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Total dominator coloring of the lexicographic product of graphs

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Abstract. A total dominator coloring of a graph which has no isolated vertex is a proper vertex coloring of the graph in which each vertex of the graph is adjacent to all vertices of some (other) color class. The total dominator chromatic number of a graph is the minimum size of color classes in a total dominator coloring of the graph.

This paper establishes the total dominator coloring of the lexicographic product of graphs. Firstly, we present some graphs whose total dominator chromatic numbers are same. Also we show some graphs whose total dominator chromatic numbers are equal to their chromatic numbers. Then, after presenting four upper bounds and a lower bound for the total dominator chromatic number of the lexicographic product of two graphs, we give a sufficient condition for that the total dominator chromatic number of the lexicographic product of two graphs to be equal to the total dominator chromatic number of the lexicographic product of the first graph to a complete graph. We next establish a Nordhaus-Gaddum-like relation for the total dominator chromatic number of the lexicographic product of two graphs, and finally find the total dominator chromatic number of the lexicographic product of a star, a wheel or a complete graph with an arbitrary graph.

1. Introduction

We consider non-empty, undirected, finite and simple graphs throughout the paper. Let $G = (V, E)$ be a graph having *vertex set* $V = V(G)$ and *edge set* $E = E(G)$. The order of G is $|V(G)|$. While the *degree*, *open* and *closed neighborhoods* of a vertex $u \in V$ are denoted respectively by $deg_G(u)$, $N_G(u)$ and $N_G[u]$, the *minimum* and *maximum degrees* of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. While a *complete p-partite graph*, a *wheel* of order $n + 1$, and the subgraph of G *induced* by a vertex set S are denoted respectively by $K_{n_1,n_2,\cdots,n_p}$, W_n and $G[S]$, a *cycle* and a *complete graph* of order n are denoted by C_n and K_n, respectively. The *complement* of G, denoted by $\overline{G}$, is a graph with vertex set $V(G)$ and for each two vertices u and v, $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. Here, $\omega(G)$ and $\alpha(G)$ denote the *clique* and *independence numbers* of G, while $[\ell]$ represents the set $\{1, 2, \cdots, \ell\}$ for some positive integer ℓ, respectively.

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The *lexicographic product* $G \circ H$ of a graph G with a graph H is defined on $V(G \circ H) = V(G) \times V(H)$, and two vertices (g_1, h_1), (g_2, h_2) of $G \circ H$ being adjacent whenever $g_1 g_2 \in E(G)$, or $g_1 = g_2$ and $h_1 h_2 \in E(H)$. Harary (1959) introduced this product as the *composition* of graphs. Even though this theory originated with Hausdorff's (1914), graph theorists initiated the investigations with Harary's paper. For any vertex $g \in V(G)$, the subgraph of $G \circ H$ induced by the vertex set $\{g\} \times V(H)$, which is isomorphic to H, is called the copy ${}_gH$ of H. The copy G_h of G for any vertex $h \in V(H)$ can be defined similarly. This product is also known as *graph substitution*, which indicates that $G \circ H$ is formed from G by substituting a copy ${}_gH$ of H for each vertex g of G and then joining each vertex of ${}_gH$ with every vertex of ${}_{g'}H$ when $gg' \in E(G)$.

Various researchers have studied domination thoroughly. Haynes, Hedetnemi, and Slater have summarized the literature in the books [5, 6]. Also, recently two new books [3, 4] are written on this topic by Haynes, Hedetniemi and Henning. Total domination is a well-known domination variant and the recent book [7] is summarized on this topic by Henning and Yeo. A *total dominating set*, TDS in short, S of a graph G is a subset of $V(G)$ which satisfy the condition $S \cap N_G(v) \neq \emptyset$ for every vertex v. The minimum size of a TDS of G is the *total domination number* $\gamma_t(G)$ of G.

Another significant concept in graph theory is (vertex, edge or both) coloring of a graph. For a positive integer ℓ, a *proper ℓ-coloring* of G is a map from the vertex set of G to an ℓ-set of colors in such a way any two neighbor vertices have distinct colors. In another words, it is a homomorphism $f : G \to K_\ell$ from G to a complete graph K_ℓ (with the vertex set $[\ell]$). The *chromatic number* $\chi(G)$ of G is the minimum size of colors required in a proper coloring of G, or equivalently the homomorphism f is an epimorphism. A *color class* in a proper coloring of a graph means that all vertices of it has the same color. We write $f = (V_1, V_2, \cdots, V_\ell)$ when f is a proper coloring of G having the coloring classes $V_1, V_2, \cdots, V_\ell$, in which each vertex in V_i is colored with color i.

The total dominator coloring was defined as follows by the relation between total domination and coloring. The reader can study Section 4 of Part I from the book [4], and the papers [1] and [8–14] for more information on this concept.

Definition 1.1. [11] A *total dominator coloring*, TDC in short, of a graph G with no isolated vertex is a proper coloring of G such that each vertex of the graph is adjacent to every vertex of some color class. The minimum number of color classes in a TDC of G is called the *total dominator chromatic number* $\chi_d^t(G)$ of G.

Let $f = (V_1, V_2, \cdots, V_\ell)$ be a TDC of a graph G. If $V_i \subseteq N(u)$, it is said that a vertex u is a *common neighbor* of V_i or u is *totally dominated* by V_i or V_i *totally dominates* u and it is denoted by $u \succ_t V_i$. Otherwise, it is denoted by $u \not\succ_t V_i$. Further, if $u \succ_t V_i$ and $u \not\succ_t V_j$ for every $j \neq i$, the vertex u is called the *private neighbor* of V_i with respect to f. The *common neighborhood* of V_i in G with respect to f, denoted by $CN_G(V_i, f)$ or $CN(V_i)$ in short, is the set of all common neighbors of V_i. Moreover, the *private neighborhood* of V_i in G with respect to f, denoted by $PN_G(V_i, f)$ or $PN(V_i)$ in short, is the set of all private neighbors of V_i. Also a TDC of G having $\chi_d^t(G)$ colors is called a *min*-TDC.

This paper concentrates on the total dominator coloring of the lexicographic product of graphs. In details, in Section 2, we determine some graphs whose total dominator chromatic numbers are same. Also we present some graphs whose total dominator chromatic numbers are equal to their chromatic numbers. Then in Section 3, after presenting four upper bounds and a lower bound for the total dominator chromatic number of the lexicographic product of two graphs, we give a sufficient condition for that the total dominator chromatic number of the lexicographic product of two graphs to be equal to the total dominator chromatic number of the lexicographic product of the first factor to a complete graph. Also we establish a Nordhaus-Gaddum-like relation for the total dominator chromatic number of the lexicographic product of two graphs. In Section 4, we give the total dominator chromatic numbers of the lexicographic product of a star, a wheel or a complete graph with an arbitrary graph. Finally we offer some open problems which can be some motivations for future works.

Here some propositions are presented which are relevant to our investigation.

Proposition 1.2. [2, 15] *For any graphs G and H, $\chi(G \circ H) \leq \chi(G)\chi(H)$, and if G is a bipartite graph or a complete graph, then $\chi(G \circ H) = \chi(G)\chi(H)$.*

Proposition 1.3. [16] *For a cycle C_n and any graph H,*

$$\chi(C_n \circ H) = \begin{cases} 2\chi(H) & \text{for even } n, \\ 2\chi(H) + \lceil \frac{2\chi(H)}{n-1} \rceil & \text{for odd } n. \end{cases}$$

Proposition 1.4. [11] *For any connected graph G with $\delta(G) \geq 1$,*

$$\chi_d^t(G) \leq \gamma_t(G) + \min_S \chi(G[V(G) - S]),$$

in which S is a min-TDS of G. And so $\chi_d^t(G) \leq \chi(G) + \gamma_t(G)$.

2. Some equalities

In this section, we show some graphs whose total dominator chromatic numbers are same. For this purpose we require the following two definitions.

Definition 2.1. Let G be a p-partite graph which has the partition $V(G) = X_1 \cup \cdots \cup X_p$ to independent sets, with the property that all vertices of X_i are adjacent to all vertices of X_j if and only if some vertex of X_i is adjacent to some vertex of X_j. Then G is called as a *semi-complete p-partite graph.*

Definition 2.2. An *n-tuple lexicographic product* of a graph G whose vertex set is $V(G) = \{v_1, v_2, \cdots, v_n\}$, with n paired disjoint graphs H_1, H_2, $\cdots$, H_n is the graph $G \circ (H_1, H_2, \cdots, H_n)$ which is formed from G by substituting H_i for vertex v_i of G and joining each vertex of H_i with every vertex of H_j when $v_iv_j \in E(G)$.

Notice that for given graphs H_1, H_2, $\cdots$, H_n and a graph G of order n, there are $n!$ n-tuple lexicographic product graphs $G \circ (H_{\sigma(1)}, H_{\sigma(2)}, \cdots, H_{\sigma(n)})$ where σ is a permutation on $[n]$, and all of them are isomorphic to the lexicographic product graph $G \circ H$ when $H \cong H_i$ for each $1 \leq i \leq n$.

Theorem 2.3. *Let G be a semi-complete p-partite graph having the partition $V(G) = X_1 \cup \cdots \cup X_p$ to independent sets, and let G_c be the contract of G which is obtained by contracting of all vertices of X_i to a vertex x_i. Then $\chi_d^t(G) = \chi_d^t(G_c)$.*

Proof. Let G be a semi-complete p-partite graph having the partition $V(G) = X_1 \cup \cdots \cup X_p$ to independent sets, and let G_c be its contract. For any min-TDC $(V_1, V_2, \cdots, V_\ell)$ of G_c, since $(\bigcup_{x_i \in V_1} X_i, \bigcup_{x_i \in V_2} X_i, \cdots, \bigcup_{x_i \in V_\ell} X_i)$ is a TDC of G, we have

$$\chi_d^t(G) \leq \chi_d^t(G_c). \tag{1}$$

For the other inequality, assume $f = (V_1, V_2, \cdots, V_\ell)$ is a min-TDC of G, and for $1 \leq j \leq \ell$ let $V'_j = \{x_i \mid X_i \cap V_j \neq \emptyset\}$. If for each $1 \leq i \leq p$, $X_i \subseteq V_j$ for some $1 \leq j \leq \ell$, then $f' = (V'_1, V'_2, \cdots, V'_\ell)$ will be a TDC of G_c, and so $\chi_d^t(G) = \chi_d^t(G_c)$ by (1). Therefore, we may assume that for some $1 \leq i \leq p$, $X_i \nsubseteq V_j$ when $1 \leq j \leq \ell$. Let $X_i \subseteq V_{i_1} \cup V_{i_2} \cup \cdots \cup V_{i_s}$ for some $1 \leq i_s < \cdots < i_1 \leq \ell$ and some $s \geq 2$. By the minimality of f, $V_{i_j} \nsubseteq X_i$ for $1 \leq j \leq s$ (because otherwise, by adding the vertices in V_{i_j} to another color class V_{i_k}, we will have $\chi_d^t(G) < \ell$). Define

$$g : V(G_c) \longrightarrow [\ell]$$
$$g(x_i) = \max\{f(x) \mid x \in X_i\}.$$

Then $g = (V''_1, V''_2, \cdots, V''_\ell)$ is a TDC of G_c, in which $V''_j \subseteq V'_j$ for each j. This fact and (1) imply $\chi_d^t(G) = \chi_d^t(G_c)$. $\square$

Corollary 2.4. *Let G be any graph of order n which has no isolated vertex. Then $\chi_d^t(G \circ (\overline{K_{p_1}}, \cdots, \overline{K_{p_n}})) = \chi_d^t(G)$ for any n positive integers $p_1, \cdots, p_n$.*

Corollary 2.5. *For any complete m-partite graph $H = K_{p_1, \cdots, p_m}$ when $m \geq 1$, if $G \circ H$ is the lexicographic product graph which is formed from $G \circ K_m$ by substituting the empty graph $\overline{K_{p_i}}$ for each vertex from $V(G) \times \{i\}$ for each $1 \leq i \leq m$, then $\chi_d^t(G \circ H) = \chi_d^t(G \circ K_m)$.*

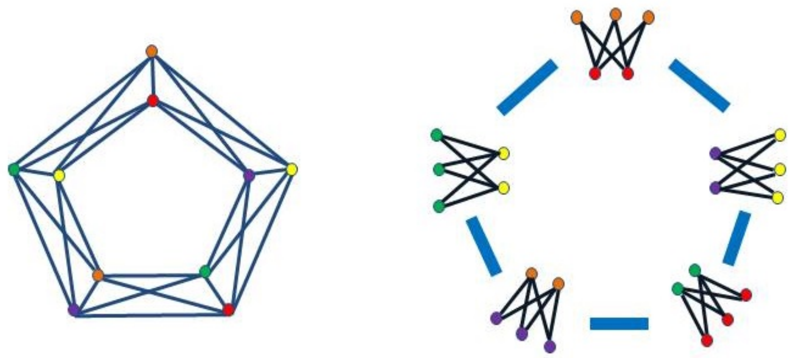

Figure 1: An example to explain Corollary 2.5. The graphs $C_5 \circ K_2$ and $C_5 \circ (K_{2,3})$ with their min-TDCs with five color classes.

3. Some bounds and a Nordhaus-Gaddum-like relation

Here, we establish four upper bounds and a lower bound for the total dominator chromatic number of the lexicographic product of two graphs. Then we give a sufficient condition for that the total dominator chromatic number of the lexicographic product of two graphs to be equal to the total dominator chromatic number of the lexicographic product of the first graph with a complete graph. Finally, we present a Nordhaus-Gaddum-like relation for the total dominator chromatic number of the lexicographic product of two graphs.

3.1. Some upper bounds

First we state a needed theorem on the total domination number of the lexicographic product of two graphs.

Theorem 3.1. *For any graph G without isolated vertex and any graph H,*

$$\gamma_t(G \circ H) = \gamma_t(G).$$

Proof. Since for any min-TDS T of G and any vertex $h \in V(H)$, the set $T \times \{h\}$ is a TDS of $G \circ H$, we have

$$\gamma_t(G \circ H) \leq \gamma_t(G). \tag{2}$$

For a min-TDS S of $G \circ H$, let $(g_i, h) \in S$.

`Fact 1.` Let $(g_j, h') \in S$ for some $g_j \in N_G(g_i)$ and some $h' \in V(H)$. Then by the minimality of S, it follows $|{}_{g_i}H \cap S| = |{}_{g_j}H \cap S| = 1$.

`Fact 2.` Let $(g_j, h') \notin S$ for each $g_j \in N_G(g_i)$ and each $h' \in V(H)$. Then by the fact that S is minimum, ${}_{g_i}H \cap S$ is a min-TDS of ${}_{g_i}H$ and it follows $|{}_{g_i}H \cap S| = \gamma_t(H) \geq 2$.

Let $S = S_1 \cup S_2$ be the partition of S in which $S_j = \{(g_i, h) \mid (g_i, h)$ satisfies in `Fact j`$\}$ for $j = 1, 2$. For each g_iH with $|{}_{g_i}H \cap S| \geq 2$, we choose only one vertex $(g_i, h)^* \in S_2$ and only one vertex $(g_{i_j}, h')^*$ such that $g_{i_j} \in N_G(g_i)$ and $h' \in V(H)$. Then, since $S' = S_1 \cup \{(g_i, h)^*, (g_{i_j}, h')^* \mid (g_i, h) \in S_2\}$ is obviously a TDS of G, it follows $\gamma_t(G \circ H) \geq \gamma_t(G)$ which implies $\gamma_t(G \circ H) = \gamma_t(G)$ by (2). □

As a result of Proposition 1.4 and Theorem 3.1, the folowing theorem is obtained.

Theorem 3.2. *For any graph G without isolated vertex and any graph H,* $\chi_d^t(G \circ H) \leq \gamma_t(G) + \chi(G \circ H)$.

The next theorem gives another upper bound for the total dominator chromatic number of the lexicographic product graphs in terms of the total dominator chromatic number of the first graph and the chromatic number of the second graph.

Theorem 3.3. *For any graph G without isolated vertex and any graph H, $\chi_d^t(G \circ H) \leq \chi_d^t(G)\chi(H)$.*

Proof. Let G and H be two graphs that G has no isolated vertex. From Corollary 2.5 when $m = 1$, we may assume $H \ncong \overline{K_n}$ and $n \geq 2$. Let $f = (V_1, V_2, \cdots, V_\ell)$ be a min-TDC of G and let $g = (U_1, U_2, \cdots, U_p)$ be a proper coloring of H where $p = \chi(H) \geq 2$. Define

$$\phi : V(G \circ H) \longrightarrow [\ell] \times [p]$$
$$\phi((a, b)) = (f(a), g(b)) \text{ for } (a, b) \in V(G \circ H).$$

On another word, $\phi = (V_1 \times U_1, \cdots, V_1 \times U_p, V_2 \times U_1, \cdots, V_2 \times U_p, \cdots, V_\ell \times U_1, \cdots, V_\ell \times U_p)$. First ϕ is a proper coloring of $G \circ H$. Because $aa' \in E(G)$ or $bb' \in E(H)$ for any two neighbors (a, b) and (a', b') in $G \circ H$, and so $f(a) \neq f(a')$ or $g(b) \neq g(b')$, respectively, which means $\phi((a, b)) \neq \phi((a', b'))$. More, ϕ is a TDC of $G \circ H$. Because if $a \succ_t V_i$ for some V_i, and U_j is a color class of g such that $U_j \cap N_H(b) \neq \emptyset$ for any vertex (a, b) in $G \circ H$, then $(a, b) \succ_t V_i \times U_j$. Thus, $\chi_d^t(G \circ H) \leq \chi_d^t(G)\chi(H)$. □

Before giving the next two upper bounds, first we call a min-TDC $f = (V_1, V_2, \cdots, V_\ell)$ of the lexicographic product graph $G \circ H$ as a *good min*-TDC if for every $(v, u) \in V(G \circ H)$, there exists a color class V_i such that $(v, u) \succ_t V_i$ and $V_i \cap V({}_vH) = \emptyset$.

Theorem 3.4. *For any graph G without isolated vertex and any graph H of order at least two,*

$$\chi_d^t(G \circ H) \leq \min_{H_1, H_2} \chi_d^t(G \circ H_1) + \chi(G \circ H_2),$$

in which H_1 and H_2 are two induced subgraphs of H such that $V(H) = V(H_1) \cup V(H_2)$ is a partition and $G \circ H_1$ has a good min-TDC.

Proof. Let G be a graph with $\delta(G) \geq 1$ and let H_1 and H_2 be two induced subgraphs of H of order at least two where $V(H) = V(H_1) \cup V(H_2)$ is a partition. Let $f = (V_1, V_2, \cdots, V_p)$ be a good min-TDC of $G \circ H_1$, and let g be a proper coloring of $G \circ H_2$ with $q = \chi(G \circ H_2)$ color classes. Define

$$\phi : V(G \circ H) \longrightarrow [p + q]$$

$$\phi((v, h)) = \begin{cases} f(v) & \text{if } h \in V(H_1), \\ p + g((v, h)) & \text{if } h \in V(H_2). \end{cases}$$

Obviously ϕ is a proper coloring of $G \circ H$. Let $(v, h) \in V(G \circ H)$. Since f is a TDC of $G \circ H_1$, $(v, h) \in V(G \circ H_1)$ implies $(v, h) \succ_t V_i$ for some $1 \leq i \leq p$. Therefore, we may assume $(v, h) \in V(G \circ H_2)$. Then, since f is a good-min-TDC of $G \circ H_1$, for a vertex $(v, h') \in V(G \circ H_1)$ there is a color class V_i such that $(v, h') \succ_t V_i$. By the condition $V_i \cap V({}_vH_1) = \emptyset$, we may conclude $(v, h) \succ_t V_i$. This means ϕ is a TDC of $G \circ H$, and then $\chi_d^t(G \circ H) \leq \chi_d^t(G \circ H_1) + \chi(G \circ H_2)$. □

Corollary 3.5. *For any graph G without isolated vertex and any graph H of order at least two,*

$$\chi_d^t(G \circ H) \leq \chi_d^t(G) + \min_{h \in V(H)} \chi(G \circ (H - h)).$$

Now we present an important upper bound for the total dominator chromatic number of the lexicographic product of two graphs in terms of the total dominator chromatic number of the lexicographic product of the first graph with a complete graph whose order is equal to the chromatic number of the second graph.

Theorem 3.6. *Let G be a graph of order at least 3 and let H be a graph of order at least 2 with chromatic number r. If $G \circ K_r$ has a good min-TDC, then $\chi_d^t(G \circ H) \leq \chi_d^t(G \circ K_r)$.*

Proof. By assumption $V(G \circ H) = \{(v_i, u_j) \mid 1 \le i \le n \text{ and } 1 \le j \le m\}$, let $f : G \circ K_r \to [\ell]$ be a good min-TDC of $G \circ K_r$ with the color classes $V_1, V_2, \cdots, V_\ell$, and let $\theta : H \to K_r$ be an epimorphism. Define $\phi = f \circ (id, \theta) : G \circ H \longrightarrow [\ell]$. Since ϕ is the composition of two epimorphisms, it is an epimorphism. By assumption $\phi = (W_1, W_2, \cdots, W_\ell)$, we have

$$\begin{array}{rcl} (id, \theta)(W_k) & = & \{(v_i, \theta(u_j)) \mid (v_i, u_j) \in W_k\} \\ & = & \{(v_i, \theta(u_j)) \mid k = \phi((v_i, u_j)) = f((v_i, \theta(u_j)))\} \\ & \subseteq & V_k \end{array}$$

for each $1 \le k \le \ell$. Let $(a, b) \in V_k$. Since $b \in V(K_r)$ and $\theta : H \to K_r$ is an epimorphism, there exists a vertex $b' \in V(H)$ with $b = \theta(b')$. Then, since $k = f((a, b)) = f((a, \theta(b'))) = \phi(a, b')$, we have $(a, b') \in W_k$. This means that every member of V_k is in the form of a member of $(id, \theta)(W_k)$, and so $(id, \theta)(W_k) = V_k$ for each $1 \le k \le \ell$. Since f is a good min-TDC of $G \circ K_r$, there is a color class V_p such that $V_p \cap V({}_{v_i}H) = \emptyset$ and $(id, \theta)(v_i, u_j) \succ_t V_p$ for each vertex $(v_i, u_j) \in V(G \circ H)$. Hence, $(v_i, \theta(u_j))$ is adjacent to every vertex $(a, b) \in V_p$, which implies $v_i \neq a$ and $v_i a \in E(G)$. By $v_i a \in E(G)$, we conclude that (v_i, u_j) is adjacent to every vertex (a, b_0) where $\theta(b_0) = b$. But $(a, b_0) \in W_k$ (because $\phi((a, b_0)) = f((a, \theta(b_0))) = k$ and $(id, \theta)(W_p) = V_p$) implies $(v_i, u_j) \succ_t W_p$. Hence ϕ is a TDC of $G \circ H$. Thus, we have $\chi_d^t(G \circ H) \le \chi_d^t(G \circ K_r)$. □

3.2. A lower bound and an equality

Now we present a lower bound for the total dominator chromatic number of the lexicographic product of two graphs, which gives a sufficient condition for the converse of Theorem 3.6.

Let $f = (V_1, V_2, \cdots, V_\ell)$ be a min-TDC of the lexicographic product $G \circ H$ and let $V(G) = \{v_1, v_2, \cdots, v_n\}$. For $1 \le k \le n$, let the number of used colors for coloring the vertices of ${}_{v_k}H$ by f be m_k. We contract the vertices with same color in each ${}_{v_k}H$ to a vertex and assign the same color to the vertex. In details, if $V({}_{v_k}H) \cap V_{i_j} \neq \emptyset$ for $1 \le j \le m_k$, we contract every $V({}_{v_k}H) \cap V_{i_j}$ by vertex $w^k_{i_j}$ and define $f'(w^k_{i_j}) = i_j$. We denote f' by $(V'_1, V'_2, \cdots, V'_\ell)$ and this graph by *Cont* $(G \circ H, f)$, and call it the *contraction* of $G \circ H$ with respect to f. Since f' is a TDC of this graph, we have the next theorem.

Theorem 3.7. *If Cont $(G \circ H, f)$ is the contraction of the lexicographic product $G \circ H$ with respect to a min-TDC f of $G \circ H$, then $\chi_d^t(\text{Cont } (G \circ H, f)) \le \chi_d^t(G \circ H)$.*

By the fact that f is minimum, we may assume that at least one vertex of ${}_{v_k}H \cap V_{i_j}$ is adjacent to at least one vertex of ${}_{v_k}H \cap V_{i_t}$ when $t \neq j$, and this guarantees that the vertices $w^k_{i_j}$ and $w^k_{i_t}$ are adjacent. Hence the complete graph K_{m_k} obtains from ${}_{v_k}H$, and the next theorem is a result of Theorem 3.7.

Theorem 3.8. *Let Cont $(G \circ H, f)$ be the contraction of the lexicographic product $G \circ H$ with respect to a min-TDC f of $G \circ H$. If $V(G) = \{v_1, \cdots, v_n\}$ and for every $1 \le k \le n$, the number of used colors for coloring of the vertices of ${}_{v_k}H$ by f is m_k, then Cont $(G \circ H, f) = G \circ (K_{m_1}, K_{m_2}, \cdots, K_{m_n})$, and so $\chi_d^t(\text{Cont } (G \circ H, f)) = \chi_d^t(G \circ (K_{m_1}, K_{m_2}, \cdots, K_{m_n}))$. More if $\chi(H) = r = m_1 = \cdots = m_n$, then $G \circ (K_{m_1}, K_{m_2}, \cdots, K_{m_n}) = G \circ K_r$ and so $\chi_d^t(G \circ K_r) \le \chi_d^t(G \circ H)$.*

As a result of Theorems 3.6 and 3.8, we have the next theorem.

Theorem 3.9. *Let G be a graph of order $n \ge 3$ and let H be a graph of order at least two with chromatic number r. If $G \circ K_r$ has a good min-TDC, then $\chi_d^t(G \circ H) = \chi_d^t(G \circ K_r)$.*

3.3. A Nordhaus-Gaddum-like relation

Nordhaus-Gaddum-like relations for any parameter in graph theory have been a traditional problem ever since Nordhaus and Gaddum proved the following theorem in 1956 [17].

Theorem 3.10. [17] *For any graph H of order n,*

- $2\sqrt{n} \le \chi(H) + \chi(\overline{H}) \le n + 1.$

- $n \leq \chi(H)\chi(\overline{H}) \leq (\frac{n+1}{2})^2$.

Since $\overline{G \circ H} = \overline{G} \circ \overline{H}$ for any graphs G and H, as a result of Theorem 3.3 it follows

$$\chi_d^t(\overline{G \circ H}) \leq \chi_d^t(\overline{G})\chi(\overline{H})$$

when $\Delta(G) \leq |V(G)| - 2$. This implies the following Nordhaus-Gaddum-like relation for total dominator coloring of the lexicographic product graphs by Theorem 3.10.

Theorem 3.11. *For any graph G having $2 \leq \delta(G) \leq \Delta(G) \leq |V(G)| - 2$ and any graph H having n vertices,*

- $\chi_d^t(G \circ H) + \chi_d^t(\overline{G \circ H}) \leq \max\{\chi_d^t(G), \chi_d^t(\overline{G})\}(n+1)$.
- $\chi_d^t(G \circ H)\chi_d^t(\overline{G \circ H}) \leq \chi_d^t(G)\chi_d^t(\overline{G})(\frac{n+1}{2})^2$.

4. The lexicographic product of a star, a wheel and a complete graph with an arbitrary graph

Here, we study the total dominator chromatic number of the lexicographic product of a star, a complete graph and a wheel with an arbitrary graph. For this purpose we need to examine total dominator coloring of the *join* of two graphs G and H, denoted by $G \vee H$, which is the union of the graphs such that each vertex of one factor is adjacent to every vertex of the other factor.

We know the total dominator chromatic number of a graph has its chromatic number as a lower bound. The next theorem shows that for the join of any two graphs these two numbers are same.

Theorem 4.1. *For any graphs G and H, $\chi_d^t(G \vee H) = \chi(G \vee H)$.*

Proof. Since the union of two coloring functions of G and H with disjoint color sets is a total dominator coloring of $G \vee H$, we have $\chi_d^t(G \vee H) \leq \chi(G) + \chi(H)$. Conversely, let $f = (V_1, V_2, \cdots, V_\ell)$ be a min-TDC of $G \vee H$. By the definition of join of two graphs, every color class is only a subset of $V(G)$ or $V(H)$. Hence, there exists an unique partition $[\ell] = I \cup J$, where $I = \{i \mid V_i \subseteq V(G)\}$ and $J = \{i \mid V_i \subseteq V(H)\}$. Since $|I| \geq \chi(G)$ and $|J| \geq \chi(H)$, we have $\chi_d^t(G \vee H) = \ell \geq \chi(G) + \chi(H)$. By knowing $\chi(G \vee H) = \chi(G) + \chi(H)$, the result is obtained. □

As a first step, since $K_{1,n} \circ H \cong H \vee nH$, the next proposition is a consequence of Theorem 4.1.

Proposition 4.2. *For any graph H and any star $K_{1,n}$, $\chi_d^t(K_{1,n} \circ H) = 2\chi(H)$.*

Also for any graph G of order n which has a vertex v of degree $n-1$, since $G \cong ({}_vK_1) \vee ((G - v) \circ K_1)$, we have the next theorem.

Theorem 4.3. *For any graph G of order n with maximum degree $n - 1$, $\chi_d^t(G) = \chi(G)$.*

Proposition 4.4. *For any wheel W_n of order $n + 1 \geq 4$ and any graph H,*

$$\chi_d^t(W_n \circ H) = \begin{cases} 3\chi(H) & \text{for even } n, \\ 3\chi(H) + \lceil \frac{2\chi(H)}{n-1} \rceil & \text{for odd } n. \end{cases}$$

Proof. Since $W_n \circ H \cong ({}_vH) \vee (C_n \circ H)$ for any wheel W_n with the vertex v which has the maximum degree n and any graph H, we have

$$\begin{aligned} \chi_d^t(W_n \circ H) &= \chi(W_n \circ H) \\ &= \chi(H) + \chi(C_n \circ H) \\ &= \begin{cases} 3\chi(H) & \text{for even } n, \\ 3\chi(H) + \lceil \frac{2\chi(H)}{n-1} \rceil & \text{for odd } n, \end{cases} \end{aligned}$$

by Theorem 4.1 and Proposition 1.3. □

Propositions 4.2 and 4.4 show that for any graph H with $\Delta(H) < |V(H)| - 1$ and any star or wheel G, it follows $\chi_d^t(G \circ H) = \chi(G \circ H)$ while $\Delta(G \circ H) < |V(G \circ H)| - 1$. This is an example that demonstrates that the reverse of Theorem 4.3 does not hold. Trivially, $\chi_d^t(K_n \circ K_m) = \chi_d^t(K_{nm}) = n\chi(K_m)$ for any integers $n, m \geq 2$. In the next proposition we show that this equality holds for any graph H.

Proposition 4.5. *For any integer $n \geq 2$, and any graph H, $\chi_d^t(K_n \circ H) = n\chi(H)$.*

Proof. By assumption $V(K_n) = \{v_1, v_2, \cdots, v_n\}$, let ${}_{v_i}H$ be the i-th copy of H in the graph $K_n \circ H$, and let f be a proper coloring of $K_n \circ H$. Since every vertex of each copy of H is adjacent to every vertex of the other copies of H, we have $f({}_{v_i}H) \cap f({}_{v_j}H) = \emptyset$ for each $1 \leq i < j \leq n$. Then

$$\begin{array}{rcll} \chi_d^t(K_n \circ H) & \geq & \chi(K_n \circ H) & \\ & \geq & \sum_{1 \leq i \leq n} |f({}_{v_i}H)| & \\ & \geq & n\chi(H) & \\ & = & \chi_d^t(K_n)\chi(H) & \\ & \geq & \chi_d^t(K_n \circ H) & \text{(by Theorem 3.3)}, \end{array}$$

which implies $\chi_d^t(K_n \circ H) = n\chi(H) = \chi(K_n \circ H)$ by Proposition 1.2. □

By Propositions 4.2, 4.4 and 4.5, the bound in Theorem 3.3 is tight when G is a star, a wheel of odd order at least 5, or a non-trivial complete graph. Also Proposition 4.5 shows the bound in Corollary 3.5 is tight when G is a non-trivial complete graph and $H = K_2$.

For any two graphs G and H, since $\chi_d^t(G \circ H) \geq \chi(G \circ H) \geq \chi(K_n \circ H) = n\chi(H)$, we have the next theorem when G has a clique of n.

Theorem 4.6. *For any two graphs G and H, $\chi_d^t(G \circ H) \geq \omega(G)\chi(H)$.*

By Propositions 4.2 and 4.4, since $\chi_d^t(G \circ H) = \omega(G)\chi(H)$ when G is any wheel of odd order at least 5 or any star and H is any graph, the bound in Theorem 4.6 is tight.

5. Some problems

We conclude our paper by proposing some problems as motivation for future research.

Problem 5.1. *Are there some graphs G and H satisfy $\chi_d^t(G \circ H) \geq \chi_d^t(G)$?*

Problem 5.2. *Find some algorithms for total dominator coloring of the lexicograph product of two graphs.*

By Theorem 3.6, finding the total dominator chromatic number of the lexicographic product graph $G \circ K_n$ which has a good min-TDC is very important. This leads us to the following problems.

Problem 5.3. *Find the total dominator chromatic number of the lexicographic product of a graph with a complete graph.*

Problem 5.4. *Find the total dominator chromatic number of the lexicographic product of the known graphs such as cycles, paths or bipartite graphs.*

The following problem is appeared from this our belief that this is not true that for any two graphs G and H, $\chi_d^t(G \circ H) \geq \chi_d^t(G \circ K_r)$ in which $\chi(H) = r$.

Problem 5.5.
- *Find some graphs G and H with $\chi_d^t(G \circ H) < \chi_d^t(G \circ K_r)$ in which $\chi(H) = r$.*
- *Find some family of graphs G and H with $\chi_d^t(G \circ H) \geq \chi_d^t(G \circ K_r)$ in which $\chi(H) = r$.*

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