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Remark on injective edge-coloring of some sparse graphs

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Abstract. A k-edge coloring of a graph G is considered injective if any two edges that are at distance 2 or reside within the same triangle receive distinct colors. The minimum integer k for which G admits a k-injective-edge coloring is referred to as the injective edge chromatic number of G, denoted by $\chi_i'(G)$. This paper presents findings on the injective edge chromatic numbers for graphs with maximum degrees of 4 and 5. In particular, we show that if the maximum average degree ($mad(G)$) of a graph G, where the maximum degree is 4, is less than $\frac{38}{11}$, then it follows that $\chi_i'(G) \leq 13$, thereby enhancing the previous result established by Bu and Qi [Discrete Math. Algorithms Appl., 2018]. Additionally, we prove that for any graph G with a maximum degree of 5, it holds true that if $mad(G) < \frac{1501}{384}$, then $\chi_i'(G) \leq 21$, furthermore, if $mad(G) < 4$, then also $\chi_i'(G) \leq 22$.

1. Introduction

In this article, we concentrate exclusively on finite, undirected simple graphs. Let ϕ represent a k-edge coloring of the graph G. If for any three consecutive edges e_1, e_2, and e_3 that lie on the same path or triangle within G, it holds true that $\phi(e_1) \neq \phi(e_3)$, then ϕ is referred to as an injective k-edge coloring of G. The smallest integer k allows for such an injective edge coloring is referred to as its injective chromatic index, denoted by $\chi_i'(G)$. It is important to note that an injective edge-coloring need not be proper. Cardoso et al. [4] introduced the notion of injective edge coloring to address challenges in packet radio networks and demonstrated that computing $\chi_i'(G)$ is NP-hard.

The injective chromatic index is intricately linked to several other concepts. A proper injective edge coloring corresponds precisely to a strong edge coloring, which divides the edges of a graph into induced matchings. The induced star arboricity $isa(G)$ of a graph G represents the minimum number of induced star forests required to cover the edges of G, as detailed in [1]. Ferdjallah, Kerdjoudj, and Raspaud [5] utilized Proposition 2.2 from [4] to note that the induced star arboricity of a graph precisely equals its injective chromatic index, leading to the following conclusion.

Theorem 1.1. *[5] Let G be a simple graph that is not a cycle, then it follows that* $\chi_i'(G) \leq 2(\Delta - 1)^2$.

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Let Δ represent the maximum degree of G. The relationships between the injective chromatic index and both the acyclic chromatic number, as well as the star chromatic number are also examined in [5].

In 2021, Lv et al. [10] investigated the list version of injective edge coloring for subcubic graphs. Further results concerning the injective chromatic index for graphs with girth and maximum degree constraints can be found in [3, 6, 8, 9, 12]. Additionally, Lu et al. [7] and Yang et al. [11] have explored the injective chromatic index in the context of graphs with small weights.

The maximum average degree of a graph G, referred to as $mad(G)$, is defined as the maximum average calculated across all subgraphs H of G; specifically, it can be expressed as $mad(G) = \max\left\{\frac{2|E(H)|}{|V(H)|} : H \subseteq G\right\}$. In 2018, Bu and Qi [2] initiated an investigation into the injective chromatic index of graphs with a maximum degree not exceeding four.

Theorem 1.2. *[2] For any graph G with $\Delta \leq 4$, it follows that $\chi_i'(G) \leq 13$ if $mad(G) < \frac{10}{3}$.*

In this article, we explore the injective edge coloring of sparse graphs with a maximum degree $\Delta \leq 4$ and establish the following theorem, which advances the findings of Bu and Qi [2].

Theorem 1.3. *For any graph G with $\Delta \leq 4$, it follows that $\chi_i'(G) \leq 13$ if $mad(G) < \frac{38}{11}$.*

It is widely recognized that for any graph with a maximum degree of 5, the trivial upper bound on the injective chromatic index is 32. A pertinent question arises regarding the possibility of reducing this bound. Zhu et al. [13] examine the $\chi_i'(G)$ for graphs with $\Delta \leq 5$, providing several sufficient conditions for those graphs that satisfy $\chi_i'(G) \leq 20$. In this paper, we will explore the injective chromatic index of graphs with a maximum degree not exceeding 5 and present the subsequent theorems.

Theorem 1.4. *Let G be a graph with $\Delta(G) = 5$,*
(1) $\chi_i'(G) \leq 21$ if $mad(G) < \frac{1501}{384}$;
(2) $\chi_i'(G) \leq 22$ if $mad(G) < 4$.

2. Injective edge coloring of sparse graphs

In this section, we begin by establishing some essential notation. Let $G = (V(G), E(G))$ denote a graph. For any vertex $v \in V(G)$, the set of all neighbors of v is denoted as $N_G(v)$. The degree of vertex v is defined as $d_G(v) = |N_G(v)|$. A vertex with degree k (resp. at least k) is referred to as a k-vertex (resp. a k^+-vertex). A $(v_1, v_2, \ldots, v_l)$*-vertex* is defined as an l-vertex whose neighbors have degrees corresponding to $v_1, v_2, \ldots, v_l$.

Let ϕ be an injective edge coloring of the subgraph G', we define the set of available colors for an edge $e \in E(G) \setminus E(G')$ as $L_\phi(e)$, and reduce $L_\phi(e)$ to $L_(e)$ if G is obvious from the context. We define the coloring ϕ as complete when it is extended from G' to encompass all edges in G.

2.1. *Proof of Theorem 1.3*

In the present section, we will demonstrate Theorem 1.3 using a proof by contradiction. Assume G is a counterexample that minimizes the sum $|V(G)| + |E(G)|$ for Theorem 1.3. Consequently, G must be connected and satisfy the conditions $\Delta(G) \leq 4$, $mad(G) < \frac{38}{11}$, and $\chi_i'(G) > 13$. Additionally, every proper subgraph of G is injective 13-edge colorable. To begin our analysis, we will first outline several properties of G.

Lemma 2.1. $\delta(G) \geq 3$.

Proof. Suppose $d(v) = 2$ and $N(v) = \{v_1, v_2\}$. Based on the minimality of G, it can be deduced that $\chi_i'(G - v) \leq 13$. Moreover, it is noted that $|L(vv_1)| \geq 1$ and $|L(vv_2)| \geq 1$, which indicates that both edges vv_1 and vv_2 can be colored, resulting in a contradiction. □

Lemma 2.2. *Every 3^+-vertex has at least one 4-neighbor.*

Proof. We will show that each 3-vertex has at least one 4-neighbor, and a similar argument holds for 4-vertex. Let $N(v) = \{v_1, v_2, v_3\}$. Assume $d(v_i) = 3$ for $i = 1, 2, 3$. Based on the minimality of G, it follows that $\chi_i^{'}(G - v) \leq 13$. If there exist an edge $v_iv_j \in E(G)$, where $1 \leq i, j \leq 3$, then $|L(vv_i)| \geq 6$, $|L(vv_j)| \geq 6$ and $|L(vv_k)| \geq 4$, where $k \in \{1, 2, 3\} - \{i, j\}$. Consequently, vv_i, vv_j, and vv_k can be sequentially colored, leading to a contradiction. Otherwise, for $1 \leq i \leq 3$, $|L(vv_i)| \geq 3$ and thus vv_i can be assigned a color distinct from the forbidden colors, resulting in a contradiction. □

Claim 2.3. *Let v be a 3-vertex with t 4-neighbors in G.*
(1) *If $t = 1$, then v is incident with a $(3, 4, 4, 4)$-vertex.*
(2) *If $t = 2$, then v has incident with at most one $(3, 3, 3, 4)$-vertex.*

Proof. (1) Assume instead that vertex v has a $(3, 3, 4, 4)$-vertex v_1. Let $N(v_1) = \{v, w_1, w_2, w_3\}$, where $d(w_1) = 3$ and $d(w_2) = d(w_3) = 4$. Based on the minimality of G, we can obtain a coloring ϕ of $G - v$. Note that $|L(vv_1)| \geq 1$, $|L(vv_2)| \geq 2$, and $|L(vv_3)| \geq 2$. Therefore, we can extend ϕ by sequentially assigning colors to vv_1, vv_2, vv_3, which leads to a contradiction.

(2) Assume instead that v has two $(3, 3, 3, 4)$-vertices v_1 and v_2. Denote $N(v_1) = \{v, u_1, u_2, u_3\}$, $N(v_2) = \{v, w_1, w_2, w_3\}$, where $d(u_1) = d(u_2) = d(w_1) = d(w_2) = 3$ and $d(u_3) = d(w_3) = 4$. Based on the minimality of G, we can derive a coloring ϕ for the $G - v$. Note that $|L(vv_1)| \geq 1$, $|L(vv_2)| \geq 1$, and $|L(vv_3)| \geq 1$. Consequently, we can extend ϕ by sequentially assigning colors to vv_1, vv_2, vv_3, which results in a contradiction. □

For every vertex $v \in V(G)$, we establish a weight function w defined as $w(v) = d(v) - \frac{38}{11}$. Let R1, R2, and R3 be three discharging rules. Our goal is to show that $w'(v) \geq 0$ for every $v \in V(G)$ upon the completion of the discharging process.

Discharging Rules

R1 Every $(3, 4, 4, 4)$-vertex sends $\frac{6}{11}$ to each of its incident 3-vertex.

R2 Every $(3, 3, 4, 4)$-vertex sends $\frac{3}{11}$ to each of its incident 3-vertices.

R3 Every $(3, 3, 3, 4)$-vertex sends $\frac{2}{11}$ to each of its incident 3-vertices.

We will now demonstrate that $w'(v) \geq 0$ for all $v \in V(G)$.

Case 1. $d(v) = 3$.

In this case, we find $w(v) = 3 - \frac{38}{11} = -\frac{5}{11}$. According to Lemma 2.2, v has at least one 4-neighbor. If there exists exactly one 4-vertex in $N(v)$, then by Claim 2.3(1), this 4-vertex is a $(3, 4, 4, 4)$-vertex. According to R1, the 4-neighbor of v sends charge $\frac{6}{11}$ to v. Consequently, we have $w'(v) = w(v) + \frac{6}{11} > 0$. If there exist exactly two 4-vertices in $N(v)$, then by Claim 2.3(2), v has incident with at most one $(3, 3, 3, 4)$-vertex. By R1-R3, 4-neighbors of v sends charge at least $\min\{\frac{6}{11}, \frac{3}{11}\} + \frac{2}{11} = \frac{5}{11}$ to v. Thus, $w'(v) = w(v) + \frac{5}{11} = 0$. If there exist three 4-vertices in $N(v)$, then by R1-R3, every 4-neighbor sends charge at least $\frac{2}{11}$ to v. Therefore, we conclude that $w'(v) = w(v) + 3 \times \frac{2}{11} > 0$.

Case 2. $d(v) = 4$.

In this case, we have $w(v) = 4 - \frac{38}{11} = \frac{6}{11}$. According to Lemma 2.2, v can be incident to at most three 3-neighbors. If v has three such neighbors, by R3, v sends charge $3 \times \frac{2}{11} = \frac{6}{11}$ to its 3-neighbors. Hence, $w'(v) = w(v) - \frac{6}{11} = 0$. If v has exactly two 3-neighbors, then v sends charge at least $2 \times \frac{3}{11} = \frac{6}{11}$ to its 3-neighbors by R2. Therefore, $w'(v) = w(v) - \frac{6}{11} = 0$. If v has at most one 3-neighbor, then by R1, v sends charge $\frac{6}{11}$ to its 3-neighbor. Hence, $w'(v) = w(v) - \frac{6}{11} = 0$.

Given that $mad(G) < \frac{38}{11}$, it follows that $\sum_{v \in V(G)} w'(v) < 0$. However, from Case 1 and Case 2, we have

$$0 \leq \sum_{v \in V(G)} w'(v) = \sum_{v \in V(G)} w(v) < 0,$$

which result in a contradiction. □

2.2. Proof of Theorem 1.4

Suppose, to the contrary, that G is a minimal counterexample to Theorem 1.4 in terms of $|V(G)|+|E(G)|$. Specifically, G is a graph that satisfies $\Delta(G) \leq 5$ and $mad(G) < l$ (l is equal to $\frac{1501}{384}, 4$ respectively), $\chi_i'(G) > k$ (k is equal to $21, 22$, respectively), but every proper subgraph of G is k-injective-edge colorable. We will begin by establishing several properties of G.

Lemma 2.4. $\delta(G) \geq 3$.

Proof. Assume that $d(v) = 2$ and let $N(v) = \{v_1, v_2\}$. Based on the minimality of G, it follows that $\chi_i'(G-v) \leq 21$. It is important to note that $|L(vv_1)| \geq 1$, $|L(vv_2)| \geq 1$, which implies that vv_1 and vv_2 can be colored. This leads to a contradiction. □

Lemma 2.5. *Every 3-vertex has at most one 3-neighbor and at least one 5-neighbor.*

Proof. We will show that each 3-vertex has at most one 3-neighbor, and a similar argument holds for 5-neighbor. Let $N(v) = \{v_1, v_2, v_3\}$, where $d(v_1) = d(v_2) = 3$. Based on the minimality of G, $\chi_i'(G - v) \leq 21$. If there exist $v_iv_j \in E(G)$, where $1 \leq i, j \leq 3$, then it follows that $|L(vv_i)| \geq 11$, $|L(vv_j)| \geq 11$ and $|L(vv_k)| \geq 2$, where $k \in \{1, 2, 3\} - \{i, j\}$. Thus, vv_i, vv_j and vv_k can be colored in order, leading to a contradiction. Otherwise, $|L(vv_1)| \geq 7$, $|L(vv_2)| \geq 7$ and $|L(vv_3)| \geq 1$, vv_1, vv_2 and vv_3 can be colored, resulting in a contradiction. □

Lemma 2.6. *Every 5-vertex is adjacent to at most four 3-neighbors. If $d(v_1) = d(v_2) = d(v_3) = d(v_4) = 3$, then $d(v_5) = 5$.*

Proof. Let $N(v) = \{v_1, v_2, v_3, v_4, v_5\}$. Suppose $d(v_i) = 3$ for $i = \{1, 2, 3, 4, 5\}$. Based on the minimality of G, we can obtain a coloring ϕ of $G - v$. Note that $|L(vv_i)| \geq 5$, where $1 \leq i \leq 5$. Therefore, we can extend ϕ by coloring vv_1, vv_2, vv_3, vv_4 and vv_5 in order, which is a contradiction.

If $d(v_1) = d(v_2) = d(v_3) = d(v_4) = 3$, assume that $d(v_5) \leq 4$. Based on the minimality of G, it follows that $\chi_i'(G - v) \leq 21$. Notice that $|L(vv_1)| \geq 4$, $|L(vv_2)| \geq 4$, $|L(vv_3)| \geq 4$, $|L(vv_4)| \geq 4$ and $|L(vv_5)| \geq 1$, vv_1, vv_2, vv_3, vv_4 and vv_5 can be colored, leading to a contradiction. Thus, we conclude that $d(v_5) = 5$. □

Lemma 2.7. *Every $(3, 5, 5)$-vertex has at most one $(3, 3, 3^+, 3^+, 5)$-vertex. If v is incident with exactly one $(3, 3, 3^+, 3^+, 5)$-vertex, than the others 5-vertex is a $(3, 4^+, 5, 5, 5)$-vertex.*

Proof. Suppose, to the contrary, that v is adjacent to two $(3, 3, 3^+, 3^+, 5)$-vertices v_2 and v_3. Based on the minimality of G, it follows that $\chi_i'(G - v) \leq 21$. Notice that $|L(vv_1)| \geq 5$, $|L(vv_2)| \geq 1$, and $|L(vv_3)| \geq 1$. Consequently, vv_1, vv_2 and vv_3 can be assigned colors without conflict, leading to a contradiction.

If v is incident to exactly one $(3, 3, 3^+, 3^+, 5)$-vertex v_2, assume that $\sum_{v \in N(v_3)} d(v) \leq 21$. Based on the minimality of G, it follows that $\chi_i'(G - v) \leq 21$. Notice that $|L(vv_1)| \geq 5$, $|L(vv_2)| \geq 1$, and $|L(vv_3)| \geq 1$. Consequently, vv_1, vv_2 and vv_3 can be colored without conflict, leading to a contradiction. Therefore, we conclude that $\sum_{v \in N(v_3)} d(v) \geq 22$. This implies that v_3 is a $(3, 4^+, 5, 5, 5)$-vertex. □

Lemma 2.8. *Every $(4, 4, 5)$-vertex has a $(3, 4^+, 5, 5, 5)$-neighbor.*

Proof. Let $N(v) = \{v_1, v_2, v_3\}$, where $d(v_1) = 4, d(v_2) = 4$ and $d(v_3) = 5$. Suppose $\sum_{v \in N(v_3)} d(v) \leq 21$. Based on the minimality of G, it follows that $\chi_i'(G - v) \leq 21$. Notice that $|L(vv_1)| \geq 2$, $|L(vv_2)| \geq 2$ and $|L(vv_3)| \geq 1$. Consequently, vv_1, vv_2 and vv_3 can be colored without conflict, leading to a contradiction. Therefore, we conclude that $\sum_{v \in N(v_3)} d(v) \geq 22$. This implies that v_3 is a $(3, 4^+, 5, 5, 5)$-vertex. □

Lemma 2.9. *Every $(5, 5, 5)$-vertex v is adjacent to at most two $(3, 3, 3, 3, 5)$-vertices. If v is incident with exactly two $(3, 3, 3, 3, 5)$-vertices, then another 5-vertex is a $(3, 3^+, 4^+, 5, 5)$-neighbor or a $(3, 4^+, 4^+, 4^+, 5)$-neighbor.*

Proof. Suppose, to the contrary, that v is adjacent to three $(3, 3, 3, 3, 5)$-vertices v_1, v_2 and v_3 in G. Based on the minimality of G, it follows that $\chi_i'(G - v) \leq 21$. Notice that $|L(vv_1)| \geq 3$, $|L(vv_2)| \geq 3$, and $|L(vv_3)| \geq 3$, vv_1, vv_2 and vv_3 can be colored, contradiction.

If v is incident to exactly two $(3, 3, 3, 3, 5)$-vertices v_1 and v_2, assume that $\sum_{v \in N(v_3)} d(v) \leq 19$. Based on the minimality of G, it follows that $\chi_i'(G - v) \leq 21$. Notice that $|L(vv_1)| \geq 3$, $|L(vv_2)| \geq 3$, and $|L(vv_3)| \geq 1$, vv_1, vv_2 and vv_3 can be colored, leading to a contradiction. Therefore, $\sum_{v \in N(v_3)} d(v) \geq 20$. This implies that v_3 is a $(3, 3^+, 4^+, 5, 5)$-neighbor or a $(3, 4^+, 4^+, 4^+, 5)$-neighbor. □

2.2.1. Proof of Theorem 1.4(1)

If $l = \frac{1501}{384}$, $k = 21$, we can demonstrate the following structural properties of G.

Lemma 2.10. *Every 4-vertex is adjacent to at most three 3-neighbors. If $d(v_1) = d(v_2) = d(v_3) = 3$, then $d(v_4) = 5$.*

Proof. Let $N(v) = \{v_1, v_2, v_3, v_4\}$. Assume that $d(v_i) = 3$ for each $i = \{1, 2, 3, 4\}$. Based on the minimality of G, it follows that $\chi_i'(G - v) \leq 21$. If there exists $v_i v_j \in E(G)$, where $1 \leq i, j \leq 4$, then $|L(vv_i)| \geq 11$, $|L(vv_j)| \geq 11$ and $|L(vv_k)| \geq 8$, where $k \in \{1, 2, 3, 4\} - \{i, j\}$. Therefore, vv_i, vv_j and vv_k can be colored sequentially without conflict, this leads to a contradiction. Otherwise, for $1 \leq i \leq 4$, $|L(vv_i)| \geq 7$, which means that we can color vv_1, vv_2, vv_3 and vv_4 in turn, a contradiction.

If $d(v_1) = d(v_2) = d(v_3) = 3$, assume that $d(v_4) \leq 4$. Based on the minimality of G, it follows that $\chi_i'(G - v) \leq 21$. For $1 \leq i \leq 3$, $|L(vv_i)| \geq 6$, $|L(vv_4)| \geq 3$, so can be colored, resulting in a contradiction. Therefore, $d(v_4) = 5$. □

Lemma 2.11. *Every $(3, 4, 5)$-vertex has a $(3, 5, 5, 5, 5)$-neighbor.*

Proof. Let $N(v) = \{v_1, v_2, v_3\}$, where $d(v_1) = 3, d(v_2) = 4$ and $d(v_3) = 5$. We assume that $\sum_{v \in N(v_3)} d(v) \leq 22$. Based on the minimality of G, it follows that $\chi_i'(G - v) \leq 21$. Notice that $|L(vv_1)| \geq 6$, $|L(vv_2)| \geq 3$ and $|L(vv_3)| \geq 1$, vv_1, vv_2 and vv_3 can be colored without conflict, this leads to a contradiction. Therefore, we conclude that $\sum_{v \in N(v_3)} d(v) \geq 23$. This implies that v_3 is a $(3, 5, 5, 5, 5)$-vertex. □

Lemma 2.12. *Every $(4, 5, 5)$-vertex is adjacent to at most one $(3, 3, 3, 3^+, 5)$-vertex. If v is incident with exactly one $(3, 3, 3, 3^+, 5)$-vertex, then another 5-vertex is a $(3, 3^+, 5, 5, 5)$-neighbor or a $(3, 4^+, 4^+, 5, 5)$-neighbor.*

Proof. Suppose, to the contrary, that v is adjacent to two $(3, 3, 3, 3^+, 5)$-vertices v_2 and v_3. Based on the minimality of G, we can obtain a coloring ϕ of $G - v$. Notice that $|L(vv_1)| \geq 1$, $|L(vv_2)| \geq 2$, and $|L(vv_3)| \geq 2$. Therefore, ϕ can be extended by coloring vv_1, vv_2 and vv_3 in order, which is a contradiction.

If v is incident with exactly one $(3, 3, 3, 3^+, 5)$-vertex v_2, we assume that $\sum_{v \in N(v_3)} d(v) \leq 20$. Based on the minimality of G, it follows that $\chi_i'(G - v) \leq 21$. It is important to note that $|L(vv_1)| \geq 1$, $|L(vv_2)| \geq 2$, and $|L(vv_3)| \geq 1$, vv_1, vv_2 and vv_3 can be colored, this leads to a contradiction. Therefore, we conclude that $\sum_{v \in N(v_3)} d(v) \geq 21$. This implies that v_3 is a $(3, 3^+, 5, 5, 5)$-neighbor or a $(3, 4^+, 4^+, 5, 5)$-neighbor. □

Lemma 2.13. *Every $(3, 3, 3, 5)$-vertex has a $(4, 4^+, 5, 5, 5)$-neighbor.*

Proof. Let $N(v) = \{v_1, v_2, v_3, v_4\}$, where $d(v_1) = 3$, $d(v_2) = 3$, $d(v_3) = 3$, $d(v_4) = 5$. Suppose $\sum_{v \in N(v_4)} d(v) \leq 22$. Based on the minimality of G, it follows that $\chi_i'(G - v) \leq 21$. Notice that $|L(vv_1)| \geq 5$, $|L(vv_2)| \geq 5$, $|L(vv_3)| \geq 5$ and $|L(vv_4)| \geq 1$, vv_1, vv_2, vv_3 and vv_4 can be colored, this leads to a contradiction. Therefore, we conclude that $\sum_{v \in N(v_4)} d(v) \geq 23$. This implies that v_4 is a $(4, 4^+, 5, 5, 5)$-vertex. □

Lemma 2.14. *Every $(3, 3, 4, 5)$-vertex has a $(3^+, 4, 5, 5, 5)$-neighbor or a $(4, 4^+, 4^+, 5, 5)$-neighbor.*

Proof. Let $N(v) = \{v_1, v_2, v_3, v_4\}$, where $d(v_1) = 3$, $d(v_2) = 3$, $d(v_3) = 4$, $d(v_4) = 5$. Suppose $\sum_{v \in N(v_4)} d(v) \leq 21$. Based on the minimality of G, it follows that $\chi_i'(G - v) \leq 21$. Notice that $|L(vv_1)| \geq 4$, $|L(vv_2)| \geq 4$, $|L(vv_3)| \geq 1$ and $|L(vv_4)| \geq 1$, vv_1, vv_2, vv_3 and vv_4 can be colored, this leads to a contradiction. Therefore, we conclude that $\sum_{v \in N(v_4)} d(v) \geq 22$. This implies that v_4 is a $(3^+, 4, 5, 5, 5)$-vertex or a $(4, 4^+, 4^+, 5, 5)$-vertex. □

Lemma 2.15. *Every $(3, 3, 5, 5)$-vertex v is adjacent to at most one $(3, 3, 3^+, 4, 4^+)$-vertex. If v is incident with exactly one $(3, 3, 3^+, 4, 4^+)$-vertex, then another 5-vertex is a $(3^+, 4, 4^+, 5, 5)$-vertex or a $(4, 4^+, 4^+, 4^+, 5)$-vertex.*

Proof. Suppose, to the contrary, that v is adjacent to two $(3, 3, 3^+, 4, 4^+)$-vertices v_3 and v_4. Based on the minimality of G, it follows that $\chi_i'(G - v) \leq 21$. Notice that $|L(vv_1)| \geq 3$, $|L(vv_2)| \geq 3$, $|L(vv_3)| \geq 1$, and $|L(vv_4)| \geq 1$, vv_1, vv_2, vv_3 and vv_4 can be colored, this leads to a contradiction.

If v is incident with exactly one $(3, 3, 3^+, 4, 4^+)$-vertex v_3, we assume that $\sum_{v \in N(v_4)} d(v) \leq 20$. Based on the minimality of G, it follows that $\chi_i'(G - v) \leq 21$. Notice that $|L(vv_1)| \geq 3$, $|L(vv_2)| \geq 3$, $|L(vv_3)| \geq 1$, and $|L(vv_4)| \geq 1$, vv_1, vv_2, vv_3 and vv_4 can be colored, this leads to a contradiction. Therefore, we conclude that $\sum_{v \in N(v_4)} d(v) \geq 21$. This implies that v_4 is a $(3^+, 4, 4^+, 5, 5)$-vertex or a $(4, 4^+, 4^+, 4^+, 5)$-vertex. □

For every vertex $v \in V(G)$, we establish a weight function w defined as $w(v) = d(v) - \frac{1501}{384}$. Let R1-R6 be three discharging rules. Our goal is to show that $w'(v) \geq 0$ for every $v \in V(G)$ after the completion of the discharging process.

Discharging Rules

R1 Every $(3, 4^+, 4^+, 4^+, 4^+)$-vertex sends $\frac{349}{384}$ to each of its incident 3-vertex.

R2 Every $(3, 3, 4^+, 4^+, 4^+)$-vertex sends $\frac{419}{768}$ to each of its incident 3-vertices.

R3 Every $(3, 3, 3, 4^+, 4^+)$-vertex sends $\frac{419}{1152}$ to each of its incident 3-vertices.

R4 Every $(3, 3, 3, 3, 4^+)$-vertex sends $\frac{419}{1536}$ to each of its incident 3-vertices.

R5 Every 5-vertex redistributes its remaining charge after applying R1-R4 equitably to each of its incident 4-vertices.

R6 Every 4-vertex redistributes its remaining charge after applying the previous rules equitably to each of its incident 3-vertices.

Applying R1 and R5, we can immediately conclude the following result.

Claim 1. *Each* $(3, 4^+, 4^+, 4^+, 4^+)$*-vertex sends* $\frac{35}{768}$ *to each of its incident* 4*-vertices.*

Proof. If v is a $(3, 4^+, 4^+, 4^+, 4^+)$-vertex, then by Theorem 1.4(1), each 5-vertex v has a remaining charge of $5 - \frac{1501}{384} = \frac{419}{384}$. According to R1, v sends $\frac{349}{384}$ to its incident 3-vertex. And by R5, every 4-vertex receives charge at least $\frac{1}{4} \times (\frac{419}{384} - \frac{349}{384}) = \frac{35}{768}$ from its incident 5-vertex. □

Now, we consider the 4-vertex.

Claim 2. *Every* $(3, 4, 4^+, 5)$*-vertex or* $(3, 3, 5, 5)$*-vertex sends* $\frac{35}{384}$ *to its incident* 3*-vertex.*

Proof. If v is a $(3, 4, 4^+, 5)$-vertex, then by Theorem 1.4(1), each 4-vertex v has a remaining charge of $4 - \frac{1501}{384} = \frac{35}{384}$. According to R6, v sends all remaining charge $\frac{35}{384}$ to its incident 3-vertex. Therefore, the incident 3-vertex receives a charge of $\frac{35}{384}$.

If v is a $(3, 3, 5, 5)$-vertex, then by Lemmas 2.15, v has at least one $(3^+, 4, 4^+, 5, 5)$-neighbor. According to R1 and R5, each 5-neighbor sends charge $\frac{349}{384}$ to its incident 3-vertex, and sends charge $\frac{1}{2} \times (5 - \frac{1501}{384} - \frac{349}{384}) = \frac{35}{384}$ to its incident 4-vertex. Due to the remaining charge of v being $\frac{35}{384}$. By R6, v distributes at least $\frac{1}{2} \times (\frac{35}{384} + \frac{35}{384}) = \frac{35}{384}$ to each of its incident 3-vertices. □

Claim 3. *Every* $(3, 3, 4, 5)$*-vertex sends* $\frac{105}{768}$ *to each of its incident* 3*-vertices.*

Proof. By Lemma 2.14, every $(3, 3, 4^+, 5)$-vertex v has a $(3^+, 4, 5, 5, 5)$-neighbor. According to R1 and R5, each 5-neighbor sends charge $\frac{349}{384}$ to its incident 3-vertex, and sends charge $5 - \frac{1501}{384} - \frac{349}{384} = \frac{35}{192}$ to its incident 4-vertex. Due to the remaining charge of v being $\frac{35}{384}$. By R6, v distributes at least $\frac{1}{2} \times (\frac{35}{192} + \frac{35}{384}) = \frac{105}{768}$ to each of its incident 3-vertices. □

Claim 4. *Every* $(3, 3, 3, 5)$*-vertex sends* $\frac{163}{768}$ *to each of its incident* 3*-vertices.*

Proof. By Lemma 2.13, every $(3, 3, 3, 5)$-vertex v has a $(4, 4^+, 5, 5, 5)$-neighbor. By R5, 5-neighbor of v sends charge at least $\frac{1}{2} \times (5 - \frac{1501}{384}) = \frac{419}{768}$ to each of its incident 4-vertices. Due to the remaining charge of vertex v being $\frac{35}{384}$. According to R6, v distributes at least $\frac{1}{3} \times (\frac{419}{768} + \frac{35}{384}) = \frac{163}{768}$ to each of its incident 3-vertices. □

We will now demonstrate that $w'(v) \geq 0$ for all $v \in V(G)$.

Case 1. $d(v) = 3$.

In this case, $w(v) = 3 - \frac{1501}{384} = -\frac{349}{384}$. According to Lemma 2.5, v has at most one 3-neighbor and at least one 5-neighbor.

If there exists only one 5-vertex in $N(v)$, then by Lemmas 2.8 and 2.11, v is incident to a $(3,4^+,5,5,5)$-neighbor. According to R1, the only 5-neighbor of v sends charge $\frac{349}{384}$ to v. Consequently, we have $w'(v) = w(v) + \frac{349}{384} = 0$.

If exactly two 5-vertices exist in $N(v)$, we assume that v is a $(3,5,5)$-vertex. By Lemma 2.7, v is incident with at most one $(3,3,3,3,5)$-neighbor, the other 5-neighbor must be a $(3,4^+,5,5,5)$-vertex. According to R1 and R4, 5-neighbors of v sends charge at least $\frac{349}{384} + \frac{419}{1536} = \frac{605}{512}$. Hence, $w'(v) = w(v) + \frac{605}{512} > 0$. Suppose that v is a $(4,5,5)$-vertex, then by Lemma 2.12, v is incident with at most one $(3,3,3,3,5)$-neighbor. We will discuss two subcases: (a) When v is not adjacent to any $(3,3,3,3,5)$-neighbors, it follows from Lemma 2.12, R1-R3 and Claims 2-4, v receives charge at least $\frac{35}{384} + \frac{419}{768} + \frac{419}{1152} = \frac{2305}{2304}$. Hence, $w'(v) = w(v) + \frac{2305}{2304} > 0$. (b) When v has exactly one $(3,3,3,3,5)$-neighbor. If v is adjacent to a $(3,3,3,5)$-vertex, then according to R1, R2, R4 and Claim 4, v receives charge at least $\frac{163}{768} + \frac{419}{1536} + \frac{419}{768} = \frac{1583}{1536}$ from its neighbors. Consequently, we have $w'(v) = w(v) + \frac{1583}{1536} > 0$. If v is incident with a $(3,3,4,5)$-vertex, applying R1, R2, R4 and Claim 3, v receives charge at least $\frac{105}{768} + \frac{419}{1536} + \frac{419}{768} = \frac{489}{512}$ of its neighbors. Thus, we find that $w'(v) = w(v) + \frac{489}{512} > 0$. If v is incident with a $(3,3,5,5)$-vertex or a $(3,4,4^+,5)$-vertex, then by R1, R2, R4 and Claim 2, v receives charge at least $\frac{35}{384} + \frac{419}{1536} + \frac{419}{768} = \frac{1397}{1536}$ of its neighbors. Therefore, we conclude that $w'(v) = w(v) + \frac{1397}{1536} > 0$.

If there are three 5-vertices in $N(v)$, then by Lemma 2.9, v is incident with at most two $(3,3,3,3,5)$-vertices. 5-neighbors of v sends charge at least $\frac{419}{1152} + 2 \times \frac{419}{1536} = \frac{2095}{2304}$ to its incident 3-vertex by R1-R4. Therefore, $w'(v) = w(v) + \frac{2095}{2304} > 0$.

Case 2. $d(v) = 4$.

In this case, $w(v) = 4 - \frac{1501}{384} = \frac{35}{384}$. According to R5 and R6, $w'(v) = w(v) = 0$.

Case 3. $d(v) = 5$.

In this case, we have $w(v) = 5 - \frac{1501}{384} = \frac{419}{384}$. Based on Lemma 2.6, it can be concluded that v has no more than four 3-neighbors. If v has four 3-neighbors, then according to Lemma 2.6, v is a $(3,3,3,3,5)$-vertex. By R4, v sends charge $4 \times \frac{419}{1536} = \frac{419}{384}$ to each of its incident 3-neighbors. Hence, $w'(v) = w(v) - \frac{419}{384} = 0$. If v has three 3-neighbors, then v sends charge at least $3 \times \frac{419}{1152} = \frac{419}{384}$ to each of its incident 3-neighbors by R3. Hence, $w'(v) = w(v) - \frac{419}{384} = 0$. If v has exactly two 3-neighbors, then v sends charge at least $2 \times \frac{419}{768} = \frac{419}{384}$ to each of its incident 3-neighbors by R2. Therefore, $w'(v) = w(v) - \frac{419}{384} = 0$. If v has at most one 3-neighbor, then by R1, v sends charge $\frac{349}{384}$ to its incident 3-neighbor. Hence, $w'(v) = w(v) - \frac{349}{384} > 0$.

Given that $mad(G) < \frac{1501}{384}$, it follows that $\sum_{v\in V(G)} w'(v) < 0$. However, from Case 1, Case 2 and Case 3, we have

$$0 \leq \sum_{v\in V(G)} w'(v) = \sum_{v\in V(G)} w(v) < 0,$$

which result in a contradiction. □

2.2.2. Proof of Theorem 1.4(2)

If $l = 4, k = 22$, we can demonstrate the following structural properties of G.

Lemma 2.16. *If the vertex of degree 3 is incident with exactly one 3-neighbor, then its other two neighbors must be 5-vertices.*

Proof. Let $N(v) = \{v_1, v_2, v_3\}$, where $d(v_1) = 3, d(v_2) \leq 4$ and $d(v_3) = 5$. Based on the minimality of G, we can obtain a coloring ϕ of $G - v$. Notice that $|L(vv_1)| \geq 7$, $|L(vv_2)| \geq 4$, $|L(vv_3)| \geq 1$. Thus, ϕ can be extended by coloring vv_1, vv_2, vv_3 in order, resulting in a contradiction. Therefore, it can be concluded that $d(v_2) = 5$. □

Lemma 2.17. *Every $(4,5,5)$-vertex is adjacent to at most one $(3,3,3,3,5)$-vertex. If v is incident with exactly one $(3,3,3,3,5)$-vertex, then another 5-vertex is a $(3,4^+,5,5,5)$-neighbor.*

Proof. We shall assume by contradiction that v is adjacent to two $(3,3,3,3,5)$-vertices v_2 and v_3. Based on the minimality of G, we can obtain a coloring ϕ of $G - v$. It should be noted that $|L(vv_1)| \geq 2$, $|L(vv_2)| \geq 5$, and $|L(vv_3)| \geq 5$. Thus, ϕ can be extended by coloring vv_1, vv_2 and vv_3 in order, which is a contradiction.

If v is incident with exactly one $(3,3,3,3,5)$-vertex v_2, assume that $\sum_{v\in N(v_3)} d(v) \leq 21$. Based on the minimality of G, it follows that $\chi_i^{'}(G-v) \leq 22$. It should be noted that $|L(vv_1)| \geq 2$, $|L(vv_2)| \geq 5$, and $|L(vv_3)| \geq 1$, vv_1, vv_2 and vv_3 can be colored, which leads to a contradiction. Therefore, we conclude that $\sum_{v\in N(v_3)} d(v) \geq 22$. This implies that v_3 is a $(3,4^+,5,5,5)$-vertex. □

Let $v \in V(G)$ be a vertex of degree 5, adjacent to at least $t \geq 1$ vertices of degree 3 in G. According to Lemma 2.6, when v has four 3-neighbors in G, the fifth neighbor must be a 5-vertex. Hence, it follows that $t \leq 4$.

For every vertex $v \in V(G)$, we establish a weight function w defined as $w(v) = d(v) - 4$. Let R1 denote the discharging rule. Our goal is to show that $w'(v) \geq 0$ for every $v \in V(G)$ after the completion of the discharging process.

R1 Each 5-vertex distribute $1/t$ charges to each 3-neighbor.

It should be noted that $1/t \geq 1/4$. Let $w'(v)$ be defined as the new charge of $v \in G$ after applying the above discharging rules. It is clear that for each $v \in V(G)$ with $d(v) = 4$ or $d(v) = 5$, it holds that $w'(v) \geq 0$.

Let $v \in V(G)$ satisfying $d(v) = 3$. Let $N(v) = \{v_1, v_2, v_3\}$ where $3 \leq d(v_1) \leq d(v_2) \leq d(v_3)$. We can consider the following situation by Lemmas 2.5 and 2.16.

If v is a $(3,5,5)$-vertex, then according to Lemma 2.7, at least one of v_2 and v_3 is a $(3,4^+,5,5,5)$-neighbor. Accordingly, by R1, we obtain $w'(v) \geq -1 + \frac{1}{4} + 1 > 0$.

If v is a $(4,4,5)$-vertex, then by Lemma 2.8, it follows that v_3 is a $(3,4^+,5,5,5)$-neighbor. Consequently, by R1, $w'(v) \geq -1 + 1 = 0$.

If v is a $(4,5,5)$-vertex, then by Lemma 2.17, at least one of v_2 and v_3 is a $(3,4^+,5,5,5)$-neighbor. Henceforth, by R1, $w'(v) \geq -1 + \frac{1}{4} + 1 > 0$.

If v is a $(5,5,5)$-vertex, then by Lemma 2.9, v has at most two $(3,3,3,3,5)$-neighbors, the third neighbor is a $(3,3^+,4^+,5,5)$-vertex. Thus, by R1, $w'(v) \geq -1 + 2 \times \frac{1}{4} + \frac{1}{2} = 0$.

Given that $mad(G) < 4$, it follows that $\sum_{v\in V(G)} w'(v) < 0$. However, based on the preceding analysis, we have

$$0 \leq \sum_{v\in V(G)} w'(v) = \sum_{v\in V(G)} w(v) < 0,$$

which result in a contradiction. □

Declaration of competing interest

The author declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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