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A new discrete operator and convergence

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Abstract. The present article deals with approximation of a new discretely defined operator based on hypergeometric functions. We study and establish some direct results exponential functions in terms of weighted moduli of continuity.

1. Introduction

Hypergeometric functions, which constitute a significant part of special functions, appear in physics, engineering, probability and especially mathematics. Also, they can be found in the structure of some operators.

The new operators have attracted major attention from researchers in constructive approximation theory. Several operators are established via producing functions, several exponential type operators are established via differential equations. Here, we will consider the moment producing function of an composition operator.

In our motivation paper, Abel-Gupta [1] proposed for $x \geq 0$ a discrete operator as follows:

$$(C_{m,n}f)(x) = \sum_{k=0}^{\infty} c_{k,m,n}(x) f\left(\frac{k}{n}\right), \tag{1}$$

where

$$c_{k,m,n}(x) = \frac{mn^k [nx+m+n]^{-m}}{(m+n)^{k-m+1}} \, {}_2F_1\left(-k, m; 1; \frac{-mx}{m+n+nx}\right)$$

and ${}_2F_1$ represents the hypergeometric function.

In the special case $m = n$, we get the approximation operator given by

$$(C_{n,n}f)(x) = \sum_{k=0}^{\infty} c_{k,n,n}(x) f\left(\frac{k}{n}\right), \tag{2}$$

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where

$$c_{k,n,n}(x) = \frac{1}{2^{k-n+1}(2+x)^n} \, {}_2F_1\left(-k, n; 1; \frac{-x}{2+x}\right).$$

This new operator is generated by composition of the Baskakov-Szász and the Szász-Mirakyan operators in that order.

The moment producing function of these operators is expressed by

$$\left(C_{n,n}\exp_A\right)(x) = \left(2 - e^{A/n}\right)^{-1}\left(1 + x\frac{1 - e^{A/n}}{2 - e^{A/n}}\right)^{-n}, \tag{3}$$

provided that $A < \log\left(\frac{2+x}{1+x}\right)^n$ and $\exp_A(t) = e^{At}$. In the recent years; Aral et al. [6], Deniz et al. [9], Gupta and Aral [11], Gupta and Tachev [15], Ozsarac and Acar [17] studied approximation of certain operators preserving exponential functions. Moreover, Gupta and Gupta [12], Gupta et al. [13] and Gupta and Srivastava [14] investigated moment generating functions and convergence properties of some composition operators. Aral presented different weighted moduli of continuity in [4] and [5] (also see [10]).

The present paper is extension of such results and here we study some estimates of convergence for $C_{n,n}$. Firstly, we calculate the moments, and then acquire a quantitative estimate. Later, we give the central moments and express a quantitative form of Voronovskaya type formula. Moreover, we state weighted uniform approximation by the composition operator $C_{n,n}$ utilizing a weighted Korovkin-type theorem. Finally, utilizing suitable moduli of continuity defined on exponential weighted space, we have the rate of convergence of $C_{n,n}$.

The important result obtained here is that its easy to have more and more compositions of Szász-Mirakyan operators to $C_{n,n}$. Also, moment producing functions can be evaluated but even if we increase compositions further and further the error increases. So, its our claim here that the composition operator $C_{n,n}$ gives better approximation than the further compositions.

2. Main Results

The class of continuous and real functions f denoted by $C^*[0,\infty)$, have finite limit, for x tending to ∞. In [7] and [16], some interesting results have been studied for a sequence of operators L_n:

Theorem 2.1. *[16] For $L_n : C^*[0,\infty) \to C^*[0,\infty)$, if we denote the norms $\left\|L_n\left(\exp_s\right) - \exp_s\right\|_{[0,\infty)}$, $s = 0, -1, -2$ as α_n, β_n and γ_n respectively, which approach to zero as $n \to \infty$, then*

$$\left\|L_n f - f\right\|_{[0,\infty)} \leq \alpha_n \left\|f\right\|_{[0,\infty)} + (2 + \alpha_n)\,\omega^*\left(f, (\alpha_n + \gamma_n + 2\beta_n)^{1/2}\right),$$

where $\omega^*(f,\delta) := \sup\limits_{\substack{|e^{-t}-e^{-u}|\leq\delta \\ t,u>0}} \left|f(u) - f(t)\right|.$

Lemma 2.2. *We obtain the following moments:*

1. $\left(C_{n,n}\exp_0\right)(y) = 1,$
2. $\left(C_{n,n}\exp_{-1}\right)(y) = \left(2 - e^{-1/n}\right)^{-1}\left(1 + y\frac{1-e^{-1/n}}{2-e^{-1/n}}\right)^{-n},$
3. $\left(C_{n,n}\exp_{-2}\right)(y) = \left(2 - e^{-2/n}\right)^{-1}\left(1 + y\frac{1-e^{-2/n}}{2-e^{-2/n}}\right)^{-n}.$

Proof. The equalities follow from (3). □

Theorem 2.3. *For $f \in C^*[0,\infty)$, there holds*

$$\left\|C_{n,n}f - f\right\|_{[0,\infty)} \leq 2\omega^*\left(f, (\gamma_n + 2\beta_n)^{1/2}\right),$$

where

$$\beta_n = \left\|C_{n,n}\exp_{-1} - \exp_{-1}\right\|_{[0,\infty)} \to 0, \quad n \to \infty$$

and

$$\gamma_n = \left\|C_{n,n}\exp_{-2} - \exp_{-2}\right\|_{[0,\infty)} \to 0, \quad n \to \infty.$$

Proof. By Lemma 2.2, the operators $C_{n,n}$ preserve constants, so $\alpha_n = 0$. We have to compute β_n and γ_n. By the software Maple, we have

$$\left(C_{n,n}\exp_{-1}\right)(x) = \exp_{-1}(x) + \frac{e^{-x}\left(x^2+3x-2\right)}{2n} + \frac{e^{-x}\left(3x^4+10x^3-21x^2-88x+36\right)}{24n^2} + O\left(n^{-3}\right).$$

Since

$$\begin{aligned}\sup_{x\geq 0} e^{-x} &= 1, \quad \sup_{x\geq 0} e^{-x}x = e^{-1},\\ \sup_{x\geq 0} x^2e^{-x} &= 4e^{-2}, \quad \sup_{x\geq 0} x^3e^{-x} = 27e^{-3}, \quad \sup_{x\geq 0} x^4e^{-x} = 256e^{-4},\end{aligned}$$

we obtain

$$\begin{aligned}\beta_n &= \sup_{x\geq 0}\left|\left(C_{n,n}\exp_{-1}\right)(x) - \exp_{-1}(x)\right|\\ &\leq \frac{1}{n}\left(\frac{2}{e^2}+\frac{3}{2e}+1\right) + \frac{1}{n^2}\left(\frac{32}{e^4}+\frac{45}{4e^3}+\frac{7}{2e^2}+\frac{11}{3e}+\frac{3}{2}\right) + O\left(n^{-3}\right)\\ &\leq O\left(n^{-1}\right).\end{aligned}$$

Similarly,

$$\left(C_{n,n}\exp_{-2}\right)(x) = \exp_{-2}(x) + \frac{2e^{-2x}\left(x^2+3x-1\right)}{n} + \frac{2e^{-2x}\left(3x^4+14x^3+3x^2-44x+9\right)}{3n^2} + O\left(n^{-3}\right).$$

Since

$$\begin{aligned}\sup_{x\geq 0} e^{-2x} &= 1, \quad \sup_{x\geq 0} e^{-2x}x = 0.5e^{-1},\\ \sup_{x\geq 0} x^2e^{-2x} &= e^{-2}, \quad \sup_{x\geq 0} x^3e^{-2x} = \frac{27}{8}e^{-3}, \quad \sup_{x\geq 0} x^4e^{-2x} = 16e^{-4},\end{aligned}$$

we get

$$\begin{aligned}\gamma_n &= \sup_{x\geq 0}\left|\left(C_{n,n}\exp_{-2}\right)(x) - \exp_{-2}(x)\right|\\ &\leq \frac{1}{n}\left(\frac{2}{e^2}+\frac{3}{e}+2\right) + \frac{1}{n^2}\left(\frac{32}{e^4}+\frac{63}{2e^3}+\frac{2}{e^2}+\frac{44}{3e}+6\right) + O\left(n^{-3}\right)\\ &\leq O\left(n^{-1}\right).\end{aligned}$$

Using Theorem 2.1, the proof follows. □

Lemma 2.4. *Recalling that $\psi_x(t) = t - x$, the first few central moments of $C_{n,n}$ are given by*

$$\left(C_{n,n}\psi_x^0\right)(x) = 1,$$

$$\left(C_{n,n}\psi_x^1\right)(x) = \frac{1}{n},$$

$$\left(C_{n,n}\psi_x^2\right)(x) = \frac{3}{n^2} + \frac{x(x+3)}{n},$$

$$\left(C_{n,n}\psi_x^3\right)(x) = \frac{13}{n^3} + \frac{2x\left(x^2+6x+11\right)}{n^2}$$

and

$$\left(C_{n,n}\psi_x^4\right)(x) = \frac{75}{n^4} + \frac{6x^4+44x^3+133x^2+181x}{n^3} + \frac{3x^2(x+3)^2}{n^2}.$$

Also, it follows

$$\lim_{n\to\infty} n\left(C_{n,n}\psi_x^1\right)(x) = 1$$

and

$$\lim_{n\to\infty} n\left(C_{n,n}\psi_x^2\right)(x) = x^2+3x.$$

Theorem 2.5. *Let f and its second derivative belong to the class $C^*[0,\infty)$, then for any $x \geq 0$, there follows*

$$\begin{aligned}&\left|n\left[\left(C_{n,n}f\right)(x) - f(x)\right] - f'(x) - (0.5)\left(x^2+3x\right)f''(x)\right|\\ \leq\ & \frac{3\left|f''(x)\right|}{2n} + 2\left[\frac{3}{n} + \left(3x+x^2\right) + a_n(x)\right]\omega^*\left(f'', n^{-1/2}\right),\end{aligned}$$

where

$$a_n(x) = n^2\left[\left(C_{n,n}\left(\exp_{-1}(x) - \exp_{-1}(t)\right)^4\right)(x)\left(C_{n,n}\psi_x^4\right)(x)\right]^{1/2}.$$

Proof. Applying Taylor's formula to $C_{n,n}$, we have

$$\left|\left(C_{n,n}f\right)(x) - f(x) - \left(C_{n,n}\psi_x^1\right)(x)f'(x) - \frac{1}{2}\left(C_{n,n}\psi_x^2\right)(x)f''(x)\right| \leq \left|\left(C_{n,n}h_{t,x}(t-x)^2\right)(x)\right|,$$

where $h_{t,x} = \frac{f''(\eta)-f''(x)}{2}$ and $x < \eta < t$. Using Lemma 2.4, we immediately have

$$\begin{aligned}&\left|n\left[\left(C_{n,n}f\right)(x) - f(x)\right] - f'(x) - \frac{x^2+3x}{2}f''(x)\right|\\ \leq\ & \left|n\left(C_{n,n}\psi_x^1\right)(x) - 1\right|\left|f'(x)\right| + \frac{1}{2}\left|n\left(C_{n,n}\psi_x^2\right)(x) - \left(x^2+3x\right)\right|\left|f''(x)\right| + \left|n\left(C_{n,n}h_{t,x}(t-x)^2\right)(x)\right|\\ \leq\ & \frac{3}{2n}\left|f''(x)\right| + \left|n\left(C_{n,n}h_{t,x}(t-x)^2\right)(x)\right|.\end{aligned}$$

Next by the property used in [3, (3.1)], we can write

$$h_{t,x} \leq 2\left(1 + \frac{\left(\exp_{-1}(x) - \exp_{-1}(t)\right)^2}{\delta^2}\right)\omega^*\left(f'',\delta\right), \qquad \delta > 0.$$

Using above and Cauchy–Schwarz inequality and selecting $\delta = n^{-1/2}$, we have

$$\begin{aligned} n\left(C_{n,n}\left|h_{t,x}\right|(t-x)^2\right)(x) &\leq 2n\omega^*\left(f'',\delta\right)\left(C_{n,n}\psi_x^2\right)(x) \\ &+\frac{2n}{\delta^2}\omega^*\left(f'',\delta\right)\left[\left(C_{n,n}\left(\exp_{-1}(x)-\exp_{-1}(t)\right)^4\right)(x)\right]^{1/2}\left[\left(C_{n,n}\psi_x^4\right)(x)\right]^{1/2} \\ &= 2\left[\frac{3}{n}+x(x+3)+a_n(x)\right]\omega^*\left(f'',n^{-1/2}\right), \end{aligned}$$

where

$$a_n(x) = n^2\left[\left(C_{n,n}\left(\exp_{-1}(x)-\exp_{-1}(t)\right)^4\right)(x)\left(C_{n,n}\psi_x^4\right)(x)\right]^{1/2}.$$

This concludes the proof of theorem. □

Remark 2.6. *By simple computation following limits hold:*

1. $\lim_{n\to\infty} n^2\left(C_{n,n}\psi_x^4\right)(x) = 3x^2(x+3)^2,$
2. $\lim_{n\to\infty} n^2\left(C_{n,n}\left(\exp_{-1}(x)-\exp_{-1}(t)\right)^4\right)(x) = 3x^2(x+3)^2e^{-4x}.$

Corollary 2.7. *Suppose f and its second derivative belong to $C^*[0,\infty)$, then immediately one obtains*

$$\lim_{n\to\infty} n\left[\left(C_{n,n}f\right)(x)-f(x)\right] = f'(x)+(0.5)x(x+3)f''(x).$$

3. Weighted Approximation

Let us take $B_{\exp_k} := \left\{f : \left|f(x)\right| \leq M\exp_k(x),\ M>0,\ x\in[0,\infty)\right\}$ and $C_{\exp_k} = \left\{f : f\in B_{\exp_k},\ f \text{ continuous}\right\}$. Also,

$$\|f\|_{\exp_k} = \sup_{x\geq 0}\frac{|f(x)|}{\exp_k(x)},$$

(see [8]).

1. An operator A_n, which is positive and linear and defined on $C_{\exp_1}$, maps $C_{\exp_1}$ into $B_{\exp_2}$ iff

   $$A_n\exp_1 \in B_{\exp_2}.$$

2. Also, we have

   $$\|A_n\|_{C_{\exp_1}\to B_{\exp_2}} = \|A_n\exp_1\|_{\exp_2}.$$

Theorem 3.1. *If $A_n : C_{\exp_1}(\mathbb{R}) \to B_{\exp_2}(\mathbb{R})$ for $\nu = 0,1,2$ satisfies*

$$\lim_{n\to\infty}\|A_n(\exp_1^\nu)-\exp_1^\nu\|_{\exp_2} = 0,$$

then

$$\lim_{n\to\infty}\left\|A_n(f)-f\right\|_{\exp_2} = 0$$

for all $f\in C_{\exp_1}$.

As an extension of the studies [2], we consider $B_{\exp_A}$, $A > 0$, the class of functions satisfying $\left|f(x)\right| \leq M \exp_A(x)$, where $M > 0$. Further, $C_{\exp A}$ is subclass of $B_{\exp_A}$ having continuous functions with the norm

$$\left\|f\right\|_{\exp_A} = \sup_{x\geq 0} \frac{\left|f(x)\right|}{\exp_A(x)}.$$

Let $C^k_{\exp_A}(\mathbb{R}^+)$ be the subspace of all functions $f \in C_{\exp_A}(\mathbb{R}^+)$ such that $\lim_{x\to\infty} \frac{|f(x)|}{\exp_A(x)} = k$, where the constant $k > 0$ and suppose $C^{(r)}_{\exp_A}(\mathbb{R}^+)$ be the class of each functions $f \in C_{\exp_A}(\mathbb{R}^+)$ such that $f^{(m)} \in C_{\exp_A}(\mathbb{R}^+)$ for $m = 1, 2, \ldots, r$.

Theorem 3.2. *Suppose* $0 < A < \frac{B}{2}$. *If* $f \in C_{\exp_A}(\mathbb{R}^+)$, *then we have*

$$\lim_{n\to\infty} \left\|C_{n,n}f - f\right\|_{\exp_B} = 0.$$

Proof. First, let's show that $C_{n,n}$ is a positive linear operator from space $C_{\exp_A}$ into $B_{\exp_B}$. The formula (3) indicates that $C_{n,n}$ is a positive and linear operator. Since

$$\begin{aligned}
\left\|C_{n,n}\exp_A\right\|_{\exp_B} &= \sup_{x\geq 0} \frac{C_{n,n}\exp_A(x)}{\exp_B(x)} \\
&= \sup_{x\geq 0} \frac{\left(2 - e^{A/n}\right)^{-1}\left(1 + x\frac{1-e^{A/n}}{2-e^{A/n}}\right)^{-n}}{\exp_B(x)} \\
&\leq \left(2 - e^{A/n}\right)^{-1},
\end{aligned}$$

the operator $C_{n,n}$ maps the space $C_{\exp_A}$ into $B_{\exp_B}$. Now, as a application of Theorem 3.1, if we show

$$\lim_{n\to\infty} \left\|C_{n,n}\left(\exp_A^{\lambda}\right) - \exp_A^{\lambda}\right\|_{\exp_B} = 0, \ \ \lambda = 0, 1 \text{ and } 2,$$

then the proof is immediate. The result when $\lambda = 0$, follows from Lemma 2.2. Next, when $\lambda = 1$ from 3, we get

$$\begin{aligned}
\left\|C_{n,n}\left(\exp_A\right) - \exp_A\right\|_{\exp_B} &= \sup_{x\geq 0} \frac{\left|C_{n,n}\exp_A(x) - \exp_A(x)\right|}{\exp_B(x)} \\
&= \sup_{x\geq 0} \frac{e^{Ax} - \left(2 - e^{A/n}\right)^{-1}\left(1 + x\frac{1-e^{A/n}}{2-e^{A/n}}\right)^{-n}}{e^{Bx}} \\
&= \sup_{x\geq 0} e^{(A-B)x}\left(1 - e^{-Ax}\left(2 - e^{A/n}\right)^{-1}\left(1 + x\frac{1 - e^{A/n}}{2 - e^{A/n}}\right)^{-n}\right) \\
&\leq \sup_{x\geq 0}\left(1 - e^{-Ax}\left(2 - e^{A/n}\right)^{-1}\left(1 + x\frac{1 - e^{A/n}}{2 - e^{A/n}}\right)^{-n}\right).
\end{aligned}$$

The expression takes the supremum value at $x = 0$ for n large enough. Thus, we have

$$\lim_{n\to\infty} \left\|C_{n,n}\left(\exp_A\right) - \exp_A\right\|_{\exp_B} = \lim_{n\to\infty}\left(1 - \left(2 - e^{A/n}\right)^{-1}\right) = 0.$$

Similarly, for $\lambda = 2$, we get

$$\begin{aligned}
\left\|C_{n,n}\left(\exp_A^2\right)-\exp_A^2\right\|_{\rho_B} &= \sup_{x\geq 0}\frac{\left|C_{n,n}\exp_A^2(x)-\exp_A^2(x)\right|}{\exp_B(x)} \\
&= \sup_{x\geq 0} e^{-Bx}\left|\left(2-e^{2A/n}\right)^{-1}\left(1+x\frac{1-e^{2A/n}}{2-e^{2A/n}}\right)^{-n}-e^{2Ax}\right| \\
&= \sup_{x\geq 0}\frac{e^{2Ax}-\left(2-e^{2A/n}\right)^{-1}\left(1+x\frac{1-e^{2A/n}}{2-e^{2A/n}}\right)^{-n}}{e^{Bx}} \\
&= \sup_{x\geq 0} e^{(2A-B)x}\left(1-e^{-2Ax}\left(2-e^{2A/n}\right)^{-1}\left(1+x\frac{1-e^{2A/n}}{2-e^{2A/n}}\right)^{-n}\right) \\
&\leq \sup_{x\geq 0}\left(1-e^{-2Ax}\left(2-e^{2A/n}\right)^{-1}\left(1+x\frac{1-e^{2A/n}}{2-e^{2A/n}}\right)^{-n}\right).
\end{aligned}$$

In a similar way, we observe that

$$\lim_{n\to\infty}\left\|C_{n,n}\left(\exp_A^2\right)-\exp_A^2\right\|_{\exp_B}=0.$$

Therefore, the desired result follows. □

4. Order of Convergence

This section deals with the estimates on convergence with regard to weighted moduli of continuity for the functions $f \in C^k_{\exp_A}(\mathbb{R}^+)$. We take into account here the weighted moduli of continuity defined by

$$\widetilde{\omega}(f;\delta)=\sup_{\substack{|t-x|\leq\delta\\ x\geq 0}}\frac{|f(t)-f(x)|}{\exp_A(t)+\exp_A(x)},\quad \delta\geq 0 \tag{4}$$

for $f \in C^k_{\exp_A}(\mathbb{R}^+)$ (see [2]). This moduli of continuity satisfy:

1. For $f \in C^k_{\exp_A}(\mathbb{R}^+)$, we have $\lim_{\delta\to 0}\widetilde{\omega}(f;\delta)=0$.
2. For any $\xi > 0$ and $f \in C^k_{\exp_A}(\mathbb{R}^+)$, we have

$$\widetilde{\omega}(f;\xi\delta)\leq 2(1+\xi)\,\widetilde{\omega}(f;\delta). \tag{5}$$

Theorem 4.1. *Let $0 < A < B$, then for $f \in C^k_{\exp_A}(\mathbb{R}^+)$ one has*

$$\left\|C_{n,n}f-f\right\|_{\exp_B}\leq 2\left(2+\left(2-e^{A/n}\right)^{-1}+\sqrt{(2-e^{2A/n})^{-1}}\right)\widetilde{\omega}\left(f;\sqrt{v_n}\right), \tag{6}$$

where $\nu_n = \frac{3}{n^2}+\frac{e^{-2}}{(B-A)^2 n}+\frac{3e^{-1}}{2(B-A)n}$.

Proof. By using (5) for $\lambda > 0$ in which we choose $\lambda = \frac{|t-x|}{\delta}$, $\delta > 0$, Cauchy-Schwarz inequality, (3) and application of Lemmas 2.2 and 2.4, for each $x \geq 0$ leads us to

$$\begin{aligned}
&\left|C_{n,n}f(x) - f(x)\right| \\
\leq\ & 2\widetilde{\omega}(f;\delta)\left\{\left(C_{n,n}\exp_A\right)(x) + e^{Ax} + \frac{e^{Ax}}{\delta}\left(\left(C_{n,n}\psi_x^2\right)(x)\right)^{1/2} + \frac{1}{\delta}\left(\left(C_{n,n}\exp_{2A}\right)(x)\right)^{1/2}\left(\left(C_{n,n}\psi_x^2\right)(x)\right)^{1/2}\right\} \\
=\ & 2\widetilde{\omega}(f;\delta)\left\{\left(C_{n,n}\exp_A\right)(x) + e^{Ax} + \frac{\left(\left(C_{n,n}\psi_x^2\right)(x)\right)^{1/2}}{\delta}\left(e^{Ax} + \left(\left(C_{n,n}\exp_{2A}\right)(x)\right)^{1/2}\right)\right\} \\
=\ & 2\widetilde{\omega}(f;\delta)\,e^{Bx}\left\{\frac{\left(C_{n,n}\exp_A\right)(x)}{\exp_B(x)} + \exp_{A-B}(x) + \frac{\left(\left(C_{n,n}\psi_x^2\right)(x)\right)^{1/2}}{\delta\exp_{B-A}(x)}\left(1 + \left(\frac{\left(C_{n,n}\exp_{2A}\right)(x)}{\exp_{2A}(x)}\right)^{1/2}\right)\right\}.
\end{aligned}$$

We can write

$$\begin{aligned}
\sup_{x\geq 0}\frac{\left(C_{n,n}\psi_x^2\right)(x)}{\exp_{2(B-A)}(x)} &= \sup_{x\geq 0}\frac{\frac{3}{n^2} + \frac{x(x+3)}{n}}{e^{2(B-A)x}} \\
&\leq \frac{3}{n^2} + \frac{e^{-2}}{(B-A)^2 n} + \frac{3e^{-1}}{2(B-A)n} := v_n,
\end{aligned}$$

$$\begin{aligned}
\sup_{x\geq 0}\left(\frac{\left(C_{n,n}\exp_{2A}\right)(x)}{\exp_{2A}(x)}\right)^{1/2} &= \sup_{x\geq 0}\left(\frac{\left(2 - e^{2A/n}\right)^{-1}\left(1 + x\frac{1-e^{2A/n}}{2-e^{2A/n}}\right)^{-n}}{\exp_{2A}(x)}\right)^{1/2} \\
&\leq \sqrt{(2 - e^{2A/n})^{-1}}
\end{aligned}$$

and

$$\begin{aligned}
\sup_{x\geq 0}\frac{\left(C_{n,n}\exp_A\right)(x)}{\exp_B(x)} &= \sup_{x\geq 0}\frac{\left(2 - e^{A/n}\right)^{-1}\left(1 + x\frac{1-e^{A/n}}{2-e^{A/n}}\right)^{-n}}{\exp_B(x)} \\
&\leq \left(2 - e^{A/n}\right)^{-1},
\end{aligned}$$

selecting $\delta = v_n^{1/2}$, the desired result follows. □

5. Further Studies

We may further extend the studies by considering another new operator $D_n := C_{n,n} \circ S_n$, where S_n is Szász-Mirakyan operator.

$$(D_n f)(x) = \frac{2^{n-1}}{(2+x)^n}\sum_{v=0}^{\infty}\frac{1}{v!}f\left(\frac{v}{n}\right)\sum_{k=0}^{\infty}\frac{k^v}{(2e)^k}\ {}_2F_1\left(n, -k; 1; \frac{-x}{2+x}\right).$$

Also, we have

$$\left(D_n\exp_A\right)(x) = \left(2 - e^{\left(e^{A/n}-1\right)}\right)^{-1}\left(1 + x\frac{1 - e^{\left(e^{A/n}-1\right)}}{2 - e^{\left(e^{A/n}-1\right)}}\right)^{-n}.$$

Under the conditions in Theorem 2.5, we have

$$\begin{aligned}
&\left|n\left[(D_n f)(x) - f(x)\right] - f'(x) - \frac{x^2+4x}{2}f''(x)\right| \\
\leq\ & \frac{2\left|f''(x)\right|}{n} + 2\left[\frac{4}{n} + x(x+4) + b_n(x)\right]\omega^*\left(f'', n^{-1/2}\right),
\end{aligned}$$

where

$$b_n(x) = n^2 \left[\left(D_n \left(\exp_{-1}(x) - \exp_{-1}(t) \right)^4 \right)(x) \left(D_n \psi_x^4 \right)(x) \right]^{1/2}.$$

Remark 5.1. *We observe that its easy to have more and more compositions of Szász-Mirakyan operators to $C_{n,n}$. Also, moment producing functions can be evaluated but even if we increase compositions further and further the error increases. So, its our claim here that the composition operator $C_{n,n}$ gives better approximation than the further compositions.*

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Availability of data and materials
Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests
The authors declare that they have no competing interest.

Authors' contributions
Each author equally contributed to this paper, read and approved the final manuscript.

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