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The relative Haagerup property of semigroup crossed products

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Abstract. In this paper, we study the relative Haagerup property of semigroup crossed products. Let G be a lattice ordered group that acts on a unital C*-algebra $\mathcal{A}$ through an action α. We show that the inclusion $\mathcal{A} \subseteq \mathcal{A} \rtimes_{\alpha,r} G^+$ has the relative Haagerup property if and only if G has the Haagerup property.

1. Introduction

The Haagerup property was first defined for groups by Haagerup in [13], as a weaker version of amenability. Many important groups were shown to have the Haagerup property (see [6]). Since then, the Haagerup property has been considered for von Neumann algebras and C*-algebras (see [4, 5, 7, 11, 15, 20, 23, 24]). In particular, Dong extended the definition of the Haagerup property from the single C*-algebra case to the relative case of inclusions of C*-algebras in [11]. Moreover, Dong and Ruan [12] also developed a Hilbert module version of the Haagerup property for general C*-algebras $\mathcal{A} \subseteq \mathcal{B}$.

It is interesting to consider the permanence of the Haagerup property under the standard constructions of C*-algebras. The structure of group crossed products is one of the most important structures in the theory of operator algebras. It is natural to try to extend the ideas of this area to a more general setting. Semigroups have algebraic structures that are more basic than groups. Hence, various authors introduced and studied the semigroup C*-algebras and semigroup crossed products (see [14, 16, 17, 21, 22]).

In this paper, we study the relative Haagerup property of semigroup crossed products. Let G be a lattice ordered group that acts on a unital C*-algebra $\mathcal{A}$ through an action α. We show that the inclusion $\mathcal{A} \subseteq \mathcal{A} \rtimes_{\alpha,r} G^+$ has the relative Haagerup property if and only if G has the Haagerup property. This result is a generalization of [19, Theorem 5.7] which is about the Haagerup property of semigroup C*-algebras.

From now on, we always consider separable unital C*-algebras and countable discrete groups. However, by replacing the sequences with nets, we can generalize the main results of this paper to unital C*-algebras and discrete groups in an analogous way.

2. The Haagerup property

Firstly, let us recall the definition of the Haagerup property for groups from [3, Definition 12.2.1]. A discrete group G has the *Haagerup property* if there exists a sequence $\{\varphi_n\}$ of positive definite functions on G with $\varphi_n(e) = 1$, such that each φ_n vanishes at infinity and $\varphi_n(g) \to 1$ for all $g \in G$.

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Then we introduce the notion of the relative Haagerup property. Let $1 \in \mathcal{C} \subseteq \mathcal{D}$ be C*-algebras, ρ be a fixed tracial state on $\mathcal{D}$. We denote by $(L^2(\mathcal{D}, \rho), \Lambda_\rho)$ the GNS-construction associated to ρ, and let $\|\cdot\|_2$ be the associated Hilbert norm. Suppose that there exists a ρ-preserving conditional expectation $\mathcal{E}_C$ from $\mathcal{D}$ onto $\mathcal{C}$. Let $\Phi : \mathcal{D} \to \mathcal{D}$ be an $\mathcal{E}_C$-preserving $\mathcal{C}$-bimodule unital completely positive (u.c.p.) map. Then there exists a contraction $T_\Phi : L^2(\mathcal{D}, \rho) \to L^2(\mathcal{D}, \rho)$ determined by

$$\Lambda_\rho(x) \mapsto \Lambda_\rho(\Phi(x))$$

for all $x \in \mathcal{D}$. Let $e_C = T_{\mathcal{E}_C}$, then it is just the associated projection from $L^2(\mathcal{D}, \rho)$ onto $L^2(\mathcal{C}, \rho)$ (see [11]). An operator ae_Cb acts on $L^2(\mathcal{D}, \rho)$ by

$$ae_Cb(\Lambda_\rho(x)) = \Lambda_\rho(a\mathcal{E}_C(bx))$$

for all $a, b, x \in \mathcal{D}$. Let

$$F_C(\mathcal{D}) = \{T \in \mathcal{C}' \cap B(L^2(\mathcal{D}, \rho)) : T = \sum_{i \in F} a_i e_C b_i, F \ \ finite \ \ set \ \ and \ \ a_i, b_i \in \mathcal{D}\}$$

and $K_C(\mathcal{D})$ be the norm closure of $F_C(\mathcal{D})$ in $B(L^2(\mathcal{D}, \rho))$.

Definition 2.1. *The inclusion $\mathcal{C} \subseteq \mathcal{D}$ is said to have the relative Haagerup property if there exists a sequence $\{\Phi_n\}$ of $\mathcal{E}_C$-preserving, $\mathcal{C}$-bimodule, u.c.p. maps from $\mathcal{D}$ to itself such that*

1. $\|\Lambda_\rho(\Phi_n(x) - x)\|_2 \to 0$ *for every* $x \in \mathcal{D}$;
2. $T_{\Phi_n} \in K_C(\mathcal{D})$ *for all* n.

Note that the tracial state ρ in this paper may not be faithful. Hence the definition 2.1 is slightly different from [11, Definition 3.1] in which the tracial state must be faithful.

3. Semigroup crossed product

In this paper, a *lattice ordered* group is a pair $(G, \leq)$ consisting of a discrete group G and a partial order $\leq$ on G such that if e is the unit of G and $G^+ = \{s \in G | e \leq s\}$, then

1. Every pair s, t of elements of G has a least common upper bound in G^+.
2. The inequality $g \leq h$ implies $sgt \leq sht$, for all $g, h, s, t \in G$.

For details of lattice ordered groups, we refer the reader to [2] and [9]. It is known that if G is lattice ordered, then G is a quasi-lattice ordered group (see [8]) and

$$G = G^+(G^+)^{-1} = (G^+)^{-1}G^+.$$

From now on, $\mathcal{A}$ is a unital C*-algebra and α is a homomorphism from G to the group Aut($\mathcal{A}$) of automorphisms on $\mathcal{A}$. Next we recall the structure of the reduced semigroup crossed product.

Let λ be the regular isometric representation of G^+ on $\ell^2(G^+)$ and $(\pi, \mathcal{H})$ be a faithful representation of $\mathcal{A}$. For $a \in \mathcal{A}$, we define $\overline{\pi}(a) \in B(\mathcal{H} \otimes \ell^2(G^+))$ as follows:

$$\overline{\pi}(a)(\xi \otimes \delta_s) = (\pi(\alpha_s^{-1}(a))(\xi)) \otimes \delta_s$$

for all $\xi \in \mathcal{H}$ and $s \in G^+$. The homomorphisms $\overline{\pi}$ and $I_\mathcal{H} \otimes \lambda$ satisfy the covariance relation

$$\overline{\pi}(\alpha_s(a))(I_\mathcal{H} \otimes \lambda_s) = (I_\mathcal{H} \otimes \lambda_s)\overline{\pi}(a)$$

for all $s \in G^+$ and $a \in \mathcal{A}$, where $I_\mathcal{H}$ is the identity operator on $\mathcal{H}$. The *reduced semigroup crossed product* $\mathcal{A} \rtimes_{\alpha,r} G^+$ is the C*-subalgebra of $B(\mathcal{H} \otimes \ell^2(G^+))$ generated by $\{\overline{\pi}(a) : a \in \mathcal{A}\}$ and $\{I_\mathcal{H} \otimes \lambda_s : s \in G^+\}$. In fact, the reduced semigroup crossed product does not depend on the choice of the faithful representation $(\pi, \mathcal{H})$.

We identify $\mathcal{A}$ as a subset of $\mathcal{A} \rtimes_{\alpha,r} G^+$ through its canonical embeddings. Let $B_0 = \text{span}\{a\lambda_s\lambda_t^* : a \in \mathcal{A}, s, t \in G^+\}$, then B_0 is dense in $\mathcal{A} \rtimes_{\alpha,r} G^+$. We can write every element x in B_0 as a finite sum

$$x = \sum_{s,t \in G^+} a_{s,t}\lambda_s\lambda_t^*,$$

where $a_{s,t} \in \mathcal{A}$. It follows from [19, Lemma 6.2] that the map

$$\mathcal{E}_{\mathcal{A}}(\sum_{s,t \in G^+} a_{s,t}\lambda_s\lambda_t^*) = \sum_{s \in G^+} a_{s,s}$$

extends to a conditional expectation from $\mathcal{A} \rtimes_{\alpha,r} G^+$ to $\mathcal{A}$.

4. The relative Haagerup property of semigroup crossed products

In this section, τ is an α-invariant tracial state on $\mathcal{A}$. Let $\tau' = \tau \circ \mathcal{E}$. It follows from [19, Lemma 6.3] that τ' is a tracial state on $\mathcal{A} \rtimes_{\alpha,r} G^+$. We start by giving some results that will be used later.

Lemma 4.1. *[21, Proposition 2.2] Let $W : G^+ \to \mathcal{B}$ be an isometric homomorphism into a unital C^*-algebra $\mathcal{B}$. Then there is a unique extension $W : G \to \mathcal{B}$ such that $W_{u^{-1}v} = W_u^* W_v$ for all $u \in G^+$ and $v \in G$. Moreover, if $g_1, \ldots, g_m \in G$, then the matrix $(W_{g_i^{-1}g_j})_{ij}$ is positive in $M_m(\mathcal{B})$.*

Therefore, the regular isometric representation of G^+ has a unique extension $\lambda : G \to \mathcal{A} \rtimes_{\alpha,r} G^+$ such that

$$\lambda_g = \lambda_u^* \lambda_v$$

for all $g = u^{-1}v \in G$, where $u, v \in G^+$. Then we give the main results of this paper by using a similar strategy to [11].

Lemma 4.2. *Let $g, h \in G$. If $g \neq h$, then*

$$\Lambda_{\tau'}(a\lambda_g) \perp \Lambda_{\tau'}(b\lambda_h)$$

for all $a, b \in \mathcal{A}$.

Proof. If $g \neq h$, then

$$\langle \Lambda_{\tau'}(a\lambda_g), \Lambda_{\tau'}(b\lambda_h) \rangle = \tau'(\lambda_h^* b^* a \lambda_g) = \tau'(b^* a \lambda_g \lambda_h^*) = 0$$

for all $a, b \in \mathcal{A}$. Hence, $\Lambda_{\tau'}(a\lambda_g) \perp \Lambda_{\tau'}(b\lambda_h)$. □

Lemma 4.3. *If $g = u^{-1}v = st^{-1}$, where $u, v, s, t, \in G^+$, then*

$$\Lambda_{\tau'}(a\lambda_g) = \Lambda_{\tau'}(a\lambda_s\lambda_t^*)$$

for all $a \in \mathcal{A}$.

Proof. If $g = st^{-1}$, we have

$$\begin{aligned}
&\tau'((a\lambda_g - a\lambda_s\lambda_t^*)^*(a\lambda_g - a\lambda_s\lambda_t^*)) \\
&= \tau'(\lambda_g^* a^* a \lambda_g - \lambda_g^* a^* a \lambda_s \lambda_t^* - \lambda_t \lambda_s^* a^* a \lambda_g + \lambda_t \lambda_s^* a^* a \lambda_s \lambda_t^*) \\
&= \tau'(a^* a \lambda_g \lambda_g^* - a^* a \lambda_s \lambda_t^* \lambda_g^* - a^* a \lambda_g \lambda_t \lambda_s^* + a^* a \lambda_s \lambda_t^* \lambda_t \lambda_s^*) \\
&= \tau(a^* a) - \tau(a^* a) - \tau(a^* a) + \tau(a^* a) = 0.
\end{aligned}$$

for all $a \in \mathcal{A}$. Hence, $\Lambda_{\tau'}(a\lambda_g) = \Lambda_{\tau'}(a\lambda_s\lambda_t^*)$. □

Lemma 4.4. *If $x \in B_0$, then*

$$\Lambda_{\tau'}(x) = \sum_{g \in G} \Lambda_{\tau'}(\lambda_g \mathcal{E}_{\mathcal{A}}(\lambda_g^* x)).$$

Proof. For any $x = \sum_{s,t \in G^+} a_{s,t}\lambda_s\lambda_t^*$, we get

$$\begin{aligned}\Lambda_{\tau'}(x) &= \sum_{s,t \in G^+} \Lambda_{\tau'}(a_{s,t}\lambda_s\lambda_t^*) = \sum_{g \in G} \sum_{g = st^{-1}} \Lambda_{\tau'}(a_{s,t}\lambda_g) \\ &= \sum_{g \in G} \sum_{g = st^{-1}} \Lambda_{\tau'}(\lambda_g \alpha_g^{-1}(a_{s,t})) = \sum_{g \in G} \Lambda_{\tau'}(\lambda_g \mathcal{E}_{\mathcal{A}}(\lambda_g^* x)).\end{aligned}$$

This completes the proof. □

Lemma 4.5. *For each normalized positive definite function φ on G, there is an $\mathcal{E}_{\mathcal{A}}$-preserving, u.c.p. map Φ from $\mathcal{A} \rtimes_{\alpha,r} G^+$ into itself such that*

$$\Phi(axb) = a\Phi(x)b$$

for all $a, b \in \mathcal{A}$ and $x \in \mathcal{A} \rtimes_{\alpha,r} G^+$.

Proof. We define $\Phi : B_0 \to B_0$ by

$$\Phi(\sum_{s,t \in G^+} a_{s,t}\lambda_s\lambda_t^*) = \sum_{s,t \in G^+} \varphi(st^{-1}) a_{s,t}\lambda_s\lambda_t^*.$$

Then it follows from [18] that Φ is a u.c.p. map and it is easy to see that

$$\mathcal{E}_{\mathcal{A}} \circ \Phi = \mathcal{E}_{\mathcal{A}}.$$

For each $x = \sum_{s,t \in G^+} a_{s,t}\lambda_s\lambda_t^* \in B_0$, we have

$$\begin{aligned}\Phi(ax) &= \Phi(\sum_{s,t \in G^+} a a_{s,t}\lambda_s\lambda_t^*) = \sum_{s,t \in G^+} \varphi(st^{-1}) a a_{s,t}\lambda_s\lambda_t^* \\ &= a \sum_{s,t \in G^+} \varphi(st^{-1}) a_{s,t}\lambda_s\lambda_t^* = a\Phi(x)\end{aligned}$$

for all $a \in \mathcal{A}$. Similarly we can get

$$\Phi(xb) = \Phi(x)b$$

for all $b \in \mathcal{A}$ and $x \in B_0$. Hence it follows from continuity that

$$\Phi(axb) = a\Phi(x)b$$

for all $a, b \in \mathcal{A}$ and $x \in \mathcal{A} \rtimes_{\alpha,r} G^+$. □

Theorem 4.6. *The inclusion $\mathcal{A} \subseteq \mathcal{A} \rtimes_{\alpha,r} G^+$ has the relative Haagerup property if and only if G has the Haagerup property.*

Proof. Suppose that the inclusion $\mathcal{A} \subseteq \mathcal{A} \rtimes_{\alpha,r} G^+$ has the relative Haagerup property. Let $\{\Phi_n\}$ be as in Definition 2.1. Define $\varphi_n(g) : G \to \mathbb{C}$ by

$$\varphi_n(g) = \tau'(\Phi_n(\lambda_g)\lambda_g^*).$$

If $\{g_1, \ldots, g_m\}$ is an arbitrary finite subset in G, we choose $h \in G^+$ such that $s_i = hg_i \in G^+$ for all $i = 1, 2 \cdots n$. For all $g_1, \ldots, g_m \in G$ and all $c_1, \ldots, c_m \in \mathbb{C}$, the positivity of τ yields

$$\begin{aligned}\sum_{i,j=1}^{m} c_i\bar{c}_j\varphi_n(g_j^{-1}g_i) &= \sum_{i,j=1}^{m} c_i\bar{c}_j\tau'((\Phi_n(\lambda_{g_j^{-1}g_i}))\lambda^*_{g_j^{-1}g_i}) \\ &= \sum_{i,j=1}^{m} c_i\bar{c}_j\tau'((\Phi_n(\lambda_{s_j^{-1}s_i}))\lambda^*_{s_j^{-1}s_i}) \\ &= \sum_{i,j=1}^{m} \tau'(\bar{c}_j\lambda_{s_j}\Phi_n(\lambda^*_{s_j}\lambda_{s_i})c_i\lambda^*_{s_i}) \geq 0.\end{aligned}$$

Hence, φ_n is positive definite on G. Moreover, as $n \to +\infty$,

$$\begin{aligned}|\varphi_n(g) - 1| &= |\tau'(\Phi_n(\lambda_g)\lambda_g^*) - 1| = |\tau'(\Phi_n(\lambda_g)\lambda_g^*) - \tau'(\lambda_g\lambda_g^*)| \\ &= |\tau'((\Phi_n(\lambda_g) - \lambda_g)\lambda_g^*)| \leq \|\Lambda_{\tau'}(\Phi_n(\lambda_g) - \lambda_g)\|_2 \to 0\end{aligned}$$

for all $g \in G$. Since $T_{\Phi_n} \in K_{\mathcal{A}}(\mathcal{A} \rtimes_{\alpha,r} G^+)$, there exist $x_1, ..., x_m$ in $\mathcal{A} \rtimes_{\alpha,r} G^+$ and $y_1, ..., y_m \in B_0$ such that

$$\|T_{\Phi_n} - \sum_{i=1}^{m} x_i e_{\mathcal{A}} y_i\| \leq \frac{\epsilon}{2}.$$

In particular, we have

$$\|\Lambda_{\tau'}(\Phi_n(\lambda_g)) - \Lambda_{\tau'}(\sum_{i=1}^{m} x_i\mathcal{E}_{\mathcal{A}}(y_i\lambda_g))\|_2 \leq \frac{\epsilon}{2}. \tag{1}$$

It follows from Lemma 4.4 that if $x \in B_0$, then

$$\sum_{g\in G} \|\Lambda_{\tau'}(\mathcal{E}_{\mathcal{A}}(\lambda_g^* x))\|_2^2 = \|\sum_{g\in G} \Lambda_{\tau'}(\lambda_g\mathcal{E}_{\mathcal{A}}(\lambda_g^* x))\|_2^2 = \|\Lambda_{\tau'}(x)\|_2^2 < +\infty. \tag{2}$$

Hence, for any $\delta > 0$, there exists a finite set $F_{x,\delta} \subseteq G$ such that

$$\sum_{g\in G\setminus F_{x,\delta}} \|\Lambda_{\tau'}(\mathcal{E}_{\mathcal{A}}(\lambda_g^* x))\|_2^2 < \delta^2.$$

Let $M = \max_{1\leq i\leq m}\{\|\Lambda_{\tau'}(x_i)\|_2\}$, $\delta = \frac{\varepsilon}{2(M+1)m}$ and $F_\epsilon = \bigcup_{i=1}^{m} F_{y_i^*,\delta}$. For any $g \in G \setminus F_\epsilon$, it follows from the inequalities 1 and 2 that

$$\begin{aligned}|\varphi_n(g)| &= |\tau'(\Phi_n(\lambda_g)\lambda_g^*)| \\ &\leq |\tau'(\Phi_n(\lambda_g) - \sum_{i=1}^{m} x_i\mathcal{E}_{\mathcal{A}}(y_i\lambda_g))\lambda_g^*| + |\tau'(\sum_{i=1}^{m} x_i\mathcal{E}_{\mathcal{A}}(y_i\lambda_g)\lambda_g^*)| \\ &\leq \|\Lambda_{\tau'}(\Phi_n(\lambda_g) - \sum_{i=1}^{m} x_i\mathcal{E}_{\mathcal{A}}(y_i\lambda_g))\|_2 + \sum_{i=1}^{m} |\tau'(x_i\mathcal{E}_{\mathcal{A}}(y_i\lambda_g)\lambda_g^*)| \\ &\leq \frac{\epsilon}{2} + \sum_{i=1}^{m} \|\Lambda_{\tau'}(x_i)\|_2\|\Lambda_{\tau'}(\mathcal{E}_{\mathcal{A}}(y_i\lambda_g))\|_2 \\ &= \frac{\epsilon}{2} + \sum_{i=1}^{m} \|\Lambda_{\tau'}(x_i)\|_2\|\Lambda_{\tau'}(\mathcal{E}_{\mathcal{A}}(\lambda_g^* y_i^*))\|_2 \\ &\leq \frac{\epsilon}{2} + \sum_{i=1}^{m} M\frac{\varepsilon}{2(M+1)m} < \epsilon.\end{aligned}$$

This implies that G has the Haagerup property.

Suppose that G has the Haagerup property. Then there exists a sequence of normalized positive definite functions $\{\varphi_n\}$ that witnesses the Haagerup property. By Lemma 4.5, there exists a sequence $\{\Phi_n\}$ of $\mathcal{E}_{\mathcal{A}}$-preserving, $\mathcal{A}$-bimodule, u.c.p. maps from $\mathcal{A} \rtimes_{\alpha,r} G^+$ to itself. For any $a\lambda_s\lambda_t^* \in B_0$, we have

$$\begin{aligned}\|\Lambda_{\tau'}(\Phi_n(a\lambda_s\lambda_t^*) - a\lambda_s\lambda_t^*)\|_2^2 &= \|\Lambda_{\tau'}((\varphi_n(st^{-1}) - 1)a\lambda_s\lambda_t^*)\|_2^2 \\ &= (\varphi_n(st^{-1}) - 1)^2\tau'(\lambda_t\lambda_s^*a^*a\lambda_s\lambda_t^*) \\ &= (\varphi_n(st^{-1}) - 1)^2\tau(a^*a) \to 0.\end{aligned}$$

It follows from linearity and continuity that

$$\|\Lambda_{\tau'}(\Phi_n(x) - x)\|_2 \to 0$$

for all $x \in \mathcal{A} \rtimes_{\alpha,r} G^+$. For simplicity of notation, we fix n and denote $\varphi = \varphi_n$, $\Phi = \Phi_n$. Let $S_1 \subseteq S_2 \subseteq \cdots \subseteq G$ be an increasing sequence of finite sets whose union is G, and let

$$T_m = \sum_{g\in S_m} \varphi(g)\lambda_g e_{\mathcal{A}}\lambda_g^*$$

for all $m \in \mathbb{N}$. For any $a\lambda_s\lambda_t^* \in B_0$, we get

$$\begin{aligned}\lambda_g e_{\mathcal{A}}\lambda_g^* b(\Lambda_{\tau'}(a\lambda_s\lambda_t^*)) &= \Lambda_{\tau'}(\lambda_g\mathcal{E}_{\mathcal{A}}(\lambda_g^* ba\lambda_s\lambda_t^*)) \\ &= \Lambda_{\tau'}(\lambda_g\mathcal{E}_{\mathcal{A}}(\alpha_g^{-1}(ba)\lambda_g^*\lambda_s\lambda_t^*)) \\ &= \begin{cases}\Lambda_{\tau'}(ba\lambda_g), & g = st^{-1}; \\ 0, & g \neq st^{-1},\end{cases}\end{aligned}$$

and

$$\begin{aligned}b\lambda_g e_{\mathcal{A}}\lambda_g^*(\Lambda_{\tau'}(a\lambda_s\lambda_t^*)) &= \Lambda_{\tau'}(b\lambda_g\mathcal{E}_{\mathcal{A}}(\lambda_g^* a\lambda_s\lambda_t^*)) \\ &= \Lambda_{\tau'}(b\lambda_g\mathcal{E}_{\mathcal{A}}(\alpha_g^{-1}(a)\lambda_g^*\lambda_s\lambda_t^*)) \\ &= \begin{cases}\Lambda_{\tau'}(ba\lambda_g), & g = st^{-1}; \\ 0, & g \neq st^{-1},\end{cases}\end{aligned}$$

for all $b \in \mathcal{A}$. Hence $T_m \in \mathcal{A}'$. Let $x \in B_0$. It follows from Lemma 4.4 that for any $\epsilon > 0$, there exists k_ε such that for all $k \geq k_\varepsilon$,

$$\begin{aligned}\|\Lambda_{\tau'}(x - \sum_{g\in S_k} \lambda_g\mathcal{E}_{\mathcal{A}}(\lambda_g^* x))\|_2^2 &= \|\Lambda_{\tau'}(\sum_{g\in G\setminus S_k} \lambda_g\mathcal{E}_{\mathcal{A}}(\lambda_g^* x))\|_2^2 \\ &= \sum_{g\in G\setminus S_k} \|\Lambda_{\tau'}(\mathcal{E}_{\mathcal{A}}(\lambda_g^* x))\|_2^2 \leq \epsilon^2\|\Lambda_{\tau'}(x)\|_2^2.\end{aligned}$$

Since φ vanishes at the infinity, there exists a subsequence $\{S_{k_m}\}$ such that $|\varphi(g)| \leq \frac{1}{m}$ for all $g \in G \setminus S_{k_m}$.

Pick an $m \in \mathbb{N}$, and choose that $k_\varepsilon > k_m$. For any $x \in B_0$, we have

$$\begin{aligned}
\|T_\Phi - T_{k_m}(\Lambda_{\tau'}(x))\|_2 &= \|\Lambda_{\tau'}(\sum_{g\in G}\varphi(g)\lambda_g\mathcal{E}_{\mathcal{A}}(\lambda_g^*x) - \sum_{g\in S_{k_m}}\varphi(g)\lambda_g\mathcal{E}_{\mathcal{A}}(\lambda_g^*x))\|_2 \\
&\le \|\Lambda_{\tau'}(\sum_{g\in G\setminus S_{k_\epsilon}}\varphi(g)\lambda_g\mathcal{E}_{\mathcal{A}}(\lambda_g^*x)\|_2 \\
&+ \|\Lambda_{\tau'}(\sum_{g\in S_{k_\epsilon}\setminus S_{k_m}}\varphi(g)\lambda_g\mathcal{E}_{\mathcal{A}}(\lambda_g^*x))\|_2 \\
&\le \epsilon\|\Lambda_{\tau'}(x)\|_2 + (\sum_{g\in G\setminus S_{k_m}}|\varphi(g)|^2\|\Lambda_{\tau'}(\mathcal{E}_{\mathcal{A}}(\lambda_g^*x))\|_2^2)^{\frac{1}{2}} \\
&\le (\epsilon + \frac{1}{m})\|\Lambda_{\tau'}(x)\|_2.
\end{aligned}$$

Therefore $T_\Phi \in K_{\mathcal{A}}(\mathcal{A} \rtimes_{\alpha,r} G^+)$. This implies that the inclusion $\mathcal{A} \subseteq \mathcal{A} \rtimes_{\alpha,r} G^+$ has the relative Haagerup property. □

On the other hand, it follows from [11, Theorem 3.4] that G has the Haagerup property if and only if the inclusion $\mathcal{A} \subseteq \mathcal{A} \rtimes_{\alpha,r} G$ has the relative Haagerup property, where $\mathcal{A} \rtimes_{\alpha,r} G$ is the reduced crossed product. Therefore, the relative Haagerup property of $\mathcal{A} \subseteq \mathcal{A} \rtimes_{\alpha,r} G^+$ is equivalent to the relative Haagerup property of $\mathcal{A} \subseteq \mathcal{A} \rtimes_{\alpha,r} G$.

Example 4.7. *Let $(\mathbb{F}_2, \mathbb{F}_2^+)$ be the free group on two generators with the total order and let (F, F^+) be the Thompson group F with the total order (see [10]). We denote by γ the conjugation action. We define a map $\tau : C_b(G) \to \mathbb{C}$ by $\tau(f) = f(e)$ for every $f \in C_b(G)$, where G is $\mathbb{F}_2$ or F. Then τ is a γ-invariant tracial state. Since $\mathbb{F}_2$ and F have the Haagerup property (see [1, Example 2.9.12]), then the inclusion $C_b(\mathbb{F}_2) \subseteq C_b(\mathbb{F}_2) \rtimes_{\gamma,r} \mathbb{F}_2^+$ and $C_b(F) \subseteq C_b(F) \rtimes_{\gamma,r} F^+$ have the relative Haagerup property.*

We conclude this paper with the following remark.

Remark 4.8. *If $\mathcal{A} = \mathbb{C}$, then $\mathcal{A} \rtimes_{\alpha,r} G^+$ is the reduced semigroup C^*-algebra $C_r^*(G^+)$, the conditional expectation $\mathcal{E}_{\mathcal{A}}$ is a tracial state which is denoted by τ_∞, $K_{\mathcal{A}}(\mathcal{A} \rtimes_{\alpha,r} G^+)$ is the set of all compact operators on $L^2(C_r^*(G^+), \tau_\infty)$. In the case, the relative Haagerup property of the inclusion $\mathbb{C} \subseteq C_r^*(G^+)$ is equivalent to the Haagerup property of $C_r^*(G^+)$ with respect to τ_∞. Hence, Theorem 4.6 is a generalization of [19, Theorem 5.7].*

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