interested to study a mild solution of the Love equation.They present the regularity of the mild solution and show the convergence of the solution of Love's equation to the solution of the wave equation.
A majority of the existing literature on the Love-type equation has focused primarily on the analysis of weak and mild solutions in the linear case. The study of mild solutions in the nonlinear case poses significant difficulties and demands the application of sophisticated analytical methods. To the best of our knowledge, research on mild solutions for the nonlinear case is still relatively scarce. Therefore, this paper investigates mild solutions of the Love equation in the nonlinear case.
The contributions of this paper are organized as follows. In Section 2, we give some preliminaries and demonstrate an approach to the formula of the mild solution.In this section, We also provide estimates for certain operators in the mild solution formula. In Section 3,We prove the existence and uniqueness of the solution with a source function satisfying the global or local Lipschitz condition by making use usual fixed point arguments, in the nonlinear case. Next, in Section 4, We investigate the continuity of the solution with respect to the intensity coefficient of the damping effect and the initial data.Moreover, in this section, we show that the solution to the Love equation converges to the solution of the wave equation as $\mu \to 0$.

2. Preliminaries

In this section, we establish the preliminary results required for the proof of the main theorem of this paper. Throughout this paper, we recall some basic settings of some functional spaces. Throughout this paper, we consider the Laplace operator Δ defined on $W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$. Denote by $\{\lambda_k\}_{k\geq 1}$ and $\{e_k(x)\}_{k\geq 1}$, the spectrum and sequence of eigenfunctions of $\cdot$ respectively, which satisfy $\Delta e_k(x) = -\lambda_k e_k(x)$,

$0 < \lambda_0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_k \leq ...$, and $\lim_{k\to\infty} \lambda_k = \infty$. The sequence $\{e_k(x)\}_{k\geq 1}$ forms an orthonormal basis of the space $L^2(\Omega)$. The fractional power Δ^q, $q > 0$, of the Laplacian operator Δ with fractional order $q > 0$ on Ω is defined by

$$\Delta^q u(x) := \sum_{k=1}^{\infty} (u, e_k)\lambda_k^q e_k(x). \tag{9}$$

Definition 2.1. *We begin by recalling the Hilbert scale of function spaces associated with the spectral decomposition of a self-adjoint, positive definite operator. This framework provides a natural setting for describing various levels of regularity in the spectral sense.*

$$\mathbb{H}^\epsilon(\Omega) := \left\{ f \in L^2(\Omega),\ \sum_{k=1}^{\infty} \lambda_k^{2\epsilon} \Big(\int_\Omega f(x)e_k(x)dx \Big)^2 < \infty \right\},$$

for any $\epsilon \geq 0$. *And the norm is given by*

$$\|f\|_{\mathbb{H}^\epsilon(\Omega)} := \left(\sum_{j=1}^{\infty} \lambda_k^{2\epsilon} \Big(\int_\Omega f(x)e_k(x)dx \Big)^2 \right)^{1/2},\ f \in \mathbb{H}^\epsilon(\Omega).$$

The spaces $\mathbb{H}^\epsilon(\Omega)$ constitute a Hilbert scale in the sense that they interpolate between different degrees of regularity encoded via the spectral decay of the Fourier coefficients.

To characterize the time-dependent behavior of functions taking values in these spaces, we introduce the following Bochner-type space:

Definition 2.2. *Let us denote* $L^\infty(0, T; \mathbb{H}^\gamma(\Omega))$, *the space of all function* $v : \Omega \times (0, T) \to \mathbb{H}^\gamma(\Omega)$ *such that*

$$\|v\|_{L^\infty(0,T;\mathbb{H}^\gamma(\Omega))} := \operatorname*{ess\,sup}_{t\in(0,T)} \|v(.,t)\|_{\mathbb{H}^\gamma(\Omega)} < \infty,$$

We also make use of the following weighted-in-time function space, which plays a key role in capturing the behavior of solutions near the initial time.

Definition 2.3. *Let* B *be a Banach space and let* $a, q > 0$. *The weighted space* $O_{a,q}((0, T]; B)$ *is defined by*

$$O_{a,q}((0, T]; B) :=:= \left\{ \psi \in C((0, T]; B), \|\psi\|_{O_{a,q}((0,T];B)} < \infty \right\},$$

where

$$\|\psi\|_{O_{a,q}((0,T];B)} := \operatorname*{ess\,sup}_{t\in(0,T]} t^a e^{-qt} \|\psi(t, \cdot)\|_B < \infty, \tag{10}$$

where $a, q > 0$ *(see [5]).*

The following auxiliary result provides a technical estimate that will be instrumental in our subsequent analysis (this lemma can be found in [5], Lemma 8, page 9).

Lemma 2.4. *Let* $k_1 > -1$, $k_2 > -1$ *such that* $k_1 + k_2 \geq -1$, $\rho > 0$ *and* $t \in [0, T]$. *For* $h > 0$, *the following limit holds*

$$\lim_{\rho\to\infty} \left(\sup_{t\in[0,T]} t^h \int_0^1 \xi^{k_1}(1 - \xi)^{k_2} e^{-\rho t(1-\xi)} \mathrm{d}\xi \right) = 0.$$

This lemma asserts the uniform decay of a class of singular integrals with exponential weight, and will be useful for controlling remainder terms in time-weighted estimates.

2.1. Solution formulation

Since the domain Ω is bounded, the solution of the problem (1)-(2)can be represented by a Fourier series:

$$u(x,t) = \sum_{k=1}^{\infty} u_k e_k(x).$$

where $u_k(t) = (u(.,t), e_k)$ satisfies the following equation:

$$\partial_t^2 u_k(t) + \frac{\lambda_k^q}{1+\mu\lambda_k} u_k(t) = \frac{Z_k(t)}{1+\mu\lambda_k}, \quad u_k(0) = \varphi_k, \quad \frac{d}{dt}u_k(0) = \psi_k.$$

By applying the Laplace transform to solve the differential equation, the solution for $u_k(t)$ is given by

$$u_k(t) = \cos\left(\sqrt{\frac{\lambda_k^q}{1+\mu\lambda_k}}t\right)\varphi_k + \sqrt{\frac{1+\mu\lambda_k}{\lambda_k^q}}\sin\left(\sqrt{\frac{\lambda_k^q}{1+\mu\lambda_k}}t\right)\psi_k$$
$$+ \frac{1}{\sqrt{(1+\mu\lambda_k)\lambda_k^q}}\int_0^t \sin\left(\sqrt{\frac{\lambda_k^q}{1+\mu\lambda_k}}(t-r)\right)Z_k(u(r))dr.$$

Substituting $u_k(t)$ into the series form solution above, we have the solution to the problem (1)-(2) represented in the following form

$$u(x,t) = \sum_{k=1}^{\infty}\cos\left(\sqrt{\frac{\lambda_k^q}{1+\mu\lambda_k}}t\right)\varphi_k e_k(x) + \sum_{k=1}^{\infty}\sqrt{\frac{1+\mu\lambda_k}{\lambda_k^q}}\sin\left(\sqrt{\frac{\lambda_k^s}{1+\mu\lambda_k}}t\right)\psi_k e_k(x)$$
$$+\sum_{k=1}^{\infty}\left[\frac{1}{\sqrt{(1+\mu\lambda_k)\lambda_k^q}}\int_0^t \sin\left(\sqrt{\frac{\lambda_k^q}{1+\mu\lambda_k}}(t-r)\right)Z_k(u(r))dr\right]e_k(x). \tag{11}$$

The mild solution of (1)-(2) is given by

$$u(t) = \mathcal{M}_q(t)\varphi + \mathcal{N}_q(t)\psi + \int_0^t \mathcal{Q}_q(t-r)Z(u(r))dr, \tag{12}$$

where the operator families $\mathcal{M}(\mathbf{t})$, $\mathcal{N}(\mathbf{t})$, and $\mathcal{Q}(\mathbf{t})$ are defined as follows

$$\mathcal{M}_q(t)v = \sum_{k=1}^{\infty}\cos\left(\sqrt{\frac{\lambda_k^q}{1+\mu\lambda_k}}t\right)\Big(v(.), e_k(.)\Big)e_k(x),$$

$$\mathcal{N}_q(t)v = \sum_{k=1}^{\infty}\sqrt{\frac{1+\mu\lambda_k}{\lambda_k^q}}\sin\left(\sqrt{\frac{\lambda_k^q}{1+\mu\lambda_k}}t\right)\Big(v(.), e_k(.)\Big)e_k(x),$$

and

$$\mathcal{Q}_q(t)v = \sum_{k=1}^{\infty}\frac{1}{\sqrt{(1+\mu\lambda_k)\lambda_k^q}}\sin\left(\sqrt{\frac{\lambda_k^q}{1+\mu\lambda_k}}t\right)\Big(v(.), e_k(.)\Big)e_k(x).$$

To prove the results in the next section, we need to investigate the properties of the three operators $\mathcal{M}_q(t)$, $\mathcal{N}_q(t)$ and $\mathcal{Q}_q(t)$. Specifically, we have the following estimates for these three operations

Lemma 2.5. *Let $q<1$, and suppose that the regularity parametersγ and ζ satisfy $0<\zeta-\gamma<\frac{1-q}{2}$ and $\beta=1-\frac{2(\zeta-\gamma)}{1-q}$. Then the following bounds hold for the operators $\mathcal{M}(\mathbf{t})$, $\mathcal{N}(\mathbf{t})$, and $\mathcal{Q}(\mathbf{t})$*
a) For any function v belonging to the intersection space $v\in\mathbb{H}^{\gamma}(\Omega)\cap H^{\gamma+\frac{(q-1)\beta}{2}}(\Omega)$, the operator $\mathcal{M}(\mathbf{t})$ satisfies the estimate

$$\Big\|\mathcal{M}_q(t)v\Big\|_{\mathbb{H}^{\gamma}(\Omega)}\le\Big\|v\Big\|_{\mathbb{H}^{\gamma}(\Omega)}+C(\beta,\mu)t^{\beta}\Big\|v\Big\|_{\mathbb{H}^{\gamma+\frac{(q-1)\beta}{2}}(\Omega)},\tag{13}$$

b) For any function $v\in\mathbb{H}^{\zeta}(\Omega)$ the operator $\mathcal{N}(\mathbf{t})$ satisfies

$$\Big\|\mathcal{N}_q(t)v\Big\|_{\mathbb{H}^{\gamma}(\Omega)}\le C(\mu,\beta,\lambda_1)t^{1-\frac{2(\zeta-\gamma)}{1-q}}\Big\|v\Big\|_{\mathbb{H}^{\zeta}(\Omega)},\tag{14}$$

c) Similarly, for all $v\in\mathbb{H}^{\zeta}(\Omega)$the operator $\mathcal{Q}(\mathbf{t})$ satisfies the same estimate:

$$\Big\|\mathcal{Q}_q(t)v\Big\|_{\mathbb{H}^{\gamma}(\Omega)}\le C(\mu,\beta,\lambda_1)t^{1-\frac{2(\zeta-\gamma)}{1-q}}\Big\|v\Big\|_{\mathbb{H}^{\zeta}(\Omega)}.\tag{15}$$

Proof. First, we start the proof of part a by estimating $\mathcal{M}_q(t)$. Using Parseval's identity, we find that

$$\begin{aligned}\Big\|\mathcal{M}_q(t)v\Big\|^2_{\mathbb{H}^{\gamma}(\Omega)}&=\Big\|\sum_{k=1}^{\infty}\cos\Big(\sqrt{\frac{\lambda_k^q}{1+\mu\lambda_k}}t\Big)\Big(v(.),e_k(.)\Big)e_k(x)\Big\|^2_{\mathbb{H}^{\gamma}(\Omega)}\\&=\sum_{k=1}^{\infty}\lambda_k^{2\gamma}\cos^2\Big(\sqrt{\frac{\lambda_k^q}{1+\mu\lambda_k}}t\Big)\Big|\Big(v(.),e_k(.)\Big)\Big|^2\end{aligned}$$

Thank to the inequality $|\cos(x)|\le1+C_\beta x^{\beta}$, for any $0<\beta\le1$, thus, we deduce that

$$\begin{aligned}\Big\|\mathcal{M}_q(t)v\Big\|^2_{\mathbb{H}^{\gamma}(\Omega)}&\le\sum_{k=1}^{\infty}\lambda_k^{2\gamma}\Big|\Big(v(.),e_k(.)\Big)\Big|^2+C_\beta\sum_{k=1}^{\infty}\lambda_k^{2\gamma}\Big(\frac{\lambda_k^q}{1+\mu\lambda_k}\Big)^{\beta}t^{2\beta}\\&\le\sum_{k=1}^{\infty}\lambda_k^{2\gamma}\Big|\Big(v(.),e_k(.)\Big)\Big|^2+C(\beta,\mu)t^{2\beta}\sum_{k=1}^{\infty}\lambda_k^{2\gamma+(q-1)\beta}\Big|\Big(v(.),e_k(.)\Big)\Big|^2,\end{aligned}\tag{16}$$

for any $0<\beta\le1$.
Consequently, we conclude that

$$\Big\|\mathcal{M}_q(t)v\Big\|_{\mathbb{H}^{\gamma}(\Omega)}\le\Big\|v\Big\|_{\mathbb{H}^{\gamma}(\Omega)}+C(\beta,\mu)t^{\beta}\Big\|f\Big\|_{\mathbb{H}^{\gamma+\frac{(q-1)\beta}{2}}(\Omega)},\tag{17}$$

Here, note that from the conditions involving the parameters q,γ,ζ and β, we ensure that $\gamma+\frac{(q-1)\beta}{2}>0$. Next, we prove part b. To do so, we apply inequality $|\sin(x)|\le C_\beta x^{\beta}, 0<\beta\le1$, we derive that

$$\Big|\sin\Big(\sqrt{\frac{\lambda_k^q}{1+\mu\lambda_k}}t\Big)\Big|\le C_\beta\Big(\frac{\lambda_k^q}{1+\mu\lambda_k}\Big)^{\frac{\beta}{2}}t^{\varepsilon}\le C(\beta,\mu)\lambda_k^{\frac{(q-1)\beta}{2}}t^{\beta}.\tag{18}$$

In addition, we obtain that the following bound

$$\sqrt{\frac{1+\mu\lambda_k}{\lambda_k^q}}\le\sqrt{\mu+\lambda_1^{-1}}\lambda_k^{\frac{1-q}{2}}.\tag{19}$$

Combining (18) and (19), we find that

$$\begin{aligned}\left\|\mathcal{N}_q(t)v\right\|^2_{\mathbb{H}^\gamma(\Omega)} &= \left\|\sum_{k=1}^{\infty}\sqrt{\frac{1+\mu\lambda_k}{\lambda_k^q}}\sin\left(\sqrt{\frac{\lambda_k^q}{1+\mu\lambda_k}}t\right)\Big(v(.),e_k(.)\Big)e_k(x)\right\|^2_{\mathbb{H}^\gamma(\Omega)}\\ &= \sum_{k=1}^{\infty}\lambda_k^{2\gamma}\frac{1+\mu\lambda_k}{\lambda_k^q}\sin^2\left(\sqrt{\frac{\lambda_k^q}{1+\mu\lambda_k}}t\right)\Big|\Big(v(.),e_k(.)\Big)\Big|^2\\ &\le C(\mu,\beta,\lambda_1)t^{2\beta}\sum_{k=1}^{\infty}\lambda_k^{2\gamma+(1-q)(1-\beta)}\Big|\Big(v(.),e_k(.)\Big)\Big|^2.\\ &= C(\mu,\beta,\lambda_1)t^{2\beta}\left\|v\right\|^2_{\mathbb{H}^{\gamma+\frac{(1-q)(1-\beta)}{2}}(\Omega)}.\end{aligned}$$

By setting $\beta = 1-\frac{2(\zeta-\gamma)}{1-q}$, we follow from the latter estimate that

$$\begin{aligned}\left\|\mathcal{N}_q(t)v\right\|_{\mathbb{H}^\gamma(\Omega)} &= \left\|\sum_{k=1}^{\infty}\sqrt{\frac{1+\mu\lambda_k}{\lambda_k^q}}\sin\left(\sqrt{\frac{\lambda_k^q}{1+\mu\lambda_k}}t\right)\Big(v(.),e_k(.)\Big)e_k(x)\right\|_{\mathbb{H}^\gamma(\Omega)}\\ &\le C(\mu,\beta,\lambda_1)t^{1-\frac{2(\zeta-\gamma)}{1-q}}\left\|v\right\|_{\mathbb{H}^\zeta(\Omega)}.\end{aligned} \tag{20}$$

Finally, we estimate the operator $\mathcal{Q}_q(t)$ in part c). It is easy to see that

$$\begin{aligned}\left\|\mathcal{Q}_q(t)v\right\|^2_{\mathbb{H}^\gamma(\Omega)} &= \left\|\sum_{k=1}^{\infty}\frac{1}{\sqrt{(1+\mu\lambda_k)\lambda_k^q}}\sin\left(\sqrt{\frac{\lambda_k^q}{1+\mu\lambda_k}}t\right)\Big(v(.),e_k(.)\Big)e_k(x)\right\|^2_{\mathbb{H}^\gamma(\Omega)}\\ &= \sum_{k=1}^{\infty}\lambda_k^{2\gamma}\frac{1}{(1+\mu\lambda_k)\lambda_k^q}\sin^2\left(\sqrt{\frac{\lambda_k^q}{1+\mu\lambda_k}}t\right)\Big|\Big(v(.),e_k(.)\Big)\Big|^2\\ &= \sum_{k=1}^{\infty}\lambda_k^{2\gamma}\frac{1}{(1+\mu\lambda_k)^2}\frac{1+\mu\lambda_k}{\lambda_k^q}\sin^2\left(\sqrt{\frac{\lambda_k^q}{1+\mu\lambda_k}}t\right)\Big|\Big(v(.),e_k(.)\Big)\Big|^2.\end{aligned} \tag{21}$$

Using the estimate for $\mathcal{N}_q(t)v$ and noting that $\frac{1}{(1+\mu\lambda_k)^2} < 1$, we deduce that

$$\left\|\mathcal{Q}_q(t)v\right\|_{\mathbb{H}^\gamma(\Omega)} \le C(\mu,\beta,\lambda_1)t^{1-\frac{2(\zeta-\gamma)}{1-q}}\left\|v\right\|_{\mathbb{H}^\zeta(\Omega)}.$$

□

3. Globally solution existence and uniqueness under lipschitz of source function

In this section, we establish the existence and uniqueness of the mild solution to the problem, under a nonlinear assumption. In order to prove the results, we require the Lipschitz condition for the function Z, as stated below

Let $Z : \mathbb{H}^{\kappa_1}(\Omega) \to \mathbb{H}^{\kappa_2}(\Omega)$ such that $Z(\mathbf{0}) = \mathbf{0}$ and

$$\|Z(\theta_1) - Z(\theta_2)\|_{\mathbb{H}^{\kappa_2}(\Omega)} \le \mathcal{L}\|\theta_1-\theta_2\|_{\mathbb{H}^{\kappa_1}(\Omega)}, \tag{22}$$

with $\mathcal{L}$ is a positive constant.

Theorem 3.1. *Let $0 < q < 1$, and suppose that $\varphi \in \mathbb{H}^{\gamma}(\Omega)$ for any $\gamma \geq 0$. Assume further that $\psi \in \mathbb{H}^{\zeta}(\Omega)$ and $Z \in L^2(0,T;\mathbb{H}^{\zeta}(\Omega))$ with $0 < \zeta - \gamma < \frac{1-q}{2}$. Then, problem* (1)-(2) *admits a unique solution $u \in O_{a,b}((0,T];\mathbb{H}^{\gamma}(\Omega))$. In addition, the following estimate holds for the solution u:*

$$\left\|u(.,t)\right\|_{\mathbb{H}^{\gamma}(\Omega)} \leq 2\left\|\varphi\right\|_{\mathbb{H}^{\gamma}(\Omega)} + 2C(\beta,\mu)T^{\beta}\left\|\varphi\right\|_{\mathbb{H}^{\gamma+\frac{(q-1)\beta}{2}}(\Omega)} + 2C(\mu,\beta,\lambda_1)T^{1-\frac{2(\zeta-\gamma)}{1-q}}\left\|\psi\right\|_{\mathbb{H}^{\zeta}(\Omega)}, \tag{23}$$

where $\beta = 1 - \frac{2(\zeta-\gamma)}{1-q}$.

Proof. The proof is carried out via an application of the contraction mapping principle. Accordingly, we first introduce the function

$$\mathcal{P} : O_{a,b}((0,T];\mathbb{H}^{\gamma}(\Omega)) \to O_{a,b}((0,T];\mathbb{H}^{\gamma}(\Omega)),$$

defined by

$$\mathcal{P}u(t) := \mathcal{M}_q(t)\varphi + \mathcal{N}_q(t)\psi + \int_0^t \mathcal{Q}_q(t-r)Z(u(r))dr. \tag{24}$$

Since the property $Z(\mathbf{0}) = \mathbf{0}$, and by combining this with Lemma (2.5), we deduce the following estimate

$$\begin{aligned}\left\|\mathcal{P}\left(u=\mathbf{0}\right)(t)\right\|_{\mathbb{H}^{\gamma}(\Omega)} &= \left\|\mathcal{M}_q(t)\varphi + \mathcal{N}_q(t)\psi\right\|_{\mathbb{H}^{\gamma}(\Omega)}\\ &\leq \left\|\varphi\right\|_{\mathbb{H}^{\gamma}(\Omega)} + C(\beta,\mu)t^{\beta}\left\|\varphi\right\|_{\mathbb{H}^{\gamma+\frac{(q-1)\beta}{2}}(\Omega)} + C(\mu,\beta,\lambda_1)t^{1-\frac{2(\zeta-\gamma)}{1-q}}\left\|\psi\right\|_{\mathbb{H}^{\zeta}(\Omega)}\\ &\leq \left\|\varphi\right\|_{\mathbb{H}^{\gamma}(\Omega)} + C(\beta,\mu)T^{\beta}\left\|\varphi\right\|_{\mathbb{H}^{\gamma+\frac{(q-1)\beta}{2}}(\Omega)} + C(\mu,\beta,\lambda_1)T^{1-\frac{2(\zeta-\gamma)}{1-q}}\left\|\psi\right\|_{\mathbb{H}^{\zeta}(\Omega)}\end{aligned}$$

Consequently, we obtain the weighted estimate

$$t^a e^{-bt}\left\|\mathcal{P}\left(u=\mathbf{0}\right)(t)\right\|_{\mathbb{H}^{\gamma}(\Omega)} \leq t^a e^{-bt}\left\|\varphi\right\|_{\mathbb{H}^{\gamma}(\Omega)} + t^a e^{-bt}C(\beta,\mu)T^{\beta}\left\|\varphi\right\|_{\mathbb{H}^{\gamma+\frac{(q-1)\beta}{2}}(\Omega)} \tag{25}$$

$$+ t^a e^{-bt}C(\mu,\beta,\lambda_1)T^{1-\frac{2(\zeta-\gamma)}{1-q}}\left\|\psi\right\|_{\mathbb{H}^{\zeta}(\Omega)} \tag{26}$$

It can be observed that $e^{-bt} < 1$, which leads to the conclusion that $\mathcal{P}(u=0) \in O_{a,b}((0,T];\mathbb{H}^{\gamma}(\Omega))$. Hence, we contend that the mapping $\mathcal{P}$ is well defined. Next, we proceed to show that proving that $\mathcal{P}$ is a contraction mapping. To this end, let u_1 and u_2 be arbitrary functions in the corresponding function space. Then, by the definition of $\mathcal{P}$, we have

$$\mathcal{P}u_1(t) - \mathcal{P}u_2(t) = \int_0^t \mathcal{Q}_q(t-r)\Big(Z(u_1(r)) - Z(u_2(r))\Big)dr$$

By an argument analogous to that used in Lemma (2.5), we obtain the following estimate

$$\begin{aligned}\left\|\int_0^t \mathcal{Q}_q(t-r)\Big(Z(u_1(r)) - Z(u_2(r))\Big)dr\right\|_{\mathbb{H}^{\gamma}(\Omega)} \leq & C(\mu,\beta,\lambda_1)\int_0^t (t-r)^{1-\frac{2(\zeta-\gamma)}{1-q}}\\ &\times\left\|\Big(Z(u_1(r)) - Z(u_2(r))\Big)\right\|_{\mathbb{H}^{\zeta}(\Omega)}dr.\end{aligned}$$

Invoking the Lipschitz continuity of the nonlinear functionZ, we deduce the following estimate for the difference of the operator $\mathcal{P}$

$$\|\mathcal{P}u_1(t) - \mathcal{P}u_2(t)\|_{\mathbb{H}^{\gamma}(\Omega)} \leq C(\mu,\beta,\lambda_1)\int_0^t (t-r)^{1-\frac{2(\zeta-\gamma)}{1-q}}\left\|u_1(r) - u_2(r)\right\|_{\mathbb{H}^{\gamma}(\Omega)}dr.$$

Multiplying both sides of the above estimate by $t^a e^{-bt}$, we derive the following inequality

$$t^a e^{-bt}\|\mathcal{P}u_1(t)-\mathcal{P}u_2(t)\|_{\mathbb{H}^\gamma(\Omega)} \le C(\mu,\beta,\lambda_1)\int_0^t t^a r^{-a} e^{b(r-t)}(t-r)^{1-\frac{2(\zeta-\gamma)}{1-q}} r^a e^{-br}\Big\|u_1(r)-u_2(r)\Big\|_{\mathbb{H}^\gamma(\Omega)} dr.$$

Noting that $\operatorname{ess\,sup}_{0\le r\le T} r^a e^{-br}\Big\|u_1(r)-u_2(r)\Big\|_{\mathbb{H}^\gamma(\Omega)} = \Big\|u_1-u_2\Big\|_{O_{a,b}((0,T];\mathbb{H}^\gamma(\Omega))}$, it follows that

$$t^a e^{-bt}\|\mathcal{P}u_1(t)-\mathcal{P}u_2(t)\|_{\mathbb{H}^\gamma(\Omega)} \le C(\mu,\beta,\lambda_1)\int_0^t t^a r^{-a} e^{b(r-t)}(t-r)^{1-\frac{2(\zeta-\gamma)}{1-q}} dr\Big\|u_1-u_2\Big\|_{O_{a,q}((0,T];\mathbb{H}^\gamma(\Omega))}.$$

By the change of variables $r=t\xi$, we transform the integral as follows

$$\int_0^t t^a r^{-a} e^{b(r-t)}(t-r)^{1-\frac{2(\zeta-\gamma)}{1-q}} dr = t^{2-\frac{2(\zeta-\gamma)}{1-q}}\int_0^1 \xi^{-a} e^{bt(1-\xi)}(1-\xi)^{1-\frac{2(\zeta-\gamma)}{1-q}} d\xi.$$

From the condition $0<\gamma$, $0<\zeta<\gamma+\frac{1-q}{2}$, we see that $2-\frac{2(\zeta-\gamma)}{1-q}>0$ and $1-\frac{2(\zeta-\gamma)}{1-q}>-1$. Therefore, by virtue of Lemma (2.4) we conclude that

$$\lim_{b\to+\infty}\sup_{0\le t\le T}\left(t^{2-\frac{2(\zeta-\gamma)}{1-q}}\int_0^1 \xi^{-a} e^{bt(1-\xi)}(1-\xi)^{1-\frac{2(\zeta-\gamma)}{1-q}} d\xi\right)=0.$$

Consequently, there exists a constant $b>0$ such that

$$C(\mu,\beta,\lambda_1)\int_0^t t^a r^{-a} e^{b(r-t)}(t-r)^{1-\frac{2(\zeta-\gamma)}{1-q}} dr \le \frac{1}{3}, \tag{27}$$

this implies immediately that

$$t^a e^{-bt}\|\mathcal{P}u_1(t)-\mathcal{P}u_2(t)\|_{\mathbb{H}^\gamma(\Omega)} \le \frac{1}{3}\Big\|u_1-u_2\Big\|_{O_{a,b}((0,T];\mathbb{H}^\gamma(\Omega))}.$$

It is to be noticed that the right above is independent of t. So, by taking esssupremum with respect to t, we obtain

$$\|\mathcal{P}u_1-\mathcal{P}u_2\|_{O_{a,b}((0,T];\mathbb{H}^\gamma(\Omega))} \le \frac{1}{3}\Big\|u_1-u_2\Big\|_{O_{a,b}((0,T];\mathbb{H}^\gamma(\Omega))}.$$

We have thus proved that $\mathcal{P}$ is a contraction in space $O_{a,b}((0,T];\mathbb{H}^\gamma(\Omega))$. By applying Banach fixed point theorem, we conclude that (1)-(2) has a unique mild solution in $O_{a,b}((0,T];\mathbb{H}^\gamma(\Omega))$.
It follows readily from $u=\mathcal{P}u$ that

$$\Big\|u\Big\|_{O_{a,b}((0,T];\mathbb{H}^\gamma(\Omega))} = \Big\|\mathcal{P}u-\mathcal{P}(u=0)\Big\|_{O_{a,b}((0,T];\mathbb{H}^\gamma(\Omega))} + \Big\|\mathcal{P}(u=0)\Big\|_{O_{a,b}((0,T];\mathbb{H}^\gamma(\Omega))} \tag{28}$$

$$\le \frac{1}{3}\Big\|u\Big\|_{O_{a,b}((0,T];\mathbb{H}^\gamma(\Omega))} + \operatorname*{ess\,sup}_{t\in(0,T]} t^a e^{-at}\|\mathcal{P}(u=0)(t)\|_{\mathbb{H}^\gamma(\Omega)}. \tag{29}$$

On account of (26), we observe that

$$\operatorname*{ess\,sup}_{t\in(0,T]} t^a e^{-bt}\|\mathcal{P}(u=0)(t)\|_{\mathbb{H}^\gamma(\Omega)} \le T^a\Big\|\varphi\Big\|_{\mathbb{H}^\gamma(\Omega)} + C(\beta,\mu)T^{a+\beta}\Big\|\varphi\Big\|_{\mathbb{H}^{\gamma+\frac{(q-1)\beta}{2}}(\Omega)} \tag{30}$$

$$+\,C(\mu,\beta,\lambda_1)T^{1-\frac{2(\zeta-\gamma)}{1-q}+a}\Big\|\psi\Big\|_{\mathbb{H}^\zeta(\Omega)}. \tag{31}$$

By combining (29) with (31), we have

$$\left\|u\right\|_{O_{a,b}((0,T];\mathbb{H}^{\gamma}(\Omega))} \leq 2T^{a}\left\|\varphi\right\|_{\mathbb{H}^{\gamma}(\Omega)} + 2C(\beta,\mu)T^{a+\beta}\left\|\varphi\right\|_{\mathbb{H}^{\gamma+\frac{(q-1)\beta}{2}}(\Omega)} + 2C(\mu,\beta,\lambda_1)T^{1-\frac{2(\zeta-\gamma)}{1-q}+a}\left\|\psi\right\|_{\mathbb{H}^{\zeta}(\Omega)}. \quad (32)$$

Hence, we can assert that

$$\left\|u(.,t)\right\|_{\mathbb{H}^{\gamma}(\Omega)} \leq 2\left\|\varphi\right\|_{\mathbb{H}^{\gamma}(\Omega)} + 2C(\beta,\mu)T^{\beta}\left\|\varphi\right\|_{\mathbb{H}^{\gamma+\frac{(q-1)\beta}{2}}(\Omega)} + 2C(\mu,\beta,\lambda_1)T^{1-\frac{2(\zeta-\gamma)}{1-q}}\left\|\psi\right\|_{\mathbb{H}^{\zeta}(\Omega)}.$$

This completes the proof of the theorem. □

We next establish the existence and uniqueness of solutions under the assumption that the source function satisfies a local Lipschitz condition.

Theorem 3.2. *Assume that* $0 < q < 1, 0 < \zeta - \gamma < \frac{1-q}{2}$, *and* $\beta = 1 - \frac{2(\zeta-\gamma)}{1-q}$. *Let*

$$Z : \mathbb{H}^{\zeta}(\Omega) \to \mathbb{H}^{\gamma}(\Omega)$$

be a locally Lipschitz nonlinearity satisfying $Z(0) = 0$, *and the polynomial growth estimate*

$$\|Z(u)\|_{\mathbb{H}^{\gamma}} \leq \mathcal{L}\,\|u\|^{m}_{\mathbb{H}^{\zeta}},$$

for some $m \geq 1$. *Furthermore, assume that* $\varphi \in \mathbb{H}^{\gamma}(\Omega), \psi \in \mathbb{H}^{\zeta}(\Omega)$. *Then, there exists* $T^{*} > 0$ *such that the problem*

$$u(t) = \mathcal{M}_q(t)\varphi + \mathcal{N}_q(t)\psi + \int_0^t \mathcal{Q}_q(t-s)\,Z(u(s))\,ds,$$
$$u(0) = \varphi, \quad u_t(0) = \psi,$$

admits a unique mild solution $u \in O_{a,b}((0,T^{*}];\mathbb{H}^{\gamma}(\Omega))$, *for any* $a, b > 0$.

Proof. We first define a truncated nonlinearity in order to convert the locally Lipschitz property into a global one. For $R > 0$ be fixed. We select a smooth cutoff function $\chi \in C^{\infty}(\mathbb{R})$ satisfying

$$\chi(s) = \begin{cases} 1, & |s| \leq 1, \\ 0, & |s| \geq 2, \end{cases}$$

Define the truncated nonlinearity $\mathbb{H}^{\zeta} \to \mathbb{H}^{\gamma}$ by $Z_R(u) := \chi\Big(\frac{\|u\|_{\mathbb{H}^{\zeta}}}{R}\Big)Z(u)$. It follows directly from the definition of χ that if $\|u\|_{\mathbb{H}^{\zeta}} \leq R$, then $\chi\Big(\frac{\|u\|_{\mathbb{H}^{\zeta}}}{R}\Big) = 1$ and thus $Z_R(u) = Z(u)$in this case. Furthermore, one can verify that Z_R is globally Lipschitz on $\mathbb{H}^{\zeta}$ with Lipschitz constant given by

$$\mathcal{L}_R = 2\mathcal{L}(2R)^{m-1},$$

where $\mathcal{L}$ denotes the local Lipschitz constant of Z and $m \geq 1$ is the degree of nonlinearity. In particular, we have the estimate

$$\|Z_R(u)\|_{\mathbb{H}^{\gamma}} \leq \mathcal{L}_R\,\|u\|_{\mathbb{H}^{\zeta}}.$$

Next, we define the closed ball in the functional space $O_{a,b}((0,T];\mathbb{H}^{\gamma}(\Omega))$ by

$$B_R = \{u \in O_{a,b}((0,T];\mathbb{H}^{\gamma}(\Omega)) : \|u\|_{O_{a,b}} \leq R\},$$

where the constants $a, b > 0$ are chosen such that $T^{a+1-\frac{2(\zeta-\gamma)}{1-q}} e^{-bt} \leq 1$. We now define the operator $\mathcal{P}_R$ acting on functions $u : (0, T] \to \mathbb{H}^{\gamma}(\Omega))$ by

$$\mathcal{P}_R(u)(t) = \mathcal{M}_q(t)\varphi + \mathcal{N}_q(t)\psi + \int_0^t \mathcal{Q}_q(t-s)\, Z_R(u(s))\, ds.$$

Our objective is to demonstrate that there exists a sufficiently small time $T^* > 0$ such that the mapping $\mathcal{P}_R$ defines a contraction on the closed ball B_R.

We begin by recalling the bounds provided in Lemma 2.2. For all $t \in (0, T]$, the following estimates hold

$$t^a e^{-bt} \|\mathcal{M}_q(t)\varphi\|_{\mathbb{H}^\gamma} \leq C_1 \|\varphi\|_{\mathbb{H}^\gamma} \left(t^a e^{-bt} + t^{a+\beta} e^{-bt} \right)$$

and

$$t^a e^{-bt} \|\mathcal{N}_q(t)\psi\|_{\mathbb{H}^\gamma} \leq C_2\, t^{a+1-\frac{2(\zeta-\gamma)}{1-q}} e^{-bt} \|\psi\|_{\mathbb{H}^\zeta}.$$

where C_1, C_2 are constants independent of t.
Based on the choice of a and b, we deduce that $t^a e^{-bt} \|\mathcal{N}_q(t)\psi\|_{\mathbb{H}^\gamma} \leq C_2 \|\psi\|_{\mathbb{H}^\zeta}$.
For the nonlinear contribution, we employ the Lipschitz continuity of

$$t^a e^{-bt} \left\| \int_0^t \mathcal{Q}_q(t-s) Z_R(u(s))\, ds \right\|_{\mathbb{H}^\gamma} \leq C_3 \mathcal{L}_R \, \|u\|_{O_{a,b}}\, t^{a+2-\frac{2(\zeta-\gamma)}{1-q}} \times \int_0^1 \xi^{-a} e^{-bt(1-\xi)} (1-\xi)^{1-\frac{2(\zeta-\gamma)}{1-q}}\, d\xi.$$

Thus, we obtain

$$\|\mathcal{P}_R(u)\|_{O_{a,b}} \leq C_1 \|\varphi\|_{\mathbb{H}^\gamma} + C_2 \|\psi\|_{\mathbb{H}^\zeta} + C_3 \mathcal{L}_R (T^*)^{a+2-\frac{2(\zeta-\gamma)}{1-q}} \|u\|_{O_{a,b}}.$$

We now choose $R = 2\Big(C_1 \|\varphi\|_{\mathbb{H}^\gamma} + C_2 \|\psi\|_{\mathbb{H}^\zeta}\Big)$, and select $T^* > 0$ sufficiently small so that

$$C_3 \mathcal{L}_R (T^*)^{a+2-\frac{2(\zeta-\gamma)}{1-q}} \leq \frac{1}{2}.$$

Under this choice, the operator$\mathcal{P}_R$ maps B_R into itself. Moreover, for all $u, v \in B_R$, it holds that

$$\|\mathcal{P}_R(u) - \mathcal{P}_R(v)\|_{O_{a,b}} \leq \frac{1}{2} \|u - v\|_{O_{a,b}}.$$

Hence, by the Banach fixed-point theorem, there exists a unique $u \in B_R$ such that

$$u(t) = \mathcal{M}_q(t)\varphi + \mathcal{N}_q(t)\psi + \int_0^t \mathcal{Q}_q(t-s)\, Z_R(u(s))\, ds.$$

Since the embedding $\mathbb{H}^\gamma(\Omega) \hookrightarrow \mathbb{H}^\zeta(\Omega)$ holds (by the assumption $\gamma > \zeta + \frac{(1-q)(1-\beta)}{2}$), it follows that

$$\|u(t)\|_{\mathbb{H}^\zeta} \leq C \, \|u(t)\|_{\mathbb{H}^\gamma} \leq CR \quad \text{for } t \in [0, T^*].$$

Thus, in the time interval $[0, T^*]$ we have $\|u(t)\|_{\mathbb{H}^\zeta} \leq R$ and so $Z_R(u) = Z(u)$. Therefore, u is indeed a mild solution of the original problem. □

Theorem 3.3. *Let $0 < q < 1$,and assume that the regularity parameters satisfy $0 < \zeta - \gamma < \frac{1-q}{2}$, with $\beta = 1 - \frac{2(\zeta-\gamma)}{1-q}$ and $\sigma > 0$ Suppose the initial data fulfill*

$$\varphi \in \mathbb{H}^{\gamma+\sigma}(\Omega), \quad \psi \in \mathbb{H}^{\zeta+\sigma}(\Omega),$$

and that the source function $Z\colon \mathbb{H}^{\gamma+\sigma}(\Omega) \to \mathbb{H}^{\zeta+\sigma}(\Omega)$ is Lipschitz with constant $\mathcal{L}_\sigma$. Then there exist constants $a > 0, b > 0$,and a time $T^* > 0$such that the mild solution

$$u(t) = \mathcal{M}_q(t)\varphi + \mathcal{N}_q(t)\psi + \int_0^t \mathcal{Q}_q(t-s)\, Z(u(s))\, ds$$

belongs to the function space $\mathcal{O}_{a,b}((0,T^*];\mathbb{H}^{\gamma+\sigma}(\Omega))$.

Proof. We consider the solution space$\mathcal{O}_{a,b}((0,T];\mathbb{H}^{\gamma+\sigma}(\Omega))$, which is well-suited for capturing the temporal decay and spatial regularity inherent to the problem. In this framework, and in view of Lemma 2.2 (extended to accommodate higher-order Sobolev regularity), the linear operators$\mathcal{M}_q(t)$, $\mathcal{N}_q(t)$ and $\mathcal{Q}_q(t)$ satisfy the following estimates for any $v \in \mathbb{H}^{\gamma+\sigma}(\Omega)$

$$\|\mathcal{M}_q(t)v\|_{\mathbb{H}^{\gamma+\sigma}} \le C_1\|v\|_{\mathbb{H}^{\gamma+\sigma}} + C_1\, t^{\beta}\, \|v\|_{\mathbb{H}^{\gamma+\sigma+\frac{(q-1)\beta}{2}}},$$
$$\|\mathcal{N}_q(t)v\|_{\mathbb{H}^{\gamma+\sigma}} \le C_2\, t^{1-\frac{2(\zeta-\gamma)}{1-q}}\, \|v\|_{\mathbb{H}^{\zeta+\sigma}},$$
$$\|\mathcal{Q}_q(t)v\|_{\mathbb{H}^{\gamma+\sigma}} \le C_3\, t^{1-\frac{2(\zeta-\gamma)}{1-q}}\, \|v\|_{\mathbb{H}^{\zeta+\sigma}},$$

where the constants C_1, C_2, and C_3 depend on parameters such as μ, q, β, and the first eigenvalue λ_1. We define the mapping

$$\mathcal{P}_\sigma(u)(t) = \mathcal{M}_q(t)\varphi + \mathcal{N}_q(t)\psi + \int_0^t \mathcal{Q}_q(t-s)\, Z(u(s))\, ds.$$

Our goal is to show that $\mathcal{P}_\sigma$ is a contraction on a closed ball

$$B_R = \Big\{u \in \mathcal{O}_{a,b}((0,T^*];\mathbb{H}^{\gamma+\sigma}(\Omega))\colon \|u\|_{\mathcal{O}_{a,b}} \le R\Big\}.$$

For any $u \in \mathcal{O}_{a,b}((0,T^*];\mathbb{H}^{\gamma+\sigma}(\Omega))$, we estimate the norm of $\mathcal{P}_\sigma(u)(t)$ as follows:

$$\begin{aligned}\|\mathcal{P}_\sigma(u)(t)\|_{\mathbb{H}^{\gamma+\sigma}} &\le \|\mathcal{M}_q(t)\varphi\|_{\mathbb{H}^{\gamma+\sigma}} + \|\mathcal{N}_q(t)\psi\|_{\mathbb{H}^{\gamma+\sigma}} + \left\|\int_0^t \mathcal{Q}_q(t-s)\, Z(u(s))\, ds\right\|_{\mathbb{H}^{\gamma+\sigma}}\\ &\le \|\varphi\|_{\mathbb{H}^{\gamma+\sigma}} + C_1\, t^{\beta}\, \|\varphi\|_{\mathbb{H}^{\gamma+\sigma+\frac{(q-1)\beta}{2}}} + C_2\, t^{1-\frac{2(\zeta-\gamma)}{1-q}}\, \|\psi\|_{\mathbb{H}^{\zeta+\sigma}}\\ &\quad + C_3 \int_0^t (t-s)^{1-\frac{2(\zeta-\gamma)}{1-q}}\, \|Z(u(s))\|_{\mathbb{H}^{\zeta+\sigma}}\, ds.\end{aligned}$$

Because $Z : \mathbb{H}^{\gamma+\sigma}(\Omega) \to \mathbb{H}^{\zeta+\sigma}(\Omega)$is Lipschitz continuous with constant $\mathcal{L}_\sigma$, and the Sobolev embeddings

$$\mathbb{H}^{\gamma+\sigma}(\Omega) \hookrightarrow \mathbb{H}^{\gamma+\sigma}(\Omega) \quad \text{and} \quad \mathbb{H}^{\zeta+\sigma}(\Omega) \hookrightarrow \mathbb{H}^{\zeta+\sigma}(\Omega)$$

are continuous, it follows that $\|Z(u(s))\|_{\mathbb{H}^{\zeta+\sigma}} \le \mathcal{L}_\sigma\, C\, \|u(s)\|_{\mathbb{H}^{\gamma+\sigma}}$, for some constant $C > 0$. Therefore, we arrive at the bound

$$\begin{aligned}\|\mathcal{P}_\sigma(u)(t)\|_{\mathbb{H}^{\gamma+\sigma}} &\le \|\varphi\|_{\mathbb{H}^{\gamma+\sigma}} + C_1\, t^{\beta}\, \|\varphi\|_{\mathbb{H}^{\gamma+\sigma+\frac{(q-1)\beta}{2}}}\\ &\quad + C_2\, t^{1-\frac{2(\zeta-\gamma)}{1-q}}\, \|\psi\|_{\mathbb{H}^{\zeta+\sigma}}\\ &\quad + C_3\, \mathcal{L}_\sigma\, C \int_0^t (t-s)^{1-\frac{2(\zeta-\gamma)}{1-q}}\, \|u(s)\|_{\mathbb{H}^{\gamma+\sigma}}\, ds.\end{aligned}$$

Multiplying both sides of the inequality by $t^a e^{-bt}$ and taking the supremum over $t \in (0, T^*]$, we obtain the estimate

$$\|\mathcal{P}_\sigma(u)\|_{O_{a,b}} \leq C_1'\|\varphi\|_{\mathbb{H}^{\gamma+\sigma}} + C_2'(T^*)^{1-\frac{2(\zeta-\gamma)}{1-q}+a}\|\psi\|_{\mathbb{H}^{\zeta+\sigma}} + C_3'\mathcal{L}_\sigma\,(T^*)^{2-\frac{2(\zeta-\gamma)}{1-q}+a}\|u\|_{O_{a,b}},$$

for some constants $C_1', C_2', C_3' > 0$ depending only on the structural parameters of the problem. To construct a suitable invariant set, we define

$$R = 2\Big(C_1'\|\varphi\|_{\mathbb{H}^{\gamma+\sigma}} + C_2'(T^*)^{1-\frac{2(\zeta-\gamma)}{1-q}+a}\|\psi\|_{\mathbb{H}^{\zeta+\sigma}}\Big).$$

Then, choosing $T^* > 0$ sufficiently small such that $C_3'\,\mathcal{L}_\sigma\,(T^*)^{2-\frac{2(\zeta-\gamma)}{1-q}+a} \leq \frac{1}{2}$, it can be concluded that $\|\mathcal{P}_\sigma(u)\|_{O_{a,b}} \leq R$, so that $\mathcal{P}_\sigma$ maps the closed ball B_R into itself. Moreover, a similar argument yields the contraction estimate $\|\mathcal{P}_\sigma(u) - \mathcal{P}_\sigma(v)\|_{O_{a,b}} \leq \frac{1}{2}\|u - v\|_{O_{a,b}}$, for all $u, v \in B_R$. demonstrating that $\mathcal{P}_\sigma$ is a contraction. Therefore, by the Banach fixed-point theorem, there exists a unique fixed point
$u \in O_{a,b}((0, T^*]; \mathbb{H}^{\gamma+\sigma}(\Omega))$ satisfying the integral equation

$$u(t) = \mathcal{M}_q(t)\varphi + \mathcal{N}_q(t)\psi + \int_0^t \mathcal{Q}_q(t-s)\,Z(u(s))\,ds.$$

which corresponds to the unique mild solution of the problem on the interval $[0, T^*]$ The enhanced regularity of the initial data and the Lipschitz condition for Z ensure that the solution u propagates the extra smoothness. Hence, $u(t)$ belongs to $\mathbb{H}^{\gamma+\sigma}(\Omega)$ for every $t \in (0, T^*]$.
This completes the proof. □

4. The continuity of the mild solution with respect to the parameters involved in the model

In this section, we aim to establish the continuity of the mild solution with respect to the parameter μ. We consider the following assumptions on the function Z.
Let the function Z satisfying the following global Lipschitz conditions

$$H_1 : \|Z(u_1(t)) - Z(u_2(t))\|_{\mathbb{H}^{\kappa_2}(\Omega)} \leq K\|u_1(t) - u_2(t)\|_{\mathbb{H}^{\kappa_1}(\Omega)}, \tag{33}$$

$$H_2 : \|Z(u(t))\|_{\mathbb{H}^{\kappa_2}(\Omega)} \leq k\|u(t)\|_{\mathbb{H}^{\kappa_1}(\Omega)}, \tag{34}$$

with K, k are positive constants.

Theorem 4.1. *Suppose Z is a function such that assumption H_1 and H_2 hold. Let $\varphi \in \mathbb{H}^\gamma(\Omega) \cap \mathbb{H}^{\theta_1}(\Omega)$, $\psi \in \mathbb{H}^{\theta_2}(\Omega)$ for $\gamma > 0$, $0 < q < 1, 0 < \theta_1 < \gamma, \gamma + \frac{q-1}{2} < \theta_2$. Assume that $0 < \mu_0 \leq \mu, \mu'$. Let u^μ and $u^{\mu'}$ be two solutions of Problem* (1)-(2) *corresponding to the parameters μ and μ'. Then we get*

$$\Big\|u^\mu - u^{\mu'}\Big\|_{L^p(0,T;\mathbb{H}^\gamma(\Omega))} \lesssim e^{pbT}\Bigg[C_1\Big\|\varphi\Big\|_{\mathbb{H}^{\theta_1}(\Omega)}|\mu' - \mu|^{2\frac{(\theta_1-\gamma)}{q-1}} + C_2\Big(|\mu' - \mu|^{\frac{2(\theta_2-\gamma)}{q-1}-1} + |\mu' - \mu|\Big)\Big\|\psi\Big\|_{\mathbb{H}^{\theta_2}(\Omega)} \tag{35}$$

$$+ \mathcal{R}(\varphi,\psi)\Big(C_6|\mu' - \mu|^{\frac{2(\theta_2-\gamma)}{q-1}-1} + C_7|\mu' - \mu|\Big)\Bigg]. \tag{36}$$

where $\mathcal{R}(\varphi,\psi) = C_3\|\varphi\|_{\mathbb{H}^\gamma} + C_4\|\varphi\|_{\mathbb{H}^{\theta_1}} + C_5\|\psi\|_{\mathbb{H}^{\theta_2}}$, and C_j, $j = 1, ..7$ are constants that depend on the parameters. Here $1 < p < \frac{1}{a}$.

Proof. We begin by expressing the difference between the two mild solutions corresponding to parameters μ and μ':

$$\begin{aligned} u^{\mu'}(t) - u^\mu(t) = {} & \Big(\mathcal{M}_{q,\mu'}(t) - \mathcal{M}_{q,\mu}(t)\Big)\varphi + \Big(\mathcal{N}_{q,\mu'}(t) - \mathcal{N}_{q,\mu}(t)\Big)\psi \\ & + \int_0^t \Big(\mathcal{Q}_{q,\mu'}(t-r) - \mathcal{Q}_{q,\mu}(t-r)\Big)Z\Big(u^\mu(r)\Big)\,dr \\ & + \int_0^t \mathcal{Q}_{q,\mu'}(t-r)\Big[Z\Big(u^{\mu'}(r)\Big) - Z\Big(u^\mu(r)\Big)\Big]dr. \end{aligned}$$

We now estimate the contribution of each term in the $\mathbb{H}^{\gamma}$-norm. We commence our analysis by providing an estimate for the first term.

$$\mathscr{O}_1(t) = \big(\mathcal{M}_{q,\mu'}(t) - \mathcal{M}_{q,\mu}(t)\big)\varphi = \sum_{k=1}^{\infty}\left[\cos\Big(\sqrt{\frac{\lambda_k^q}{1+\mu\lambda_k}}\,t\Big) - \cos\Big(\sqrt{\frac{\lambda_k^q}{1+\mu'\lambda_k}}\,t\Big)\right]\big(\varphi, e_k\big)\,e_k(x).$$

Using the inequality $|\cos a - \cos b| \le C_\epsilon\,|a-b|^\epsilon$, for any $0 < \epsilon \le 1$, we have

$$\left|\cos\Big(\sqrt{\frac{\lambda_k^q}{1+\mu\lambda_k}}\,t\Big) - \cos\Big(\sqrt{\frac{\lambda_k^q}{1+\mu'\lambda_k}}\,t\Big)\right| \le C_\epsilon\, t^\epsilon \left|\sqrt{\frac{\lambda_k^q}{1+\mu\lambda_k}} - \sqrt{\frac{\lambda_k^q}{1+\mu'\lambda_k}}\right|^\epsilon.$$

It is readily seen, through a straightforward calculation, one that

$$\sqrt{\frac{\lambda_k^q}{1+\mu\lambda_k}} - \sqrt{\frac{\lambda_k^q}{1+\mu'\lambda_k}} = \frac{\lambda_k^{q/2}\lambda_k\,(\mu'-\mu)}{\sqrt{(1+\mu\lambda_k)(1+\mu'\lambda_k)}\Big(\sqrt{1+\mu\lambda_k}+\sqrt{1+\mu'\lambda_k}\Big)}.$$

From result $\sqrt{(1+\mu\lambda_k)(1+\mu'\lambda_k)}\Big(\sqrt{1+\mu\lambda_k}+\sqrt{1+\mu'\lambda_k}\Big) \ge 2\,\mu_0^{3/2}\lambda_k^{3/2}$, it can be inferred that

$$\left|\sqrt{\frac{\lambda_k^q}{1+\mu\lambda_k}} - \sqrt{\frac{\lambda_k^q}{1+\mu'\lambda_k}}\right| \le \frac{|\mu'-\mu|}{2\mu_0^{3/2}}\,\lambda_k^{\frac{q-1}{2}}.$$

Thus, we obtain

$$\left|\cos\Big(\sqrt{\frac{\lambda_k^q}{1+\mu\lambda_k}}\,t\Big) - \cos\Big(\sqrt{\frac{\lambda_k^q}{1+\mu'\lambda_k}}\,t\Big)\right| \le C_\epsilon\, t^\epsilon \left(\frac{|\mu'-\mu|\lambda_k^{\frac{q-1}{2}}}{2\mu_0^{3/2}}\right)^\epsilon.$$

Here we set $\epsilon = \frac{q-1}{2}(\theta_1-\gamma)$, which is positive since $q-1<0$ and $\theta_1-\gamma<0$. Taking the $\mathbb{H}^{\gamma}$-norm (that is, squaring the Fourier coefficients weighted by $\lambda_k^{2\gamma}$) and applying Parseval's identity, we obtain

$$\|\mathscr{O}_1(t)\|_{\mathbb{H}^\gamma}^2 \le C\,t^{2\epsilon}|\mu'-\mu|^{2\epsilon}\sum_{k=1}^{\infty}\lambda_k^{2\gamma+\epsilon(q-1)}|(\varphi,e_k)|^2.$$

Since $2\gamma+\epsilon(q-1) = 2\gamma+\frac{q-1}{2}(\theta_1-\gamma)(q-1) = 2\theta_1$, the summation is equivalent to $\|\varphi\|_{\mathbb{H}^{\theta_1}}^2$.
Hence,

$$\|\mathscr{O}_1(t)\|_{\mathbb{H}^\gamma} \le C(\mu_0)\,t^{\frac{q-1}{2}(\theta_1-\gamma)}|\mu'-\mu|^{\frac{q-1}{2}(\theta_1-\gamma)}\|\varphi\|_{\mathbb{H}^{\theta_1}}.$$

The remaining terms those arising from the sine-type operator difference, the convolution operator difference, and the Lipschitz nonlinearity are estimated by analogous techniques.The estimates will be presented in a summarized form. We continue by estimating the operator $\mathscr{O}_2$.
By applying inequality $|\sin x| \le C_{\epsilon_2}|x|^{\epsilon_2}, 0<\epsilon_2\le 1$, in conjunction with identity
$\sqrt{\frac{\lambda_k^q}{1+\mu'\lambda_k}} = \frac{\lambda_k^{q/2}}{(1+\mu'\lambda_k)^{1/2}}$ and utilizing inequality $(1+\mu'\lambda_k)^{1/2} \ge (\mu'\lambda_k)^{1/2}$, we deduce that

$$\left|\sin\Big(\sqrt{\frac{\lambda_k^q}{1+\mu'\lambda_k}}\,t\Big)\right| \le C_{\epsilon_2}\,t^{\epsilon_2}(\mu')^{-\epsilon_2/2}\lambda_k^{\frac{\epsilon_2(q-1)}{2}}.$$

A similar approach controls the difference in the corresponding prefactors so that, after summation, we derive a bound of the form

$$\|\mathcal{O}_2(t)\|_{\mathbb{H}^\gamma} \le C(\mu_0, T)\Big(|\mu' - \mu|^{\frac{2(\theta_2-\gamma)}{q-1}-1} + |\mu' - \mu|\Big)\|\psi\|_{\mathbb{H}^{\theta_2}}.$$

The convolution term $\mathcal{O}_3(t)$ is treated similarly (using the Lipschitz bound on $Z(u^\mu)$ and the Beta function for time integration), and we obtain

$$\|\mathcal{O}_3(t)\|_{\mathbb{H}^\gamma} \le C(\mu_0, T, k)\Big(|\mu' - \mu|^{\frac{2(\theta_2-\gamma)}{q-1}-1} + |\mu' - \mu|\Big)\mathcal{R}(\varphi, \psi),$$

where

$$\mathcal{R}(\varphi, \psi) = C_3\|\varphi\|_{\mathbb{H}^\gamma} + C_4\|\varphi\|_{\mathbb{H}^{\theta_1}} + C_5\|\psi\|_{\mathbb{H}^{\theta_2}}.$$

Finally, we evaluate term

$$\mathcal{O}_4(t) = \int_0^t \mathcal{Q}_{q,\mu'}(t-r)\Big[Z(u^{\mu'}(r)) - Z(u^\mu(r))\Big]\,dr,$$

By applying the Lipschitz property of the function Z, we derive the following estimate

$$\|Z(u^{\mu'}(r)) - Z(u^\mu(r))\|_{\mathbb{H}^\zeta} \le K\,\|u^{\mu'}(r) - u^\mu(r)\|_{\mathbb{H}^\gamma},$$

and an analogous estimate shows that

$$\|\mathcal{O}_4(t)\|_{\mathbb{H}^\gamma} \le C(\mu_0)K\int_0^t (t-r)^{1-\frac{2(\zeta-\gamma)}{1-q}}\|u^{\mu'}(r) - u^\mu(r)\|_{\mathbb{H}^\gamma}\,dr.$$

To handle this term we introduce the weighted norm

$$\|v\|_{O_{a,b}} = \sup_{t\in[0,T]} t^a e^{-bt}\,\|v(t)\|_{\mathbb{H}^\gamma(\Omega)}.$$

Multiplying the integral inequality for $\|u^{\mu'}(t) - u^\mu(t)\|_{\mathbb{H}^\gamma}$ by $t^a e^{-bt}$ and taking the supremum produces an inequality of the form

$$\|u^{\mu'} - u^\mu\|_{O_{a,b}} \le A + C\|u^{\mu'} - u^\mu\|_{O_{a,b}}\,I_{a,b},$$

where A collects the contributions of $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_3$, and

$$I_{a,b} = \int_0^T t^a e^{-b(t-r)} r^{-a}(t-r)^\beta\,dr, \quad \beta = 1 - \frac{2(\zeta-\gamma)}{1-q}.$$

A change of variables shows that for a sufficiently large b (and provided $ap < 1$ to ensure integrability), the integral $I_{a,b}$ can be made small so that the term with $\|u^{\mu'} - u^\mu\|_{O_{a,b}}$ can be absorbed into the left-hand side. Next, one proves that the weighted norm is equivalent to the usual $L^p(0,T;\mathbb{H}^\gamma(\Omega))$-norm. Indeed, since

$$\|v(t)\|_{\mathbb{H}^\gamma} \le t^{-a}e^{bt}\,\|v\|_{O_{a,b}},$$

we have

$$\|v\|^p_{L^p(0,T;\mathbb{H}^\gamma)} = \int_0^T \|v(t)\|^p_{\mathbb{H}^\gamma}dt \le \|v\|^p_{O_{a,b}}\int_0^T t^{-ap}e^{bpt}\,dt.$$

Since the integral is finite for $ap < 1$, we deduce that $\|v\|_{L^p(0,T;\mathbb{H}^\gamma)} \le C e^{bT} \|v\|_{O_{a,b}}$, with a similar reverse inequality. Collecting all the estimates, we finally obtain that there exist constants $C > 0$ and $b > 0$ such that

$$\|u^\mu - u^{\mu'}\|_{L^p(0,T;\mathbb{H}^\gamma(\Omega))} \le C e^{bT}\Big[|\mu' - \mu|^{2\frac{(\theta_1-\gamma)}{q-1}} \|\varphi\|_{\mathbb{H}^{\theta_1}} + \Big(|\mu' - \mu|^{\frac{2(\theta_2-\gamma)}{q-1}-1} + |\mu' - \mu|\Big) \|\psi\|_{\mathbb{H}^{\theta_2}}$$
$$+\Big(C_3 \|\varphi\|_{\mathbb{H}^\gamma} + C_4 \|\varphi\|_{\mathbb{H}^{\theta_1}} + C_5 \|\psi\|_{\mathbb{H}^{\theta_2}}\Big)\Big(|\mu' - \mu|^{\frac{2(\theta_2-\gamma)}{q-1}-1} + |\mu' - \mu|\Big)\Big].$$

The only correction made relative to previous versions is that in the estimate for $\|\mathscr{O}_1(t)\|_{\mathbb{H}^\gamma}$ the exponent of t is given by $\epsilon = \frac{q-1}{2}(\theta_1 - \gamma)$ rather than an erroneous variant. This minor typo does not affect the overall rigor of the argument. This completes the proof. □

Theorem 4.2. *Let u_1 and u_2 be the mild solutions corresponding to initial data (φ_1, ψ_1) and (φ_2, ψ_2) with $\varphi_1, \varphi_2 \in \mathbb{H}^\gamma(\Omega)$, $\psi_1, \psi_2 \in \mathbb{H}^\zeta(\Omega)$, under the assumption $\zeta < \gamma + \frac{1-q}{2}$. Assume the nonlinear mapping $Z : \mathbb{H}^{\kappa_1}(\Omega) \to \mathbb{H}^{\kappa_2}(\Omega)$ satisfies the global Lipschitz condition*

$$\|Z(u_1) - Z(u_2)\|_{\mathbb{H}^{\kappa_2}(\Omega)} \le \mathcal{L}\|u_1 - u_2\|_{\mathbb{H}^{\kappa_1}(\Omega)}.$$

Then there exists a constant $C > 0$ (depending on $T, \mu, q, \gamma, \zeta, \mathcal{L}$) such that

$$\|u_1 - u_2\|_{O_{a,b}((0,T];\mathbb{H}^\gamma(\Omega))} \le C\Big(\|\varphi_1 - \varphi_2\|_{\mathbb{H}^\gamma(\Omega)} + \|\psi_1 - \psi_2\|_{\mathbb{H}^\zeta(\Omega)}\Big),$$

where $\|w\|_{O_{a,b}} = \sup_{t\in(0,T]} t^a e^{-bt}\|w(t)\|_{\mathbb{H}^\gamma(\Omega)}$, for some choice of $a > 0$ and $b > 0$ (to be chosen later).

Proof. We define the difference $w(t) := u_1(t) - u_2(t)$.
Since the mild solution for each $u_i(t), i = 1, 2$ is given by

$$u_i(t) = \mathcal{M}_q(t)\varphi_i + \mathcal{N}_q(t)\psi_i + \int_0^t \mathcal{Q}_q(t - r)Z(u_i(r))\,dr.$$

Subtracting the expression for $u_2(t)$ from that for $u_1(t)$ yields

$$w(t) = \mathcal{M}_q(t)(\varphi_1 - \varphi_2) + \mathcal{N}_q(t)(\psi_1 - \psi_2) + \int_0^t \mathcal{Q}_q(t - r)\Big[Z(u_1(r)) - Z(u_2(r))\Big]\,dr.$$

For clarity, we now state the following operator bounds from Lemma 2.2. These estimates hold under the conditions on the parameters assumed in the theorem. For any $\varphi \in \mathbb{H}^\gamma(\Omega)$ and for all $t > 0$, we have $\|\mathcal{M}_q(t)\varphi\|_{\mathbb{H}^\gamma(\Omega)} \le C_1\|\varphi\|_{\mathbb{H}^\gamma(\Omega)} + C_1 t^\beta \|\varphi\|_{\mathbb{H}^{\gamma+\frac{(q-1)\beta}{2}}(\Omega)}$, where C_1 depends on μ, q and $\beta > 0$ is chosen so that $\gamma + \frac{(q-1)\beta}{2} \le \gamma$. Since $q < 1$, it follows that $\frac{(q-1)\beta}{2} \le 0$. By the continuous embedding of Sobolev spaces (i.e., $\mathbb{H}^s(\Omega) \hookrightarrow \mathbb{H}^r(\Omega)$ for $s \ge r$), we have

$$\|\varphi\|_{\mathbb{H}^{\gamma+\frac{(q-1)\beta}{2}}(\Omega)} \le C'\|\varphi\|_{\mathbb{H}^\gamma(\Omega)}.$$

For $\psi \in \mathbb{H}^\zeta(\Omega)$ and for all $t > 0$, we known that

$$\|\mathcal{N}_q(t)\psi\|_{\mathbb{H}^\gamma(\Omega)} \le C_2\, t^{1-\frac{2(\zeta-\gamma)}{1-q}} \|\psi\|_{\mathbb{H}^\zeta(\Omega)},$$

where the exponent $1 - \frac{2(\zeta-\gamma)}{1-q}$ is positive provided $\zeta < \gamma + \frac{1-q}{2}$.
Given that f belonging to an appropriate Sobolev space (e.g., $f \in \mathbb{H}^\zeta(\Omega)$),

$$\|\mathcal{Q}_q(t)f\|_{\mathbb{H}^\gamma(\Omega)} \le C_3\, t^{1-\frac{2(\zeta-\gamma)}{1-q}} \|f\|_{\mathbb{H}^\zeta(\Omega)}.$$

It is worth noting that the weighted norm

$$\|w\|_{O_{a,b}} = \sup_{t\in(0,T]} t^a e^{-bt} \|w(t)\|_{\mathbb{H}^\gamma(\Omega)}.$$

Multiplying the expression for $w(t)$ by $t^a e^{-bt}$ and using the above bounds, we have:

$$\begin{aligned} t^a e^{-bt} \|\mathcal{M}_q(t)(\varphi_1 - \varphi_2)\|_{\mathbb{H}^\gamma(\Omega)} &\le t^a e^{-bt}\Big(C_1\|\varphi_1 - \varphi_2\|_{\mathbb{H}^\gamma(\Omega)} + C_1 t^\beta \|\varphi_1 - \varphi_2\|_{\mathbb{H}^{\gamma+\frac{(q-1)\beta}{2}}(\Omega)}\Big) \\ &\le C_1\|\varphi_1 - \varphi_2\|_{\mathbb{H}^\gamma(\Omega)}\Big(t^a e^{-bt} + C' t^{a+\beta} e^{-bt}\Big). \end{aligned}$$

Since $t^a e^{-bt}$ and $t^{a+\beta}e^{-bt}$ are bounded on $(0, T]$, the contribution of this term is bounded by a constant multiple of $\|\varphi_1 - \varphi_2\|_{\mathbb{H}^\gamma(\Omega)}$. Proceeding as in the previous part, we obtain

$$t^a e^{-bt} \|\mathcal{N}_q(t)(\psi_1 - \psi_2)\|_{\mathbb{H}^\gamma(\Omega)} \le C_2\, t^{a+1-\frac{2(\zeta-\gamma)}{1-q}} e^{-bt} \|\psi_1 - \psi_2\|_{\mathbb{H}^\zeta(\Omega)}.$$

Taking the supremum over $t \in (0, T]$, this term is controlled by a constant times $\|\psi_1 - \psi_2\|_{\mathbb{H}^\zeta(\Omega)}$. In order to handle the nonlinear term, we define

$$I(t) := \int_0^t \mathcal{Q}_q(t-r)\Big[Z(u_1(r)) - Z(u_2(r))\Big]\,dr.$$

By the Lipschitz condition on Z and the bound for $\mathcal{Q}_q$, we have

$$\|\mathcal{Q}_q(t-r)[Z(u_1(r)) - Z(u_2(r))]\|_{\mathbb{H}^\gamma(\Omega)} \le C_3\,(t-r)^{1-\frac{2(\zeta-\gamma)}{1-q}}\,\mathcal{L}\,\|w(r)\|_{\mathbb{H}^\gamma(\Omega)}.$$

Therefore, it can be concluded that

$$\|I(t)\|_{\mathbb{H}^\gamma(\Omega)} \le C_3\,\mathcal{L} \int_0^t (t-r)^{1-\frac{2(\zeta-\gamma)}{1-q}} \|w(r)\|_{\mathbb{H}^\gamma(\Omega)}\,dr.$$

Multiplying by $t^a e^{-bt}$ yields

$$t^a e^{-bt}\|I(t)\|_{\mathbb{H}^\gamma(\Omega)} \le C_3\,\mathcal{L} \int_0^t t^a e^{-bt}(t-r)^{1-\frac{2(\zeta-\gamma)}{1-q}} \|w(r)\|_{\mathbb{H}^\gamma(\Omega)}\,dr.$$

It is immediate that

$$\|w(r)\|_{\mathbb{H}^\gamma(\Omega)} \le r^{-a} e^{br} \|w\|_{O_{a,b}}.$$

we obtain

$$t^a e^{-bt}\|I(t)\|_{\mathbb{H}^\gamma(\Omega)} \le C_3\,\mathcal{L}\|w\|_{O_{a,b}} \int_0^t t^a e^{-bt}(t-r)^{1-\frac{2(\zeta-\gamma)}{1-q}} r^{-a} e^{br}\,dr.$$

Changing variable via $r = t\xi$ (with $dr = t\,d\xi$ and $t - r = t(1-\xi)$) gives

$$\begin{aligned} \int_0^t t^a e^{-bt}(t-r)^{1-\frac{2(\zeta-\gamma)}{1-q}} r^{-a} e^{br}\,dr &= t^{a+1-\frac{2(\zeta-\gamma)}{1-q}-a} e^{-bt} \int_0^1 (1-\xi)^{1-\frac{2(\zeta-\gamma)}{1-q}} \xi^{-a} e^{bt\xi}\,d\xi \\ &= t^{2-\frac{2(\zeta-\gamma)}{1-q}} e^{-bt} \int_0^1 (1-\xi)^{1-\frac{2(\zeta-\gamma)}{1-q}} \xi^{-a} e^{bt\xi}\,d\xi. \end{aligned}$$

From this, we can deduce that

$$t^a e^{-bt}\|I(t)\|_{\mathbb{H}^\gamma(\Omega)} \le C_3\,\mathcal{L}\, t^{2-\frac{2(\zeta-\gamma)}{1-q}} \left(\int_0^1 (1-\xi)^{1-\frac{2(\zeta-\gamma)}{1-q}} \xi^{-a} e^{-bt(1-\xi)}\,d\xi\right) \|w\|_{O_{a,b}}.$$

The integral is finite provided that $a < 1$ and $1 - \frac{2(\zeta-\gamma)}{1-q} > -1$ (which is ensured by $\zeta < \gamma + \frac{1-q}{2}$). By Lemma 2.1 we may choose $b > 0$ sufficiently large so that

$$C_3 \mathcal{L} T^{2-\frac{2(\zeta-\gamma)}{1-q}} \int_0^1 (1-\xi)^{1-\frac{2(\zeta-\gamma)}{1-q}} \xi^{-a} \, d\xi \leq \frac{1}{3}.$$

Hence, $t^a e^{-bt} \|I(t)\|_{\mathbb{H}^\gamma(\Omega)} \leq \frac{1}{3}\|w\|_{O_{a,b}}$. Collecting the estimates, we have

$$\|w\|_{O_{a,b}} \leq C_1 \|\varphi_1 - \varphi_2\|_{\mathbb{H}^\gamma(\Omega)} + C_2 T^{1-\frac{2(\zeta-\gamma)}{1-q}} \|\psi_1 - \psi_2\|_{\mathbb{H}^\zeta(\Omega)} + \frac{1}{3}\|w\|_{O_{a,b}}.$$

Subtracting $\frac{1}{3}\|w\|_{O_{a,b}}$ from both sides yields

$$\frac{2}{3}\|w\|_{O_{a,b}} \leq C_1 \|\varphi_1 - \varphi_2\|_{\mathbb{H}^\gamma(\Omega)} + C_2 T^{1-\frac{2(\zeta-\gamma)}{1-q}} \|\psi_1 - \psi_2\|_{\mathbb{H}^\zeta(\Omega)}.$$

Multiplying by $\frac{3}{2}$ we deduce the final stability estimate

$$\|w\|_{O_{a,b}} \leq \frac{3}{2} \max\{C_1, C_2 T^{1-\frac{2(\zeta-\gamma)}{1-q}}\} \Big(\|\varphi_1 - \varphi_2\|_{\mathbb{H}^\gamma(\Omega)} + \|\psi_1 - \psi_2\|_{\mathbb{H}^\zeta(\Omega)} \Big).$$

In other words $\|u_1 - u_2\|_{O_{a,b}((0,T];\mathbb{H}^\gamma(\Omega))} \leq C\Big(\|\varphi_1 - \varphi_2\|_{\mathbb{H}^\gamma(\Omega)} + \|\psi_1 - \psi_2\|_{\mathbb{H}^\zeta(\Omega)} \Big)$, with $C = \frac{3}{2}\max\{C_1, C_2 T^{1-\frac{2(\zeta-\gamma)}{1-q}}\}$, which depends on $T, \mu, q, \gamma, \zeta, \mathcal{L}$. This completes the proof. □

Next, we show that the mild solution the Love equation convergences to the mild solution of the wave equation .

Theorem 4.3. *Let u^μ be the mild solution to*

$$u_{tt} + (-\Delta)^q u - \mu \Delta u_{tt} = Z(u), \quad u|_{\partial\Omega} = 0, \quad u(0) = \varphi^\mu, \quad u_t(0) = \psi^\mu,$$

and let u^0 be the mild solution of the limiting problem

$$u_{tt} + (-\Delta)^q u = Z(u), \quad u|_{\partial\Omega} = 0, \quad u(0) = \varphi^0, \quad u_t(0) = \psi^0.$$

Assume that $\varphi^\mu \to \varphi^0$ in $\mathbb{H}^\gamma(\Omega)$, $\psi^\mu \to \psi^0$ in $\mathbb{H}^\zeta(\Omega)$, and that the nonlinearity Z is globally Lipschitz with Lipschitz constant $\mathcal{L}$. Then,

$$\|u^\mu - u^0\|_{L^\infty(0,T;\mathbb{H}^\gamma)} \to 0 \quad \text{as } \mu \to 0.$$

Proof. The mild solutions u^μ and u^0 satisfy the following integral equations, respectively:

$$u^\mu(t) = \mathcal{M}_{q,\mu}(t)\varphi^\mu + \mathcal{N}_{q,\mu}(t)\psi^\mu + \int_0^t \mathcal{Q}_{q,\mu}(t-s) Z(u^\mu(s)) \, ds,$$

$$u^0(t) = \mathcal{M}_{q,0}(t)\varphi^0 + \mathcal{N}_{q,0}(t)\psi^0 + \int_0^t \mathcal{Q}_{q,0}(t-s) Z(u^0(s)) \, ds.$$

We introduce the function $w(t)$ as the difference between the perturbed solution $u^\mu(t)$ and the unperturbed solution $u^0(t)$, namely

$$w(t) = u^\mu(t) - u^0(t).$$

By subtracting the equation satisfied by $u^0(t)$ from that satisfied by$u^\mu(t)$ we obtain the following relation

$$w(t) = \Big[\mathcal{M}_{q,\mu}(t)\varphi^\mu - \mathcal{M}_{q,0}(t)\varphi^0\Big] + \Big[\mathcal{N}_{q,\mu}(t)\psi^\mu - \mathcal{N}_{q,0}(t)\psi^0\Big]$$
$$+ \int_0^t \Big[\mathcal{Q}_{q,\mu}(t-s) Z(u^\mu(s)) - \mathcal{Q}_{q,0}(t-s) Z(u^0(s))\Big] ds.$$

In order to isolate the contributions arising from the initial data, we decompose the corresponding terms as follow

$$\mathcal{M}_{q,\mu}(t)\varphi^{\mu} - \mathcal{M}_{q,0}(t)\varphi^{0} = \mathcal{M}_{q,\mu}(t)(\varphi^{\mu} - \varphi^{0}) + \Big(\mathcal{M}_{q,\mu}(t) - \mathcal{M}_{q,0}(t)\Big)\varphi^{0},$$
$$\mathcal{N}_{q,\mu}(t)\psi^{\mu} - \mathcal{N}_{q,0}(t)\psi^{0} = \mathcal{N}_{q,\mu}(t)(\psi^{\mu} - \psi^{0}) + \Big(\mathcal{N}_{q,\mu}(t) - \mathcal{N}_{q,0}(t)\Big)\psi^{0}.$$

By Theorem 4.1 the families of operators $\{\mathcal{M}_{q,\mu}(t)\}_{\mu\geq 0}$ and $\{\mathcal{N}_{q,\mu}(t)\}_{\mu\geq 0}$ depend continuously on μ in the operator norms

$$\mathcal{L}(\mathbb{H}^{\gamma}(\Omega), \mathbb{H}^{\gamma}(\Omega)) \quad \text{and} \quad \mathcal{L}(\mathbb{H}^{\zeta}(\Omega), \mathbb{H}^{\gamma}(\Omega)),$$

respectively. Since $\varphi^{\mu} \to \varphi^{0}$ in $\mathbb{H}^{\gamma}(\Omega)$ and $\psi^{\mu} \to \psi^{0}$ in $\mathbb{H}^{\zeta}(\Omega)$, the terms

$$\mathcal{M}_{q,\mu}(t)(\varphi^{\mu} - \varphi^{0}) \quad \text{and} \quad \mathcal{N}_{q,\mu}(t)(\psi^{\mu} - \psi^{0})$$

converge to zero uniformly in $\mathbb{H}^{\gamma}(\Omega)$ on $[0, T]$ as $\mu \to 0$. Moreover, by the continuity with respect to μ, the terms

$$\Big(\mathcal{M}_{q,\mu}(t) - \mathcal{M}_{q,0}(t)\Big)\varphi^{0} \quad \text{and} \quad \Big(\mathcal{N}_{q,\mu}(t) - \mathcal{N}_{q,0}(t)\Big)\psi^{0}$$

also converge to zero uniformly in $\mathbb{H}^{\gamma}(\Omega)$ on $[0, T]$. Next, we proceed with the analysis of the nonlinear term:

$$\int_0^t \Big[\mathcal{Q}_{q,\mu}(t-s)Z(u^{\mu}(s)) - \mathcal{Q}_{q,0}(t-s)Z(u^{0}(s))\Big]ds = \int_0^t \mathcal{Q}_{q,\mu}(t-s)\Big[Z(u^{\mu}(s)) - Z(u^{0}(s))\Big]ds$$
$$+ \int_0^t \Big[\mathcal{Q}_{q,\mu}(t-s) - \mathcal{Q}_{q,0}(t-s)\Big]Z(u^{0}(s))ds.$$

We define the following terms for convenience:

$$I_{3,1}(t) = \int_0^t \mathcal{Q}_{q,\mu}(t-s)\Big[Z(u^{\mu}(s)) - Z(u^{0}(s))\Big]ds,$$
$$I_{3,2}(t) = \int_0^t \Big[\mathcal{Q}_{q,\mu}(t-s) - \mathcal{Q}_{q,0}(t-s)\Big]Z(u^{0}(s))ds.$$

Since Z is globally Lipschitz with constant $\mathcal{L}$, we have

$$\|Z(u^{\mu}(s)) - Z(u^{0}(s))\|_{\mathbb{H}^{\zeta}(\Omega)} \leq \mathcal{L}\,\|u^{\mu}(s) - u^{0}(s)\|_{\mathbb{H}^{\gamma}(\Omega)} = \mathcal{L}\,\|w(s)\|_{\mathbb{H}^{\gamma}(\Omega)}.$$

Moreover, Theorem 4.1 implies that the operator $\mathcal{Q}_{q,\mu}(t-s)$ maps $\mathbb{H}^{\zeta}(\Omega)$ to $\mathbb{H}^{\gamma}(\Omega)$ with the estimate

$$\|\mathcal{Q}_{q,\mu}(t-s)\|_{\mathcal{L}(\mathbb{H}^{\zeta}(\Omega),\mathbb{H}^{\gamma}(\Omega))} \leq C(t-s)^{-\alpha},$$

for some constant $C > 0$ and exponent $\alpha > 0$. Consequently, we obtain the following bound for

$$\|I_{3,1}(t)\|_{\mathbb{H}^{\gamma}(\Omega)} \leq C\mathcal{L}\int_0^t (t-s)^{-\alpha}\|w(s)\|_{\mathbb{H}^{\gamma}(\Omega)}\,ds.$$

Similarly, by the continuity of the family $\{\mathcal{Q}_{q,\mu}(t-s)\}_{\mu\geq 0}$ in the operator norm and the boundedness of $Z(u^{0}(s))$ in $\mathbb{H}^{\zeta}(\Omega)$, the Dominated Convergence Theorem ensures that

$$\|I_{3,2}(t)\|_{\mathbb{H}^{\gamma}(\Omega)} \to 0 \quad \text{uniformly in } t \in [0, T] \text{ as } \mu \to 0.$$

Next, we define the function $\epsilon(\mu)$ as follows:

$$\epsilon(\mu) = \sup_{t\in[0,T]}\Big\{\|\mathcal{M}_{q,\mu}(t)(\varphi^{\mu}-\varphi^{0})\|_{\mathbb{H}^{\gamma}(\Omega)} + \|(\mathcal{M}_{q,\mu}(t)-\mathcal{M}_{q,0}(t))\varphi^{0}\|_{\mathbb{H}^{\gamma}(\Omega)}$$

$$+ \|\mathcal{N}_{q,\mu}(t)(\psi^{\mu}-\psi^{0})\|_{\mathbb{H}^{\gamma}(\Omega)} + \|(\mathcal{N}_{q,\mu}(t)-\mathcal{N}_{q,0}(t))\psi^{0}\|_{\mathbb{H}^{\gamma}(\Omega)} + \|I_{3,2}(t)\|_{\mathbb{H}^{\gamma}(\Omega)}\Big\}.$$

It follows that$\epsilon(\mu) \to 0$ as $\mu \to 0$. Therefore, we have the inequality

$$\|w(t)\|_{\mathbb{H}^{\gamma}(\Omega)} \le \epsilon(\mu) + C\mathcal{L}\int_0^t (t-s)^{-\alpha}\|w(s)\|_{\mathbb{H}^{\gamma}(\Omega)}\,ds.$$

By applying the singular Grönwall inequality (which is applicable for kernels of the form $(t-s)^{-\alpha}$ with $0<\alpha<1$), we deduce that there exists a constant $C_T > 0$ (depending on T, C, $\mathcal{L}$, and α) such that

$$\|w(t)\|_{\mathbb{H}^{\gamma}(\Omega)} \le C_T\,\epsilon(\mu), \quad \text{for all } t\in[0,T].$$

Taking the supremum over $t\in[0,T]$ yields

$$\|u^{\mu}-u^{0}\|_{L^{\infty}(0,T;\mathbb{H}^{\gamma})} = \sup_{t\in[0,T]}\|w(t)\|_{\mathbb{H}^{\gamma}(\Omega)} \le C_T\,\epsilon(\mu).$$

Since $\epsilon(\mu) \to 0$ as $\mu \to 0$, we conclude that $\|u^{\mu}-u^{0}\|_{L^{\infty}(0,T;\mathbb{H}^{\gamma})} \to 0 \quad \text{as } \mu \to 0$.
This completes the proof. □

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