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Approximation by a new sequence of generalized α-Szász-Gamma operators

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Abstract. The goal of this article is to present the Szász-integral type sequences of operators using the gamma function and Hermite polynomials. We also compute certain estimates at key moments and test functions. Additionally, we go over the uniform convergence theorem, which is derived from the first-order modulus of smoothness, and the order of approximation via the Korovkin theorem. We then examine pointwise approximation results in the context of Lipschitz-type space and Peetre's K-functional, second-order modulus of smoothness. Furthermore, their convergence outcomes are examined in a weighted space.

1. Introduction and preliminaries

The approximation with the sequences of Bernstein-operators [5] is restricted to bounded functions on $[0,1]$. To approximate on $[0,\infty)$, Szász [31] gave a modification of the sequences of Bernstein-operators, which play an important role in the development of operator theory, as below:

$$S_n(f;y) = e^{-ny}\sum_{l=0}^{\infty}\frac{(ny)^l}{l!}f\left(\frac{l}{n}\right),\ n\in\mathbb{N}, \tag{1}$$

where the real-valued function $f\in C[0,\infty)$. The linear positive operators introduced in (1) are restricted to functions that belong to the space of continuous functions only. To approximate in longer class of

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functions, i.e., the space of functions that are measurable in the Lebesgue sense, several integral versions of these sequences of operators are introduced, e.g., Jakimovski-Leviatan-Beta type integral operators [1], Phillips operators via q-Dunkl generalization [2], Szász-Kantorovich [31], Szász-Durremeyer operators [9], Szász operators to bivariate functions [20], Szász-Jakimovski-Leviatan operators [21], parametric extension of Szász-Mirakjan-Kantorovich operators [22], Jakimovski-Leviatan-Beta operators [23], Stancu-Bernstein-Kantorovich operators [15], q-Bernstein shifted operators [18], q-Bernstein-Stancu operators [19], Baskakov-Kantorovich operators [14], Szász–Mirakjan operators to bivariate [24], Szász-type operators [33], Szász-type operators using Charlier polynomials [34] and Szász-Mirakjan-Durrmeyer operators [26]. Recent relevant studies on statistical convergence and Korovkin theorems to attract a wider audience we see [6, 12, 16, 17, 27–29, 32]

In 2016, Grażyna [10] presented a class of sequence of operators $G_n^\alpha(.;.)$ presented by the relation

$$G_n^\alpha(f;y) = e^{-(ny+\alpha y^2)} \sum_{l=0}^{\infty} \frac{y^l}{l!} H_l(n,\alpha) f\Big(\frac{l}{n}\Big), \quad n \in \mathbb{N},\ \alpha \geq 0,\ \ y \in \mathbb{R}_0^+, \tag{2}$$

where $H_l(.;.)$ is the two variable Hermite polynomial (see [11]) given by

$$H_l(n,\alpha) = l! \sum_{m=0}^{[\frac{l}{2}]} \frac{n^{l-2m}\alpha^m}{(l-2m)!m!}. \tag{3}$$

The Hermite polynomials and their properties were investigated in many papers, for example in ([4], [7]). Integrals of these polynomials are ubiquitous in problems involving classical and quantum optics as well as quantum physics (see [7]). The operators (2) are linear and positive. Basic facts on positive linear operators, their generalizations, and applications can be found in [8].

Remark 1.1. *The sequences of operators presented in* (2) *are restricted for continuous functions only.*

Motivated by the above foundation of the present research work, we introduce a sequence of positive linear operators to give approximations in a bigger class of functions, i.e., the space of Lebesgue measurable functions, which is named as Szász-gamma operators given the Hermite Polynomial as:

$$H_m^\alpha(g;y) \quad = \quad \sum_{s=0}^{\infty} P_{m,s}^\alpha(y) \int_0^\infty Q_s(t) g(t) dt, \quad for \quad y \in \mathbb{R}_0^+, \tag{4}$$

where

$$P_{m,s}^\alpha(y) = e^{-(my+\alpha y^2)} \frac{y^s}{s!} H_s(m,\alpha) \ \ and \ \ Q_s(t) = \frac{m^{s+\beta+1}}{\Gamma(s+\beta+1)} \Big[t^{s+\beta} e^{-mt} \Big],$$

with Γ (Gamma) function which is given by as:

$\Gamma u = \int_0^\infty m^{u-1} e^{-m} dz, \quad \Gamma u = (u-1)\Gamma(u-1) = (u-1)!.$

Lemma 1.2. *[10] Let $G_m^\alpha(.;.)$ be the sequence of operators presented by* (2)*. Then, we have*

$$\begin{aligned} G_m^\alpha(1;y) &= 1, \\ G_m^\alpha(e_1;y) &= y + \frac{2\alpha y^2}{m}, \\ G_m^\alpha(e_2;y) &= y^2 + \frac{4\alpha y^3 + y}{m} + \frac{4\alpha^2 y^4 + 4\alpha y^2}{m^2}, \\ *G_m^\alpha(e_p;y) &= y^p + O(m^{-1}). \end{aligned}$$

Lemma 1.3. *Let $H_m^{\alpha}(.;.)$ be the sequence of operators given by* (4) *and $e_i(t) = t^i$, $i \in \{0, 1, 2\}$. Then, one get*

$$\begin{aligned} H_m^{\alpha}(1;y) &= 1, \\ H_m^{\alpha}(e_1;y) &= y + \frac{1}{m}(2\alpha y^2 + \beta + 1), \\ H_m^{\alpha}(e_2;y) &= y^2 + \frac{1}{m}(4\alpha y^3 + 2\beta y + 4y) + \frac{1}{m^2}(4\alpha^2 y^4 + 2\alpha(2\beta+5)y^2 + \beta^2 + 3\beta + 2), \end{aligned}$$

for each $y \in \mathbb{R}_0^+$.

Proof. From the Eq. (4), we have

$$H_m^{\alpha}(f;y) = \sum_{s=0}^{\infty} P_{m,s}^{\alpha}(y) \int_0^{\infty} Q_s(t) f(t) dt.$$

Now, for $i = 0$,

$$\begin{aligned} H_m^{\alpha}(e_0;y) &= \sum_{s=0}^{\infty} P_{m,s}^{\alpha}(y) \frac{m^{s+\beta+1}}{\Gamma(s+\beta+1)} \int_0^{\infty} t^{s+\beta} e^{-mt}(1) dt \\ &= \sum_{s=0}^{\infty} P_{m,s}^{\alpha}(y) \frac{m^{s+\beta+1}}{\Gamma(s+\beta+1)} \frac{\Gamma(s+\beta+1)}{m^{s+\beta+1}} \\ &= 1. \end{aligned}$$

For $i = 1$,

$$\begin{aligned} H_m^{\alpha}(e_1;y) &= \sum_{s=0}^{\infty} P_{m,s}^{\alpha}(y) \frac{m^{s+\beta+1}}{\Gamma(s+\beta+1)} \int_0^{\infty} t^{s+\beta} e^{-mt}(t) dt \\ &= \sum_{s=0}^{\infty} P_{m,s}^{\alpha}(y) \frac{m^{s+\beta+1}}{\Gamma(s+\beta+1)} \int_0^{\infty} t^{s+\beta+1} e^{-mt} dt \\ &= \sum_{s=0}^{\infty} P_{m,s}^{\alpha}(y) \frac{m^{s+\beta+1}}{\Gamma(s+\beta+1)} \frac{\Gamma(s+\beta+2)}{m^{s+\beta+2}} \\ &= \sum_{s=0}^{\infty} P_{m,s}^{\alpha}(y) \frac{1}{m}(s+\beta+1) \\ &= G_m^{\alpha}(e_1;y) + \frac{\beta+1}{m} \\ &= y + \frac{1}{m}(2\alpha y^2 + \beta + 1). \end{aligned}$$

For $i = 2$,

$$\begin{aligned} H_m^{\alpha}(e_2;y) &= \sum_{s=0}^{\infty} P_{m,s}^{\alpha}(y) \frac{m^{s+\beta+1}}{\Gamma(s+\beta+1)} \int_0^{\infty} t^{s+\beta} e^{-mt}(t^2) dt \\ &= \sum_{s=0}^{\infty} P_{m,s}^{\alpha}(y) \frac{m^{s+\beta+1}}{\Gamma(s+\beta+1)} \int_0^{\infty} t^{s+\beta+2} e^{-mt} dt \end{aligned}$$

$$
\begin{aligned}
&= \sum_{s=0}^{\infty} P_{m,s}^{\alpha}(y)\frac{m^{s+\beta+1}}{\Gamma(s+\beta+1)}\frac{\Gamma(s+\beta+3)}{m^{s+\beta+3}}\\
&= \sum_{s=0}^{\infty} P_{m,s}^{\alpha}(y)\frac{1}{m^2}(s+\beta+2)(s+\beta+1)\\
&= \sum_{s=0}^{\infty} P_{m,s}^{\alpha}(y)\Big[\frac{s^2}{m^2}+\frac{s}{m^2}(2\beta+3)+\frac{1}{m^2}(\beta^2+3\beta+2)\Big]\\
&= G_m^{\alpha}(e_2;y)+\frac{2\beta+3}{m}G_m^{\alpha}(e_1;y)+\frac{\beta^2+3\beta+2}{m^2}G_m^{\alpha}(e_0;y)\\
&= y^2+\frac{1}{m}(4\alpha y^3+2\beta y+4y)+\frac{1}{m^2}(4\alpha^2y^4+2\alpha(2\beta+5)y^2+\beta^2+3\beta+2).
\end{aligned}
$$

□

Lemma 1.4. *Let $H_m^{\alpha}(.;.)$ be the operators given by* (4) *and central moments $\eta_i(t;y)=(t-y)^i$, $i\in\{0,1,2\}$. Then, one get*

$$
\begin{aligned}
H_m^{\alpha}(\eta_0;y) &= 1,\\
H_m^{\alpha}(\eta_1;y) &= \frac{1}{m}(2\alpha y^2+\beta+1),\\
H_m^{\alpha}(\eta_2;y) &= \frac{1}{m}\Big[4\alpha y^3-2\alpha y^2+2(\beta+2)y-\beta-1\Big]+\frac{1}{m^2}\Big[4\alpha^2y^4+2\alpha(2\beta+5)y^2+\beta^2+3\beta+2\Big],
\end{aligned}
$$

for each $y\in\mathbb{R}_0^+$.

Proof. Using the definition of $H_m^{\alpha}(.;.)$, we get for $i=0$, it is obvious that

$$H_m^{\alpha}(\eta_0;y)=1.$$

Now, we consider for $i=1$, that is $H_m^{\alpha}(\eta_1;y)$ as follows:

$$
\begin{aligned}
H_m^{\alpha}(\eta_1;y) &= \sum_{s=0}^{\infty} P_{m,s}^{\alpha}(y)\frac{m^{s+\beta+1}}{\Gamma(s+\beta+1)}\int_0^{\infty} t^{s+\beta}e^{-mt}(y-t)dt\\
&= y\sum_{s=0}^{\infty} P_{m,s}^{\alpha}(y)\frac{m^{s+\beta+1}}{\Gamma(s+\beta+1)}\int_0^{\infty} t^{s+\beta}e^{-mt}dt-\sum_{s=0}^{\infty} P_{m,s}^{\alpha}(y)\frac{m^{s+\beta+1}}{\Gamma(s+\beta+1)}\int_0^{\infty} t^{s+\beta+1}e^{-mt}dt\\
&= yH_m^{\alpha}(e_0;y)-H_m^{\alpha}(e_1;y)\\
&= \frac{1}{m}(2\alpha y^2+\beta+1).
\end{aligned}
$$

Further, for $i=2$, that is $H_m^{\alpha}(\eta_2;y)$ as follows:

$$
\begin{aligned}
H_m^{\alpha}(\eta_2;y) &= \sum_{s=0}^{\infty} P_{m,s}^{\alpha}(y)\frac{m^{s+\beta+1}}{\Gamma(s+\beta+1)}\int_0^{\infty} t^{s+\beta}e^{-mt}(y-t)^2dt\\
&= y^2H_m^{\alpha}(e_0;y)-2yH_m^{\alpha}(e_1;y)+H_m^{\alpha}(e_2;y)\\
&= y^2(1)-2y\Big[y+\frac{1}{m}(2\alpha y^2+\beta+1)\Big]+\Big[y^2+\frac{1}{m}(4\alpha y^3+2\beta y+4y)\\
&\quad+\frac{1}{m^2}(4\alpha^2y^4+2\alpha(2\beta+5)y^2+\beta^2+3\beta+2)\Big]\\
&= \frac{1}{m}\Big[4\alpha y^3-2\alpha y^2+2(\beta+2)y-\beta-1\Big]+\frac{1}{m^2}\Big[4\alpha^2y^4+2\alpha(2\beta+5)y^2+\beta^2+3\beta+2\Big].
\end{aligned}
$$

□

In subsequent sections, we deal with the rate of convergence and order of approximation for our operators. Fuhrer, direct results are discussed locally and globally in different spaces. In the last section, we proposed a bivariant version of these sequences of linear positive operators. Moreover, uniform convergence rate and approximation order are investigated.

2. Rate of convergence and order of approximation

There are numerous uses for Korovkin's theorem in mathematical science and other academic disciplines [3]. In order to apply Korovkin's theorem, we note that this section has to supply the approximation properties to new operators H_m^{α} by (4). Next, we may additionally provide the weighted space approximation for these operators here. The mathematical notation $C_{\beta}(\mathbb{R}^{+})$ is designated for the class of these functions that are bounded on the semi-axis and continuous. The supremum norm and $\mathbb{R}^{+}$ are defined on $C_{\beta}(\mathbb{R}^{+})$, and we write $\| f \|_{C_{\beta}(\mathbb{R}^{+})} = \sup_{y\in\mathbb{R}^{+}} \mid f(y) \mid$. In addition, we consider $\mathcal{H} = \{f : y \in \mathbb{R}^{+}, \frac{f(y)}{1+y^2}$ is converges when $y \to \infty\}$.

Theorem 2.1. *For any $f \in C[0,\infty) \cap \mathcal{H}$ and $y \in [0,\infty)$*

$$\lim_{\mu\to\infty} H_{\mu}^{\alpha}(f;y) = f(y)$$

is uniformly converges.

Proof. Using Korovkin's theorem, we confirm that $\lim_{\mu\to\infty} H_{\mu}^{\alpha}(t^{\mu};y) = y^{\mu}$ is uniformly on $[0,\infty)$ for $\mu = 0,1,2$. If $\mu \to \infty$, then $\frac{1}{\mu} \to 0$. By taking into account the Lemma 1.3, it is very easy to conclude that $\lim_{\mu\to\infty} H_{\mu}^{\alpha}(1;y) = 1$, $\lim_{\mu\to\infty} H_{\mu}^{\alpha}(t;y) = y$, and $\lim_{\mu\to\infty} H_{\mu}^{\alpha}(t^2;y) = y^2$. This observation completes the proof. □

Theorem 2.2. *Let $H_m^{\alpha}(.;.)$ be operators introduced in Eq. (4). Then, for all $g \in C_B[0,\infty)$, $H_m^{\alpha}(g;y) \rightrightarrows g$ on each closed and bounded subset of $[0,\infty)$ where $\rightrightarrows$ represents uniform convergent.*

Proof. In view of the Korovkin-type theorem, which regards the uniform convergence of the sequence of operators (linear and positive), it is adequate to see that

$$\lim_{m\to\infty} H_m^{\alpha}(t^i;y) = y^i, \; i \in \{0,1,2\},$$

uniformly on each closed and bounded subset of $[0,\infty)$. In the light of Lemma 1.3, this result can easily be proved. □

Definition 2.3. *Let g be a continuous function given on positive semi-axes. Then the modulus of smoothness is defined as:*

$$\omega(h;\delta) = \sup_{|u_1-u_2|\leq\delta} |h(u_1) - h(u_2)|, \qquad u_1, u_2 \in [0,\infty).$$

In view of result given by Shisha et al. [30], we can prove the approximation order via of Ditzian-Totik modulus of continuity.

Theorem 2.4. *For $g \in C_B[0,\infty)$ and the operators $H_m^{\alpha}(.;.)$ introduced in Eq. (4), one has*

$$|H_m^{\alpha}(g;y) - g(y)| \leq 2\omega(g;\delta),$$

where $\delta = \sqrt{H_m^{\alpha}((t-y)^2;y)}$.

Proof. We can easily prove above result in the light of modulus of continuity and result given by Shisha et al. [30]. □

In this section, we think back to some functional spaces and functional relations as $C_B[0,\infty)$ represents a space of bound and continuous real-valued functions. Now, Peetre's K-functional is given by

$$K_2(g,\delta) = \inf_{h\in C_B^2[0,\infty)} \left\{ ||g-h||_{C_B[0,\infty)} + \delta ||h''||_{C_B^2[0,\infty)} \right\},$$

where $C_B^2[0,\infty) = \{h \in C_B[0,\infty) : h', h'' \in C_B[0,\infty)\}$ provided with the norm $||g|| = \sup_{0\le y<\infty} |g(y)|$ and Ditzian-Totik modulus of smoothness of second order is given by

$$\omega_2(g;\sqrt{\delta}) = \sup_{0<k\le\sqrt{\delta}} \sup_{y\in[0,\infty)} |f(y+2k) - 2f(y+k) + f(y)|.$$

We recall a relation from DeVore and Lorentz ([8] page no. 177, Theorem 2.4) as:

$$K_2(g;\delta) \le C\omega_2(g;\sqrt{\delta}), \tag{5}$$

where C is a constant absolute. Now in view to prove the further result, we take the auxiliary operator as:

$$\widehat{H}_m^\alpha(g;y) = H_m^\alpha(g;y) + g(y) - g\left(y + \frac{2\alpha y^2 + \beta + 1}{m}\right) \tag{6}$$

where $g \in C_B[0,\infty)$, $y \ge 0$ and $n > 2$. From Eq. (6), one can yield

$$\widehat{H}_m^\alpha(1;y) = 1,\ \widehat{H}_m^\alpha(\eta_1;x) = 0 \text{ and } |\widehat{H}_m^\alpha(g;y)| \le 3||g||. \tag{7}$$

Lemma 2.5. *For $n > 2$ and $y \ge 0$, one yield*

$$|\widehat{H}_m^\alpha(h;y) - h(y)| \le \theta(y)||h''||,$$

where $h \in C_B^2[0,\infty)$ and $\theta(y) = \widehat{H}_m^\alpha(\eta_2;y) + (\widehat{H}_m^\alpha(\eta_1;y))^2$.

Proof. For $h \in C_B^2[0,\infty)$ and in view of relation Taylor's expansion, we get

$$h(t) = h(y) + (t-y)h'(y) + \int_y^t (t-v)h''(v)dv. \tag{8}$$

Now, applying the auxiliary operators $\widehat{H}_m^\alpha(.;.)$ given in Eq.(6) on both the sides in above Eq. (8), we get

$$\widehat{H}_m^\alpha(h;y) - h(y) = h'(y)\widehat{H}_m^\alpha(\eta_1;y) + \widehat{H}_m^\alpha\Big(\int_y^t (t-v)h''(v)dv; y\Big).$$

Using the Eqs. (7) and (8), we get

$$\begin{aligned}\widehat{H}_m^\alpha(h;y) - h(y) &= \widehat{H}_m^\alpha\Big(\int_y^t (t-v)h''(v)dv; y\Big)\\ &= H_m^\alpha\Big(\int_y^t (t-v)h''(v)dv; y\Big) - \int_y^{(y+\frac{2\alpha y^2+\beta+1}{m}} \left((y + \frac{2\alpha y^2+\beta+1}{m} - v\right)h''(v)dv,\end{aligned}$$

$$|\widehat{H}_m^{\alpha}(h;y)-h(y)|\leq\left|H_m^{\alpha}\Big(\int\limits_y^t(t-v)h''(v)dv;y\Big)\right|+\left|\int\limits_y^{(y+\frac{2\alpha y^2+\beta+1}{m}}\Big((y+\frac{2\alpha y^2+\beta+1}{m}-v\Big)h''(v)dv\right|. \tag{9}$$

Since,

$$\left|\int\limits_y^t(t-v)h''(v)dv\right|\leq(t-y)^2\parallel h''\parallel, \tag{10}$$

then

$$\left|\int\limits_y^{(y+\frac{2\alpha y^2+\beta+1}{m}}\Big((y+\frac{2\alpha y^2+\beta+1}{m}-v\Big)h''(v)dv\right|\leq\Big((y+\frac{2\alpha y^2+\beta+1}{m}-y\Big)^2\parallel h''\parallel. \tag{11}$$

In view of (9), (10) and (11), we find

$$\begin{aligned}|\widehat{H}_m^{\alpha}(h;y)-h(y)|&\leq\left\{\widehat{H}_m^{\alpha}(\eta_2;y)+\Big((y+\frac{2\alpha y^2+\beta+1}{m}-y\Big)^2\right\}\|h''\|\\&=\theta(y)\|h''\|.\end{aligned}$$

Which proves the required result. □

Theorem 2.6. *For $g\in C_B^2[0,\infty)$, there corresponds a non-negative $\tilde{C}>0$ as:*

$$|\,H_m^{\alpha}(g;y)-g(y)\,|\leq\tilde{C}\omega_2\Big(g;\sqrt{\theta(y)}\Big)+\omega(g;H_m^{\alpha}(\eta_1;y)),$$

where $\theta(y)$ is presented in Lemma 2.5.

Proof. For $h\in C_B^2[0,\infty)$ and $g\in C_B[0,\infty)$ and in view of $\widehat{H}_m^{\alpha}(.;.)$, one has

$$|H_m^{\alpha}(g;y)-g(y)|\leq|\widehat{H}_m^{\alpha}(g-h;y)|+|(g-h)(y)|+|\widehat{H}_m^{\alpha}(h;y)-h(y)|+\left|g\Big((y+\frac{2\alpha y^2+\beta+1}{m}\Big)-g(y)\right|.$$

In the light of Lemma 2.5 and inequalities in Eq. (7), one get

$$\begin{aligned}|H_m^{\alpha}(g;y)-g(y)|&\leq4\|g-h\|+|\widehat{H}_m^{\alpha}(h;y)-h(y)|+\left|g\Big((y+\frac{2\alpha y^2+\beta+1}{m}\Big)-g(y)\right|\\&\leq4\|g-h\|+\theta(y)\|h''\|+\omega\Big(g;H_m^{\alpha}((t-y);y)\Big).\end{aligned}$$

Using Eq. (5), we yield the desired result. □

Now, we discuss the next result in Lipschitz type space [25], which is given as:

$$Lip_{\tilde{M}}^{\zeta_1,\zeta_2}(\gamma):=\Big\{g\in C_B[0,\infty):|g(t)-g(y)|\leq\tilde{M}\frac{|t-y|^{\gamma}}{(t+\zeta_1y+\zeta_2y^2)^{\frac{\gamma}{2}}}:y,t\in(0,\infty)\Big\},$$

where $\tilde{M}>0$, $0<\gamma\leq1$ and $\zeta_1,\zeta_2>0$.

Theorem 2.7. *Let $H_m^\alpha(.;.)$ be the operator given by (4). Then, for $g \in Lip_M^{\zeta_1,\zeta_2}(\gamma)$, one has*

$$|H_m^\alpha(g;y) - g(y)| \leq \tilde{M}\left(\frac{\lambda(y)}{\zeta_1 y + \zeta_2 y^2}\right)^{\frac{\gamma}{2}}, \tag{12}$$

where $0 < \gamma \leq 1$, $\zeta_1, \zeta_2 \in (0,\infty)$ and $\lambda(y) = H_m^\alpha(\eta_2; y)$.

Proof. For $\gamma = 1$ and $y \geq 0$, one yield

$$\begin{aligned}|H_m^\alpha(g;y) - g(y)| &\leq H_m^\alpha(|g(t) - g(y)|; y)\\ &\leq \tilde{M} H_m^\alpha\left(\frac{|t-y|}{(t + \zeta_1 y + \zeta_2 y^2)^{\frac{1}{2}}}; y\right).\end{aligned}$$

Since $\dfrac{1}{t + \zeta_1 y + \zeta_2 y^2} < \dfrac{1}{\zeta_1 y + \zeta_2 y^2}$, for all $y \in (0,\infty)$, we yield

$$\begin{aligned}|H_m^\alpha(g;y) - g(y)| &\leq \frac{\tilde{M}}{(\zeta_1 y + \zeta_2 y^2)^{\frac{1}{2}}}(H_m^\alpha(\eta_2; y))^{\frac{1}{2}}\\ &\leq \tilde{M}\left(\frac{\lambda(y)}{\zeta_1 y + \zeta_2 y^2}\right)^{\frac{1}{2}},\end{aligned}$$

which implies that Theorem 2.7 works for $\gamma = 1$. Next, we consider for $\gamma \in (0,1)$ and in view of Hölder's inequality using $p = \frac{2}{\gamma}$ and $q = \frac{2}{2-\gamma}$, one get

$$\begin{aligned}|H_m^\alpha(g;y) - g(y)| &\leq \left(H_m^\alpha(|g(t) - g(y)|^{\frac{2}{\gamma}}; y)\right)^{\frac{\gamma}{2}}\\ &\leq \tilde{M}\left(H_m^\alpha\left(\frac{|t-y|^2}{(t + \zeta_1 y + \zeta_2 y^2)}; y\right)\right)^{\frac{\gamma}{2}}.\end{aligned}$$

Since $\dfrac{1}{t + \zeta_1 y + \zeta_2 y^2} < \dfrac{1}{\zeta_1 y + \zeta_2 y^2}$, for all $y \in (0,\infty)$, one get

$$|H_m^\alpha(g;y) - g(y)| \leq \tilde{M}\left(\frac{H_m^\alpha(|t-y|^2; y)}{\zeta_1 y + \zeta_2 y^2}\right)^{\frac{\gamma}{2}} \leq \tilde{M}\left(\frac{\lambda(y)}{\zeta_1 y + \zeta_2 y^2}\right)^{\frac{\gamma}{2}}.$$

Hence, we yield the required result. □

Next, we deal with the approximation locally given r^{th} order modulus of smoothness and then the Lipschitz-type maximal function which is introduced by Lenze [13] as:

$$\widetilde{\omega}_r(g;y) = \sup_{t \neq y, t \in (0,\infty)} \frac{|g(t) - g(y)|}{|t-y|^r},\ y \in [0,\infty) \text{ and } r \in (0,1]. \tag{13}$$

Theorem 2.8. *Let $g \in C_B[0,\infty)$ and $r \in (0,1]$. Then, for all $y \in [0,\infty)$, we have*

$$|H_m^\alpha(g;y) - g(y)| \leq \widetilde{\omega}_r(g;y)\big(\lambda(y)\big)^{\frac{r}{2}}.$$

Proof. It is noted that

$$|H_m^\alpha(g;y) - g(y)| \leq H_m^\alpha(|g(t) - g(y)|; y).$$

Using Eq. (13), one get

$$|H_m^\alpha(g;y) - g(y)| \leq \widetilde{\omega}_s(g;y) H_m^\alpha(|t-y|^r;y).$$

Then using Hölder's inequality with $p = \frac{2}{r}$ and $q = \frac{2}{2-r}$, we have

$$|H_m^\alpha(g;y) - g(y)| \leq \widetilde{\omega}_r(g;y)\Big(H_m^\alpha(|t-y|^2;y)\Big)^{\frac{r}{2}}.$$

Hence, we completes the proof. □

3. Approximation in weighted space

We employ the weighted space and recollect the weighted spaces in the domain of $\mathbb{R}^+$ by using the following mathematical equivalence to provide further approximation theorems in light of Korovkin's theorem.

$$\begin{aligned}
\mathcal{U}_\sigma(\mathbb{R}^+) &= \Big\{f \text{ such that } \mid f(y) \mid\leq M_f\sigma(y)\Big\}, \\
\mathcal{V}_\sigma(\mathbb{R}^+) &= \{f \text{ such that } f \in C[0,\infty) \cap \mathcal{U}_\sigma(\mathbb{R}^+)\}, \\
\mathcal{W}_\sigma^k(\mathbb{R}^+) &= \left\{f \text{ such that } f \in \mathcal{V}_\sigma(\mathbb{R}^+) \text{ and } \lim_{y\to\infty} \frac{f(y)}{\sigma(y)} = \nu(\text{a positive number})\right\},
\end{aligned}$$

with the weight function $\sigma(y) = 1 + y^2$ and the positive real numbers M_f depends on f and the norm is calculated as $\| f \|_\sigma = \sup_{y\geq 0} \frac{|f(y)|}{\sigma(y)}$.

Theorem 3.1. *For all $f \in \mathcal{W}_\sigma^k(\mathbb{R}^+)$ and $y \in [0,\infty)$ we get*

$$\lim_{\mu\to\infty} H_\mu^\alpha(f) - f \|_\sigma = 0$$

is uniformly converges.

Proof. Take $f(t) = t^\mu$ and $f \in \mathcal{W}_\sigma^k(\mathbb{R}^+)$, use the Korovkin's theorem to easily get that $H_\mu^\alpha(t^\mu;y) \to y^\mu$ is uniformly, if μ approaches to ∞. Thus the Lemma 1.3, gives $H_\mu^\alpha(1;y) = 1$, therefore

$$\lim_{\mu\to\infty} \left\|H_\mu^\alpha(1;y) - 1\right\|_\sigma = 0, \tag{14}$$

In similar way,

$$\begin{aligned}
\left\|H_\mu^\alpha(t;y) - y\right\|_\sigma &= \sup_{y\in[0,\infty)} \frac{\left|H_\mu^\alpha(t;y) - y\right|}{1+y^2} \\
&= \sup_{y\in[0,\infty)} \frac{\left|\frac{1}{\mu}(2\alpha y^2 + \beta + 1)\right|}{1+y^2} \\
&= \frac{2\alpha}{\mu} \sup_{y\in[0,\infty)} \frac{y^2}{1+y^2} + \frac{\beta+1}{\mu} \sup_{y\in[0,\infty)} \frac{1}{1+y^2}.
\end{aligned}$$

Then clearly $\frac{1}{\mu} \to 0$ as $\mu \to \infty$ implies that

$$\lim_{\mu\to\infty} \left\|H_\mu^\alpha(t;y) - y\right\|_\sigma = 0, \tag{15}$$

$$\left\|H_\mu^\alpha\left(t^2;y\right) - y^2\right\|_\sigma = \sup_{y\in[0,\infty)} \frac{\left|H_\mu^\alpha(t^2;y) - y^2\right|}{1+y^2}$$

$$
\begin{aligned}
&= \frac{1}{\mu} \sup_{y\in[0,\infty)} \frac{\left|4\alpha y^3 + 2\beta y + 4y\right|}{1+y^2} \\
&+ \frac{1}{\mu^2} \sup_{y\in[0,\infty)} \frac{\left|4\alpha^2 y^4 + 2\alpha(2\beta+5)y^2\right|}{1+y^2} \\
&+ \frac{1}{\mu^2} \sup_{y\in[0,\infty)} \frac{\left|\beta^2 + 3\beta + 2\right|}{1+y^2} \\
&\leq \frac{1}{\mu}\left|2\beta+4\right| + \frac{1}{\mu^2}\left|4\alpha(2\beta+5)\right| \\
&\leq \frac{1}{\mu}\max\left(|\alpha|, \left|\beta\right|\right).
\end{aligned}
$$

Thus, for $\frac{1}{\mu} \to 0$ we have

$$\lim_{\mu\to\infty} \left\| H_\mu^\alpha\left(t^2; y\right) - y^2 \right\|_\sigma = 0, \tag{16}$$

which complete the required proof. □

Theorem 3.2. *For all $\varphi \in C^\tau[0,\infty)$, $\tau \in \mathbb{N}$ and any $\nu \in [0,\infty)$ the operators H_μ^α have*

$$\lim_{\mu\to\infty} \sup_{y\in[0,\infty)} \frac{|H_\mu^\alpha(\varphi; y) - \varphi(y)|}{(1+y^2)^{1+\nu}} = 0,$$

where $C^\tau[0,\infty)$ be the set of τ-th order continuous functions.

Proof. Taking into account the inequality $|\varphi(y)| \leq \|\varphi\|_\sigma(1+y^2)$ then for any positive number y_0 it is easy to get that

$$
\begin{aligned}
\lim_{\mu\to\infty} \sup_{y\in[0,\infty)} \frac{|H_\mu^\alpha(g; y) - g(y)|}{(1+y^2)^{1+\nu}} &\leq \sup_{y\leq y_0} \frac{|H_\mu^\alpha(\varphi; y) - \varphi(y)|}{(1+y^2)^{1+\nu}} + \sup_{y\geq y_0} \frac{|H_\mu^\alpha(\varphi; y) - \varphi(y)|}{(1+y^2)^{1+\nu}} \\
&\leq \|H_\mu^\alpha(\varphi) - \varphi(y)\|_\sigma C[0, y_0] \\
&+ \|g\|_\sigma \sup_{y\geq y_0} \frac{|(1+t^2; y) - \varphi(y)|}{(1+y^2)^{1+\nu}} + \sup_{y\geq y_0} \frac{|\varphi(y)|}{(1+y^2)^{1+\nu}} \\
&= C_1 + C_2 + C_3 (\text{suppose}).
\end{aligned}
$$

Thus

$$C_3 = \sup_{y\geq y_0} \frac{|\varphi(y)|}{(1+y^2)^{1+\nu}} \leq \sup_{y\geq y_0} \frac{\|\varphi\|_\sigma (1+y^2)}{(1+y^2)^{1+\nu}} \leq \frac{\|\varphi\|_\sigma}{(1+y_0^2)^\nu}. \tag{17}$$

From the Lemma 1.3, it have

$$\lim_{\mu\to\infty} \sup_{y\geq y_0} \frac{H_\mu^\alpha(1+t^2; y)}{1+y^2} = 1.$$

Now, we suppose that for any given $\epsilon^* > 0$, there exists a new positive integer $\mu_1 \in \mathbb{N}$ and $\mu \geq \mu_1$ satisfying

$$\sup_{y\geq y_0} \frac{H_\mu^\alpha(1+t^2; y)}{1+y^2} \leq \frac{(1+y_0^2)^\nu}{\|\varphi\|_\sigma} \frac{\epsilon^*}{3} + 1,$$

and for $\mu \geq \mu_1$

$$C_2 = \|g\|_\sigma \sup_{y \geq y_0} \frac{H_\mu^\alpha(1+t^2;y)}{(1+y^2)^{1+\nu}} \leq \frac{\|\varphi\|_\sigma}{(1+y_0^2)^\nu} + \frac{\epsilon^*}{3}. \tag{18}$$

Taking in account of (17) and (18), we see

$$C_2 + C_3 \leq \frac{2\|\varphi\|_\sigma}{(1+y_0^2)^\nu} + \frac{\epsilon^*}{3}.$$

Choose very large y_0, such that $\frac{\|\varphi\|_\sigma}{(1+y_0^2)^\nu} \leq \frac{\epsilon^*}{6}$, then we get

$$C_2 + C_3 \leq \frac{2\epsilon^*}{3}, \quad \text{for } \mu \geq \mu_1. \tag{19}$$

Similarly, for $\mu \geq \mu_2$ we have

$$C_1 = \|H_\mu^\alpha(\varphi) - \varphi(y)\|_{C[0,y_0]} \leq \frac{\epsilon^*}{3}. \tag{20}$$

Lastly, we take $\mu_3 = \max(\mu_1, \mu_2)$ and combining (19) and (20), we get

$$\sup_{y\in[0,\infty)} \frac{|H_\mu^\alpha(\varphi;y) - \varphi(y)|}{(1+y^2)^{1+\nu}} < \epsilon^*.$$

Which gives the desired proof. □

4. Conclusion and observations

In this paper, we introduce a sequence of linear positive operators in integral form via Hermite Polynomial to approximate the functions that belong to Lebesgue measurable space named as Szász-Gamma type operators defined by (4). Further, we calculate some estimates that are used to prove the convergence rate and approximation order. Moreover, the various approximation results, e.g., local approximation results. In the last section, we present two two-dimensional versions of these sequences of positive linear operators. Moreover, their order of approximation and rate of convergence are discussed.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

Authors' contributions
All authors contributed equally to writing this paper. All authors read and approved the manuscript.

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