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On pseudo-orbit, shadowing, statistical shadowing and orbits of non-Newtonian numbers

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Abstract. In this article, we are investigated on the concepts of pseudo-orbits, asymptotic pseudo- orbits, statistical pseudo-orbits, and different shadowing properties in a dynamical system for a continuous function f on a compact set, and also we are obtained some important results.

1. Introduction

Grossman and Katz [1, 4, 8] introduced the concept of non-Newtonian numbers, a new algebraic structure, as a substitute for the classical calculus or Newtonian calculus. Following them, Çakmak and Basar [3] who defined the set $\mathbb{R}^{(N)}$ of non-Newtonian real numbers, non-Newtonian metric space $(\mathbb{R}^{(N)}, d_N)$, non-Newtonian normed space $(\mathbb{R}^{(N)}, \|.\|_N)$, and a number of well-known inequalities.

A generator is a bijective function α whose range is a subset of $\mathbb{R}$ and whose domain is $\mathbb{R}$. Consider $\mathbb{R}^{(N)} = \{\alpha(x) : x \in R\}$. The set $\mathbb{R}^{(N+)}$ represents the $\alpha-$ positive real numbers, and $\mathbb{R}^{(N-)}$ represents the α-negative real numbers. For every $n \in \mathbb{Z}$, we write $\alpha(n) = \dot{n}$.
An α arithmetic is the arithmetic whose domain is $\mathbb{R}^{(N)}$ whose operations are defined as follows: considering $x, y \in \mathbb{R}$ and for any generator α,

$$\alpha - \text{addition } x \dot{+} y = \alpha\{\alpha^{-1}(x) + \alpha^{-1}(y)\}$$
$$\alpha - \text{subtraction } x \dot{-} y = \alpha\{\alpha^{-1}(x) - \alpha^{-1}(y)\}$$
$$\alpha - \text{multiplication } x \dot{\times} y = \alpha\{\alpha^{-1}(x) \times \alpha^{-1}(y)$$
$$\alpha - \text{division } x \dot{/} y = \alpha\{\alpha^{-1}(x)/\alpha^{-1}(y)\}, \alpha^{-1}(x) \neq 0$$
$$\alpha - \text{order } x \dot{\leq} y = \{\alpha^{-1}(x) \leq \alpha^{-1}(y)\},$$

The set $\mathbb{R}^{(N)}$, with the above operations, forms a non-Newtonian complete order field.

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Let D be an infinite compact subset of $\mathbb{R}^{(N)}$, and $d_N : D\dot{\times}D \rightarrow D$ be a function on D is satisfied the following axioms: for all $t, s, z \in D$,
(M1) $d_N(t,s)\dot{\geq}\dot{0}$,
(M2) $d_N(t,s) = \dot{0}$, iff $t = s$,
(M3) $d_N(t,s) = d_N(s,t)$,
(M3) $d_N(t,s)\dot{\leq}d_N(s,z)\dot{+}d_N(z,t)$.
Then (D, d_N) is a compact non-Newtonian metric space.

Independently, Fast [5] and Steinhaus [11] introduced the notion of statistical convergence of $\mathbb{N}$. Further, this notion was more developed by Salat [10], Fridy [6, 7], Tripathy [12, 13], and others.

Let M be a subset of $\mathbb{N}$. The density of M is defined by

$$\eta(M) = \lim_{n\to\infty} \frac{|M_n|}{n},$$

where $M_n = \{k \leq n : k \in M\}$, and $|M_n|$ is the cardinality of M_n.

Example 1.1. *(i) The set of even natural numbers has density zero. (ii) The set of p^{th} power of natural numbers has density zero, where $p \geq 2$.*

A sequence $\{x_k\}$ of non-Newtonian numbers is called statistically convergent to some point β in $\mathbb{R}^{(N)}$, if for every $\varepsilon\dot{>}\dot{0}$, we have

$$\eta(\{k \in \mathbb{N} : d(x_k, \beta)\dot{>}\dot{\varepsilon}\}) = 0.$$

2. Definitions and Notations

Let (D, d_N) be a compact non-Newtonian metric space. If $f : D \rightarrow D$ is a continuous function, then (D, f) is referred to as a dynamical system and the function $f : D \rightarrow D$ has a fixed point in D. Let (D, f) be a dynamical system for the map $f : D \rightarrow D$ on the compact set D and for each $t \in D$, the forward orbit of the point t is given by

$$O^+_f(t) = \{t, f(t), f^2(t), ..., f^k(t), ..\} \subset D,$$

and the backward orbit of t is given by

$$O^-_f(t) = \{\{t_{-i}\}_{i\geq 0}\} \subset D, \text{where } f(t_{-i}) = t_{-i+1}, \text{for } i \geq 0.$$

Definition 2.1. *[2] Let (D, f) be a dynamical system (in Newtonian sense). A sequence of points $\{t_0, t_1, t_2, ...\} \subset D$ is called an $\varepsilon-$pseudo-orbit if for $\varepsilon > 0$,*

$$d(f(t_i), t_{i+1}) < \varepsilon, \text{for every } i \geq 0,$$

where d is a Newtonian metric, and $\mathbb{P}(f, \varepsilon)$ be the set of all $\varepsilon-$pseudo-orbits of f.

Definition 2.2. *An $\varepsilon-$pseudo-orbit $\{t_0, t_1, t_2, ...\} \subset D$ is said to be converges statistically to p if for $\varepsilon\dot{>}\dot{0}$,*

$$\eta(\{i \in \mathbb{N}_0 : d_N(t_i, p)\dot{>}\varepsilon\}) = 0,$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Example 2.3. *Consider the sequence (t_i) is follows:*

$$t_i = \begin{cases} \dot{1} & \text{if } k = i^2, i \in \mathbb{N} \\ \dot{0} & \text{if otherwise} \end{cases}.$$

Then for every $\varepsilon \dot{>} \dot{0}$, we have

$$\eta(\{i \in \mathbb{N} : d_N(t_i, \dot{0}) \dot{>} \varepsilon\}) = \lim_{n\to\infty} \frac{|i \leq n : d_N(t_i, \dot{0}) \dot{>} \varepsilon|}{n} \leq \lim_{n\to\infty} \frac{|i \leq n : t_i \neq \dot{0}|}{n} \leq \lim_{n\to\infty} \frac{n^{1/2}}{n} = 0.$$

Therefore, the sequence (t_i) is statistically converges to $\dot{0}$ but not usually converges to $\dot{0}$.

Definition 2.4. *[2] A sequence of points $\{t_0, t_1, t_2, ...\} \subset D$ is called an asymptotic pseudo-orbit if*

$$lim_{i\to\infty} d(f(t_i), t_{i+1}) = 0.$$

Definition 2.5. *An asymptotic pseudo-orbit $\{t_0, t_1, t_2, ...\} \subset D$ is said to be statistical converges to p if*

$$st - lim\ d_N(t_i, p) = \dot{0},$$

where $\dot{0}$ is non-Newtonian zero.

Let L_f be the set of all limits of all the $\varepsilon-$pseudo-orbit in D, and ω_f be the set of all limits of all the forward orbits in D for the map f. Also, we denote S as the set of all statistical limits of all pseudo-orbits in D, and S_f as the set of all statistical limits of all the forward orbits in D.

Throughout the paper, we denote "shadowing property " by "*s*-property"

Definition 2.6. *[9] Let (D, f) be a dynamical system. The function f is said to have the s-property, if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that, for each $\delta-$pseudo-orbit $\{t_i\}_{i\geq 0}$, there exists $x \in D$ such that*

$$d(f^i(x), t_i) < \varepsilon, \text{ for all } i \geq 0,$$

where f^0 is an identity map, and d is a Newtonian (usual) metric.

Definition 2.7. *Let (D, f) be a dynamical system. The function f is said to have the statistical s-property, if for any $\varepsilon \dot{>} \dot{0}$, there exists a $\delta \dot{>} \dot{0}$ such that, for each $\delta-$pseudo-orbit $\{t_i\}_{i\geq 0}$, there exists $x \in D$ such that*

$$\eta(\{i \in \mathbb{N} : d_N(f^i(x), t_i) \dot{<} \varepsilon\}) = 0,$$

where f^0 is an identity map.

Definition 2.8. *[9] Let $\mathbb{P}(f, \delta)$ be the set of all $\delta-$pseudo-orbits of f and let (D, f) be a dynamical system. Then, we say that the function f has the continuous s-property if there exists a $\delta > 0$ and a continuous map $r : \mathbb{P}(f, \delta) \to D$ for every $\varepsilon > 0$, such that*

$$d(f^i(r(p), t_i) < \varepsilon, \text{ for all } i \geq 0,$$

for $p = \{t_i\}_{i\geq 0} \in \mathbb{P}(f, \delta)$, where d is a Newtonian metric.

Definition 2.9. *Let (D, f) be a dynamical system. If a $\delta \dot{>} \dot{0}$ exists for each $\varepsilon \dot{>} 0$, then the function f is said to have pseudo-orbit corresponding property, if there exists $x, y \in D$, a continuous map $r : \mathbb{P}(f, \delta) \to D$ such that for each pair of $\delta-$pseudo-orbits $p = \{t_i\}_{i\geq 0}$ and $q = \{s_i\}_{i\geq 0}$*

$$d_N(f^i(r(p)), f^i(r(q)) \dot{<} \varepsilon, \text{for all } i \geq 0,$$

where $r(p) = x, r(q) = y$, and f^0 is an identity map.

Definition 2.10. *Let (D, f) be a dynamical system. If a $\delta \dot{>} \dot{0}$ exists for each $\varepsilon \dot{>} \dot{0}$, then the function f is called statistical pseudo-orbit corresponding property, if there exists $x, y \in D$, a continuous map $r : \mathbb{P}(f, \delta) \to D$ such that for each pair of $\delta-$pseudo-orbits $p = \{t_i\}_{i\geq 0}$ and $q = \{s_i\}_{i\geq 0}$*

$$\eta(\{i \in \mathbb{N}_0 : d_N(f^i(r(p)), f^i(r(q)) \dot{>} \varepsilon\}) = 0,$$

where $r(p) = x, r(q) = y$, and f^0 is an identity map.

Definition 2.11. *Let (D, f), and (D, g) be two dynamical systems for a continuous map $f : D \to D$, and $g : D \to D$. The product of two orbits $O^+_f(t)$, and $O^+_g(s)$ defined by*

$$O^+_f(t) \dot{\times} O^+_g(s) = \{(f^i(t), g^j(s)) : f^i(t) \in O^+_f(t), \text{and } g^j(s) \in O^+_g(s)\}$$

Also, the orbit at (t, s) of the product of two functions f and g defined by

$$O^+_{f\dot{\times}g}(t, s) = \{(f^i(t), g^i(s)) : f^i(t) \in O^+_f(t), \text{and } g^i(s) \in O^+_g(s)\}.$$

3. Main Results

Theorem 3.1. *Let $\{t_i\}_{i\geq 0}$ and $\{s_i\}_{i\geq 0}$ be two $\delta-$pseudo-orbits in $\mathbb{P}(f, \delta)$ with satisfies the following conditions:*

(*i*) $d_N((t_i, s_i) \dot{>} d(f(t_i), f(s_i))$, *for all* $i \geq 0$,

(*ii*) $d_N((t_i, s_i) \to \dot{0}$, *as* $i \to \infty$.

Then

(*a*)*If* $\{t_i\}_{i\geq 0}$ *is* $\delta-$ *pseudo-orbit, then so is* $\{s_i\}_{i\geq 0}$,

(*b*)*If* $\{t_i\}_{i\geq 0}$ *is asymptotic pseudo-orbit, then so is* $\{s_i\}_{i\geq 0}$,

(*c*)*If* $\{t_i\}_{i\geq 0}$ *is statistical* $\delta-$ *pseudo-orbit, then so is* $\{s_i\}_{i\geq 0}$.

Proof: We prove only the statement (c). The cases (a) and (b) can be established following standard techniques.
We assume that $\{t_i\}_{i\geq 0}$ is statistical pseudo-orbit i.e., for every $\varepsilon \dot{>} \dot{0}$ such that

$$\eta(A) = 0, \text{where } A = \{i \in \mathbb{N}_0 : d_N(f(t_i), t_{i+1}) \dot{>} \varepsilon\},$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
Let B= $\{i \in \mathbb{N}_0 : d_N(f(s_i), s_{i+1}) \dot{>} \varepsilon\}$, and C= $\{i \in \mathbb{N}_0 : 2d_N(t_i, s_i) \dot{+} d_N(f(t_i), t_{i+1}) \dot{>} \varepsilon\}$.

From the inequality

$$\begin{aligned} d_N((f(s_i), s_{i+1}) &\dot{\leq} d_N(f(s_i), f(t_i) \dot{+} d_N(f(t_i), t_{i+1}) \dot{+} d_N(t_{i+1}, s_{i+1}) \\ &\dot{<} d_N(s_i, t_i) \dot{+} d_N(f(t_i), t_{i+1}) \dot{+} d_N(t_i, s_i) \\ &\dot{<} 2d_N(s_i, t_i) \dot{+} d_N(f(t_i), t_{i+1}). \end{aligned}$$

We have

$$\eta(\{i \in \mathbb{N}_0 : d_N((f(s_i), s_{i+1}) \dot{>} \varepsilon\}) \leq \eta(\{i \in \mathbb{N}_0 : 2_N d(s_i), t_i) \dot{+} d_N(f(t_i), t_{i+1}) \dot{>} \varepsilon\})$$

It is cleared that $B \subseteq C$. We claim $\eta(C) = 0$.
Further, let $B_{\varepsilon_1} = \{i \in \mathbb{N}_0 : d_N(t_i, s_i) \dot{>} \varepsilon\}$, and $B_{\varepsilon - 2\varepsilon_1} = \{i \in \mathbb{N}_0 : d_N(f(t_i), t_{i+1}) \dot{>} \varepsilon - 2\varepsilon_1\}$, where $\dot{0} \dot{<} \varepsilon_1 \dot{<} \varepsilon$.
Then we obtain $C = \cup_{\dot{0} \dot{<} \varepsilon_1 \dot{<} \varepsilon}(B_{\varepsilon_1} \cap B_{\varepsilon - 2\varepsilon_1})$. For each $\varepsilon_1 \in (\dot{0}, \varepsilon)$, we have

$$\eta(B_{\varepsilon - 2\varepsilon_1}) = 0,$$

which implies $\eta(\cup_{\dot{0}\dot{<}\varepsilon_1\dot{<}\varepsilon}(B_{\varepsilon_1}\cap B_{\varepsilon-2\varepsilon_1}))=0$ i.e., $\eta(C)=0$.
Therefore, $\eta(B)=0$, as B is a subset of C.

Theorem 3.2. *If $f:D\to D$ is a continuous map and let (D,f) be the dynamical system. If the limit of the forward orbit exists, then it is a fixed point in D under the associated map f.*

Proof: We have to prove that if x is the limit of some forward orbit, then it is a fixed point i.e., $f(x)=x$. Suppose x is a limit of a forward orbit, there is point $t\in D$ and a forward orbit $O_f^+(t)=\{t,f(t),f^2(t),...,f^k(t),...\}$ such that

$$f^k(t)\to x,\ \text{as}\ k\to\infty. \tag{1}$$

Since f is a continuous map , then

$$f(f^k(t))\to f(x),\ \text{as}\ k\to\infty,$$
$$i.e., f^{k+1}(t)\to f(x),\ \text{as}\ k\to\infty. \tag{2}$$

Again from the Eq(1), we have

$$f^{k+1}(t)\to x,\ \text{as}\ k\to\infty. \tag{3}$$

Therefore, from Eq(2) and Eq(3), we have

$$f(x)=x$$

Hence, x is a fixed point under the associated map f.
This following corollary immediately follows from the above theorem.

Theorem 3.3. *Let f be a continuous map on the compact metric space D.*
If (i) f has s-property,
(ii) every forward orbit is convergent,
then every $\varepsilon-$pseudo-orbit is also convergent.

Proof: Suppose for every $t\in D$, the forward orbit at t is convergent,then for every $\varepsilon\dot{>}\dot{0}$, there exists a natural number n_0 satisfying

$$d_N(f^m(t),f^n(t))\dot{<}\tfrac{\varepsilon}{3},\ \text{for all}\ \ n,m\geq n_0.$$

Since f has shadowing property, for every pair of $\varepsilon-$pseudo-orbits $\{s_i\}_{i\geq 0}$, there exists $z\in D$ such that

$$d_N(f^i(z),s_i)\dot{<}\tfrac{\varepsilon}{3},\ \text{for all}\ i\geq 0.$$

Now, for all $n,m\geq n_0$, we have

$$d_N(s_m,s_n)\dot{\leq}\ d_N(t_m,f^m(z))\dot{+}d_N(f^m(z),f^n(z))\dot{+}d_N(f^n(z),t_n)$$

From the above two equations, we have

$$d_N(s_m,s_n)\dot{\leq}\varepsilon,\ \text{for all}\ \ n,m\geq n_0.$$

Therefore, $\{s_i\}_{i\geq 0}$ is Cauchy sequence, hence convergent.

Remark 3.4. *In veiw of the above Theorem, we have "every fixed point is also statistical fixed point in D.*

Theorem 3.5. *Let $f:D\to D$ be a continuous map, and let (D,f) be the dynamical system. If $s\in\overline{O_f^+(x)}$ and $t\in\overline{O_f^+(s)}$, then $t\in\overline{O_f^+(x)}$.*

Proof: Assume $s \in \overline{O_f^+(x)}$ i.e., s is a limit of $O_f^+(x)$, then for every $\varepsilon \dot{>} \dot{0}$, there exist a $N \in \mathbb{N}$, and a sequence $\{f^{n_k}(x)\}$ such that

$$d_N(f^{n_k}(x), s) \dot{<} \frac{\varepsilon}{2}, \text{for all } n_k \geq N. \tag{4}$$

Further, $t \in \overline{O_f^+(s)}$, for every $\varepsilon \dot{>} \dot{0}$, there is a $N_1 \in \mathbb{N}$, and a subsequence $\{f^{n_m}(s)\}$ such that

$$d_N(f^{n_m}(s), t) \dot{<} \frac{\varepsilon}{2}, \text{for all } n_m \geq N_1. \tag{5}$$

Since f is continuous, then from Eq(4), we have

$$d_N(f^{n_k+n_m}(x), f^{n_m}(s)) \dot{<} \frac{\varepsilon}{2}, \text{for all } n_k \geq N. \tag{6}$$

Now, using the Eq(5) and Eq(6), we obtain

$$\begin{aligned} d_N(f^{n_k+n_m}(x), t) &\dot{\leq} d_N(f^{n_k+n_m}(x), f^{n_m}(s)) \dot{+} d_N(f^{n_m}(s), t) \\ &\dot{<} \frac{\varepsilon}{2} \dot{+} \frac{\varepsilon}{2} \\ &= \varepsilon, \end{aligned}$$

for all $n_k, n_m \geq K = max\{N, N_1\}$. Hence, $t \in \overline{O_f^+(x)}$.

Theorem 3.6. *Let (D, f) be a dynamical system for a continuous map $f : D \to D$. Then under the associated map f, the statements hold:*
(i) the set S_f is closed ,
(ii) the set S is closed ,
(iii) the set S is an invariant,
(iv) The set S_f is an invariant.

Proof: Let (D, f) be a dynamical system. We have to prove that (i) $S_f = \overline{S_f}$, and (ii) $\overline{S} = S$, (iii) S is invariant set,(iv)S_f is invariant set.
(i)Let y be any element in $\overline{S_f}$, then there exists a forward orbit $O_f^+(t) = \{t, f(t), f^2(t), ..., f^k(t), ..\}$ such that

$$\eta(\{i \in \mathbb{N}_0 : d_N(f^k(t), y) \dot{>} \varepsilon\}) = 0$$

We have, by the Theorem 2., y is a fixed point under the associated map f
that is, $f(y) = y$. Then y is statistical fixed point in D.
Now, consider the orbit $O_f^+(y) = \{y, f(y), f^2(y), ..., f^k(y), ...\} \subset D$ such that

$$\eta(\{i \in \mathbb{N}_0 : d_N(f^k(y), y) \dot{>} \varepsilon\}) = 0$$

Therefore, y is a limit of some forward orbit in D i.e., $y \in S_f$.
Hence, S_f is a closed set.
(ii) We claim S is closed. Suppose L is an element of the closure of S, then there is a sequence $\{L_1, L_2, L_3, ..., L_i, ..\}$ of limits of the different $\varepsilon-$ pseudo-orbits such that

$$\eta(\{i \in \mathbb{N} : d_N(L_i, l) \dot{>} \varepsilon\}) = 0$$

Then, for $\varepsilon \dot{>} \dot{0}$, there is a subset $K = \{k_1 < k_2 < ... < k_n < ...\} \subset \mathbb{N}$ with $\eta(K) = 0$ such that

$$\eta(\{i \in \mathbb{N} : d_N(L_{k_i}, L) \dot{>} \varepsilon\}) = 0. \tag{7}$$

Further, for each $L_k \in S$, there exists a $\varepsilon-$ pseudo-orbit $\{t_i\}_{i \geq 0}$ and a $M_k \subset \mathbb{N}$ with $\eta(M_k) = 0$ such that

$$\eta(\{i \in \mathbb{M}_k : d_N(t_i, L_k) \dot{>} \varepsilon\}) = 0. \tag{8}$$

From the inequality,for all $i \in M = K \cap M_k$,

$$d_N(t_i, L)\dot{\leq} d_N(t_i, L_k)\dot{+}d_N(L_k, L)$$

and from Eq(7) and Eq(8), we have

$$\eta(\{i \in M : d_N(t_i, L)\dot{>}\varepsilon\}) \leq \eta(\{i \in M : d_N(t_k, L_k)\dot{>}\varepsilon\}) + \eta(\{i \in M : d_N(L_i, L)\dot{>}\varepsilon\})$$

which implies

$$\eta(\{i \in M : d_N(t_i, L)\dot{>}\varepsilon\}) = 0.$$

That is, we can find a $\varepsilon-$ pseudo-orbits $\{t_i\}_{i\geq 0}$ such that $\{t_i\}_{i\geq 0}$ statistically converges to L. Hence, $L \in S$.
(iii) We prove that $m \in S \implies f(m) \in S$. Suppose $m \in S$, for every $\varepsilon\dot{>}\dot{0}$, there is a $\varepsilon-$ pseudo-orbit $\{s_i\}_{i\geq 0}$ and a subset $K = \{k_1 < k_2 < ... < k_n < ...\} \subset \mathbb{N}$ with $\eta(K) = 0$ such that

$$\eta(\{i \in K : d_N(s_i, m)\dot{>}\varepsilon\}) = 0. \tag{9}$$

$$\eta(\{i \in \mathbb{N}_0 : d(f(s_i), s_{i+1})\dot{>}\varepsilon\}) = 0. \tag{10}$$

From Eq(9), and Eq(10), we have

$$\eta(\{i \in K : d_N(f(s_i), m)\dot{>}\varepsilon\}) = 0.$$

Since f is a continuous function, then from Eq(9), we get

$$\eta(\{i \in K : d_N(f(s_i), f(m))\dot{>}\varepsilon\}) = 0$$

Thus, for given $\varepsilon\dot{>}\dot{0}$, by using above two equations we obtain

$$\eta(\{i \in K : d_N(f(m), s_{k+1})\dot{>}\varepsilon\}) = 0.$$

That is, $f(m)$ is a statistical limit of an $\varepsilon-$ pseudo-orbit $\{s_i\}_{i\geq 0}$. Therefore, $f(m) \in S$.
(iv) For any $y \in S_f$, by Remark 1, y is statistical fixed point in D.
Hence, $f(y) \in S_f$.
The following result follows from the above theorem.

Corollary 3.7. *Let (D, f) be a dynamical system. Let ω_f and L_f be the sets of all limits of all forward orbits and $\varepsilon-$ pseudo-orbits in D respectively, for the map f. Then*
(i) ω_f is a closed and invariant set,
(ii) L_f is a closed and invariant set.

Theorem 3.8. *Let (D, f), and (D, g) be two dynamical systems for continuous maps $f : D \to D$, and $g : D \to D$, respectively. Then*

$$\overline{O^+_{f\dot{\times}g}(t, s)} = \overline{O^+_f(t)}\dot{\times}\overline{O^+_g(s)}$$

iff $(t, g(s)) \in \overline{O^+_{f\dot{\times}g}(t, s)}$.

Proof: The necessity part is immediately held from the equality condition. For the sufficient part, we assume $(t, g(s)) \in \overline{O^+_{f\dot{\times}g}(t, s)}$.
We shall show that

$$\overline{O^+_{f\dot{\times}g}(t, s)} = \overline{O^+_f(t)}\dot{\times}\overline{O^+_g(s)}.$$

Suppose $(f^i(t), g^i(s)) \in O^+_{f\times g}(t, s)$, which implies $f^i(t) \in O^+_f(t)$, and $g^i(s) \in O^+_g(s)$, for all $i \geq 0$. Then for all $i, j \geq 0$, we obtain

$$(f^i(t), g^j(s)) \in O^+_f(t)\dot{\times}O^+_g(s).$$

This implies

$$O^{+}_{f\dot{\times}g}(t,s) \subseteq O^{+}_{f}(t)\dot{\times}O^{+}_{g}(s).$$

Therefore, we have

$$\overline{O^{+}_{f\dot{\times}g}(t,s)} \subseteq \overline{O^{+}_{f}(t)}\dot{\times}\overline{O^{+}_{g}(s)}.$$

For the reverse inclusion, we consider for $j > i > 0$,

$$(f^{i}(t), g^{j}(s)) \in O^{+}_{f}(t)\dot{\times}O^{+}_{g}(s).$$

Then $f^{i}(t) \in O^{+}_{f}(t)$, and $g^{j}(s) \in O^{+}_{g}(s)$.
So, there is a natural number p such that $j = i + p$, and also, by our assumption $(t, g(s)) \in \overline{O^{+}_{f\dot{\times}g}(t,s)}$, and

$$\begin{aligned}(f^{i}(t), g^{j}(s)) &= (f^{i}(t), g^{i+p}(s))\\ &= (f^{i}(t), g^{i}(s))(t, g^{p}(s)).\end{aligned}$$

So, by our assumption, we have

$$(f^{i}(t), g^{j}(s)) \in \overline{O^{+}_{f\dot{\times}g}(t,s)},$$

Therefore, we get

$$O^{+}_{f}(t)\dot{\times}O^{+}_{g}(s) \subseteq \overline{O^{+}_{f\dot{\times}g}(t,s)}$$

which implies

$$\overline{O^{+}_{f}(t)}\dot{\times}\overline{O^{+}_{g}(s)} \subseteq \overline{O^{+}_{f\dot{\times}g}(t,s)}.$$

Hence,

$$\overline{O^{+}_{f}(t)}\dot{\times}\overline{O^{+}_{g}(s)} = \overline{O^{+}_{f\dot{\times}g}(t,s)}.$$

Theorem 3.9. *Let (D, f) be a dynamical system. If there is a $\delta\dot{>}\dot{0}$ for each $\varepsilon\dot{>}\dot{0}$, and there exists $x, y \in D$ with $d_N(t_i, s_i)\dot{<}\delta$, for all $i \geq 0$, for each pair of $\delta-$pseudo-orbits $\{t_i\}_{i\geq 0}$ and $\{s_i\}_{i\geq 0}$ such that f has s-property on D, then*

$$d_N(f^{i}(x), f^{i}(y))\dot{<}\varepsilon, \text{for all } i \geq 0,$$

where f^{0} is an identity map.

Proof: For given $\varepsilon\dot{>}\dot{0}$, there exists a $\delta\dot{>}\dot{0}$ and for every pair of $\delta-$pseudo-orbits $\{t_i\}_{i\geq 0}$ and $\{s_i\}_{i\geq 0}$. Now taking $\varepsilon = \delta$, we have

$$d_N(f(t_i), t_{i+1})\dot{<}\frac{\varepsilon}{5}, \text{for all } i \geq 0. \tag{11}$$

$$d_N(f(s_i), s_{i+1})\dot{<}\frac{\varepsilon}{5}, \text{for all } i \geq 0. \tag{12}$$

Since f has s-property, for every pair of $\delta-$pseudo-orbits $\{t_i\}_{i\geq 0}$ and $\{s_i\}_{i\geq 0}$, there exists $x, y \in D$ such that

$$d_N(f^{i}(x), t_i)\dot{<}\frac{\varepsilon}{5}, \text{ for all } i \geq 0, \tag{13}$$

$$d_N(f^{i}(y), s_i)\dot{<}\frac{\varepsilon}{5}, \text{ for all } i \geq 0, \tag{14}$$

Further, f is uniformly continuous on D, for given $\varepsilon\dot{>}\dot{0}$, there is a $\delta\dot{>}\dot{0}$ such that

$$d_N(t_i, s_i)\dot{<}\delta \implies d_N(f(t_i), f(s_i))\dot{<}\frac{\varepsilon}{5}, \tag{15}$$

for all $i \geq 0$.
Now, from the Equations (11), (12), (13), (14), and (15), we obtain

$$\begin{aligned} d_N(f^i(x), f^i(y))&\dot{\leq} d_N(f^i(x), t_i)\dot{+}d_N(t_i, f(t_{i-1}))\dot{+}d_N(f(t_{i-1}), f(s_{i-1})) \\ &\dot{+}d(f(s_{i-1}), s_i)\dot{+}d(s_i, f^i(y)) \\ &\dot{<}\frac{\varepsilon}{5}\dot{+}\frac{\varepsilon}{5}\dot{+}\frac{\varepsilon}{5}\dot{+}\frac{\varepsilon}{5}\dot{+}\frac{\varepsilon}{5} \\ &= \varepsilon, \text{for all } i \geq 0. \end{aligned}$$

The proof is complete.
The following corollary can be established from the previous result.

Corollary 3.10. *Let (D, f) be a dynamical system. If there is a $\delta\dot{>}\dot{0}$ for each $\varepsilon\dot{>}\dot{0}$, and there exists $x, y \in D$ with $d_N(t_i, s_i)\dot{<}\delta$, for all $i \geq 0$ for each pair of $\delta-$pseudo-orbits $\{t_i\}_{i\geq 0}$ and $\{s_i\}_{i\geq 0}$ such that f has continuous s-property on D then f has pseudo-orbit corresponding property i.e.,*

$$d_N(f^i(r(p)), f^i(r(q)))\dot{<}\varepsilon, \textit{ for all } i \geq 0,$$

where $r(p) = x, r(q) = y$, and f^0 is an identity map.

Theorem 3.11. *Let (D, f) be a dynamical system. If there is a $\delta\dot{>}\dot{0}$ for each $\varepsilon\dot{>}\dot{0}$, and there exists $x, y \in D$ with $\eta(\{i \in \mathbb{N}_0 : d_N(t_i, s_i)\dot{>}\delta\}) = 0$, for each pair of $\delta-$pseudo-orbits $\{t_i\}_{i\geq 0}$ and $\{s_i\}_{i\geq 0}$ such that f has statistical s-property on D, then*

$$\eta(\{i \in \mathbb{N}_0 : d(f^i(x), f^i(y))\dot{>}\varepsilon\}) = 0,$$

where f^0 is an identity map.

Proof: For given $\varepsilon\dot{>}\dot{0}$, there exists a $\delta\dot{>}\dot{0}$ and for every pair of $\delta-$pseudo-orbits $\{t_i\}_{i\geq 0}$ and $\{s_i\}_{i\geq 0}$. Now, taking $\delta = \varepsilon$, we have

$$\eta(\{i \in \mathbb{N}_0 : d_N(f(t_i), t_{i+1})\dot{>}\varepsilon\}) = 0. \tag{16}$$

$$\eta(\{i \in \mathbb{N}_0 : d_N(f(s_i), s_{i+1})\dot{>}\varepsilon\}) = 0. \tag{17}$$

Now, f has statistical s-property, then for every pair of $\delta-$pseudo-orbits $\{t_i\}_{i\geq 0}$ and $\{s_i\}_{i\geq 0}$, there exists $x, y \in D$ such that

$$\eta(\{i \in \mathbb{N}_0 : d_N(f^i(x), t_i)\dot{>}\varepsilon\}) = 0. \tag{18}$$

$$\eta(\{i \in \mathbb{N}_0 : d_N(f^i(y), s_i)\dot{>}\varepsilon\}) = 0. \tag{19}$$

Further, f is uniformly continuous on D, for given $\varepsilon\dot{>}\dot{0}$, there exists a $\delta(= \varepsilon)\dot{>}\dot{0}$, and for every pair of $\delta-$pseudo-orbits $\{t_i\}_{i\geq 0}$ and $\{s_i\}_{i\geq 0}$ such that

$$\eta(\{i \in \mathbb{N}_0 : d_N(t_i, s_i)\dot{>}\delta\}) = 0 \implies \eta(\{i \in \mathbb{N}_0 : d_N(f(t_i), f(s_i))\dot{>}\varepsilon\} = 0. \tag{20}$$

Now, from the Equations (16), (17), (18), (19), and (20), we obtain

$$\begin{aligned} d_N(f^i(x), f^i(y))&\dot{\leq} d_N(f^i(x), t_i)\dot{+}d_N(t_i, f(t_{i-1}))\dot{+}d_N(f(t_{i-1}), f(s_{i-1})) \\ &\dot{+}d_N(f(s_{i-1}), s_i)\dot{+}d_N(s_i), f^i(y)) \end{aligned}$$

$$\implies \eta(\{i \in \mathbb{N} : d_N(f^i(x), f^i(y)) \dot{>} \varepsilon\}) \leq \eta(\{i \in \mathbb{N} : d_N(f^i(x), t_i) \dot{>} \varepsilon\}) + \eta(\{i \in \mathbb{N} : d_N(t_i, f(t_{i-1})) \dot{>} \varepsilon\}) + \eta(\{i \in \mathbb{N} : d_N(f(t_{i-1}), f(s_{i-1})) \dot{>} \varepsilon\}) + \eta(\{i \in \mathbb{N} : d_N(f(s_{i-1}), s_i) \dot{>} \varepsilon\}) + \eta(\{i \in \mathbb{N} : d_N(s_i, f^i(y)) \dot{>} \varepsilon\})$$

From the above equations, we get the required result
i.e.,

$$\eta(\{i \in \mathbb{N} : d_N(f^i(x), f^i(y)) \dot{>} \varepsilon\}) = 0.$$

The following corollary can be established from the previous result.

Corollary 3.12. *Let (D, f) be a dynamical system. If there is a $\delta \dot{>} \dot{0}$ for each $\varepsilon \dot{>} \dot{0}$, and there exists $x, y \in D$ with $\eta(\{i \in \mathbb{N}_0 : d_N(t_i, s_i) \dot{>} \delta\}) = 0$, for each pair of $\delta-$pseudo-orbits $\{t_i\}_{i\geq 0}$ and $\{s_i\}_{i\geq 0}$ such that f has statistical continuous s-property on D, then f has statistical pseudo-orbit corresponding property i.e.,*

$$\eta(\{i \in \mathbb{N} : d_N(f^i(r(p)), f^i(r(q))) \dot{>} \varepsilon\}) = 0,$$

where $r(p) = x, r(q) = y$, and f^0 is an identity map.

4. Conclusion

In this paper, we have studied various types of shadowing that is, different types of approximation methods to approximate pseudo-orbit by forward orbit under a continuous map. This study shall help the readers to find out some other approximation methods that approximate pseudo orbits in less errors.

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