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Hyper-Leonardo numbers: Combinatorial interpretation and some positivities

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Abstract. In this paper, we establish the combinatorial interpretations of the hyper-Leonardo numbers $Le_n^{(r)}$ and Leonardo numbers Le_n. We investigate the log-concavity of the Leonardo numbers for $n \geq 3$ and the hyper-Leonardo numbers for $n, r \geq 1$. In addition, we prove the log-balancedness of the hyper-Leonardo numbers for $r = 1, 2$. Furthermore, we prove the q-log-concavity of the polynomial $\sum_{k=0}^{n} Le_k^{(r)} q^k$ for $n, r \geq 1$.

1. Introduction

The sequence $\{Le_n\}_{n\geq 0}$ of *Leonardo numbers* was introduced by Catarino and Borges [9] and is defined recursively by

$$Le_n = Le_{n-1} + Le_{n-2} + 1, \quad n \geq 2, \tag{1}$$

with initial values $Le_0 = Le_1 = 1$.

This sequence has the following generating function:

$$\sum_{n\geq 0} Le_n x^n = \frac{1 - x + x^2}{(1-x)(1-x-x^2)}. \tag{2}$$

Its Binet formula is:

$$Le_n = \frac{2(\alpha^{n+1} - \beta^{n+1})}{\alpha - \beta} - 1, \tag{3}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

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The previous authors [9] proved that this sequence has some relations with the Fibonacci $\{F_n\}$, Lucas $\{L_n\}$ sequences such as

$$Le_n = 2F_n - 1 \tag{4}$$

and

$$Le_n = L_{n+2} - F_{n+2} - 1. \tag{5}$$

Alp, and Kocer obtained Cassini's identity for Leonardo numbers as

$$(Le_n)^2 - Le_{n-1}Le_{n+1} = Le_{n-1} - Le_{n-2} + 4(-1)^n. \tag{6}$$

For other identities and properties of Leonardo number, see the reference [5, 10].

As an extension of Leonardo numbers, Kuhapatanakul and Chobsorn [15] defined the generalized Leonardo numbers $\mathcal{L}_{k,n}$ recursively by

$$\mathcal{L}_{k,n} = \mathcal{L}_{k,n-1} + \mathcal{L}_{k,n-2} + k, \quad n \geq 2, \tag{7}$$

with initial values $\mathcal{L}_{k,0} = \mathcal{L}_{k,1} = 1$, and the parameter k can be a fixed positive integer or an indeterminate. When $k = 1$, we obtain the classical Leonardo numbers $\mathcal{L}_{1,n} = Le_n$.

They established [15] the following generating function of $\mathcal{L}_{k,n}$:

$$\sum_{n\geq 0} \mathcal{L}_{k,n}x^n = \frac{1 - x + kx^2}{(1 - x)(1 - x - x^2)}. \tag{8}$$

Kuhapatanakul and Chobsorn [15] proposed also some identities and an incomplete version of this number $\mathcal{L}_{k,n}$. Shattuck [18] interpreted it as the enumerator of two classes of linear colored tilings of length n. Furthermore, he [18] proposed a (p, q)-generalization of $\mathcal{L}_{k,n}$ by considering the joint distribution of a pair of statistics on one of the aforementioned classes of colored tilings.

The linear recurrence sequences like Fibonacci and Lucas sequences are generalized in many ways in the literature. For example, Dil and Mező [13] introduced hyper-Fibonacci and hyper-Lucas numbers as the generalizations of the Fibonacci and Lucas numbers, by the following formulas

$$F_n^{(r)} = \sum_{k=0}^{n} F_k^{(r-1)} \text{ with } F_n^{(0)} = F_n, \; F_0^{(r)} = 0, \; F_1^{(r)} = 1 \tag{9}$$

and

$$L_n^{(r)} = \sum_{k=0}^{n} L_k^{(r-1)} \text{ with } L_n^{(0)} = L_n, \; L_0^{(r)} = 2, \; L_1^{(r)} = 2r + 1. \tag{10}$$

They obtained also that the hyper-Fibonacci and hyper-Lucas numbers satisfy the following recurrence relations, respectively:

$$F_n^{(r)} = F_{n-1}^{(r)} + F_n^{(r-1)} \tag{11}$$

and

$$L_n^{(r)} = L_{n-1}^{(r)} + L_n^{(r-1)}. \tag{12}$$

In the same way, Bahsi and Mersin [17] introduced the hyper-Leonardo numbers as follows

$$Le_n^{(r)} = \sum_{k=0}^{n} Le_k^{(r-1)} \text{ with } Le_n^{(0)} = Le_n,\ Le_0^{(r)} = 1,\ Le_1^{(r)} = r+1. \tag{13}$$

This relation yields the recurrence relation

$$Le_n^{(r)} = Le_{n-1}^{(r)} + Le_n^{(r-1)}. \tag{14}$$

The previous authors [17] gave the generating function of the hyper-Leonardo numbers as follows

$$\sum_{n\geq 0} Le_n^{(r)} x^n = \frac{1-x+x^2}{(1-2x+x^3)(1-x)^r}. \tag{15}$$

They presented also [17] some combinatorial properties of these numbers using the Euler-Seidel symmetric algorithm. And they gave the recurrence relations and summation formulas.

The first few values of the Leonardo and hyper-Leonardo numbers as follows:

n	0	1	2	3	4	5	6	7	8	9	10
Le_n	1	1	3	5	9	15	25	41	67	109	177
$Le_n^{[1]}$	1	2	5	10	19	34	59	100	167	276	453
$Le_n^{[2]}$	1	3	8	18	37	71	130	230	397	673	1126

Let n and s be two positive integers, the bi^{s}nomial coefficients $\binom{n}{k}_s$ are defined as the k^{th} coefficients in the expansion,

$$(1+x+\cdots+x^s)^n = \sum_{k=0}^{sn} \binom{n}{k}_s x^k,$$

with $\binom{n}{k}_s = 0$ unless $sn \geq k \geq 0$.

The bi^{s}nomial coefficients satisfy other properties, see for instance [4, 6, 8]. These coefficients, as for usual binomial coefficients, are built trough the Pascal triangle, known as s-Pascal triangle or generalized Pascal triangle. The initial values of this triangle can be found in the OEIS as sequence A027907 for $s = 2$, which is also known as the triangle of trinomial coefficients [19].

The outline of the paper is as follows: Section 2 contains the combinatorial interpretations of the hyper-Leonardo and Leonardo numbers, and the generating function proof of the relation between the hyper-Leonardo number and Leonardo number. Section 3 contains three subsections, in Subsection 3.1 we show that $\{Le_n\}_{n\geq 3}$ and $\{Le_n^{(r)}\}_{n\geq 1}$ for $r \geq 1$ are log-concave. In Subsection 3.2 we establish that $\{Le_n^{(1)}\}_{n\geq 1}$ and $\{Le_n^{(2)}\}_{n\geq 1}$ are log-balanced. Finally, Subsection 3.3 contains the proof of the q-log-concavity of the polynomial $\sum_{k=0}^{n} Le_k^{(r)} q^k$ for $n, r \geq 1$.

2. Combinatorial interpretations

Recall to the combinatorial interpretation of the generalized Leonardo number $\mathcal{L}_{k,n}$ given by Shattuck [18]. To do this, he proposed the following definition:

Definition 2.1. *A k-tile is a rectangular piece coming in one of k colors which*

- *must occur as the first piece in a tiling, if it occurs at all,*
- *has arbitrary length greater than or equal two.*

A k-tile of length l will be denoted by k_l for all $l \geq 2$.

According to this definition, he established the following combinatorial interpretation for the generalized Leonardo numbers $\mathcal{L}_{k,n}$.

Proposition 2.2. *[18, Proposition 2.1] The generalized Leonardo number $\mathcal{L}_{k,n}$ counts the number of n-board tilings using squares, dominos and k-tiles.*

This proposition allows us to propose the following combinatorial interpretation for the hyper-Leonardo numbers $Le_n^{(r)}$.

Theorem 2.3. *The hyper-Leonardo number $Le_n^{(r)}$ counts the number of $n+2r$-board tilings using squares, one 1-tile and at least r dominos.*

Proof. Let $Le_n^{(r)} = l_{n+2r}^{(r)}$. So, in the first step of the proof we show that the sequence of numbers l_{n+2r} obeys the same recurrence relation (14),

$$l_{n+2r}^{(r)} = l_{n-1+2r}^{(r)} + l_{n+2r-2}^{(r-1)}.$$

We consider the last tile in an $(n+2r)$-board tiling. If the $(n+2r)$-board ends with a square, then the remaining $(n-1+2r)$-board can be tiled in $l_{n-1+2r}^{(r)}$ ways. If it ends with a domino, the remaining $(n+2r-2)$-board can be tiled in $l_{n+2r-2}^{(r-1)}$ ways. Otherwise, if it ends with a 1-tile, then from Definition 2.1 this 1-tile takes all cases of $(n+2r)$-board tiling, then it remains zero 0 ways to tile. Thus, board tilings satisfy the same recurrence relation (14) as hyper-Leonardo numbers.

Now we are testing the initial condition. For $n = 0$, there is one $2r$-board tiling, with at least r dominos, thus $l_{2r}^{(r)} = 1$ and consequently $l_{2r}^{(r)} = Le_0^{(r)}$. For $n = 1$, the $(1+2r)$-board can be tiled in $\binom{r+1}{r} = r+1 = l_{1+2r}^{(r)} = Le_1^{(r)}$ ways with at least r-dominos. For $r = 0$ there is no constraint on the number of dominoes and clearly we have $l_n^{(0)} = Le_n^{(0)} = Le_n$. This completes the proof. □

By setting $r = 0$ in Theorem 2.3, we obtain immediately the tilling interpretation of the Leonardo numbers.

Corollary 2.4. *The Leonardo number Le_n counts the number of n-board tilings using squares, one 1-tile and dominos.*

Bahsi and Mersin [17, Theorem 2] gave the relation between the hyper-Leonardo number and Leonardo number using recurrence relation (14):

$$Le_n^{(r)} = \sum_{k=0}^{n} \binom{n+r-k-1}{r-1} Le_k. \tag{16}$$

For $r = 1$ and $r = 2$, the previous authors [17] established the following two identities:

$$Le_n^{(1)} = Le_{n+2} - (n+2) \tag{17}$$

and

$$Le_n^{(2)} = Le_{n+4} - \frac{1}{2}(n^2 + 7n + 16). \tag{18}$$

In Proposition 2.6, we present a generalization of the identities 17 and 18. We begin with the following lemma [11].

Lemma 2.5. *For a positive integer r, we have*

$$Le_n^{(r)} = 2F_{n+1}^{(r)} - \binom{n+r}{r} \tag{19}$$

and

$$F_n^{(r)} = F_{n+2r} - \sum_{k=0}^{r-1} \binom{n+r+k}{r-1-k}. \tag{20}$$

This lemma gives us the following result.

Proposition 2.6. *For a positive integer r, we have*

$$Le_n^{(r)} = Le_{n+2r} - 2\sum_{k=0}^{r-1}\binom{n+r+k+1}{r-1-k} - \binom{n+r}{r} + 1. \tag{21}$$

Proof. From relations (19) and (20) in Lemma 2.5, we have

$$\begin{aligned} Le_n^{(r)} &= 2\left(F_{n+1+2r} - \sum_{k=0}^{r-1}\binom{n+r+k+1}{r-1-k}\right) - \binom{n+r}{r} \\ &= Le_{n+2r} - 2\sum_{k=0}^{r-1}\binom{n+r+k+1}{r-1-k} - \binom{n+r}{r} + 1. \end{aligned}$$

This completes the proof. □

In the next proposition, we give the generating function proof of the identity (16) using relation (15).

Proposition 2.7. *If $n \geq 1$ and $r \geq 1$, we have*

$$Le_n^{(r)} = \sum_{k=0}^{n}\binom{n+r-k-1}{r-1}Le_k. \tag{22}$$

Proof. Form (2) and (15), the generating functions of Leonardo and hyper-Leonardo numbers are respectively:

$$\sum_{n\geq 0} Le_n x^n = \frac{1-x+x^2}{(1-2x+x^3)}$$

and

$$\sum_{n\geq 0} Le_n^{(r)} x^n = \frac{1-x+x^2}{(1-2x+x^3)(1-x)^r}.$$

We know that

$$\sum_{k\geq 0}\binom{r+j-1}{r-1}x^k = \frac{1}{(1-x)^r}.$$

Thus, using this equation together with (15), we readily obtain the desired identity via convolution. □

3. Log-concavity, log-balancedness and q-log-concavity

Let $\{a_n\}_{n\geq 0}$ be a sequence of nonnegative numbers. The sequence is called log-concave (resp. log-convex) if $a_n a_{n+2} \leq a_{n+1}^2$ (resp. $a_n a_{n+2} \geq a_{n+1}^2$) for all $n \geq 0$.

We say that $\{a_n\}_{n\geq 0}$ is log-balanced [14] if $\{a_n\}_{n\geq 0}$ is log-convex and $\{a_n/n!\}_{n\geq 0}$ is log-concave.

The log-convexity, log-concavity are important properties of combinatorial sequences, and they play an important role in many fields such as quantum physics, white noise theory,probability, economics and mathematical biology. Clearly, log-balancedness implies log-convexity. The log-concave and log-convex sequences arise often in combinatorics and have been extensively investigated. We refer the reader to[7, 20, 21] for the log-concavity and [16, 23] for the log-convexity.

For two polynomials with real coefficients $f(q)$ and $g(q)$, denote $f(q) \geq_q g(q)$ if the difference $f(q) - g(q)$ has only nonnegative coefficients. For a polynomial sequence $(f_n(q))_{n\geq 0}$, it is called q-log-concave (resp. q-log-convex, introduced by Liu and Wang [16]), first suggested by Stanley [20], if

$$f_n(q)^2 \geq_q f_{n-1}(q)f_{n+1}(q) \quad (\text{resp. } f_n(q)^2 \leq_q f_{n-1}(q)f_{n+1}(q)).$$

The first work about the log-concavity, log-convexity and log-balancedness of the hyper-numbers is due to Zheng and Liu [22], they proved that the hyper-Fibonacci numbers and the hyper-Lucas numbers satisfy these properties under some conditions. In addition, they extend their work to the generalized hyper-Fibonacci and hyper-Lucas numbers. After, Ahmia et *al.* in [1, 2] do the same for the hyper-Pell, the hyperpell-Lucas, the hyper-Jacobsthal and the hyerjacobsthal-Lucas numbers.

Motivated by these work, we establish in this section the log-concavity of Leornardo numbers Le_n and hyper-Leonardo numbers $Le_n^{(r)}$, then we investigate the log-balancedness property of hyper-Leonardo numbers $Le_n^{(r)}$ for $r = 1, 2$. Finally, we prove that the polynomial $\sum_{k=0}^{n} Le_k^{(r)} q^k$ is q-log-concave for $r, n \geq 1$.

3.1. Log-concavity property

First of all, we start by the following lemmas.

Lemma 3.1. *[12, 21] If the sequences $\{a_n\}_n$ and $\{b_n\}_n$ are log-concave, then so is their ordinary convolution $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.*

Lemma 3.2. *[21] If the sequence $\{x_n\}_n$ is log-concave, then so is their binomial convolution $y_n = \sum_{k=0}^{n} \binom{n}{k} x_k$.*

Lemma 3.3. *[3] If the sequence $\{z_n\}_n$ is log-concave, then so is their bi^{s}nomial convolution $t_n = \sum_{k=0}^{n} \binom{n}{k}_s z_k$.*

Now, we prove in the following theorem the log-concavity of Leonardo numbers.

Theorem 3.4. *The Leonardo numbers $\{Le_n\}_n$ form a log-concave sequence for $n \geq 3$.*

Proof. It is clear that the sequence Le_0, Le_1, Le_2 is not log-concave. So, to proof that the sequence of Leonardo number is log-concave for $n \geq 3$, it suffices to check the following inequality:

$$(Le_{n+1})^2 - Le_n Le_{n-1} \geq 0, \quad \text{for } n \geq 4.$$

From relation (4), we obtain

$$\begin{aligned} (Le_{n+1})^2 - Le_n Le_{n-1} &= Le_n - Le_{n-1} + 4(-1)^2 \\ &= Le_{n-2} + 4(-1)^2 + 1 \\ &\geq 0, \end{aligned}$$

which completes the proof. □

By Lemma 3.2, Lemma 3.3 and above theorem, we obtain the following results.

Corollary 3.5. *The sequences $\{\sum_{k=0}^{n} \binom{n}{k} Le_k\}_{n\geq 3}$ and $\{\sum_{k=0}^{sn} \binom{n}{k}_s Le_k\}_{n\geq 3}$ are log-concave.*

From (14), we note that $\{Le_n^{(r)}\}_n$ is the convolution of $\{Le_n^{(r)}\}_n$ and $\{1\}_n$. Then, we can prove that the hyper-Leonardo numbers verify the log-concavity property as follows.

Theorem 3.6. *The hyper-Leonardo numbers $\{Le_n^{(r)}\}_{n\geq 1}$ $(r \geq 1)$ form a log-concave sequence.*

Proof. Using equation (17), we show that the sequence $\{Le_n^{(1)}\}_{n\geq 1}$ satisfies the log-concavity property. Consider the following computation:

$$\begin{aligned}\left(Le_{n+1}^{(1)}\right)^2 - Le_n^{(1)}Le_{n+2}^{(1)} &= (Le_{n+3}-(n+3))^2 - (Le_{n+2}-(n+2))(Le_{n+4}-(n+4)) \\ &= \left((Le_{n+3})^2 - Le_{n+2}Le_{n+4}\right) + (n^2+6n+9) - (n^2+6n+8) \\ &\quad + ((n+4)Le_{n+2} + (n+2)Le_{n+4} - 2(n+3)Le_{n+3}) \\ &= \left((Le_{n+3})^2 - Le_{n+2}Le_{n+4}\right) - 1 \\ &\quad +(n+2)(Le_{n+2}-Le_{n+1}) - 2Le_{n+1}.\end{aligned}$$

From equation (6), we obtain the simplified expression:

$$\left(Le_{n+1}^{(1)}\right)^2 - Le_n^{(1)}Le_{n+2}^{(1)} = (n+3)Le_n - 2Le_{n+1} + 4(-1)^{n+3} + n + 2. \tag{23}$$

To verify positivity, we first consider the case $n = 1$:

$$\left(Le_2^{(1)}\right)^2 - Le_1^{(1)}Le_3^{(1)} = 4Le_1 - 2Le_2 + 4(-1)^4 + 3 = 4\times 1 - 2\times 3 + 7 = 5 > 0.$$

Now, assuming by induction that the expression in (23) is positive for some $n \geq 2$, we show it remains positive for $n+1$, that is:

$$\left(Le_{n+2}^{(1)}\right)^2 - Le_{n+1}^{(1)}Le_{n+3}^{(1)} = (n+4)Le_{n+1} - 2Le_{n+2} + 4(-1)^{n+4} + n + 3 \geq 0.$$

Using equation (1), we compute:

$$\begin{aligned}\left(Le_{n+2}^{(1)}\right)^2 - Le_{n+1}^{(1)}Le_{n+3}^{(1)} &= \left((n+3)Le_n - 2Le_{n+1} + 4(-1)^{n+3} + n + 2\right) \\ &\quad + \left((n+2)Le_{n-1} - 2Le_n + 4(-1)^{n+2} + n + 1\right) \\ &\quad + Le_n + 2Le_{n-1} + 4(-1)^{n+4} + 2 \\ &\geq 0.\end{aligned}$$

Thus, we conclude that the sequence $\{Le_n^{(1)}\}_{n\geq 1}$ is log-concave. Then, by Lemma 3.1, it follows that the sequence $\{Le_n^{(r)}\}_{n\geq 1}$ is also log-concave for all $r \geq 2$. This completes the proof of the theorem. □

Using Theorem 3.6, Lemma 3.2 and Lemma 3.3, we obtain also the following results.

Corollary 3.7. *The sequences* $\{\sum_{k=0}^{n}\binom{n}{k}Le_k^{(r)}\}_{n\geq 1}$ *and* $\{\sum_{k=0}^{sn}\binom{n}{k}_s Le_k^{(r)}\}_{n\geq 1}$ *($r \geq 1$) are log-concave.*

3.2. The log-balancedness property

Theorem 3.8. *The sequence* $\{n!Le_n^{(1)}\}_{n\geq 1}$ *is log-balanced.*

Proof. By Theorem 3.6, to prove that the sequence $n!Le^{(1)}nn \geq 1$ is log-balanced, it suffices to show that it is log-convex.

From equations (17) and (23), we obtain

$$\begin{aligned}(n+1)\left(Le_{n+1}^{(1)}\right)^2 - (n+2)Le_n^{(1)}Le_{n+2}^{(1)} &= (n+1)\,((n+3)Le_n - 2Le_{n+1} \\ &\quad +4(-1)^{n+3} + n + 2\big) - (Le_{n+3}-(n+3))^2.\end{aligned}$$

Let us define

$$E_n = (n+1)\left((n+3)Le_n - 2Le_{n+1} + 4(-1)^{n+3} + n + 2\right) - (Le_{n+3}-(n+3))^2.$$

It is straightforward to verify that $E_n < 0$ for $1 \leq n \leq 4$. Next, we prove by induction the following two inequalities:

$$Le_n \geq 2n, \quad \text{for } n \geq 4 \tag{24}$$

and

$$Le_n \geq n^2, \quad \text{for } n \geq 8. \tag{25}$$

It remains to show that $E_n < 0$ for all $n \geq 5$.
For $n \geq 5$, applying inequalities (25) and the identity (23), we have:

$$\begin{aligned} E_n &= (n^2+4n+3)Le_n + 2(n+1)Le_{n+2} + 2(n+1) + 4Le_{n+3} + 4(n+1)(-1)^{n+3} \\ &\quad -(Le_{n+3})^2 - 3n - 7 \\ &\leq (n^2+4n+3)Le_n + 2(n+1)Le_{n+2} + 4Le_{n+3} - (Le_{n+3})^2 + 3n - 1 \\ &\leq (3n-1) - (Le_{n+3} - (n+3)^2)Le_{n+3} \\ &< 0. \end{aligned}$$

Furthermore, observe that

$$\left((n+1)!Le^{(1)}_{n+1}\right)^2 - n!(n+2)!Le^{(1)}_n Le^{(1)}_{n+2} = n!(n+1)!E_n,$$

which implies

$$\left((n+1)!Le^{(1)}_{n+1}\right)^2 - n!(n+2)!Le^{(1)}_n Le^{(1)}_{n+2} < 0, \quad \text{for } n \geq 1.$$

Therefore, the sequence $\{n!Le^{(1)}_n\}_{n\geq 1}$ is log-convex, and hence, by Theorem 3.6, it is log-balanced. □

Lemma 3.9. *For $n \geq 0$, we have*

$$\begin{aligned} \left(Le^{(2)}_{n+1}\right)^2 - Le^{(2)}_n Le^{(2)}_{n+2} &= \frac{1}{2}(n^2+11n+16)Le_{n+2} - (2n+7)Le_{n+3} \\ &\quad + \frac{1}{2}(3n^2+21n+36) + 4(-1)^{n+5}. \end{aligned} \tag{26}$$

Proof. From equation (18), we begin with the expression

$$\left(Le^{(2)}_{n+1}\right)^2 - Le^{(2)}_n Le^{(2)}_{n+2},$$

which can be expanded as

$$\left(Le_{n+5} - \frac{1}{2}(n^2+9n+24)\right)^2 - \left(Le_{n+4} - \frac{1}{2}(n^2+7n+16)\right)\left(Le_{n+6} - \frac{1}{2}(n^2+11n+34)\right).$$

Simplifying this expression yields

$$2Le_{n+2} + 4(-1)^{n+5} + 2 + \frac{1}{2}(n^2+5n + \frac{1}{2}(n^2+7n+16)(Le_{n+2}+1) - (2n+7)Le_{n+3}.$$

Combining like terms and simplifying further, we obtain

$$\frac{1}{2}(n^2+11n+16)Le_{n+2} - (2n+7)Le_{n+3} + (n^2+6n+10) + 4(-1)^{n+5}.$$

This completes the proof. □

Theorem 3.10. *The sequence $\{n!Le^{(2)}_n\}_{n\geq 1}$ is log-balanced.*

Proof. By Theorem 3.6, to establish that the sequence $\{n!Le_n^{(2)}\}_{n\geq 1}$ is log-balanced, it suffices to prove that it is log-convex.

From equation (18) and Lemma 3.9, we obtain the following expression:

$$\begin{aligned}(n+1)\left(Le_{n+1}^{(2)}\right)^2-(n+2)Le_n^{(2)}Le_{n+2}^{(2)} &= (n+2)\left(\frac{1}{2}(n^2+11n+16)Le_{n+2}\right.\\ &\quad \left.-(2n+7)Le_{n+3}+(n^2+6n+10)+4(-1)^{n+5}\right)\\ &\quad -\left(Le_{n+5}-\frac{1}{2}(n^2+9n+24)\right)^2.\end{aligned}$$

Define

$$\begin{aligned}S_n &= (n+2)\left(\frac{1}{2}(n^2+11n+16)Le_{n+2}-(2n+7)Le_{n+3}+(n^2+6n+10)+4(-1)^{n+5}\right)\\ &\quad -\left(Le_{n+5}-\frac{1}{2}(n^2+9n+24)\right)^2.\end{aligned}$$

It is straightforward to verify that $S_n<0$ for $1\leq n\leq 4$. We now show that $S_n<0$ for all $n\geq 5$.

Expanding S_n, we obtain:

$$\begin{aligned}S_n &= \frac{1}{2}(n^3+15n^2+56n+72)Le_{n+2}+(7n+26)Le_{n+3}+(4n+8)(-1)^{n+5}\\ &\quad -4Le_{n+2}Le_{n+3}-(Le_{n+2})^2-4(Le_{n+3})^2-\frac{1}{4}(n^4+14n^3+89n^2+272n+320).\end{aligned}$$

Now, observe that:

$$\begin{aligned}S_n &< \left(\frac{1}{2}(n^3+15n^2+56n+72)Le_{n+2}-(Le_{n+2})^2-4(Le_{n+3})^2\right)\\ &\quad +((7n+26)Le_{n+3}-4Le_{n+2}Le_{n+3})-\frac{1}{4}(n^4+14n^3+89n^2+256n+288)\\ &< 0.\end{aligned}$$

Therefore, $S_n<0$ for all $n\geq 1$, which implies that the sequence $\{n!Le_n^{(2)}\}_{n\geq 1}$ is log-convex. By Theorem 3.6, it follows that the sequence is log-balanced. □

3.3. *q-Log-concavity property*

For $r\geq 1$, let

$$Le_{n,r}(q)=\sum_{k=0}^{n}Le_k^{(r)}q^k.$$

Theorem 3.11. *The polynomial $Le_{n,r}(q)$ is q-log-concave for $n\geq 0$.*

Proof. When $n\geq 1$ and $r\geq 1$, we have

$$\begin{aligned}Le_{n+1,r}^2(q)-Le_{n,r}(q)Le_{n+2,r}(q) &= \left(\sum_{k=0}^{n+1}Le_k^{(r)}q^k\right)^2-\sum_{k=0}^{n}Le_k^{(r)}q^k\sum_{k=0}^{n+2}Le_k^{(r)}q^k\\ &= \sum_{k=0}^{n+1}Le_k^{(r)}q^k\left(\sum_{k=0}^{n}Le_k^{(r)}q^k+Le_{n+1}^{(r)}q^{n+1}\right)-\sum_{k=0}^{n}Le_k^{(r)}q^k\left(\sum_{k=0}^{n+1}Le_k^{(r)}q^k+Le_{n+2}^{(r)}q^{n+2}\right)\\ &= \sum_{k=0}^{n+1}Le_k^{(r)}Le_{n+1}^{(r)}q^{n+k+1}-\sum_{k=0}^{n}Le_k^{(r)}Le_{n+2}^{(r)}q^{n+k+2}\\ &= \sum_{k=0}^{n}\left(Le_{k+1}^{(r)}Le_{n+1}^{(r)}-Le_k^{(r)}Le_{n+2}^{(r)}\right)q^{n+k+2}+Le_{n+1}^{(r)}q^{n+1}.\end{aligned}$$

As $\left\{Le_n^{(r)}\right\}_{n\geq 1}$ is log-concave, then the polynomial $Le_{n,r}(q)$ is q-log-concave for $n \geq 1$. $\square$

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