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Singular value inequalities for functions of matrices

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Abstract. In this paper, we prove several singular value inequalities for functions of matrices. As special cases of our results, we give some applications involving the spectral norms and numerical radii of matrices. Among other results, we prove that if A and B are $n \times n$ complex matrices and f is a nonnegative increasing concave function on $[0, \infty)$ such that $f(0) = 0$, then for $a, b \geq 0$, we have

$$s_j\left(f(|aA^*B + bB^*A|)\right)$$

$$\leq \quad s_i\left(f\left(\frac{a|A|^2 + b|B|^2}{2}\right) \oplus f\left(\frac{b|A|^2 + a|B|^2}{2}\right)\right) + s_{j-i+1}\left(f\left(\frac{|bA^*B + aB^*A|}{2}\right) \oplus f\left(\frac{|aA^*B + bB^*A|}{2}\right)\right)$$

for $1 \leq i \leq j \leq n$. A special case of this inequality is related to recent inequalities given in [10] and [12]. Also, we prove that

$$\|\text{Re } A\| \leq \frac{1}{2}\left(\|A\| + w(A)\right) \leq \|A\|.$$

Here, Re T, $s_j(T)$, $\|T\|$, and $w(T)$ are the real part, the j^{th} singular value, the spectral norm, and the numerical radius of the matrix T, respectively.

1. Introduction

Let $\mathbb{M}_n(\mathbb{C})$ be the C^*-algebra of all $n \times n$ complex matrices. A matrix $A \in \mathbb{M}_n(\mathbb{C})$ is said to be positive semidefinite if $x^*Ax \geq 0$ for all $x \in \mathbb{C}^n$. The absolute value of a complex matrix $A \in \mathbb{M}_n(\mathbb{C})$ is $|A| = (A^*A)^{1/2}$. For $A \in \mathbb{M}_n(\mathbb{C})$ the singular values of A, denoted by $s_1(A), ..., s_n(A)$, are the eigenvalues of $|A| = (A^*A)^{1/2}$ arranged in decreasing order and repeated according to multiplicity.

A matrix norm $|||\cdot|||$ on $\mathbb{M}_n(\mathbb{C})$ is said to be unitarily invariant if $|||UAV||| = |||A|||$ for all $A \in \mathbb{M}_n(\mathbb{C})$ and for all unitary matrices $U, V \in \mathbb{M}_n(\mathbb{C})$. One of the most common examples of unitarily invariant norms is the spectral norm, which is defined by $\|A\| = \max_{\|x\|=1} \|Ax\|$ for $A \in \mathbb{M}_n(\mathbb{C})$ and $x \in \mathbb{C}^n$. It is known that the spectral norm of $A \in \mathbb{M}_n(\mathbb{C})$ is equal to the largest singular value of A, i.e, $\|A\| = s_1(A)$. For $A, B \in \mathbb{M}_n(\mathbb{C})$, $A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is the direct sum of A and B.

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The numerical radius of $A \in \mathbb{M}_n(\mathbb{C})$ is denoted by $\omega(A)$ and is defined by

$$\omega(A) = \max_{\|x\|=1}\{|\langle Ax, x\rangle| : x \in \mathbb{C}^n\}.$$

The author in [16] proved that the numerical radius of $A \in \mathbb{M}_n(\mathbb{C})$ can be represented by

$$\omega(A) = \sup_{\theta\in\mathbb{R}} \left\|\operatorname{Re}\left(e^{i\theta}A\right)\right\|.$$

Here, $\operatorname{Re} A = \frac{A+A^*}{2}$ is the real part of the matrix A.

It is clear that

$$\omega(A) \leq \|A\|. \tag{1}$$

If A is normal, then inequality (1) becomes an equality, that is

$$\omega(A) = \|A\|.$$

Some of known basic relations involving singular values, spectral norms, and numerical radii of matrices that we need in our paper can be stated as follows: For $A, B \in \mathbb{M}_n(\mathbb{C})$, we have

$$s_j(|A|) = s_j(A) = s_j(A^*),$$

$$s_j\left(\begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}\right) = s_j\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right),$$

$$\left\|\begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}\right\| = \left\|\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right\| = \max\left(\|A\|, \|B\|\right), \tag{2}$$

$$w\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \max\left(w\left(A\right), w\left(B\right)\right),$$

and

$$w\left(\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}\right) = w\left(A\right).$$

For more facts, definitions, theorems about matrices, we refer the reader to [9], [13], [17], and [18].

It was proved in [12] that if $A, B \in \mathbb{M}_n(\mathbb{C})$, then

$$s_j(AB^* + BA^*) \leq s_j(\left(|A|^2 + |B|^2\right) \oplus \left(|A|^2 + |B|^2\right)) \tag{3}$$

for $j = 1, 2, ..., n$. Related to inequality (3), the authors in [10] proved that

$$s_j(A^*B + B^*A) \leq s_j(\left(|A|^2 + |B|^2\right) \oplus \left(|A|^2 + |B|^2\right)) \tag{4}$$

for $j = 1, 2, ..., n$.

In this paper, we give new singular value inequalities for functions of matrices in which the special cases of our results are related to inequalities (3) and (4) and give several new upper bounds for $s_j(A^*B + B^*A)$. As an application of our work, we give a refinement of a well-known basic inequality by involving the numerical radii of matrices. For more recent results concerning numerical radii of matrices, we refer the reader to [2], [8], [11], and [14]. Also, for very recent results concerning singular values and norms of matrices, we refer the reader to [1], [3], [4], [5], [6], and [7].

2. Main results

To state our first main result, we need the following lemmas. For the first lemma, see, e.g., [9, p. 291], the second lemma can be found in [15] and for the third lemma, see, e.g., [18, p. 275].

Lemma 2.1. *Let $A \in \mathbb{M}_n(\mathbb{C})$ and let f be a nonnegative increasing function on $[0, \infty)$. Then*

$$s_j(f(|A|) = f\left(s_j(A)\right)$$

for $j = 1, 2, ..., n$.

Lemma 2.2. *Let $M \in \mathbb{M}_m(\mathbb{C})$ and $N \in \mathbb{M}_n(\mathbb{C})$ be such that $\begin{bmatrix} M & K \\ K^* & N \end{bmatrix}$ is positive semidefinite and let $r = \min(m, n)$. Then*

$$2s_j(K) \leq s_j\left(\begin{bmatrix} M & K \\ K^* & N \end{bmatrix}\right)$$

for $j = 1, 2, ..., r$.

Lemma 2.3. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then*

$$s_j(A + B) \leq s_i(A) + s_{j-i+1}(B) \tag{5}$$

for $1 \leq i \leq j \leq n$. In particular, if $i = j$, then

$$s_j(A + B) \leq s_j(A) + \|B\|. \tag{6}$$

Theorem 2.4. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ and let f be a nonnegative increasing concave function on $[0, \infty)$ such that $f(0) = 0$. Then for $a, b \geq 0$, we have*

$$\begin{aligned} & s_j(f(|aA^*B + bB^*A|)) \\ \leq \; & s_i\left(f\left(\frac{a|A|^2 + b|B|^2}{2}\right) \oplus f\left(\frac{b|A|^2 + a|B|^2}{2}\right)\right) \\ & + s_{j-i+1}\left(f\left(\frac{|bA^*B + aB^*A|}{2}\right) \oplus f\left(\frac{|aA^*B + bB^*A|}{2}\right)\right) \end{aligned} \tag{7}$$

for $1 \leq i \leq j \leq n$. In particular, we have

$$\begin{aligned} & s_j(aA^*B + bB^*A) \\ \leq \; & s_i\left(\left(\frac{a|A|^2 + b|B|^2}{2}\right) \oplus \left(\frac{b|A|^2 + a|B|^2}{2}\right)\right) \\ & + s_{j-i+1}\left(\left(\frac{bA^*B + aB^*A}{2}\right) \oplus \left(\frac{aA^*B + bB^*A}{2}\right)\right) \end{aligned} \tag{8}$$

and

$$s_j(aA^*B + bB^*A)$$

$$\leq s_j\left(\left(\frac{a|A|^2 + b|B|^2}{2}\right) \oplus \left(\frac{b|A|^2 + a|B|^2}{2}\right)\right) + \frac{1}{2}\|aA^*B + bB^*A\|. \tag{9}$$

Proof. Let $X = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} B & A \\ 0 & 0 \end{bmatrix}$. Then

$$\begin{aligned} aX^*X + bY^*Y &= a\begin{bmatrix} A^* & 0 \\ B^* & 0 \end{bmatrix}\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} B^* & 0 \\ A^* & 0 \end{bmatrix}\begin{bmatrix} B & A \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} a\left|A\right|^2 & aA^*B \\ aB^*A & a\left|B\right|^2 \end{bmatrix} + \begin{bmatrix} b\left|B\right|^2 & bB^*A \\ bA^*B & b\left|A\right|^2 \end{bmatrix} \\ &= \begin{bmatrix} a\left|A\right|^2 + b\left|B\right|^2 & aA^*B + bB^*A \\ bA^*B + aB^*A & b\left|A\right|^2 + a\left|B\right|^2 \end{bmatrix}, \end{aligned}$$

which is a positive semidefinite matrix. So, we have

$$\begin{aligned} & s_j\left(f(|aA^*B + bB^*A|)\right) \\ &= f\left(s_j(aA^*B + bB^*A)\right) \text{ (by Lemma 2.1)} \\ &\leq f\left(\frac{1}{2}s_j\left(\begin{bmatrix} a\left|A\right|^2 + b\left|B\right|^2 & aA^*B + bB^*A \\ bA^*B + aB^*A & b\left|A\right|^2 + a\left|B\right|^2 \end{bmatrix}\right)\right) \qquad (10) \\ &\qquad \text{(by Lemma 2.2)} \\ &= f\left(\frac{1}{2}s_j\left(\begin{array}{c} \begin{bmatrix} a\left|A\right|^2 + b\left|B\right|^2 & 0 \\ 0 & b\left|A\right|^2 + a\left|B\right|^2 \end{bmatrix} \\ + \begin{bmatrix} 0 & aA^*B + bB^*A \\ bA^*B + aB^*A & 0 \end{bmatrix} \end{array}\right)\right) \\ &\leq f\left(\begin{array}{c} \frac{1}{2}s_i\left(\begin{bmatrix} a\left|A\right|^2 + b\left|B\right|^2 & 0 \\ 0 & b\left|A\right|^2 + a\left|B\right|^2 \end{bmatrix}\right) \\ + \frac{1}{2}s_{j-i+1}\left(\begin{bmatrix} 0 & aA^*B + bB^*A \\ bA^*B + aB^*A & 0 \end{bmatrix}\right) \end{array}\right) \\ &\qquad \text{(by inequality (5))} \\ &\leq \left(\begin{array}{c} f\left(\frac{1}{2}s_i\left(\begin{bmatrix} a\left|A\right|^2 + b\left|B\right|^2 & 0 \\ 0 & b\left|A\right|^2 + a\left|B\right|^2 \end{bmatrix}\right)\right) \\ + f\left(\frac{1}{2}s_{j-i+1}\left(\begin{bmatrix} 0 & aA^*B + bB^*A \\ bA^*B + aB^*A & 0 \end{bmatrix}\right)\right) \end{array}\right) \qquad (11) \\ &\qquad \text{(by the concavity of the function } f) \\ &= \left(\begin{array}{c} f\left(\frac{1}{2}s_i\left(\begin{bmatrix} a\left|A\right|^2 + b\left|B\right|^2 & 0 \\ 0 & b\left|A\right|^2 + a\left|B\right|^2 \end{bmatrix}\right)\right) \\ + f\left(\frac{1}{2}s_{j-i+1}\left(\begin{bmatrix} bA^*B + aB^*A & 0 \\ 0 & aA^*B + bB^*A \end{bmatrix}\right)\right) \end{array}\right) \\ &= \left(\begin{array}{c} s_i\left(\begin{bmatrix} f\left(\frac{a|A|^2+b|B|^2}{2}\right) & 0 \\ 0 & f\left(\frac{b|A|^2+a|B|^2}{2}\right) \end{bmatrix}\right) \\ + s_{j-i+1}\left(f\left|\begin{bmatrix} \frac{bA^*B+aB^*A}{2} & 0 \\ 0 & \frac{aA^*B+bB^*A}{2} \end{bmatrix}\right|\right) \end{array}\right) \\ &= \left(\begin{array}{c} s_i\left(\begin{bmatrix} f\left(\frac{a|A|^2+b|B|^2}{2}\right) & 0 \\ 0 & f\left(\frac{b|A|^2+a|B|^2}{2}\right) \end{bmatrix}\right) \\ + s_{j-i+1}\left(f\left(\begin{bmatrix} \frac{|bA^*B+aB^*A|}{2} & 0 \\ 0 & \frac{|aA^*B+bB^*A|}{2} \end{bmatrix}\right)\right) \end{array}\right) \end{aligned}$$

$$
\begin{aligned}
&= \left(\begin{array}{c} s_i\left(\begin{bmatrix} f\left(\frac{a|A|^2+b|B|^2}{2}\right) & 0 \\ 0 & f\left(\frac{b|A|^2+a|B|^2}{2}\right)\end{bmatrix}\right) \\ +s_{j-i+1}\left(\begin{bmatrix} f\left(\frac{|bA^*B+aB^*A|}{2}\right) & 0 \\ 0 & f\left(\frac{|aA^*B+bB^*A|}{2}\right)\end{bmatrix}\right) \end{array} \right) \\
&= s_i\left(f\left(\frac{a|A|^2+b|B|^2}{2}\right)\oplus f\left(\frac{b|A|^2+a|B|^2}{2}\right)\right) \\
&\quad +s_{j-i+1}\left(f\left(\frac{|bA^*B+aB^*A|}{2}\right)\oplus f\left(\frac{|aA^*B+bB^*A|}{2}\right)\right),
\end{aligned}
$$

which proves inequality (7). The inequality (8) follows by letting $f(t) = t$ in inequality (7), while inequality (9) follows by letting $i = j$ in inequality (8) and using relation (2). □

Remark 2.5. *Inequality (9) is related to inequality (4) and they are equivalent if $j = 1$ and $a = b = 1$.*

Theorem 2.6. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ and let f be a nonnegative increasing concave function on $[0,\infty)$ such that $f(0) = 0$. Then for $a, b \geq 0$, we have*

$$
\begin{aligned}
&s_j\left(f(|aA^*B+bB^*A|)\right) \\
&= s_i\left(f\left(\frac{a}{2}|A|^2\right)\oplus f\left(\frac{a}{2}|B|^2\right)\right)+\max\left(\left\|f\left(\frac{b}{2}|B|^2\right)\right\|,\left\|f\left(\frac{b}{2}|A|^2\right)\right\|\right) \\
&\quad +s_{j-i+1}\left(f\left(\frac{|bA^*B+aB^*A|}{2}\right)\oplus f\left(\frac{|aA^*B+bB^*A|}{2}\right)\right) \qquad (12)
\end{aligned}
$$

for $1 \leq i \leq j \leq n$. In particular, we have

$$
\begin{aligned}
s_j(aA^*B+bB^*A) &\leq s_i\left(\frac{a|A|^2\oplus a|B|^2}{2}\right)+\frac{b}{2}\max\left(\|B\|^2,\|A\|^2\right) \\
&\quad +s_{j-i+1}\left(\frac{(bA^*B+aB^*A)\oplus(aA^*B+bB^*A)}{2}\right) \qquad (13)
\end{aligned}
$$

and

$$
\begin{aligned}
s_j(aA^*B+bB^*A) &\leq s_j\left(\frac{a|A|^2\oplus a|B|^2}{2}\right)+\frac{b}{2}\max\left(\|B\|^2,\|A\|^2\right) \\
&\quad +\frac{1}{2}\|aA^*B+bB^*A\|. \qquad (14)
\end{aligned}
$$

Proof. By inequality (11), we have

$$
s_j\left(f(|aA^*B+bB^*A|)\right)
$$

$$\leq \begin{pmatrix} f\left(\frac{1}{2}s_i\left(\begin{bmatrix} a|A|^2+b|B|^2 & 0 \\ 0 & a|B|^2+b|A|^2 \end{bmatrix}\right)\right) \\ +f\left(\frac{1}{2}s_{j-i+1}\left(\begin{bmatrix} 0 & aA^*B+bB^*A \\ bA^*B+aB^*A & 0 \end{bmatrix}\right)\right) \end{pmatrix}$$

$$\leq \begin{pmatrix} f\left(\frac{1}{2}s_i\left(\begin{bmatrix} a|A|^2 & 0 \\ 0 & a|B|^2 \end{bmatrix}+\frac{1}{2}\left\|\begin{bmatrix} b|B|^2 & 0 \\ 0 & b|A|^2 \end{bmatrix}\right\|\right)\right) \\ +f\left(\frac{1}{2}s_{j-i+1}\left(\begin{bmatrix} 0 & aA^*B+bB^*A \\ bA^*B+aB^*A & 0 \end{bmatrix}\right)\right) \end{pmatrix}$$

(by inequality (6))

$$\leq \begin{pmatrix} s_i\left(f\left(\begin{bmatrix} \frac{a}{2}|A|^2 & 0 \\ 0 & \frac{a}{2}|B|^2 \end{bmatrix}\right)+f\left(\frac{1}{2}\left\|\begin{bmatrix} b|B|^2 & 0 \\ 0 & b|A|^2 \end{bmatrix}\right\|\right)\right) \\ +f\left(s_{j-i+1}\left(\begin{bmatrix} \frac{bA^*B+aB^*A}{2} & 0 \\ 0 & \frac{aA^*B+bB^*A}{2} \end{bmatrix}\right)\right) \end{pmatrix}$$

$$= \begin{pmatrix} s_i\left(\begin{bmatrix} f\left(\frac{a}{2}|A|^2\right) & 0 \\ 0 & f\left(\frac{a}{2}|B|^2\right) \end{bmatrix}+\left\|\begin{bmatrix} f\left(\frac{b}{2}|B|^2\right) & 0 \\ 0 & f\left(\frac{b}{2}|A|^2\right) \end{bmatrix}\right\|\right) \\ +s_{j-i+1}\left(\begin{bmatrix} f\left(\frac{|bA^*B+aB^*A|}{2}\right) & 0 \\ 0 & f\left(\frac{|aA^*B+bB^*A|}{2}\right) \end{bmatrix}\right) \end{pmatrix}$$

$$= s_i\left(f\left(\frac{a}{2}|A|^2\right)\oplus f\left(\frac{a}{2}|B|^2\right)\right)+\max\left(\left\|f\left(\frac{b}{2}|B|^2\right)\right\|,\left\|f\left(\frac{b}{2}|A|^2\right)\right\|\right)$$
$$+s_{j-i+1}\left(f\left(\frac{|bA^*B+aB^*A|}{2}\right)\oplus f\left(\frac{|aA^*B+bB^*A|}{2}\right)\right),$$

which proves inequality (12). The inequality (13) follows by letting $f(t) = t$ in inequality (12), while inequality (14) follows by letting $i = j$ in inequality (13) and using relation (2). □

Theorem 2.7. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ and let f be a nonnegative increasing concave function on $[0,\infty)$ such that $f(0) = 0$. Then for $a, b \geq 0$, we have*

$$s_j\left(f(|aA^*B+bB^*A|)\right)$$
$$\leq s_i\left(f\left(\frac{|bA^*B+aB^*A|}{2}\right)\oplus f\left(\frac{|aA^*B+bB^*A|}{2}\right)\right)$$
$$+s_{j-i+1}\left(f\left(\frac{a|A|^2+b|B|^2}{2}\right)\oplus f\left(\frac{b|A|^2+a|B|^2}{2}\right)\right) \tag{15}$$

for $1 \leq i \leq j \leq n$. In particular, we have

$$s_j(aA^*B+bB^*A)$$
$$\leq s_i\left(\left(\frac{bA^*B+aB^*A}{2}\right)\oplus\left(\frac{aA^*B+bB^*A}{2}\right)\right)$$
$$+s_{j-i+1}\left(\left(\frac{a|A|^2+b|B|^2}{2}\right)\oplus\left(\frac{b|A|^2+a|B|^2}{2}\right)\right) \tag{16}$$

and

$$s_j(aA^*B+bB^*A)$$

$$\begin{aligned}
&\le s_j\left(\left(\frac{bA^*B+aB^*A}{2}\right)\oplus\left(\frac{aA^*B+bB^*A}{2}\right)\right)\\
&\quad+\max\left(\left\|\frac{a|A|^2+b|B|^2}{2}\right\|,\left\|\frac{b|A|^2+a|B|^2}{2}\right\|\right).
\end{aligned} \tag{17}$$

Proof. By inequality (10), we have

$$s_j\left(f(|aA^*B+bB^*A|)\right)$$

$$\begin{aligned}
&\le f\left(\frac{1}{2}s_j\left(\begin{bmatrix} a|A|^2+b|B|^2 & aA^*B+bB^*A\\ bA^*B+aB^*A & b|A|^2+a|B|^2\end{bmatrix}\right)\right)\\
&= f\left(\frac{1}{2}s_j\left(\begin{bmatrix} 0 & aA^*B+bB^*A\\ bA^*B+aB^*A & 0\end{bmatrix}+\begin{bmatrix} a|A|^2+b|B|^2 & 0\\ 0 & b|A|^2+a|B|^2\end{bmatrix}\right)\right)\\
&\le f\left(s_i\left(\begin{bmatrix} 0 & \frac{aA^*B+bB^*A}{2}\\ \frac{bA^*B+aB^*A}{2} & 0\end{bmatrix}\right)+s_{j-i+1}\left(\begin{bmatrix} \frac{a|A|^2+b|B|^2}{2} & 0\\ 0 & \frac{a|B|^2+b|A|^2}{2}\end{bmatrix}\right)\right)\\
&= f\left(s_i\left(\begin{bmatrix} \frac{bA^*B+aB^*A}{2} & 0\\ 0 & \frac{aA^*B+bB^*A}{2}\end{bmatrix}\right)+s_{j-i+1}\left(\begin{bmatrix} \frac{a|A|^2+b|B|^2}{2} & 0\\ 0 & \frac{a|B|^2+b|A|^2}{2}\end{bmatrix}\right)\right)\\
&\le f\left(s_i\left(\begin{bmatrix} \frac{bA^*B+aB^*A}{2} & 0\\ 0 & \frac{aA^*B+bB^*A}{2}\end{bmatrix}\right)\right)+f\left(s_{j-i+1}\left(\begin{bmatrix} \frac{a|A|^2+b|B|^2}{2} & 0\\ 0 & \frac{a|B|^2+b|A|^2}{2}\end{bmatrix}\right)\right)
\end{aligned}$$

$$\begin{aligned}
&= s_i\left(f\left(\begin{bmatrix} \frac{|bA^*B+aB^*A|}{2} & 0\\ 0 & \frac{|aA^*B+bB^*A|}{2}\end{bmatrix}\right)\right)+s_{j-i+1}\left(f\left(\begin{bmatrix} \frac{a|A|^2+b|B|^2}{2} & 0\\ 0 & \frac{a|B|^2+b|A|^2}{2}\end{bmatrix}\right)\right)\\
&= s_i\left(\begin{bmatrix} f\left(\frac{|bA^*B+aB^*A|}{2}\right) & 0\\ 0 & f\left(\frac{|aA^*B+bB^*A|}{2}\right)\end{bmatrix}\right)\\
&\quad+s_{j-i+1}\left(\begin{bmatrix} f\left(\frac{a|A|^2+b|B|^2}{2}\right) & 0\\ 0 & f\left(\frac{b|A|^2+a|B|^2}{2}\right)\end{bmatrix}\right)\\
&= s_i\left(f\left(\frac{|bA^*B+aB^*A|}{2}\right)\oplus f\left(\frac{|aA^*B+bB^*A|}{2}\right)\right)\\
&\quad+s_{j-i+1}\left(f\left(\frac{a|A|^2+b|B|^2}{2}\right)\oplus f\left(\frac{b|A|^2+a|B|^2}{2}\right)\right),
\end{aligned}$$

which proves inequality (15). The inequality (16) follows by letting $f(t)=t$ in inequality (15), while inequality (17) follows by letting $i=j$ in inequality (16) and using relation (2). □

Corollary 2.8. *Let $A,B\in\mathbb{M}_n(\mathbb{C})$. Then*

$$s_j\left(A^*B+B^*A\right)$$

$$\le s_j\left(\left(\frac{A^*B+B^*A}{2}\right)\oplus\left(\frac{A^*B+B^*A}{2}\right)\right)+\|A\|\,\|B\|$$

for $j=1,2,\dots,n$.

Proof. Letting $a = b = 1$ in inequality (17), we have

$$\begin{aligned} s_j\left(A^*B + B^*A\right) &\leq s_j\left(\left(\frac{A^*B + B^*A}{2}\right) \oplus \left(\frac{A^*B + B^*A}{2}\right)\right) + \left\|\frac{|A|^2 + |B|^2}{2}\right\| \\ &\leq s_j\left(\left(\frac{A^*B + B^*A}{2}\right) \oplus \left(\frac{A^*B + B^*A}{2}\right)\right) + \frac{1}{2}\left(\|A\|^2 + \|B\|^2\right). \end{aligned}$$

Replacing A by $\sqrt{t}A$ and B by $\frac{1}{\sqrt{t}}B$, where $t > 0$, and taking the minimum over t, we have

$$s_j\left(A^*B + B^*A\right)$$

$$\leq s_j\left(\left(\frac{A^*B + B^*A}{2}\right) \oplus \left(\frac{A^*B + B^*A}{2}\right)\right) + \|A\|\,\|B\|.$$

□

An application of our work, can be seen in the following corollary.

Corollary 2.9. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then*

$$2s_j\left(\mathrm{R}e\, A\right) \leq s_j\left(A \oplus A\right) + \|\mathrm{R}e\, A\| \tag{18}$$

for $j = 1, 2, ..., n$, and

$$2s_j\left(\mathrm{Re}\, A\right) \leq s_j\left(A \oplus A\right) + w(A). \tag{19}$$

Proof. Letting $B = I$ and $a = b = 1$ in inequality (9), we have

$$\begin{aligned} s_j\left(2\mathrm{Re}\, A\right) &\leq s_j\left(\left(\frac{|A|^2 + I}{2}\right) \oplus \left(\frac{|A|^2 + I}{2}\right)\right) + \|\mathrm{Re}\, A\| \\ &= s_j\left(\frac{1}{2}\begin{bmatrix} |A|^2 + I & 0 \\ 0 & |A|^2 + I \end{bmatrix}\right) + \|\mathrm{Re}\, A\| \\ &= \frac{1}{2}s_j\left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} |A|^2 & 0 \\ 0 & |A|^2 \end{bmatrix}\right) + \|\mathrm{Re}\, A\| \\ &= \frac{1}{2}\left(1 + s_j\left(\begin{bmatrix} |A|^2 & 0 \\ 0 & |A|^2 \end{bmatrix}\right)\right) + \|\mathrm{Re}\, A\| \\ &= \frac{1}{2}\left(1 + s_j\left(|A|^2 \oplus |A|^2\right)\right) + \|\mathrm{Re}\, A\| \\ &= \frac{1}{2}\left(1 + s_j^2\left(A \oplus A\right)\right) + \|\mathrm{Re}\, A\|. \end{aligned}$$

Replacing A by tA, where $t > 0$, and taking the minimum over t, we have

$$2s_j\left(\mathrm{Re}\, A\right) \leq s_j\left(A \oplus A\right) + \|\mathrm{Re}\, A\|,$$

which proves inequality (18). To prove inequality (19), we have

$$\begin{aligned} 2s_j\left(\mathrm{Re}\, A\right) &\leq s_j\left(A \oplus A\right) + \|\mathrm{Re}\, A\| \\ &\leq s_j\left(A \oplus A\right) + \max_{\theta \in \mathbb{R}}\left\|\mathrm{Re}\left(e^{i\theta}A\right)\right\| \\ &= s_j\left(A \oplus A\right) + w(A). \end{aligned}$$

□

It follows by the triangle inequality that for $A \in \mathbb{M}_n(\mathbb{C})$,

$$\|\mathrm{Re}\, A\| \leq \|A\| . \tag{20}$$

As an application of Corollary 2.9, we give a refinement of inequality (20). This can be seen as follows: let $j = 1$ in inequality (19), we have

$$2\, \|\mathrm{Re}\, A\| \leq \|A\| + w(A),$$

or equivalently,

$$\|\mathrm{Re}\, A\| \leq \frac{1}{2} \left(\|A\| + w(A)\right),$$

which is a refinement of inequality (20). In fact, we have

$$\begin{aligned} \|\mathrm{Re}\, A\| &\leq \frac{1}{2} \|A\| + \frac{1}{2} w(A) \\ &\leq \frac{1}{2} \|A\| + \frac{1}{2} \|A\| \\ &\quad \text{(by inequality (1))} \\ &= \|A\| . \end{aligned}$$

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