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A Lyapunov approach to the stability of stochastic time delay systems

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Abstract. In this paper, we conduct a comprehensive stability analysis of time-delay systems influenced by various types of stochastic perturbations, including standard Brownian motion, randomly varying coefficients, and functions governed by stochastic processes. The primary objective is to assess mean-fourth stability and stochastic stability. To this end, we employ specifically designed Lyapunov functionals to derive sufficient conditions that ensure these stability criteria are met. We further extend the analysis to systems characterized by randomly changing coefficients, contributing new theoretical insights to the literature. The proposed results are substantiated with illustrative examples and a detailed exploration of the corresponding stability regions.

1. Introduction

The stability theory of dynamical systems owes much to the pioneering work of A. M. Lyapunov, who introduced both direct and indirect analytical methods [1]. Together with LaSalle's invariance principle, these methods offer powerful tools for assessing and ensuring the stability of nonlinear systems and feedback controllers. Practical uses of Lyapunov theory include robust back-stepping control strategies for vehicle steering [2] and adaptive control in satellite orientation systems [3]. Stability analysis for systems with time delays has also been addressed using Lyapunov functions, which help establish exponential stability criteria [4]. In stochastic systems and functional differential equations, carefully constructed Lyapunov functions support the analysis of mean square and global stability [5]. Broader applications involve fuzzy control systems and neural networks [7–10]. In [11], mean-square stability and probabilistic convergence are studied using Lyapunov techniques tailored to stochastic and perturbed systems.

Using randomness in differential equations provides a powerful way to describe real-world phenomena with inherent uncertainty [12]. Stochastic differential equations include terms driven by stochastic processes such as white noise, which is formally the derivative of Brownian motion $B(t)$. Since Brownian motion is not classically differentiable, a specialized form of calculus, different from standard calculus, is used to handle these cases [12–14]. Studies on exact solutions can be found in [14, 15], while the existence and uniqueness of solutions are treated in [13, 14]. Various numerical methods have been proposed to deal with noise and random coefficients in such systems [16–19]. The behavior of the resulting solutions depends significantly

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on whether the randomness comes from white noise, random parameters, or both [20].

In recent decades, interest in the analysis and design of systems affected by time delays has grown substantially. Scholars have approached this subject from two distinct angles: one rooted in rigorous functional analysis and the other focused on practical applications. The former has advanced the theoretical foundations of functional differential equations, while the latter has addressed challenges in controlling and stabilizing delayed systems. Time delays are inherent in numerous real-world systems—such as process control, biological networks, and communication systems—resulting from sensor lag, data processing time, or intrinsic system dynamics. These delays are particularly prominent in networked and distributed control systems, where fixed, variable, or random delays in communication channels are inevitable. Such delays can severely compromise system performance and may even trigger instability. Consequently, since the 1960s, substantial research has been dedicated to developing control strategies and algorithms capable of mitigating the negative impacts of time delays, as evidenced in studies like [21–25].

A stochastic delay differential equation with a deterministic drift term and a stochastic diffusion term can be written as:

$$dx(t) = f\Big(x(t), x(t-\tau)\Big)\,dt + \sigma\Big(x(t), x(t-\tau)\Big)\,dW(t), \tag{1}$$

where:

- $x(t)$: The state variable of the system at time t.
- $x(t-\tau)$: The delayed state, reflecting the value of the state variable at time $t-\tau$.
- τ: The time delay, which can be constant or time-varying.
- $f\Big(x(t), x(t-\tau)\Big)$: The deterministic drift function that governs the primary dynamics.
- $\sigma\Big(x(t), x(t-\tau)\Big)$: The stochastic diffusion function, which describes how randomness affects the system over time.
- $W(t)$: A Wiener process or Brownian motion, representing a source of randomness.

This paper investigates the stochastic stability of the trivial solution to equation (1), driven by Brownian motion, through the use of Lyapunov functions. It also explores both the mean fourth stability and the asymptotic mean fourth stability of equation (1), considering cases where f and σ are measurable deterministic functions that satisfy the Lipschitz condition, as well as when random functions act as stochastic processes. Furthermore, we examine stochastic time-delay systems involving random variable coefficients. Section 3 is dedicated to the analysis of asymptotic mean fourth stability for equation (1), under the assumption that f and σ are measurable deterministic functions, stochastic processes, or random coefficient variables. Our results are complemented by several illustrative examples.
Consider the stochastic delay differential equation (1) with initial condition $X(t_0) = X_0$, where $X(t)$ is the solution process. The drift f and diffusion σ are deterministic, measurable functions defined on $[t_0, \infty) \times \mathbb{R}$, and $W(t)$ denotes a one-dimensional Brownian motion. Continuity in t and compliance with the Lipschitz condition are assumed for both f and σ [15]. Assuming that the positive constants k_1 and k_2 exist such that $t \in [t_0, \infty)$ and $X_1, X_2 \in \mathbb{R}^+ \times \mathbb{R}$, we have:

$$\begin{aligned} &|f(t-\tau, X_1) - f(t-\tau, X_2)| \le k_1 |X_1 - X_2|, \\ &|\sigma(t-\tau, X_1) - \sigma(t-\tau, X_2)| \le k_2 |X_1 - X_2|. \end{aligned} \tag{2}$$

2. Preliminaries

In this section, we outline essential definitions, results, and key concepts that form the basis of the paper. These foundational aspects will provide the necessary background for comprehending the main discussions and analyses presented later.

Definition 2.1. *A stochastic process $\{X(t),\ t \in T\}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is said to be a fourth-order stochastic process if its fourth moment exists and is finite for all $t \in T$, that is,*

$$\mathbb{E}[X(t)^4] < \infty.$$

The process is considered to belong to the fourth-order normed space if it satisfies

$$\|X\|_4^4 = \mathbb{E}\left[\int_0^T |X(t)|^4\, dt\right] < \infty.$$

Furthermore, the process is called square-integrable in the fourth moment sense if

$$\int_0^\infty \mathbb{E}[X(t)^4]\, dt < \infty.$$

Here, $\mathbb{E}[\cdot]$ denotes the expectation operator. These conditions ensure that the process has sufficient regularity for analysis in higher-moment functional spaces.

Definition 2.2. *[14, 26] The trivial solution of the time-delay differential equation is said to possess the following types of stability:*

- ***Stochastic stability*** *means that for every $\epsilon \in (0, 1)$ and $l > 0$, one can find a $\delta = \delta(\epsilon, l) > 0$ so that whenever $|X_0| < \delta$, the inequality $\mathbb{P}[|X(t, X_0)| < l] \geq 1 - \epsilon$ holds.*
- ***Mean-fourth stability*** *requires that for any $\epsilon > 0$, there is a $\delta > 0$ such that $\mathbb{E}[|X(t, X_0)|^4] < \epsilon$ whenever $|X_0| < \delta$.*
- ***Asymptotic mean-fourth stability*** *implies that the solution is mean fourth stable and, in addition,*

  $$\lim_{t\to\infty} \mathbb{E}[|X(t, X_0)|^4] = 0.$$

- ***Exponential mean-fourth stability*** *means that the solution is mean-fourth stable, and there exist constants $\kappa > 0$, $\lambda > 0$ such that*

  $$\mathbb{E}[|X(t, X_0)|^4] < \kappa e^{-\lambda t}.$$

Definition 2.3. *[27] For real numbers $n, m > 1$ such that $\frac{1}{n} + \frac{1}{m} = 1$, Hölder's inequality gives:*

$$\mathbb{E}[|XY|] \leq (\mathbb{E}[|X|^n])^{1/n} (\mathbb{E}[|Y|^m])^{1/m}.$$

Setting $n = \frac{1}{4}$, $m = \frac{4}{3}$, we obtain:

$$\mathbb{E}[|XY|] \leq \left(\mathbb{E}[|X|^{1/4}]\right)^4 \left(\mathbb{E}[|Y|^{4/3}]\right)^{3/4}. \tag{3}$$

The integral form for the functions f, g over $[a, b]$ becomes:

$$\left|\int_a^b f(x)g(x)\, dx\right| \leq \left(\int_a^b |f(x)|^4\, dx\right)^{1/4} \left(\int_a^b |g(x)|^{4/3}\, dx\right)^{3/4}.$$

Lemma 2.4. *[27] Assume that $X(t)$ is a mean fourth stochastic process. We want to show that this implies it is also a mean-square process.*

Proof: *By the Cauchy–Schwarz inequality, we know:*

$$\left(\mathbb{E}[X(t)^2]\right)^2 \leq \mathbb{E}[X(t)^4] \cdot \mathbb{E}[1] = \mathbb{E}[X(t)^4],$$

since $\mathbb{E}[1] = 1$. *Taking the square root of both sides gives:*

$$\mathbb{E}[X(t)^2] \leq \sqrt{\mathbb{E}[X(t)^4]}.$$

Because we are given that $\mathbb{E}[X(t)^4] < \infty$, *it follows that* $\mathbb{E}[X(t)^2] < \infty$, *so* $X(t)$ *is indeed a mean-square process. Let us assume that* $X(t)$ *has rapidly decaying tails, characterized by sub-exponential or sub-Gaussian distributions, and that it possesses a finite higher moment* $p > 4(e.g., .\mathbb{E}[X(t)^6] < \infty)$ *so that the converse of lemma (2.4) holds.*

Definition 2.5. *[28] We say* $\mathcal{V}(t, X)$ *is positive definite if there exists a positive definite function* $\nu(X)$ *such that*

$$\mathcal{V}(t, X) \geq \nu(X).$$

It is negative definite if

$$\mathcal{V}(t, X) \leq -\nu(X).$$

Lemma 2.6. *[29] In Itô calculus, the chain rule describes the differential of a function of a stochastic process. Let*

$$U(t, X) : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$$

be a function with continuous partial derivatives:

$$\frac{\partial U}{\partial t}, \quad \frac{\partial U}{\partial X}, \quad \frac{\partial^2 U}{\partial X^2}.$$

Let L be the generator of the stochastic process. Then the SDE is:

$$dX = LU(t, X)\,dt + \sigma(t, X)\,\frac{\partial U}{\partial X}\,dW(t),$$

and the Itô formula yields:

$$dU(t, X) = \left[\frac{\partial U}{\partial t} + f(t, X)\frac{\partial U}{\partial X} + \frac{1}{2}\sigma^2(t, X)\frac{\partial^2 U}{\partial X^2}\right]dt + \sigma(t, X)\frac{\partial U}{\partial X}\,dW(t).$$

Definition 2.7. *[30] Let* $\{X(t), t \in \mathbb{T}\}$ *be a stochastic process adapted to a filtration* $\{\mathcal{F}_t\}$. *Then:*

- $X(t)$ *is a* ***martingale*** *if* $\mathbb{E}[X(t) \mid \mathcal{F}_s] = X(s)$ *for* $s < t$.
- *It is a* ***supermartingale*** *if* $\mathbb{E}[X(t) \mid \mathcal{F}_s] \leq X(s)$.
- *It is a* ***submartingale*** *if* $\mathbb{E}[X(t) \mid \mathcal{F}_s] \geq X(s)$.

Definition 2.8. *[6] The Itô integral*

$$\int_{t_0}^{t_1} X(\tau)\,dW_\tau$$

is defined in a complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_{W_t}, P)$, *where* $W(t)$ *is Brownian motion and* $\mathcal{F}_{W_t}$ *is its natural filtration. If* $X(t)$ *is an* $\mathcal{F}_{W_t}$*-adapted process (e.g., including delayed terms like* $X(t-\tau)$*), then:*

- *Zero-mean:* $\mathbb{E}\left[\int_{t_0}^{t_1} X(\tau)\,dW_\tau\right] = 0$
- *Isometry:* $\mathbb{E}\left[\left(\int_{t_0}^{t_1} X(\tau)\,dW_\tau\right)^2\right] = \mathbb{E}\left[\int_{t_0}^{t_1} X(\tau)^2\,d\tau\right]$

Assume $(\Omega, \mathcal{F}, P)$ is the underlying space and $\mathcal{F}_t \subseteq \mathcal{F}$ is a filtration with $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$. A process $\{X(t), X(t-\tau)\} \subset \mathbb{R}^+$ is adapted if it is $\mathcal{F}_t$-measurable and satisfies

$$\mathbb{E}[|X(t)|^4] < \infty \quad \forall t.$$

Definition 2.9. *[6] Consider a continuous, non-negative function $u(t)$ on $[0,T]$ that satisfies*

$$u(t) \leq u(0) + \int_0^t \phi(s)u(s)\,ds,$$

for all $t \in [0,T]$, where $\phi(s)$ is a non-negative, integrable function. Then Gronwall's inequality gives the bound

$$u(t) \leq u(0)\exp\left(\int_0^t \phi(s)\,ds\right),$$

for all $t \in [0,T]$.

3. Main results

This section is devoted to studying the mean-fourth stability of stochastic and random systems by employing Lyapunov-based methods. We define a Lyapunov function $\mathcal{V}(t,X)$, which is assumed to be continuous and twice differentiable, as

$$\mathcal{V}\colon Q_h \times [t_0,\infty) \to \mathbb{R}^+,$$

where the set $Q_h = \{X \in \mathbb{R} \,|\, \|X(t)\| < h,\ t \geq t_0,\ h > 0\}$ defines a region of interest around the origin. Using this function, we establish several results that characterize the conditions under which the zero solution of the system remains stable in both a probabilistic sense and in terms of its fourth statistical moment.

Theorem 3.1. *The zero solution of equation (1) is said to be mean-fourth stable and asymptotically mean-fourth stable if there exist positive constants k_1, k_2, and k_3 such that the following conditions are satisfied:*

- *The expectation of the Lyapunov function satisfies $\mathbb{E}[\mathcal{V}(t,X)] \geq k_1\mathbb{E}[X^4(t)]$,*
- *The initial condition satisfies $\mathbb{E}[\mathcal{V}(t_0,X_0)] \leq k_2X_0^4$,*
- *The difference in the Lyapunov function's expectation evolves as $\mathbb{E}[\mathcal{V}(t,X)-\mathcal{V}(t_0,X_0)] \leq -k_3\int_{t_0}^t \mathbb{E}[|X^4(s)|]\,ds$.*

Proof. Applying the Ito Lemma (2.6) to the Lyapunov function $\mathcal{V}(t,X) = X^4(t)$ implies that

$$\begin{aligned} d[X^4(t)] &= \Big[\frac{\partial X^4(t)}{\partial t} + \frac{\partial X^4(t)}{\partial X}f(x(t),X(t-\tau)) + \frac{1}{2}\sigma^2(X(t),X(t-\tau))\frac{\partial^2 X^4(t)}{\partial X^2}\Big]dt \\ &\quad + \sigma(X(t),X(t-\tau))\frac{\partial X^2(t)}{\partial X}dW(t) \\ &= \mathcal{L}[X(t)]^4dt + \sigma(X(t),X(t-\tau))\frac{\partial X(t)^4}{\partial X}dW(t), \end{aligned}$$

where $\mathcal{L}$ denotes the differential operator that governs the dynamics of the system

$$\mathcal{L} = \partial t + f\partial_x + \frac{1}{2}\sigma^2\partial_{xx}.$$

Subsequently, it can be integrated starting from t_0 to t to obtain,

$$X^4(t) - X_o^4 = \int_{t_o}^t \mathcal{L}[X^4(s)]ds + \int_{t_o}^t \sigma(X(s),X(s-\tau))\frac{\partial X^4(s)}{\partial X}dW(s).$$

Utilizing the zero-mean property of the stochastic integral, we obtain:
$\mathbb{E}\left[\int_{t_o}^{t}\sigma(X(s),X(s-\tau)\frac{\partial X^4(s)}{\partial X}dB(s)\right]=0$. Therefore,

$$\mathbb{E}[X^4(t)-X_o^2]=\int_{t_o}^{t}\mathbb{E}\mathcal{L}[X^4(s)]ds. \tag{4}$$

From definition (2.5), we can assume

$$\mathcal{V}(t,X)\geq K_1X^4(t),\quad \mathcal{V}(t_0,X_0)\leq K_2X_0^4$$

This condition guarantees that $\mathcal{V}$ is bounded, ensuring that it stays within a certain limit at $t=t_o$ and

$$\mathcal{L}\mathcal{V}(t,X)\leq -K_3X^4(t).$$

Therefore, from (4)

$$\mathbb{E}[X^4(t)-X_0^4]\leq -k_3\int_{t_0}^{t}\mathbb{E}[X^4(s)]\,ds$$

Now, condition (3) implies

$$\mathbb{E}\mathcal{V}(t,X)+k_3\int_{t_0}^{t}\mathbb{E}[X^4(s)]\,ds\leq \mathbb{E}\mathcal{V}(t_0,X_0).$$

then

$$\int_{t_0}^{t}\mathbb{E}[X^4(s)]\,ds\leq \frac{1}{k_3}\mathbb{E}\mathcal{V}(t_0,X_0), \tag{5}$$

condition (2) with inequality (5) imply

$$\int_{t_0}^{t}\mathbb{E}[X^4(s)]\,ds\leq \frac{k_2}{k_3}X_0^4<\infty.$$

Thus, $X(t)$ is a process that is integrable with respect to its mean fourth moment and is stable in terms of its fourth moment.

$$\mathbb{E}[X^4(t)]\leq \frac{1}{k_1}\mathbb{E}\mathcal{V}(t,X)\leq \frac{1}{k_1}\mathbb{E}\mathcal{V}(t_0,X_0)\leq \frac{k_2}{k_1}X_0^4.$$

Following (4), and from equation (1), we get

$$d[X^4(t)]=\mathcal{L}[X^4(t)]\,dt+\frac{\partial X^4(t)}{\partial X}\sigma(X(t),X(t-\tau))\,dW(t). \tag{6}$$

where

$$\mathcal{L}[X^4(t)]=4X^3(t)f(X(t),X(t-\tau))+\sigma^2(X(t),X(t-\tau)). \tag{7}$$

By substituting (7) into (6) and taking expectation we get

$$d\mathbb{E}[X^4(t)]=\left[4\mathbb{E}[X^3(t)f(X(t),X(t-\tau))]+\mathbb{E}[\sigma^2(X(t),X(t-\tau))]\right]dt+4\mathbb{E}[X^3(t)\sigma(X(t),X(t-\tau))]dW(t).$$

By condition (2), and for positive constants C_1, C_2, we have

$$\begin{aligned}4\mathbb{E}[X^3(t)f(X(t),X(t-\tau))]&\leq \mathbb{E}\left[X^4(t)+f^2(X(t),X(t-\tau))\right]\\&\leq K_1\mathbb{E}[X^4(t)]=K_1\|X(t)\|_4^4\leq C_1.\end{aligned}$$

and

$$\mathbb{E}[\sigma^2(X(t), X(t-\tau))] \leq K_2 \mathbb{E}[X^4(t)] = K_2 \|X(t)\|_4^4 \leq C_2.$$

Hence, for $C > 0$, $\quad d\mathbb{E}[X^4(t)] \leq C$, i.e,

$$\mathbb{E}\left[\|X(t_2)\|^4 - \|X(t_1)\|^4\right] \leq C(t_2 - t_1).$$

If $0 \leq t_1 \leq t_2$ and $\mathbb{E}[X^4(t)]$ satisfies the Lipschitz condition with $\lim_{x\to\infty} \mathbb{E}[X^4(t)]$, the zero solution of equation (1) is mean-fourth asymptotically stable. □

Theorem 3.2. *The zero solution of equation (1) is said to be* exponentially mean-fourth stable *if there exist positive constants K_1 and K_2 such that the following conditions are satisfied:*

- *The expectation of the Lyapunov function satisfies* $\mathbb{E}[\mathcal{L}(t, X)] \geq K_1 e^{\lambda t} \mathbb{E}[X^4(t)]$,
- *The initial value of the Lyapunov function is bounded as* $\mathbb{E}[\mathcal{L}(t_0, X_0)] \leq K_2 X_0^4$,
- *The Lyapunov function's expectation is non-positive, i.e.,* $\mathbb{E}[\mathcal{L}(t, X)] \leq 0$.

Proof. Let $\mathcal{V}$ be a sufficiently smooth function of $(t, X(t-\tau), X(t))$. Then, in light of equation (1), we obtain the following Itô differential:

$$\begin{aligned} d\mathcal{V}(t, X(t-\tau), X(t)) &= \frac{\partial \mathcal{V}}{\partial t}(X(t-\tau), X(t)) + f(X(t-\tau), X(t)) \frac{\partial \mathcal{V}}{\partial X}(X(t-\tau), X(t)) \\ &+ \frac{1}{2}\sigma^2(X(t-\tau), X(t)) \frac{\partial^2 \mathcal{V}}{\partial X^2}(X(t-\tau), X(t)) \\ &+ \sigma(X(t-\tau), X(t)) \frac{\partial \mathcal{V}}{\partial X}(X(t-\tau), X(t))\, dW(t). \end{aligned}$$

where,

$$\begin{aligned} &\mathcal{V}_t(X(t-\tau), X(t)) + f(X(t-\tau), X(t)) \mathcal{V}_X(X(t-\tau), X(t)) \\ &+ \frac{1}{2!}\sigma^2(X(t-\tau), X(t)) \mathcal{V}_{XX}(X(t-\tau), X(t)) = \mathcal{L}\mathcal{V}(X(t-\tau), X(t)). \end{aligned}$$

Consider the Lyapunov functional $\mathcal{V}(t, X(t-\tau), X(t))$, and let $\mathcal{L}$ be the associated differential operator. From Itô's lemma, we can write:

$$d\mathcal{V} = \mathcal{L}\mathcal{V}(X(t-\tau), X(t))\, dt + \sigma(X(t-\tau), X(t)) \mathcal{V}_X(X(t-\tau), X(t))\, dW(t).$$

Integration over $[t_0, t]$, followed by taking expectations and using the zero-mean property of the stochastic integral (see (2.8)), gives:

$$\mathbb{E}[\mathcal{V}(t, X(t-\tau), X(t))] = \mathbb{E}[\mathcal{V}(t_0, X_0)] + \int_{t_0}^{t} \mathbb{E}[\mathcal{L}\mathcal{V}(X(s-\tau), X(s))]\, ds.$$

Under assumption (3), $\mathcal{L}\mathcal{V} \leq 0$, and thus:

$$\mathbb{E}[\mathcal{V}(t, X(t-\tau), X(t))] \leq \mathbb{E}[\mathcal{V}(t_0, X_0)].$$

Invoking inequalities from (1) and (2), we derive:

$$k_1 \mathbb{E}[X^4(t)] \leq e^{-\lambda t} \mathbb{E}[\mathcal{V}(t_0, X_0)] \leq k_2 e^{-\lambda t} X_0^4,$$

which establishes exponential stability in the fourth moment:

$$\mathbb{E}[|X(t)|^4] \leq M e^{-\lambda t}, \quad \text{for some constant } M > 0.$$

□

In the following theorem, we assume that the functions f and σ are random, each defined as a stochastic process. Consequently, the solution process is denoted by $X(t) := X(t, \omega)$, where $\omega \in \Omega$ represents the inherent randomness. The process $X(t)$ is defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Theorem 3.3. *The trivial solution of equation (1) is said to be* asymptotically mean-fourth stable *if the process $X(t) := X(t-\tau, \omega)$, together with the coefficient functions $f(\omega, t-\tau)$ and $\sigma(\omega, t-\tau)$, are all stochastic processes that possess finite fourth moments; that is, they are fourth-integrable over the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.*

Proof. Applying Itô's lemma (cf. equation (2.6)) to the function $X^4(t)$, and using the stochastic differential equation (1), we have:

$$d[X^4(t)] = \frac{\partial X^4(t)}{\partial t} dt + \frac{\partial X^4(t)}{\partial X} dX(t) + \frac{1}{2} \frac{\partial^2 X^4(t)}{\partial X^2} (dX(t))^2.$$

Substituting the expression for $dX(t)$ from equation (1), we obtain:

$$dX^4(t) = \frac{d}{dt} X^4(t)\, dt + \frac{d}{dX} X^4(t)\, [f(\omega, t-\tau)\, dt + \sigma(\omega, t-\tau)\, dW(t)] + \frac{1}{2} \frac{d^2}{dX^2} X^4(t) \cdot \sigma^2(\omega, t-\tau)\, (dW(t))^2.$$

Using the identities:

$$\frac{d}{dt} X^4(t) = 0, \quad \frac{d}{dX} X^4(t) = 4X^3(t), \quad \frac{d^2}{dX^2} X^4(t) = 12X^2(t),$$

we simplify:

$$d[X^4(t)] = 4X^3(t) f(\omega, t-\tau)\, dt + 4X^3(t)\sigma(\omega, t-\tau)\, dW(t) + 6X^2(t)\sigma^2(\omega, t-\tau)\, dt.$$

As $B(t) \sim N(0, t)$ and from the quadratic variation of the Brownian motion, the property holds.

$$\mathbb{E}\left[dW^2(t)\right] = dt, \quad \mathbb{E}\left[dt^2\right] = 0, \quad \mathbb{E}\left[dW(t)\, dt\right] = 0. \tag{8}$$

we obtain:

$$d\mathbb{E}[X^4(t)] = \mathbb{E}\left[4X^3(t) f(\omega, t-\tau) + 6X^2(t)\sigma^2(\omega, t-\tau)\right] dt.$$

Now, applying Hölder's inequality (cf. equation (2.3)), there exist positive constants C_1, C_2 such that:

$$\mathbb{E}\left|f(\omega, t-\tau) X^3(t)\right| \le \left\|f(\omega, t-\tau)\right\|_m \left\|X^3(t)\right\|_n \le C_1,$$

and

$$\mathbb{E}\left|\sigma(\omega, t-\tau)\right|^4 \le C_2.$$

Thus, the expectation of $X^4(t)$ is Lipschitz continuous in time:

$$\mathbb{E}\left|X^4(t_2) - X^4(t_1)\right| \le M(t_2 - t_1), \quad \text{for all } 0 \le t_1 \le t_2,$$

where $M > 0$ is a constant.

Moreover, under the assumption that $\lim_{t\to\infty} \mathbb{E}\left[X^4(t)\right] = 0$, we conclude that the zero solution is **asymptotically mean-fourth stable**:

$$\lim_{t\to\infty} \mathbb{E}\left[|X(t)|^4\right] = 0.$$

□

Theorem 3.4. *The zero solution of equation (1) is stable in probability if there exist positive constants K_1, K_2, and K_3 such that the following conditions hold:*

- $\mathbb{E}[\mathcal{V}(t, X)] \geq K_1 \mathbb{E}[X^4(t)]$,
- $\mathbb{E}[\mathcal{V}(t_0, X_0)] \leq K_2 X_0^2$,
- $\mathbb{E}[\mathcal{V}(t, X) - \mathcal{V}(t_0, X_0)] \leq 0$.

In this case, the zero solution is stable in probability.

Proof. In the stochastic system given by equation (1), mean-fourth stability implies stability in probability. From this, it follows that for initial conditions $|X_0| < \delta$ (where $\delta > 0$), the solution will be stable in probability.
Moreover, from condition (3), we know that $L\mathcal{V}(X(t), X(t))$ is a super-martingale. Applying this super-martingale property together with conditions (1) and (2) yields the following inequality:

$$P\left(\sup_t |X(t, X_0)| > \epsilon_1 \mid \mathcal{F}_{t_0}\right) \leq P\left(\sup_t [\mathcal{V}(t, X(t))]^{\frac{1}{4}} K_1 > \epsilon_1 \mid \mathcal{F}_{t_0}\right).$$

This simplifies to:

$$P\left(\sup_t \mathcal{V}(t, X(t)) > K_1 \epsilon_1^4 \mid \mathcal{F}_{t_0}\right).$$

By using the properties of $\mathcal{V}(t, X(t))$, we obtain an upper bound for the probability:

$$\leq \frac{\mathcal{V}(t_0, X_0)}{K_1 \epsilon_1^4} \leq \frac{K_2 X_0^4}{K_1 \epsilon_1^4} \leq \frac{K_2 \delta^4}{K_1 \epsilon_1^4} < \epsilon.$$

Therefore, the zero solution is stable in probability for this stochastic time-delay system. Consider the case where the coefficients in the stochastic time-delay system are random variables

$$\{dX(t, \omega) = a(\omega) f(X(t), X(t-\tau))\, dt + b(\omega)\sigma(X(t), X(t-\tau)) dW(t), X(t_0) = X_0.\} \tag{9}$$

- $a(\omega)$, $b(\omega)$ that are random variables satisfying specific conditions.
- $W(t)$ is a one-dimensional Brownian motion.
- $f(X(t), X(t-\tau))$, $\sigma(X(t), X(t-\tau))$ are stochastic processes.
- τ is time delay.

□

Theorem 3.5. *The zero solution of equation (9) is asymptotically mean-square stable if the following conditions hold:*

1. $\mathbb{E}[a^2(\omega)] < \infty$ *and* $\mathbb{E}[b^2(\omega)] < \infty$,
2. $f(X(t), X(t-\tau))$ *and* $\sigma(X(t), X(t-\tau))$ *satisfy the Lipschitz condition,*
3. $\lambda > 0$.

Proof. We begin by applying the Ito formula (2.6) to the function $X^4(t)$, we obtain

$$d\left(X^4(t)\right) = \mathcal{L}\left(X^4(t)\right) dt + 4X^3(t)\, b(\omega)\, \sigma\left(X(t), X(t-\tau)\right) dW(t),$$

where $\mathcal{L}$ is the infinitesimal generator given by

$$\mathcal{L}\left(X^4(t)\right) = 4X^3(t)\, a(\omega)\, f\left(X(t), X(t-\tau)\right) + 6X^2(t)\, b^2(\omega)\, \sigma^2\left(X(t), X(t-\tau)\right).$$

Taking the expectation of both sides, we get

$$\mathbb{E}\left[d\left(X^4(t)\right)\right] = \mathbb{E}\left[\mathcal{L}\left(X^4(t)\right)\right] dt.$$

This simplifies to

$$\mathbb{E}\left[\mathcal{L}\left(X^4(t)\right)\right] = 4\mathbb{E}\left[X^3(t)\, a(\omega)\, f\left(X(t), X(t-\tau)\right)\right] + 6\mathbb{E}\left[X^2(t)\, b^2(\omega)\, \sigma^2\left(X(t), X(t-\tau)\right)\right].$$

By applying the Lipschitz condition on f and σ, we obtain the following bounds:

$$\left|f\left(X(t), X(t-\tau)\right)\right| \le K\left(1 + |X(t)| + |X(t-\tau)|\right),$$

$$\left|\sigma\left(X(t), X(t-\tau)\right)\right| \le \mathcal{L}\left(1 + |X(t)| + |X(t-\tau)|\right).$$

Hence, we can bound the expectation as

$$\mathbb{E}\left[\mathcal{L}\left(X^4(t)\right)\right] \le 4K\mathbb{E}\left[a^2(\omega)\right]\mathbb{E}\left[X^4(t)\right] + 6\mathcal{L}\mathbb{E}\left[b^2(\omega)\right]\mathbb{E}\left[X^4(t)\right].$$

Applying Grownwall's lemma (2.9), we arrive at the following inequality:

$$\mathbb{E}\left[X^4(t)\right] \le e^{4K\mathbb{E}\left[a^2(\omega)\right]+6\mathcal{L}\mathbb{E}\left[b^2(\omega)\right]}\mathbb{E}\left[X^4(0)\right].$$

This shows that the system exhibits mean-fourth stability, implying that

$$\limsup_{t\to\infty} \mathbb{E}\left[|X(t)|^4\right] = 0$$

if the condition

$$4K\mathbb{E}\left[a^2(\omega)\right] + 6\mathcal{L}\mathbb{E}\left[b^2(\omega)\right] < -\lambda$$

holds for some $\lambda > 0$. Therefore, the following conditions must be satisfied:

$$\mathbb{E}\left[a^2(\omega)\right] < \infty, \quad \mathbb{E}\left[b^2(\omega)\right] < \infty,$$

and the functions $f(X(t), X(t-\tau))$ and $\sigma(X(t), X(t-\tau))$ must satisfy the Lipschitz condition, with $\lambda > 0$. □

4. Numerical examples

Here, we provide a number of examples to illustrate the practical relevance of the stability criteria developed in this study. These case studies are intended to verify the theoretical results and show their performance under a range of stochastic conditions. Additionally, numerical simulations are carried out to graphically depict system behavior and further substantiate the theoretical analysis.

Example 4.1. *Consider the modified nonlinear stochastic scalar differential equation:*

$$dX(t) = -aX^3(t)\, dt + b\cos(X(t))\, dB(t), \quad a > 0.$$

For analyzing mean-fourth stability, we introduce the Lyapunov function:

$$V(t-\tau, X(t)) = X^4(t).$$

Using Itô's lemma for the function $V(t-\tau, X(t)) = X^4(t)$, we get:

$$dV(t, X(t)) = LV(t-\tau, X(t))\, dt + V_x(t-\tau, X(t))\, b\cos(X(t))\, dB(t),$$

where $LV(t-\tau, X(t))$ is given by:

$$LV(t-\tau, X(t)) = V_t(t-\tau, X(t)) + V_x(t-\tau, X(t))\,(-aX^3(t)) + \frac{1}{2}V_{xx}(t-\tau, X(t))\,b^2\cos^2(X(t)).$$

We have the following partial derivatives:

$$V_x(t-\tau, X(t)) = 4X^3(t), \quad V_{xx}(t-\tau, X(t)) = 12X^2(t).$$

Thus,

$$\begin{aligned} LV(t-\tau, X(t)) &= 4X^3(t)\,(-aX^3(t)) + \frac{1}{2}\cdot 12X^2(t)\,b^2\cos^2(X(t)) \\ &= -4aX^6(t) + 6b^2X^2(t)\cos^2(X(t)). \end{aligned}$$

Since $\cos^2(X(t)) \leq 1$, *we get:*

$$LV(t-\tau, X(t)) \leq -4aX^6(t) + 6b^2X^2(t).$$

For mean-fourth stability, the Lyapunov function $E[X^4(t)]$ *must remain bounded. Thus, the condition for asymptotic mean-fourth stability becomes:*

$$6b^2 - 4a < 0 \quad \Rightarrow \quad b^2 < \frac{2}{3}a.$$

By introducing the Lyapunov function $V(t-\tau, X(t)) = X^4(t)$, *we derive the condition*

$$b^2 < 2a$$

for mean-fourth stability. If this condition holds, the zero equilibrium point of the system is stable in the mean-fourth sense. To illustrate the conditions for stability, we plot the regions defined by the inequalities $b^2 < 2a$ *and* $b^2 < \frac{2}{3}a$.

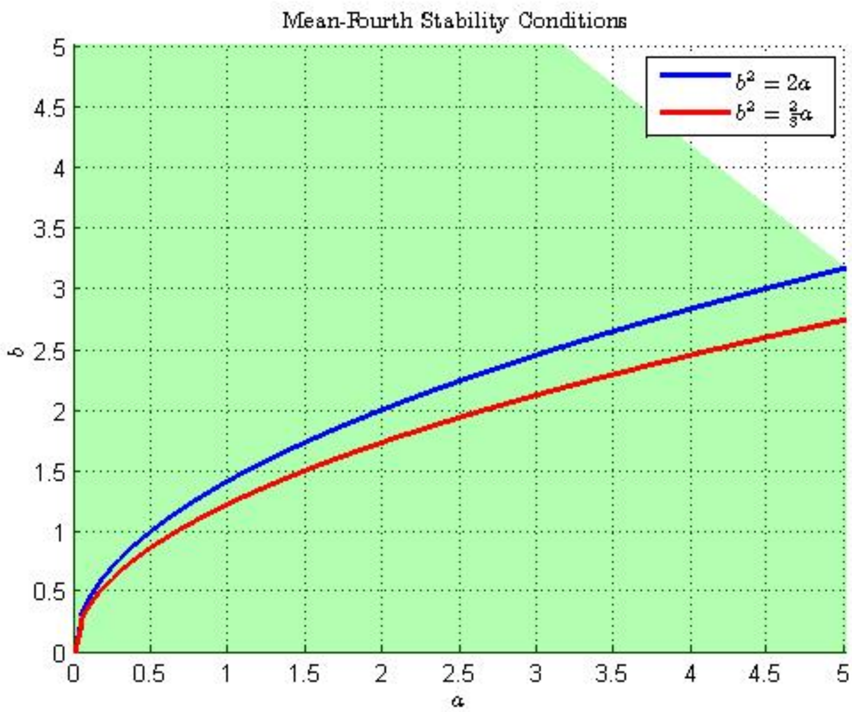

Figure 1: Mean Fourth Stability Condition

The figure above displays the stability regions in the (a, b^2) parameter space. The boundaries defined by the inequalities $b^2 < \frac{2}{3}a$ and $b^2 < 2a$ are shown as two distinct curves, demonstrating how different values of a and b impact the system's stability. The intersection of the regions beneath both curves represents the parameter space where mean-fourth asymptotic stability is guaranteed. Notably, increasing the damping coefficient a expands the range of noise intensities b for which stability is maintained. This visual clearly supports the analytical results by illustrating the stability conditions derived in the theoretical analysis.

Example 4.2. *Consider a nonlinear stochastic time-delay system governed by the following stochastic differential equation with random coefficients $A(\omega)$ and $B(\omega)$:*

$$dX(t,\omega) = -A(\omega)X(t,\omega)dt + B(\omega)\sin(X(t-\tau,\omega))dW(t),$$

where $W(t)$ is the one-dimensional Brownian motion, and τ denotes the time delay.

We define the Lyapunov function as:

$$V(t, X(t,\omega)) = X^4(t,\omega),$$

and apply Itô's lemma to obtain the differential of $V(t, X(t,\omega))$:

$$dV(t, X(t,\omega)) = LV(t, X(t,\omega))dt + V_x(t, X(t,\omega))B(\omega)\sin(X(t-\tau,\omega))dW(t),$$

where L represents the generator of the system, and the expression for $LV(t, X(t,\omega))$ is given by:

$$LV(t, X(t,\omega)) = -4A(\omega)X^6(t,\omega) + 6B^2(\omega)X^2(t,\omega)\sin^2(X(t-\tau,\omega)).$$

By taking expectations and applying the Cauchy-Schwarz inequality, we arrive at the following upper bound for the expectation of $LV(t, X(t,\omega))$:

$$E[LV(t, X(t,\omega))] \leq -4E[A(\omega)X^6(t,\omega)] + \left[E[B^4(\omega)]\right]^{1/2}\left[E[X^4(t,\omega)]\right]^{1/2}.$$

If $X(t,\omega)$ is independent of $A(\omega)$ and $B(\omega)$, we can simplify the expectation as:

$$E[LV(t, X(t,\omega))] \leq \left[-4E[A(\omega)] + E[B^2(\omega)]\right]E[X^6(t,\omega)].$$

However, when there is dependency between $X(t,\omega)$ and the random coefficients $A(\omega)$ and $B(\omega)$, the upper bound becomes:

$$E[LV(t, X(t,\omega))] \leq \left[\left[E[B^4(\omega)\right]^{1/2} - 2\left[E[A^2(\omega)\right]^{1/2}\right]E[X^2(t,\omega)].$$

For the system to exhibit asymptotic mean-fourth stability, the following condition must hold:

$$||B(\omega)||_4^4 - 2||A(\omega)||_2^2 < 0,$$

where $A(\omega)$ follows a nonnegative bounded distribution with finite support. This inequality defines the stability region for the system's dynamics, which can be visualized by plotting the region corresponding to the inequality $||B(\omega)||_4^4 < 2||A(\omega)||_2^2$.

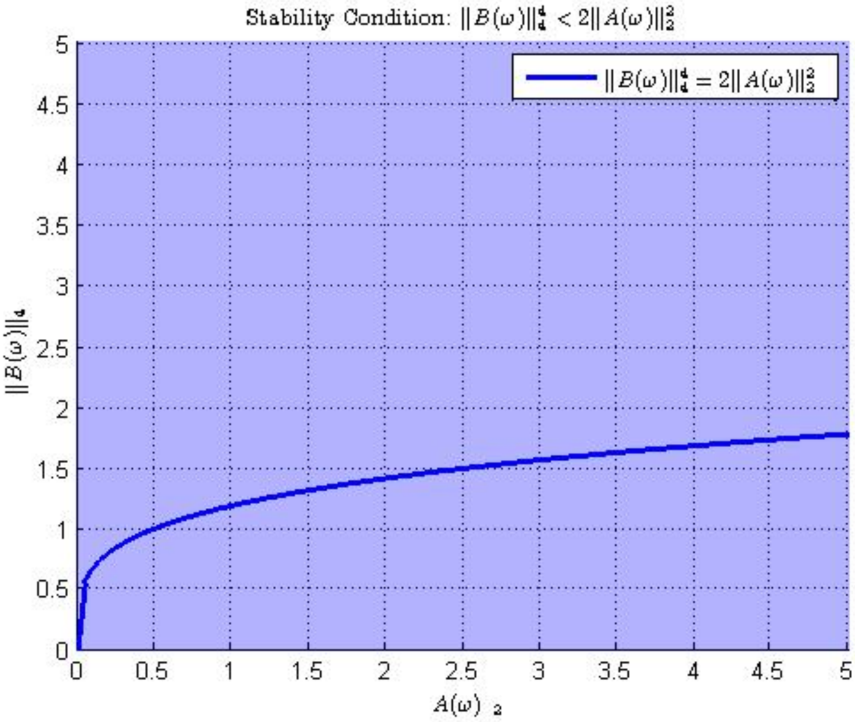

Figure 2: Mean Fourth Stability Condition

Figure 2 depicts the region in which the inequality $\|B(\omega)\|_4^4 < 2\|A(\omega)\|_2^2$ holds, ensuring asymptotic mean-fourth stability. The boundary curve clearly distinguishes between the stable and unstable domains based on the statistical properties of the system's coefficients. The requirement that the fourth moment of the noise term $B(\omega)$ be sufficiently small relative to the squared second moment of $A(\omega)$ reflects the balancing effect between noise and damping. This graphical representation confirms that, despite inherent randomness, stability is achievable when these moment conditions are satisfied. Overall, the plot effectively supports the theoretical stability criterion derived from moment analysis.

Example 4.3. *Consider the following modified system of stochastic differential equations with time delays:*

$$dX_1(t) = aX_2(t-\tau)dt + X_1(t)dB_1(t),$$
$$dX_2(t) = bX_1(t-\tau)dt + X_2(t)dB_2(t),$$

where $B(t) = (B_1(t), B_2(t))^T$ is a standard Wiener process, and $\tau > 0$ is the time delay. This system can be represented in matrix form as:

$$dX(t) = \begin{pmatrix} aX_2(t-\tau) \\ bX_1(t-\tau) \end{pmatrix} dt + \begin{pmatrix} X_1(t) & 0 \\ 0 & X_2(t) \end{pmatrix} dB(t).$$

To analyze mean-fourth stability, we define a Lyapunov function that incorporates fourth moments. A suitable choice is:

$$L(t, X(t)) = E[X_1^4(t)] + E[X_2^4(t)].$$

Using Ito's lemma for stochastic processes, we can compute the time derivative of the Lyapunov function:

$$dL(t, X(t)) = L_L(t, X(t))dt + L_{X_1}(t, X(t))X_1(t)dB_1(t) + L_{X_2}(t, X(t))X_2(t)dB_2(t),$$

where

$$L_{X_i} = \frac{\partial L}{\partial X_i}.$$

First, we compute the derivatives:

$$L_{X_1} = 4E[X_1^3(t)],$$
$$L_{X_2} = 4E[X_2^3(t)].$$

Now, we can compute the deterministic part $L_L(t, X(t))$:

$$\begin{aligned} L_L(t, X(t)) &= L_{X_1}(t, X(t))\,(aE[X_2(t-\tau)]) + L_{X_2}(t, X(t))\,(bE[X_1(t-\tau)]) \\ &= 4E[X_1^3(t)]aE[X_2(t-\tau)] + 4E[X_2^3(t)]bE[X_1(t-\tau)]. \end{aligned}$$

Next, we consider the stochastic parts. The quadratic variations will contribute:

$$dB_1(t)dB_1(t) = dt,$$
$$dB_2(t)dB_2(t) = dt.$$

Thus, the Ito term becomes:

$$\frac{1}{2}\left(E[X_1^4(t)]dt + E[X_2^4(t)]dt\right).$$

Putting everything together, we have:

$$\begin{aligned} dL(t, X(t)) = &\left(4\left(aE[X_1^3(t)]E[X_2(t-\tau)] + bE[X_2^3(t)]E[X_1(t-\tau)]\right) + 2E[X_1^4(t)] + 2E[X_2^4(t)]\right) \\ &dt + 4E[X_1^3(t)]X_1(t)dB_1(t) + 4E[X_2^3(t)]X_2(t)dB_2(t). \end{aligned}$$

Taking expectations, we arrive at:

$$\begin{aligned} E[dL(t, X(t))] \leq\ &4\left(aE[X_1^3(t)]E[X_2(t-\tau)] + bE[X_2^3(t)]E[X_1(t-\tau)]\right) \\ &+ 2E[X_1^4(t)] + 2E[X_2^4(t)]. \end{aligned}$$

To ensure stability, we require conditions on a and b. Specifically, we want:

$$4(a + b) < -2.$$

Thus, we derive the condition for asymptotic mean-fourth stability:

$$a + b < -\frac{1}{2}.$$

Let's take specific values for parameters to illustrate stability. Set

$$a = -1 \quad \text{and} \quad b = -0.6.$$

Then we have:

$$a + b = -1 - 0.6 = -1.6 < -\frac{1}{2},$$

which satisfies the stability condition. This example illustrates a modified two-dimensional stochastic time-delay system and the conditions for its asymptotic mean-fourth stability. To illustrate the stability condition, we plot the region defined by the inequality $a + b < -\frac{1}{2}$.

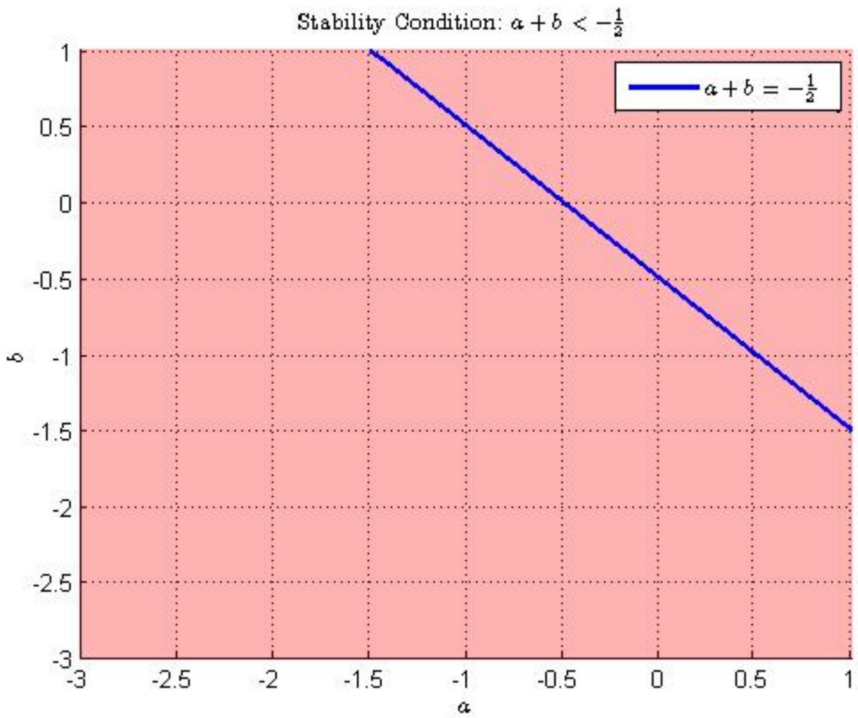

Figure 3: Mean Fourth Stability Condition

Figure 3 illustrates the stability region defined by the inequality $a+b<-\frac{1}{2}$. *This region identifies the combinations of* a *and* b *that ensure asymptotic mean-fourth stability of the system. The requirement for both parameters to be negative reflects their collective damping influence, which is crucial for mitigating stochastic effects. The figure demonstrates that even modest levels of delay and coupling can lead to instability if the condition is not met. Therefore, careful adjustment of both parameters is essential when dealing with delay-coupled stochastic systems.*

5. Conclusion

This study explores the stability of systems influenced by randomness and time delays using Lyapunov theory. It identifies conditions that keep the system's behavior stable over time, particularly in terms of mean-fourth moments. Examples illustrate how changes in parameters and the nature of random variables affect stability, highlighting the role of probability distributions. Overall, the findings offer both theoretical insights and practical guidance for managing systems with stochastic elements and delays.

Conflict of interest

The authors declare that they have no conflicts of interest.

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