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One-sided generalized Drazin-Riesz and one-sided generalized Drazin-meromorphic invertible operators

Snežana Č. Živković-Zlatanović[a]

[a] *University of Niš, Faculty of Sciences and Mathematics, P.O. Box 224, 18000 Niš, Serbia*

Abstract. The aim of this paper is to introduce and study left and right versions of the class of generalized Drazin-Riesz invertible operators, as well as left and right versions of the class of generalized Drazin-meromorphic invertible operators.

1. Introduction and background

This paper builds on papers published in recent years on the topic of the left and right version of the class of Drazin invertible as well as the class of generalized Drazin invertible operators. Each of these one-sided versions of invertibility has its own algebraic definition-see the papers [12], [6], [23], [10], [21]. Considering the classes of generalized Drazin-Riesz and generalized Drazin-meromorphic invertible operators, it was natural to set the question whether there are left and right versions of these classes. In this paper, they are introduced by algebraic definitions and various characterizations are shown.

The basic concepts of this topic are Drazin invertibility and generalized Drazin invertibility. In 1958 Drazin introduced a new kind of a generalized inverse [9]: an element a of an associative ring (or a semigroup) $\mathcal{A}$ is Drazin invertible if there exists an element $b \in \mathcal{A}$ such that $ab = ba$, $bab = b$, $a^n ba = a^n$, for some nonnegative integer n. The concept of generalized Drazin invertible elements was introduced by Koliha [18]: an element a of a Banach algebra $\mathcal{A}$ is generalized Drazin invertible if there exists $b \in \mathcal{A}$ such that $ab = ba$, $bab = b$, $aba - a$ is quasinilpotent.

In the algebra of bounded linear operators $L(X)$ on a complex Banach space X this notation has been further generalized by introducing the concept of generalized Drazin-Riesz invertibility: an operator $A \in L(X)$ is generalized Drazin-Riesz invertible, if there exists $B \in L(X)$ such that $AB = BA$, $BAB = B$, $ABA - A$ is Riesz [27]. By replacing the third condition in the previous definitions by the condition that $ABA - A$ is meromorphic, we get the concept of generalized Drazin-meromorphic invertible operators [28].

Necessary and sufficient for $A \in L(X)$ to be Drazin invertible is that T can be decomposed into a direct sum of an invertible operator and a nilpotent one [17], while A is generalized Drazin invertible if and only

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Email address: mladvlad@mts.rs, snezana.zivkovic-zlatanovic@pmf.edu.rs (Snežana Č. Živković-Zlatanović)
ORCID iD: https://orcid.org/0000-0001-8480-4898 (Snežana Č. Živković-Zlatanović)

if it can be decomposed into a direct sum of an invertible operator and a quasinilpotent one [18]. Recently, Ghorbel and Mnif [12] introduced the concept of left (resp., right) generalized Drazin invertible operators in $L(X)$, and proved that this kind of operators is characterized by the property that they can be decomposed into a direct sum of a nilpotent operator and a left invertible (resp., right invertible) operator. On a Hilbert space the class of left (resp., right) Drazin invertible operators coincides with the class bearing the same name in existing literature (see [4], [2]), though they do not coincide on an arbitrary Banach space [27, p. 170]. Berkani, Ren and Jiang introduced the concept of left (resp., right) Drazin invertible elements in an associative ring R with a unit: an element $a \in R$ is left (resp., right) Drazin invertible if and only if there exists an element $b \in R$ such that $bab = b^2a = b$ and $a - aba = a - ba^2$ is nilpotent (resp. $bab = ab^2 = b$ and $a - aba = a - a^2b$ is nilpotent) if and only if there exists an idempotent p commuting with a such that ap is nilpotent and $a+p$ is left (resp., right) invertible [6, Theorems 2.5 and 2.6], [23, Lemma 2.1, Theorems 2.1 and 2.2]. From [12, Theorems 3.4 and 3.8] it follows that the set of left (resp., right) Drazin invertible operators in $L(X)$, in the sense of Ghorbel and Mnif, coincides with the set of left (resp., right) Drazin invertible elements in the Banach algebra $L(X)$, in the sense of Berkani, Ren and Jiang.

In [10] Ferreyra, Levis and Thome introduced the concept of left (resp., right) generalized Drazin invertible operators in $L(X)$ in terms of generalized Kato decomposition. They proved that this kind of operators is characterized by the property that they can be decomposed into a direct sum of a quasinilpotent operator and a left invertible (resp., right invertible) operator. In the case of Hilbert spaces the class of left (resp., right) generalized Drazin invertible operators coincides with the class bearing the same name and introduced by Hocine, Benharrat and Messirdi in [15]. These classes do not coincide on an arbitrary Banach space. In [21, Propositions 2.1 and 2.2] Ounandjela et al. proved that an operator $A \in L(X)$, where X is a Hilbert space, is left (resp., right) generalized Drazin invertible if and only if there exists $B \in L(X)$ such that $ABA = BA^2$, $B^2A = B = BAB$, $A - ABA$ is quasinilpotent (resp., $ABA = A^2B$, $AB^2 = B = BAB$, $A - ABA$ is quasinilpotent).

The purpose of this paper is to introduce the concept of one-sided generalized Drazin-Riesz and one-sided generalized Drazin-meromorphic invertibility in the algebra $L(X)$: an operator $A \in L(X)$ is left (resp., right) generalized Drazin-Riesz invertible if there is $B \in L(X)$ such that $ABA = BA^2$, $B^2A = B$, $A - ABA$ is Riesz (resp., $ABA = A^2B$, $AB^2 = B$, $A - ABA$ is Riesz); if the third condition in the previous definition is replaced by the condition that $A - ABA$ is meromorphic then we get the concept of left (resp., right) generalized Drazin-meromorphic invertible operators. Starting from these algebraic definitions we obtain various characterizations of left (resp., right) generalized Drazin-Riesz invertible and left (resp., right) generalized Drazin-meromorphic invertible operators, among others that an operator $A \in L(X)$ is left (resp., right) generalized Drazin-Riesz invertible if and only if it can be decomposed in a direct sum of a left (resp., right) invertible operator and a Riesz operator, while A is left (resp., right) generalized Drazin-meromorphic invertible if and only if it can be decomposed in a direct sum of a left (resp., right) invertible operator and a meromorphic operator. Some of the characterizations of left and right generalized Drazin-meromorphic invertible operators include among others the concepts of left and right Drazin invertible operators introduced and considered in [12], [6] and [23], while some of the characterizations of left and right generalized Drazin-Riesz invertible operators include the concepts of left and right Browder operators.

The introduction of the notation of generalized Saphar-Riesz decomposition as well as the notation of generalized Saphar-meromorphic decomposition allowed us to obtain some characteristics of these classes of operators, which results in obtaining various properties of the corresponding spectra. By using the concept of generalized Saphar-meromorphic decomposition it is proved that on an arbitrary Banach space the class of left (resp., right) generalized Drazin-Riesz invertible operators does not coincide with the class of operators that are characterized by the property that they can be decomposed into a direct sum of a bounded bellow (resp., surjective) operator and a Riesz operator. Also it is proved that on an arbitrary Banach space the class of left (resp., right) generalized Drazin-meromorphic invertible operators does not coincide with the class of operators that are characterized by the property that they can be decomposed into a direct sum of a bounded bellow (resp., surjective) operator and a meromorphic operator (Remark 5.16).

The paper is organized as follows. The second section contains some definitions and the third section contains some preliminary results. In the forth section we introduce and investigate left (right) generalized Drazin-Riesz invertible operators. The fifth section is devoted to left (right) generalized Drazinmerpmorphic invertible operators. The sixth section contains various properties of corresponding spectra.

2. Definitions

Let $L(X)$ be the Banach algebra of bounded linear operators acting on an infinite dimensional complex Banach space X. The dual of X is denoted by X', and the dual of $A \in L(X)$ by A'. Here $\mathbb{N}$ ($\mathbb{N}_0$) denotes the set of all positive (non-negative) integers, $\mathbb{C}$ denotes the set of all complex numbers.

Let $A \in L(X)$. By $\sigma(A)$, $\sigma_l(A)$, $\sigma_r(A)$ and $\sigma_p(A)$ we denote its spectrum, left spectrum, right spectrum and point spectrum, respectively. The compression spectrum of A, denoted by $\sigma_{cp}(A)$, is the set of all complex λ such that $A - \lambda I$ does not have dense range. We use $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively, to denote the null-space and the range of A.

For $n \in \mathbb{N}_0$ set $\alpha_n(A) = \dim(\mathcal{N}(A) \cap \mathcal{R}(A^n))$ and $\beta_n(A) = \operatorname{codim}(\mathcal{R}(A) + \mathcal{N}(A^n))$, and set $\alpha(A) = \alpha_0(A)$ and $\beta(A) = \beta_0(A)$. We define the infimum of the empty set to be ∞. The ascent $a(A)$ and the descent $d(A)$ of A are defined by $a(A) = \inf\{n \in \mathbb{N}_0 : \alpha_n(A) = 0\}$ and $d(A) = \inf\{n \in \mathbb{N}_0 : \beta_n(A) = 0\}$. The essential ascent $a_e(A)$ and essential descent $d_e(A)$ of A are defined by $a_e(A) = \inf\{n \in \mathbb{N}_0 : \alpha_n(A) < \infty\}$ and $d_e(A) = \inf\{n \in \mathbb{N}_0 : \beta_n(A) < \infty\}$. The descent spectrum of A is $\sigma_{dsc}(A) = \{\lambda \in \mathbb{C} : d(T - \lambda I) = \infty\}$. The degree of stable iteration $\operatorname{dis}(A)$ is defined by:

$$\operatorname{dis}(A) = \inf\{n \in \mathbb{N}_0 : m \geq n, m \in \mathbb{N} \Longrightarrow \mathcal{N}(A) \cap \mathcal{R}(A^n) = \mathcal{N}(A) \cap \mathcal{R}(A^m)\}.$$

A subspace M of X is complemented in X if it is closed and there is a closed subspace N of X such that $X = M \oplus N$. If $\alpha(A) < \infty$, $a(A) < \infty$ and $\mathcal{R}(A)$ is complemented, then A is called *left Browder*. If $\beta(A) < \infty$, $d(A) < \infty$ and $\mathcal{N}(A)$ is complemented, then A is called *right Browder*. The set of left (right) Browder operators is denoted by $\mathcal{B}_l(X)$ ($\mathcal{B}_r(X)$), while $\mathcal{B}(X) = \mathcal{B}_l(X) \cap \mathcal{B}_r(X)$ is the set of Browder operators. It is well known that A is Drazin invertible if and only if $a(A) < \infty$ and $d(A) < \infty$. $A \in L(X)$ is called *left Drazin invertible* if $a(A) < \infty$ and $\mathcal{R}(A) + \mathcal{N}(A^{a(A)})$ is a complemented subspace of X. If $d(A) < \infty$ and $\mathcal{N}(A) \cap \mathcal{R}(A^{d(A)})$ is complemented in X, then A is called *right Drazin invertible* [12]. An operator $A \in L(X)$ is *upper Drazin invertible* operator if $a(A) < \infty$ and $\mathcal{R}(A^{a(A)+1})$ is closed. If $d(A) < \infty$ and $\mathcal{R}(A^{d(A)})$ is closed, then A is called *lower Drazin invertible*. The Browder spectrum, the left Browder spectrum, the right Browder spectrum, the Drazin spectrum, the left Drazin spectrum, the right Drazin spectrum and the upper Drazin spectrum of A are denoted by $\sigma_{\mathcal{B}}(A)$, $\sigma^l_{\mathcal{B}}(A)$, $\sigma^r_{\mathcal{B}}(A)$, $\sigma_D(A)$, $\sigma^l_D(A)$, $\sigma^r_D(A)$ and $\sigma^+_D(A)$, respectively.

If $K \subset \mathbb{C}$, then ∂K is the boundary of K, $\operatorname{acc} K$ is the set of accumulation points of K, $\operatorname{int} K$ is the set of interior points of K and $\operatorname{iso} K$ is the set of isolated points of K. The *connected hull* of a compact set $K \subset \mathbb{C}$, denoted by ηK, is the complement of the unbounded component of $\mathbb{C} \setminus K$ [14, Definition 7.10.1].We recall that, for compact subsets $H, K \subset \mathbb{C}$, the following implication holds ([14, Theorem 7.10.3]):

$$\partial H \subset K \subset H \Longrightarrow \partial H \subset \partial K \subset K \subset H \subset \eta K = \eta H\,. \tag{2.1}$$

An operator $A \in L(X)$ has the single-valued extension property at $\lambda_0 \in \mathbb{C}$ (SVEP at λ_0 for brevity) if for every open disc D_{λ_0} centered at λ_0 the only analytic function $f : D_{\lambda_0} \to X$ satisfying $(A - \lambda)f(\lambda) = 0$ for all $\lambda \in \mathcal{D}_{\lambda_0}$ is the function $f \equiv 0$. An operator $A \in L(X)$ is said to have the SVEP if A has the SVEP at every point $\lambda \in \mathbb{C}$.

An operator $A \in L(X)$ is a *Riesz operator*, if $\alpha(A - \lambda I) < \infty$ and $\beta(A - \lambda I) < \infty$ for every non-zero $\lambda \in \mathbb{C}$. An operator $A \in L(X)$ is called *meromorphic* if its non-zero spectral points are poles of its resolvent, and in that case we shall write $A \in (\mathcal{M})$. Every Riesz operator is meromorphic (see [29, Corollary 3.1]). We say that A is *polinomially Riesz* (*polinomially meromorphic*) if there exists non-trivial polynomial p such that $p(A)$ is Riesz (meromorphic).

If M is an A-invariant subspace of X, we define $A_M : M \to M$ as $A_M x = Ax$, $x \in M$. If M and N are two closed A-invariant subspaces of X such that $X = M \oplus N$, we say that A is *completely reduced* by the pair (M, N) denoting it by $(M, N) \in Red(A)$. In this case we write $A = A_M \oplus A_N$.

An operator $A \in L(X)$ is said to be *Kato* if $\mathcal{R}(A)$ is closed and $\mathcal{N}(A) \subset \mathcal{R}(A^n)$ for every $n \in \mathbb{N}$. An operator $T \in L(X)$ is said to admit a *generalized Kato-Riesz decomposition*, abbreviated as *GKRD*, if there exists a pair

$(M, N) \in Red(T)$ such that T_M is Kato and T_N is Riesz. An operator $A \in L(X)$ is said to admit a *generalized Kato-meromorphic decomposition*, abbreviated to $GK(\mathcal{M})D$, if there exists a pair $(M, N) \in Red(T)$ such that T_M is Kato and T_N is meromorphic. The generalized Kato-Riesz spectrum and the generalized Kato-meromorphic spectrum of $A \in L(X)$ are denoted by $\sigma_{gKR}(A)$ and $\sigma_{gK\mathcal{M}}(A)$, respectively (see [27], [28]).

If A is Kato operator and $\mathcal{N}(A)$ and $\mathcal{R}(A)$ are complemented in X, then A is called a *Saphar operator*. An operator $A \in L(X)$ is called *essentially Saphar* if $\mathcal{N}(A) \overset{e}{\subset} \cap_{n=1}^{\infty} \mathcal{R}(A^n)$, and $\mathcal{N}(A)$ and $\mathcal{R}(A)$ are complemented in X [20, p. 233]. From [22, Theorem 2.1] and [25, Lemma 2.1] it follows that A is essentially Saphar if and only if there exists $(M, N) \in Red(A)$ such that $\dim N < \infty$, A_N is nilpotent and A_M is Saphar. The degree of a nilpotent operator A is the smallest $d \in \mathbb{N}_0$ such that $A^d = 0$. We say that $A \in L(X)$ is *of Saphar type of degree* d if there exists a pair $(M, N) \in Red(A)$ such that A_M is Saphar and A_N is nilpotent of degree d [30]. In that case $\text{dis}(A) = d$. We recall that $A \in L(X)$ is of Saphar type of degree d if and only if $\text{dis}(A) = d$ is finite and the subspaces $\mathcal{R}(A) + \mathcal{N}(A^d)$ and $\mathcal{N}(A) \cap \mathcal{R}(A^d)$ are complemented [30, Theorem 4.2]. It is said that $A \in L(X)$ admits a *generalized Saphar decomposition* if there exists a pair $(M, N) \in Red(A)$ such that A_M is Saphar and A_N is quasinilpotent [19]. The Saphar spectrum, the essentially Saphar spectrum, the Saphar type spectrum and the generalized Saphar spectrum of A are denoted by $\sigma_S(A)$, $\sigma_{eS}(A)$, $\sigma_{St}(A)$ and $\sigma_{gS}(A)$, respectively.

3. Preiliminary results

Lemma 3.1. *[30, Lemma 3.2] Let $X = X_1 \oplus X_2 \oplus \cdots \oplus X_n$ where $X_1, X_2, \dots, X_n$ are closed subspaces of X and let M_i be a subset of X_i, $i = 1, \dots, n$. Then the set $M_1 + M_2 + \cdots + M_n$ is closed if and only if M_i is closed for each $i \in \{1, \dots, n\}$.*

Lemma 3.2. *[30, Lemma 3.3] Let $X = X_1 \oplus X_2 \oplus \cdots \oplus X_n$ where $X_1, X_2, \dots, X_n$ are closed subspaces of X and let M_i be a subspace of X_i, $i = 1, \dots, n$. Then the subspace $M_1 \oplus M_2 \oplus \cdots \oplus M_n$ is a complemented subspace of X if and only if M_i is a complemented subspace of X_i for each $i \in \{1, \dots, n\}$.*

Lemma 3.3. *[19, Lemma 2.6] If M is a complemented subspace of X, then $M^{\perp}$ is a complemented subspace of X'.*

Lemma 3.4. *[30, Lemma 3.5] If M is complemented subspace of X and M_1 is a closed subspace of X such that $M \subset M_1$, then M is complemented in M_1.*

Lemma 3.5. *[30, Lemma 3.11] For $A \in L(X)$ let there exists a pair $(M, N) \in Red(A)$. Then A is left (resp., right) invertible if and only if A_M and A_N are left (resp., right) invertible.*

Proof. From the equalities $\mathcal{N}(A) = \mathcal{N}(A_M) \oplus \mathcal{N}(A_N)$, $\mathcal{R}(A) = \mathcal{R}(A_M) \oplus \mathcal{R}(A_N)$ and Lemma 3.2 it follows that $\mathcal{N}(A)$ (resp. $\mathcal{R}(A)$) is complemented if and only if $\mathcal{N}(A_M)$ and $\mathcal{N}(A_N)$ (resp. $\mathcal{R}(A_M)$ and $\mathcal{R}(A_N)$) are complemented subspaces in M and N, respectively. Since a bounded linear operator is left (resp. right) invertible if and only if it is injective (resp., surjective) with a complemented range (resp. null-space), we get the desired conclusion. □

Lemma 3.6. *For $A \in L(X)$ let there exists a pair $(M, N) \in Red(A)$. Then*

(i) *[26, Lemma 2.11] A is Riesz if and only if A_M and A_N are Riesz.*

(ii) *[28, Lemma 2] A is meromorphic if and only if A_M and A_N are meromorphic.*

(iii) *[30, Lemma 3.11] A is Saphar if and only if A_M and A_N are Saphar.*

(iv) *[25, Lemma 2.1] A is left (resp., right) Browder if and only if A_M and A_N are left (resp., right) Browder.*

Lemma 3.7. *For $A \in L(X)$ let there exists a pair $(M, N) \in Red(A)$. Then A is of Saphar type if and only if A_M and A_N are of Saphar type.*

Proof. For $n \in \mathbb{N}_0$ it holds

$$\begin{aligned} \mathcal{N}(A) \cap \mathcal{R}(A^n) &= (\mathcal{N}(A_M) \cap \mathcal{R}(A_M^n)) \oplus (\mathcal{N}(A_N)) \cap \mathcal{R}(A_N^n)), \\ \mathcal{R}(A) + \mathcal{N}(A^n) &= (\mathcal{R}(A_M) + \mathcal{N}(A_M^n)) \oplus (\mathcal{R}(A_N)) + \mathcal{N}(A_N^n)). \end{aligned} \tag{3.1}$$

which implies that $\text{dis}(A) < \infty$ if and only if $\text{dis}(A_M) < \infty$ and $\text{dis}(A_N) < \infty$, and in that case $\text{dis}(A) = \max\{\text{dis}(A_M), \text{dis}(A_N)\}$. Let $d = \text{dis}(A) < \infty$. From (3.1) and Lemma 3.2 it follows that $\mathcal{N}(A) \cap \mathcal{R}(A^d)$ and $\mathcal{R}(A) + \mathcal{N}(A^d)$ are complemented if and only if $\mathcal{N}(A_M) \cap \mathcal{R}(A_M^d)$ and $\mathcal{R}(A_M) + \mathcal{N}(A_M^d)$ are complemented subspaces of M, and $\mathcal{N}(A_N) \cap \mathcal{R}(A_N^d)$ and $\mathcal{R}(A_N) + \mathcal{N}(A_N^d)$ are complemented subspaces of N. Now according to [30, Theorem 4.2] we conclude that A is of Saphar type if and only if A_M and A_N are of Saphar type. □

Lemma 3.8. *For $A \in L(X)$ let there exists a pair $(M, N) \in Red(A)$. Then A is left (right) Drazin invertible if and only if A_M and A_N are left (right) Drazin invertible.*

Proof. From [30, Corollary 4.23] we have that A is left Drazin invertible if and only if A is of Saphar type and $a(A) < \infty$. As $a(A) < \infty$ if and only if $a(A_M) < \infty$ and $a(A_N) < \infty$, by using Lemma 3.7 and again [30, Corollary 4.23] we conclude A is left Drazin invertible if and only if A_M and A_N are left Drazin invertible.

The assertion for the case of right Drazin invertible operators is proved in a similar way by using [30, Corollary 4.24]. □

Lemma 3.9. *Let $A, B \in L(X)$. Then:*

(i) *If A and B are left Drazin invertible and $P \in L(X)$ is a projector commuting with A and B, then $AP + B(I - P)$ is left Drazin invertible.*

(ii) *If A and B are right Drazin invertible and $P \in L(X)$ is a projector commuting with A and B, then $AP + B(I - P)$ is right Drazin invertible.*

Proof. (i): Suppose that A and B are left Drazin invertible, and that $P \in L(X)$ is a projector commuting with A and B. Since $(\mathcal{R}(P), \mathcal{N}(P)) \in Red(A)$ and $(\mathcal{R}(P), \mathcal{N}(P)) \in Red(B)$, according to Lemma 3.8 it follows that $A_{\mathcal{R}(P)}$ and $B_{\mathcal{N}(P)}$ are left Drazin invertible. As $(\mathcal{N}(P), \mathcal{R}(P)) \in Red(AP + B(I - P))$, we have that

$$AP + B(I - P) = (AP + B(I - P))_{\mathcal{R}(P)} \oplus (AP + B(I - P))_{\mathcal{N}(P)} = A_{\mathcal{R}(P)} \oplus B_{\mathcal{N}(P)}. \tag{3.2}$$

Again using Lemma 3.9, from (3.2) we conclude that $AP + B(I - P)$ is left Drazin invertible.

(ii) It is proved in an analogous way as (i). □

Lemma 3.10. *Let $A, B \in L(X)$. Then:*

(i) *If A and B are left Browder and $P \in L(X)$ is a projector commuting with A and B, then $AP + B(I - P)$ is left Browder.*

(ii) *If A and B are right Browder and $P \in L(X)$ is a projector commuting with A and B, then $AP + B(I - P)$ is right Browder.*

Proof. The proof can be demonstrated by using Lemma 3.6 (iv) in a similar way as the proof of Lemma 3.9. □

4. Left and right generalized Drazin-Riesz invertible operators

The aim of this section is to introduce and explore the concept of generalized Saphar-Riesz decomposion and the concept of left and right generalized Drazin-Riesz invertible operators.

Definition 4.1. Let $A \in L(X)$. If there exists a pair $(M, N) \in Red(A)$ such that A_M is Saphar and A_N is Riesz, then A admits a *generalized Saphar-Riesz decomposition*, or shortly A admits a GSRD(M, N) (or A admits a GSRD).

Theorem 4.2. *Let $A \in L(X)$. The following conditions are equivalent:*

(i) *A admits a GSRD;*

(ii) *There exists a projector $P \in L(X)$ commuting with A such that $A + P$ is essentially Saphar and AP is Riesz;*

(iii) *There exists a projector $P \in L(X)$ commuting with A such that $A + P$ is of Saphar type and AP is Riesz.*

Proof. (i)$\Longrightarrow$(ii): Let A admit a GSRD. Then there exists $(M, N) \in Red(A)$ such that A_M is Saphar and A_N is Riesz. Let $P \in L(X)$ be a projector such that $\mathcal{R}(P) = N$ and $\mathcal{N}(P) = M$. Then P commutes with A, and hence $(M, N) \in Red(AP)$, $(M, N) \in Red(A + P)$. Thus $AP = (AP)_M \oplus (AP)_N = 0 \oplus A_N$, and since A_N is Riesz, according to Lemma 3.6 (i) we obtain that AP is Riesz, while from [29, Theorem 3.2] we conclude that $(A+P)_N = A_N + I_N$ is Browder. From [29, Theorem 2.32] it follows that there exists closed subspaces N_1 and N_2 of N such that $(N_1, N_2) \in Red((A + P)_N)$, $\dim N_2 < \infty$, $(A + P)_{N_1}$ is invertible and $(A + P)_{N_2}$ is nilpotent. According to Lemma 3.1 we have that the subspace $M \oplus N_1$ is closed. As M and N_1 are $A + P$-invariant, we have that $M \oplus N_1$ is $A + P$-invariant. Since $M \oplus N_1$ is complemented in X with N_2, we have that $(M \oplus N_1, N_2) \in Red(A + P)$. As $(A + P)_M = A_M$ and $(A + P)_{N_1}$ are Saphar, Lemma 3.6 (iii) ensures that $(A + P)_{M \oplus N_1}$ is Saphar. Consequently, $A + P$ is essentially Saphar.

(ii)$\Longrightarrow$(iii): It is clear.

(iii)$\Longrightarrow$(i): Let there exist a projector $P \in L(X)$ commuting with A such that $A + P$ is of Saphar type and AP is Riesz. Then $(\mathcal{R}(P), \mathcal{N}(P)) \in Red(AP)$, $(\mathcal{R}(P), \mathcal{N}(P)) \in Red(A + P)$. From Lemma 3.6 (i) it follows that $A_{\mathcal{R}(P)} = (AP)_{\mathcal{R}(P)}$ is Riesz, while from Lemma 3.7 it follows that $A_{\mathcal{N}(P)} = (A + P)_{\mathcal{N}(P)}$ is of Saphar type. Hence there exist closed subspaces N_1, N_2 of $\mathcal{N}(P)$ such that $(N_1, N_2) \in Red(A_{\mathcal{N}(P)})$, A_{N_1} is Saphar and A_{N_2} is nilpotent. According to Lemma 3.1 the subspace $N_2 \oplus \mathcal{R}(P)$ is closed. Since $(N_1, N_2 \oplus \mathcal{R}(P)) \in Red(A)$ and $A_{N_2 \oplus \mathcal{R}(P)} = A_{N_2} \oplus A_{\mathcal{R}(P)}$ is Riesz according to Lemma 3.6 (i), we obtain that A admits a GSRD$(N_1, N_2 \oplus \mathcal{R}(P))$. □

Definition 4.3. An operator $A \in L(X)$ is *left generalized Drazin-Riesz invertible* if there is $B \in L(X)$ such that

$$ABA = BA^2,\ B^2A = B,\ A - ABA \text{ is Riesz.} \tag{4.1}$$

An operator $A \in L(X)$ is *right generalized Drazin-Riesz invertible* if there is $B \in L(X)$ such that

$$ABA = A^2B,\ AB^2 = B,\ A - ABA \text{ is Riesz.} \tag{4.2}$$

In [14, Definition 7.5.2] Harte introduced the notation of a quasipolar element in a Banach algebra: an element a of a Banach algebra $\mathcal{A}$ is *quasipolar* if there is an idempotent $q \in \mathcal{A}$ such that $aq = qa$, $a(1 - q)$ is quasinilpotent and $q \in (\mathcal{A}a) \cap (a\mathcal{A})$. The notation of Risz-quasipolar operators was introduced in [27]: an operator $A \in L(X)$ is *Riesz-quasipolar* if there exists a bounded projection Q commuting with A and satisfying $A(I - Q)$ is Riesz, $Q \in (L(X)A) \cap (AL(X))$.

Definition 4.4. An operator $A \in L(X)$ is *left Risz-quasipolar* if there exists a projector $Q \in L(X)$ satisfying

$$AQ = QA,\ A(I - Q) \text{ is Riesz},\ Q \in L(X)A. \tag{4.3}$$

An operator $A \in L(X)$ is *right Riesz-quasipolar* if there exists a projector $Q \in L(X)$ satisfying

$$AQ = QA,\ A(I - Q) \text{ is Riesz},\ Q \in AL(X). \tag{4.4}$$

In the following theorem we give some characterizations of left generalized Drazin-Riesz invertible operators.

Theorem 4.5. *For $A \in L(X)$ the following conditions are equivalent:*

(i) *A is left generalized Drazin-Riesz invertible;*

(ii) *There exists $C \in L(X)$ such that*

$$ACA = CA^2,\ C^2A = C = CAC,\ A - ACA \text{ is Riesz};$$

(iii) *There exists* $(M, N) \in Red(A)$ *such that* A_M *is left invertible and* A_N *is Riesz;*

(iv) *There exists a projector* $P \in L(X)$ *commuting with* A *such that* $A + P$ *is left Browder and* AP *is Riesz;*

(v) *There exists a projector* $P \in L(X)$ *commuting with* A *such that* $A(I - P) + P$ *is left Browder and* AP *is Riesz;*

(vi) *There exist* $C, D, P \in L(X)$ *such that* C *is left Browder,* D *is Riesz,* P *is a projector commuting with* C *and* D*, and* $A = C(I - P) + DP$*;*

(vii) A *is left Riesz-quasipolar;*

(viii) A *admits a GSRD and* $0 \notin \operatorname{int} \sigma_l(A)$*;*

(ix) A *admits a GSRD and* A *has the SVEP at* 0*;*

(x) A *admits a GSRD and* $0 \notin \operatorname{acc} \sigma^l_{\mathcal{B}}(A)$*;*

(xi) A *admits a GSRD and* $0 \notin \operatorname{int} \sigma^l_{\mathcal{B}}(A)$*;*

(xii) A *admits a GSRD and* $0 \notin \operatorname{acc} \sigma^+_D(A)$*;*

(xiii) A *admits a GSRD and* $0 \notin \operatorname{int} \sigma^+_D(A)$*;*

(xiv) A *admits a GSRD and* $0 \notin \operatorname{int} \sigma_p(A)$*.*

Proof. (i)$\Longrightarrow$(ii): Let A be left generalized Drazin-Riesz invertible. Then there exists $B \in L(X)$ such that the conditions (4.1) hold. Set $C = BAB$. Then

$$\begin{aligned}
&ACA = ABABA = BA^2BA = BABA^2 = CA^2,\\
&C^2A = (BAB)^2A = BA(BBA)BA = BABBA = BAB = C,\\
&CAC = BABABAB = BABBA^2B = BABAB = B^2A^2B = BAB = C,\\
&A - ACA = A - ABABA = A - AB^2A^2 = A - ABA \text{ is Riesz.}
\end{aligned}$$

(ii)$\Longrightarrow$(i): It is obvious.

(ii)$\Longrightarrow$(iii): Suppose that there exists $C \in L(X)$ and $R \in L(X)$ which is Riesz such that $ACA = CA^2$, $C^2A = C = CAC$, $A - ACA = R$. Then $(CA)^2 = CACA = CA$, $A(CA) = CA^2 = (CA)A$, $C(CA) = C = (CA)C$, and so CA is a bounded projector which commutes with A, C and R. It implies that $(\mathcal{R}(CA), \mathcal{N}(CA)) \in Red(A)$, $(\mathcal{R}(CA), \mathcal{N}(CA)) \in Red(C)$ and $(\mathcal{R}(CA), \mathcal{N}(CA)) \in Red(R)$. According to Lemma 3.6 (i) we conclude that $R_{\mathcal{N}(CA)}$ is Riesz. Further, $A_{\mathcal{N}(CA)} = (A(I - CA))_{\mathcal{N}(CA)} = R_{\mathcal{N}(CA)}$ and so $A_{\mathcal{N}(CA)}$ is Riesz. As $C_{\mathcal{R}(CA)}A_{\mathcal{R}(CA)} = (CA)_{\mathcal{R}(CA)} = I_{\mathcal{R}(CA)}$, it follows that $A_{\mathcal{R}(CA)}$ is left invertible.

(iii)$\Longrightarrow$(i): Suppose that there exists $(M, N) \in Red(A)$ such that A_M is left invertible and A_N is Riesz. Let B_M is a left inverse of A_M, i.e. $B_MA_M = I_M$. Set $B = B_M \oplus 0_N$. Then

$$B^2A = (B_M \oplus 0_N)^2(A_M \oplus A_N) = B_M^2A_M \oplus 0_N = B_M \oplus 0_N = B,$$

$$ABA = (A_M \oplus A_N)(B_M \oplus 0_N)(A_M \oplus A_N) = A_MB_MA_M \oplus 0_N = A_M \oplus 0_N,$$
$$BA^2 = (B_M \oplus 0_N)(A_M \oplus A_N)^2 = B_MA_M^2 \oplus 0_N = A_M \oplus 0_N,$$

and hence $ABA = B^2A$ and $A - ABA = 0_M \oplus A_N$ is Riesz according to Lemma 3.6 (i). Therefore, A is left generalized Drazin-Riesz invertible.

(iii)$\Longrightarrow$(iv): Suppose that there exists $(M, N) \in Red(A)$ such that A_M is left invertible and A_N is Riesz. Let P be the projector of X onto N along M. Then P commutes with A, and hence P commutes with AP and $A + P$. Thus $(M, N) \in Red(AP)$ and $(M, N) \in Red(A + P)$. From

$$AP = (AP)_M \oplus (AP)_N = 0 \oplus A_N$$

and Lemma 3.6 (i) it follows that AP is Riesz. As A_N is Riesz, we have that $A_N + I_N$ is Browder [29, Theorem 3.2], and hence it is left Browder. From

$$A + P = (A + P)_M \oplus (A + P)_N = A_M \oplus (A_N + I_N),$$

according to Lemma 3.6 (iv) we conclude that $A+P$ is left Browder.

(iv)⟹(iii): Suppose that there exists a projector $P \in L(X)$ commuting with A such that $A+P$ is left Browder and AP is Riesz. Let $M = \mathcal{N}(P)$ and $N = \mathcal{R}(P)$. Then $(M,N) \in Red(A)$, $(M,N) \in Red(AP)$ and $(M,N) \in Red(A+P)$. From Lemma 3.6 (i) we obtain that $A_N = (AP)_N$ is Riesz, while from Lemma 3.6 (iv) we have that $A_M = (A+P)_M$ is left Browder. From [24, Theorem 5] it follows that there exist closed subspace M_1, M_2 of M such that $\dim M_2 < \infty$, $(M_1, M_2) \in Red(A_M)$, $A_M = A_{M_1} \oplus A_{M_2}$, A_{M_1} is left invertible and A_{M_2} is nilpotent. Then $N_1 = M_2 \oplus N$ is a closed subspace of X, $(M_1, N_1) \in Red(A)$, $A_{N_1} = A_{M_2} \oplus A_N$ is Riesz according to Lemma 3.6 (i).

(iv)⟹(v): Let there exist a projector $P \in L(X)$ commuting with A such that $A+P$ is left Browder and AP is Riesz. Then P commutes with $A+P$ and I, and since $A+P$ and I are left Browder, from Lemma 3.10 (i) it follows that $A(I-P)+P = (A+P)(I-P) + I \cdot P$ is left Browder.

(v)⟹(vi): Let P be a projector which commutes with A and such that $A(I-P)+P$ is left Browder and AP is Riesz. Set $C = A(I-P)+P$ and $D = AP$. Then C is left Browder, D is Riesz, P commutes with C and D, and

$$C(I-P) + DP = (A(I-P)+P)(I-P) + AP = A.$$

(vi)⟹(iii): Suppose that there exist $C, D, P \in L(X)$ such that C is left Browder, D is Riesz, P is a projector commuting with C and D, and $A = C(I-P) + DP$. Then P is commuting with $C(I-P)$ and DP, and hence $(\mathcal{R}(P), \mathcal{N}(P)) \in Red(C(I-P))$ and $(\mathcal{R}(P), \mathcal{N}(P)) \in Red(DP)$. Consequently,

$$\begin{aligned} A &= C(I-P) + DP = ((C(I-P))_{\mathcal{R}(P)} \oplus (C(I-P))_{\mathcal{N}(P)}) + ((DP)_{\mathcal{R}(P)} \oplus (DP)_{\mathcal{N}(P)}) \\ &= (0_{\mathcal{R}(P)} \oplus C_{\mathcal{N}(P)}) + (D_{\mathcal{R}(P)} \oplus 0_{\mathcal{N}(P)}) = D_{\mathcal{R}(P)} \oplus C_{\mathcal{N}(P)}. \end{aligned}$$

Lemma 3.6 (i) ensures that $D_{\mathcal{R}(P)}$ is Riesz, while Lemma 3.6 (iv) ensures that $C_{\mathcal{N}(P)}$ is left Browder. Now as in the proof of the implication (iv)⟹(iii) we conclude that A is a direct sum of a Riesz operator and a left invertible operator.

(i)⟹(vii): Suppose that A is left generalized Drazin-Riesz invertible. Then there exists $B \in L(X)$ such that $ABA = BA^2$, $B^2A = B$, $A - ABA$ is Riesz. Set $Q = BA$. Then

$$Q^2 = BABA = B^2A^2 = BA = Q, \quad AQ = ABA = BA^2 = QA,$$

$A(I-Q) = A - ABA$ is Riesz and $Q \in L(X)A$. Therefore, A is left Riesz-quasipolar.

(vii)⟹(i): Let A be left Riesz-quasipolar. Thus there exists a projector $Q \in L(X)$ such that $AQ = QA$, $A(I-Q)$ is Riesz, $Q \in L(X)A$. Then there exists $T \in L(X)$ such that $Q = TA$. Set $B = QTQ$. Then $ABA = BA^2$, $B^2A = B$ and $A - ABA = A - AQ = A(I-Q)$ is Riesz, and so A is left generalized Drazin-Riesz invertible.

(iii)⟹(viii): Suppose that there exists $(M,N) \in Red(A)$ such that A_M is left invertible and A_N is Riesz. Then A_M is Saphar, and so A admits a GSRD(M,N). Further there exists an $\epsilon > 0$ such that $D(0,\epsilon) \cap \sigma_l(A_M) = \emptyset$. Since A_N is Riesz, its left spectrum $\sigma_l(A_N)$ is at most countable with 0 as its only possible limit point. As $\sigma_l(A) = \sigma_l(A_M) \cup \sigma_l(A_N)$ according Lemma 3.5, we conclude that $0 \notin \operatorname{int} \sigma_l(A)$.

(viii)⟹(ix): Let $0 \notin \operatorname{int} \sigma_l(A)$. This implies that $0 \notin \sigma_l(A)$ or $0 \in \partial \sigma_l(A)$. In both cases A has SVEP at 0 by the identity theorem for analytic functions.

(ix)⟹(iii): Suppose that A admits a GSRD and A has the SVEP at 0. Then there exists $(M,N) \in Red(A)$ such that A_M is Saphar and A_N is Riesz. Further we conclude that A_M has the SVEP at 0 and according [1, Theorem 2.49] it follows that A_M is left invertible.

(iii)⟹(x): Suppose that there exists $(M,N) \in Red(A)$ such that A_M is left invertible and A_N is Riesz. Then A admits a GSRD(M,N). As A_M is left invertible, there exists $\epsilon > 0$ such that for every $\lambda \in \mathbb{C}$ satisfying $|\lambda| < \epsilon$ we have $A_M - \lambda I_M$ is left invertible. Since A_N is Riesz, it follows that $A_N - \lambda I_N$ is left Browder for every $\lambda \in \mathbb{C}$ such that $0 < |\lambda| < \epsilon$ [29, Theorem 3.2]. Lemma 3.6 (iv) ensures that $A - \lambda I$ is left Browder for every $\lambda \in \mathbb{C}$ such that $0 < |\lambda| < \epsilon$, and so $0 \notin \operatorname{acc} \sigma_{\mathcal{B}}^l(A)$.

The implications (x)⟹(xi)⟹(xiii), (x)⟹(xii)⟹(xiii), (viii)⟹(xiv) are obvious.

(xiii)$\Longrightarrow$(iii), (xiv)$\Longrightarrow$(iii): Suppose that A admits GSRD and $0 \notin \operatorname{int} \sigma_D^+(A)$ (resp., $0 \notin \operatorname{int} \sigma_p(A)$). Then there exists a decomposition $(M, N) \in Red(A)$ such that A_M is Saphar and A_N is Riesz. By using Grabiner's punctured neighborhood theorem [11, Theorem 4.7], from the proof of of the implication (xii)$\Longrightarrow$(i) (resp., (xiv)$\Longrightarrow$(i)) in [19, Theorem 3.19] we conclude that A_M is left invertible. □

Characterizations of right generalized Drazin-Riesz invertible operators are obtained in a similar way as those of left generalized Drazin-Riesz invertible operators in Theorem 4.5.

Theorem 4.6. *For $A \in L(X)$ the following conditions are equivalent:*

(i) *A is right generalized Drazin-Riesz invertible;*

(ii) *There exists $C \in L(X)$ such that*

$$ACA = A^2C,\ AC^2 = C = CAC,\ A - ACA \text{ is Riesz};$$

(iii) *There exists $(M, N) \in Red(A)$ such that A_M is right invertible and A_N is Riesz;*

(iv) *There exists a projector $P \in L(X)$ commuting with A such that $A + P$ is right Browder and AP is Riesz;*

(v) *There exists a projector $P \in L(X)$ commuting with A such that $A(I - P) + P$ is right Browder and AP is Riesz;*

(vi) *There exist $C, D, P \in L(X)$ such that C is right Browder, D is Riesz, P is a projector commuting with C and D, and $A = C(I - P) + DP$;*

(vii) *A is right Riesz-quasipolar;*

(viii) *A admits a GSRD and $0 \notin \operatorname{int} \sigma_r(A)$;*

(ix) *A admits a GSRD and A' has the SVEP at 0;*

(x) *A admits a GSRD and $0 \notin \operatorname{acc} \sigma^r_{\mathcal{B}}(A)$;*

(xi) *A admits a GSRD and $0 \notin \operatorname{int} \sigma^r_{\mathcal{B}}(A)$;*

(xii) *A admits a GSRD and $0 \notin \operatorname{acc} \sigma_{dsc}(A)$;*

(xiii) *A admits a GSRD and $0 \notin \operatorname{int} \sigma_{dsc}(A)$;*

(xiv) *A admits a GSRD and $0 \notin \operatorname{int} \sigma_{cp}(A)$.*

Remark 4.7. From Theorem 4.5, Theorem 4.6 and [10, Theorems 3.3 and 3.4], [19, Theorems 3.19 and 3.20] it follows that every left (right) generalized Drazin invertible operator is left generalized Drazin-Riesz invertible.

The condition that $0 \notin \operatorname{int} \sigma_l(A)$ ($0 \notin \operatorname{int} \sigma_r(A)$) in the statement (viii) in Theorem 4.5 (Theorem 4.6) can not be replaced with the stronger condition that $0 \notin \operatorname{acc} \sigma_l(A)$ ($0 \notin \operatorname{acc} \sigma_r(A)$). Also, the condition that $0 \notin \operatorname{int} \sigma_p(A)$ ($0 \notin \operatorname{int} \sigma_{cp}(A)$) in the statement (xiv) in Theorem 4.5 (Theorem 4.6), can not be replaced with the stronger condition that $0 \notin \operatorname{acc} \sigma_p(A)$ ($0 \notin \operatorname{acc} \sigma_{cp}(A)$). Namely, if $A \in L(X)$ is a Riesz operator with infinite spectrum, then A is left and right generalized Drazin-Riesz invertible, but $\sigma(A) = \sigma_l(A) = \sigma_r(A)$, $\sigma(A) \setminus \{0\} \subset \sigma_p(A)$, $\sigma(A) \setminus \{0\} \subset \sigma_{cp}(A)$, and $0 \in \operatorname{acc} \sigma_l(A) = \operatorname{acc} \sigma_r(A)$, as well as $0 \in \operatorname{acc} \sigma_p(A)$ and $0 \in \operatorname{acc} \sigma_{cp}(A)$.

According to [19, Theorems 3.19 and 3.20] we conclude that A is neither left generalized Drazin invertible nor right generalized Drazin invertible. Hence the class of left (right) generalized Drazin invertible operators is strictly contained in the class of left (right) generalized Drazin-Riesz invertible operators.

Corollary 4.8. *Let $A \in L(X)$. Then A is generalized Drazin-Riesz invertible if and only if A is both left and right generalized Drazin-Riesz invertible.*

Proof. ($\Longrightarrow$): It is obvious.

($\Longleftarrow$): Let A be both left and right generalized Drazin-Riesz invertible. Then from Theorems 4.5 and 4.6 it follows that A admits a GSRD, and $0 \notin \operatorname{acc} \sigma^l_{\mathcal{B}}(A)$ and $0 \notin \operatorname{acc} \sigma^r_{\mathcal{B}}(A)$. Since $\operatorname{acc} \sigma^l_{\mathcal{B}}(A) \cup \operatorname{acc} \sigma^r_{\mathcal{B}}(A) = \operatorname{acc}(\sigma^l_{\mathcal{B}}(A) \cup \sigma^r_{\mathcal{B}}(A)) = \operatorname{acc} \sigma_{\mathcal{B}}(A)$, we get that $0 \notin \operatorname{acc} \sigma_{\mathcal{B}}(A)$. Now from [27, Theorem 2.3] it follows that A is generalized Drazin-Riesz invertible. □

Remark 4.9. Every left invertible operator which is not right invertible is left generalized Drazin-Riesz invertible, but it is not right generalized Drazin-Riesz invertible. Indeed, if $A \in L(X)$ is left invertible and if it is not right invertible, then A is left generalized Drazin-Riesz invertible, $0 \in \sigma_r(A)$ and $0 \notin \sigma_l(A)$. Since $\partial\,\sigma_r(A) \subset \sigma_l(A)$, we conclude that 0 cannot be a boundary point of $\sigma_r(A)$. Consequently, $0 \in \operatorname{int}\sigma_r(A)$, and from Theorem 4.6 it follows that A is not right generalized Drazin-Riesz invertible.

Similarly, every right invertible operator which is not left invertible is right generalized Drazin-Riesz invertible, but it is not left generalized Drazin-Riesz invertible. Therefore, the class of generalized Drazin-Riesz invertible operators is strictly contained in the class of left (right) generalized Drazin-Riesz invertible operators.

Corollary 4.10. *For $A \in L(X)$ the following conditions are equivalent:*

(i) *A is generalized Drazin-Riesz invertible;*

(ii) *A is either left generalized Drazin-Riesz invertible or right generalized Drazin-Riesz invertible, and $0 \notin \operatorname{int}\sigma(A)$;*

(iii) *A admits a GSRD and $0 \notin \operatorname{int}\sigma(A)$.*

Proof. (i)⟹(ii): Let A be generalized Drazin-Riesz invertible. Then A is both left and right generalized Drazin-Riesz invertible and from [27, Theorem 2.3] it follows that $0 \notin \operatorname{int}\sigma(A)$.

(ii)⟹(iii): Let A be either left generalized Drazin-Riesz invertible or right generalized Drazin-Riesz invertible, and $0 \notin \operatorname{int}\sigma(A)$. Then from Theorems 4.5 and 4.6 it follows that A admits a GSRD

(iii)⟹(i): Let A admit a GSRD and $0 \notin \operatorname{int}\sigma(A)$. Then A admits a GKRD and using [27, Theorem 2.3] we conclude that A is generalized Drazin-Riesz invertible. □

Corollary 4.11. *Let $A \in L(X)$. If $0 \notin \operatorname{int}\sigma(A)$, then A admits a GKRD if and only if A admits a GSRD.*

Proof. It follows from Corollary 4.10 and [27, Theorem 2.3]. □

Proposition 4.12. *Let $A \in L(X)$. Then:*

(i) *If there exists $(M, N) \in Red(A)$ such that A_M is left invertible and A_N is Riesz, then $(N^\perp, M^\perp) \in Red(A')$, $A'_{N^\perp}$ is right invertible and $A'_{M^\perp}$ is Riesz.*

(ii) *If there exists $(M, N) \in Red(A)$ such that A_M is right invertible and A_N is Riesz, then $(N^\perp, M^\perp) \in Red(A')$, $A'_{N^\perp}$ is left invertible and $A'_{M^\perp}$ is Riesz.*

Proof. (i): Suppose that $(M, N) \in Red(A)$, A_M is left invertible and A_N is Riesz. Let P be the projector of X such that $\mathcal{R}(P) = M$ and $\mathcal{N}(P) = N$. Then P' is a projector which commutes with A'. As $\mathcal{R}(P') = N^\perp$ and $\mathcal{N}(P') = M^\perp$, we obtain that $(N^\perp, M^\perp) \in Red(A')$. Since A_M is left invertible, we have that $\mathcal{N}(A_M) = \{0\}$ and so

$$\mathcal{R}(A'_{N^\perp}) = \mathcal{R}(A') \cap N^\perp = \mathcal{N}(A)^\perp \cap N^\perp = (\mathcal{N}(A) + N)^\perp = (\mathcal{N}(A_M) \oplus N)^\perp = N^\perp,$$

i.e. $A'_{N^\perp}$ is surjective. Further we have that

$$\mathcal{N}(A'_{N^\perp}) = \mathcal{N}(A') \cap N^\perp = \mathcal{R}(A)^\perp \cap N^\perp = (\mathcal{R}(A) + N)^\perp = (\mathcal{R}(A_M) \oplus N)^\perp \tag{4.5}$$

Since A_M is left invertible, it follows that $\mathcal{R}(A_M)$ is complemented in M. According to Lemma 3.2 we conclude that $\mathcal{R}(A_M) \oplus N$ is complemented in X and by using Lemma 3.3 we get that $(\mathcal{R}(A_M) \oplus N)^\perp$ is complemented in X'. As $N^\perp$ is a closed subspace of X' which contains $(\mathcal{R}(A_M) \oplus N)^\perp$, applying Lemma 3.4 we conclude that $(\mathcal{R}(A_M) \oplus N)^\perp$ is complemented in $N^\perp$. According to (4.5) we have that $\mathcal{N}(A'_{N^\perp})$ is complemented in $N^\perp$, and hence $A'_{N^\perp}$ is right invertible.

Let $Q = I - P$. Then Q is a projector, $\mathcal{R}(Q) = N$, $\mathcal{N}(Q) = M$, $(M, N) \in Red(AQ)$, $AQ = 0_M \oplus A_N$, and so Lemma 3.6 (i) ensures that AQ is Riesz. Also we have that $(N^\perp, M^\perp) \in Red(A'Q')$. From [29, Corollary 3.10] it follows that $A'Q' = Q'A'$ is Riesz. As $\mathcal{R}(Q') = \mathcal{N}(Q)^\perp = M^\perp$, using Lemma 3.6 (i) we obtain that $A'Q'_{M^\perp} = A'_{M^\perp}$ is Riesz.

(ii): Suppose that $(M, N) \in Red(A)$, A_M is right invertible and A_N is Riesz. Then $\mathcal{N}(A_M)$ is a complemented subspace of M and $\mathcal{R}(A_M) = M$. As in the proof of (i) we conclude that $A'_{M^\perp}$ is Riesz,

$$\mathcal{N}(A'_{N^\perp}) = (\mathcal{R}(A_M) \oplus N)^\perp = (M \oplus N)^\perp = \{0\},$$

that is $A'_{N^\perp}$ is injective, and

$$\mathcal{R}(A'_{N^\perp}) = (\mathcal{N}(A_M) \oplus N)^\perp.$$

As $\mathcal{N}(A_M)$ is complemented in M, from Lemma 3.2 it follows that $\mathcal{N}(A_M) \oplus N$ is complemented in X. Using Lemma 3.3 we get that $(\mathcal{N}(A_M) \oplus N)^\perp$ is complemented in X'. As $N^\perp$ is a closed subspace of X' which contains $(\mathcal{N}(A_M) \oplus N)^\perp$, applying Lemma 3.4 we conclude that $(\mathcal{N}(A_M) \oplus N)^\perp$ is complemented in $N^\perp$. As $\mathcal{R}(A'_{N^\perp})$ is complemented in $N^\perp$ and $A'_{N^\perp}$ is injective, we get that $A'_{N^\perp}$ is left invertible. □

Corollary 4.13. *Let $A \in L(X)$. Then:*

(i) *If A is left generalized Drazin-Riesz invertible, then A' is right generalized Drazin-Riesz invertible.*

(ii) *If A is right generalized Drazin-Riesz invertible, then A' is left generalized Drazin-Riesz invertible.*

Proof. It follows from Proposition 4.12 and the equivalence (i)⟺(iii) in Theorem 4.5 and Theorem 4.6. □

Proposition 4.14. *Let $A \in L(X)$. If A admits a $GSRD(M, N)$, then A' admits a $GSRD(N^\perp, M^\perp)$.*

Proof. Let A admit a $GSRD(M, N)$. Then $(M, N) \in Red(A)$, A_M is Saphar and A_N is Riesz. As in the proof of Proposition 4.12 we get that $(N^\perp, M^\perp) \in Red(A')$, $A'_{M^\perp}$ is Riesz and the subspaces $\mathcal{N}(A'_{N^\perp}) = (\mathcal{R}(A_M) \oplus N)^\perp$ and $\mathcal{R}(A'_{N^\perp}) = (\mathcal{N}(A_M) \oplus N)^\perp$, are complemented in $N^\perp$. From the proof of [1, Theorem 1.43] it follows that $A'_{N^\perp}$ is Kato. Therefore, $A'_{N^\perp}$ is Saphar, and so A' admits a $GSRD(N^\perp, M^\perp)$. □

5. Left and right generalized Drazin-meromorphic invertible operators

In this section we introduce and explore the concept of generalized Saphar-meromorphic decomposition and the concept of left and right generalized Drazin-meromorphic invertible operators. Also we show that on an arbitrary Banach space the class of left (resp., right) generalized Drazin-Riesz invertible operators does not coincide with the class of upper (resp., lower) generalized Drazin-Riesz invertible operators, and the class of left (resp., right) generalized Drazin-meromorphic invertible operators does not coincide with the class of upper (resp., lower) generalized Drazin-meromorphic invertible operators (see definitions on page 15).

Definition 5.1. Let $A \in L(X)$. If there exists a pair $(M, N) \in Red(A)$ such that A_M is Saphar and A_N is meromorphic, then we say that A admits a *generalized Saphar-meromorphic decomposition*, or shortly A admits a GS($\mathcal{M}$)D(M, N) (or A admits a GS($\mathcal{M}$)D).

Theorem 5.2. *Let $A \in L(X)$. The following conditions are equivalent:*

(i) *A admits a GS($\mathcal{M}$)D;*

(ii) *There exists a projector $P \in L(X)$ commuting with A such that $A + P$ is of Saphar type and AP is meromorphic.*

Proof. It follows from Lemma 3.6 (ii) analogously to the proof of Theorem 4.2. □

Definition 5.3. An operator $A \in L(X)$ is *left generalized Drazin-meromorphic invertible* if there is $B \in L(X)$ such that

$$ABA = BA^2, \; B^2A = B, \; A - ABA \text{ is meromorphic.}$$

An operator $A \in L(X)$ is *right generalized Drazin-meromorphic invertible* if there is $B \in L(X)$ such that

$$ABA = A^2B, \; AB^2 = B, \; A - ABA \text{ is meromorphic.}$$

It is clear that every left (right) generalized Drazin-Riesz invertible operators is left (right) generalized Drazin-meromorphic invertible. If A is meromorphic, then it is left (right) generalized Drazin-meromorphic invertible and 0 is a left (right) generalized Drazin-meromorphic inverse of A.

Definition 5.4. An operator $A \in L(X)$ is *left meromorphic-quasipolar* if there exists a projector $Q \in L(X)$ satisfying

$$AQ = QA,\ A(I-Q) \text{ is meromorphic},\ Q \in L(X)A.$$

An operator $A \in L(X)$ is *right meromorphic-quasipolar* if there exists a projector $Q \in L(X)$ satisfying

$$AQ = QA,\ A(I-Q) \text{ is meromorphic},\ Q \in AL(X).$$

In the following theorem we give some characterizations of left generalized Drazin-meromorphic invertible operators.

Theorem 5.5. *For $A \in L(X)$ the following conditions are equivalent:*

(i) *A is left generalized Drazin-meromorphic invertible;*

(ii) *There exists $C \in L(X)$ such that*

$$ACA = CA^2,\ C^2A = C = CAC,\ A - ACA \text{ is meromorphic};$$

(iii) *There exists $(M, N) \in Red(A)$ such that A_M is left invertible and A_N is meromorphic;*

(iv) *There exists a projector $P \in L(X)$ commuting with A such that $A + P$ is left Drazin invertible and AP is meromorphic;*

(v) *There exists a projector $P \in L(X)$ commuting with A such that $A(I - P) + P$ is left Drazin invertible and AP is meromorphic;*

(vi) *There exist $C, D, P \in L(X)$ such that C is left Drazin invertible, D is meromorphic, P is a projector commuting with C and D, and $A = C(I - P) + DP$;*

(vii) *A is left meromorphic-quasipolar;*

(viii) *A admits a GS($\mathcal{M}$)D and $0 \notin \operatorname{int} \sigma_l(A)$;*

(ix) *A admits a GS($\mathcal{M}$)D and A has the SVEP at 0;*

(x) *A admits a GS($\mathcal{M}$)D and $0 \notin \operatorname{acc} \sigma_D^l(A)$;*

(xi) *A admits a GS($\mathcal{M}$)D and $0 \notin \operatorname{int} \sigma_D^l(A)$;*

(xii) *A admits a GS($\mathcal{M}$)D and $0 \notin \operatorname{acc} \sigma_D^+(A)$;*

(xiii) *A admits a GS($\mathcal{M}$)D and $0 \notin \operatorname{int} \sigma_D^+(A)$;*

(xiv) *A admits a GS($\mathcal{M}$)D and $0 \notin \operatorname{int} \sigma_p(A)$.*

Proof. (iii)$\Longrightarrow$(iv): Let there exist $(M, N) \in Red(A)$ such that A_M is left invertible and A_N is meromorphic. Let P be the projector of X such that $\mathcal{R}(P) = N$ and $\mathcal{N}(P) = M$. Then P commutes with A, AP and $A + P$ and hence $(M, N) \in Red(AP)$ and $(M, N) \in Red(A + P)$. From

$$AP = (AP)_M \oplus (AP)_N = 0 \oplus A_N$$

according to Lemma 3.6 (ii) we get that AP is meromorphic. Since A_N is meromorphic, it follows that $A_N + I_N$ is Drazin invertible and hence it is left Drazin invertible. From

$$A + P = (A + P)_M \oplus (A + P)_N = A_M \oplus (A_N + I_N),$$

using Lemma 3.8 we conclude that $A + P$ is left Drazin invertible.

(iv)$\Longrightarrow$(iii): Suppose that there exists a projector $P \in L(X)$ commuting with A such that $A+P$ is left Drazin invertible and AP is Riesz. Let $M = \mathcal{N}(P)$ and $N = \mathcal{R}(P)$. Then $(M,N) \in Red(A)$, $(M,N) \in Red(AP)$ and $(M,N) \in Red(A+P)$. From Lemma 3.6 (ii) we obtain that $A_N = (AP)_N$ is meromorphic, while from Lemma 3.8 we get that $A_M = (A+P)_M$ is left Drazin invertible. From [12, Theorem 3.4] it follows that there exist closed subspace M_1, M_2 of M such that $(M_1, M_2) \in Red(A_M)$, $A_M = A_{M_1} \oplus A_{M_2}$, A_{M_1} is left invertible and A_{M_2} is nilpotent. From Lemma 3.1 it follows that $N_1 = M_2 \oplus N$ is a closed subspace of X, and so $(M_1, N_1) \in Red(A)$. Lemma 3.6 (ii) ensures that $A_{N_1} = A_{M_2} \oplus A_N$ is meromorphic.

The proof of the rest of the implication can be obtained by using Lemmas 3.9, 3.8 and 3.6 (ii) adapting the appropriate parts of the proof of Theorem 4.5. □

The following theorem provides characterizations of right generalized Drazin-meromorphic invertible operators and is obtained in a similar way.

Theorem 5.6. *For $A \in L(X)$ the following conditions are equivalent:*

(i) *A is right generalized Drazin-meromorphic invertible;*

(ii) *There exists $C \in L(X)$ such that*

$$ACA = A^2C,\ AC^2 = C = CAC,\ A - ACA \text{ is meromorphic;}$$

(iii) *There exists $(M,N) \in Red(A)$ such that A_M is right invertible and A_N is meromorphic;*

(iv) *There exists a projector $P \in L(X)$ commuting with A such that $A+P$ is right Drazin invertible and AP is meromorphic;*

(v) *There exists a projector $P \in L(X)$ commuting with A such that $A(I-P)+P$ is right Drazin invertible and AP is meromorphic;*

(vi) *There exist $C, D, P \in L(X)$ such that C is right Drazin invertible, D is meromorphic, P is a projector commuting with C and D, and $A = C(I-P) + DP$;*

(vii) *A is right meromorphic-quasipolar;*

(viii) *A admits a GS($\mathcal{M}$)D and $0 \notin \operatorname{int}\sigma_r(A)$;*

(ix) *A admits a GS($\mathcal{M}$)D and A' has the SVEP at 0;*

(x) *A admits a GS($\mathcal{M}$)D and $0 \notin \operatorname{acc}\sigma_D^r(A)$;*

(xi) *A admits a GS($\mathcal{M}$)D and $0 \notin \operatorname{int}\sigma_D^r(A)$;*

(xii) *A admits a GS($\mathcal{M}$)D and $0 \notin \operatorname{acc}\sigma_{dsc}(A)$;*

(xiii) *A admits a GS($\mathcal{M}$)D and $0 \notin \operatorname{int}\sigma_{dsc}(A)$;*

(xiv) *A admits a GS($\mathcal{M}$)D and $0 \notin \operatorname{int}\sigma_{cp}(A)$.*

From Remark 4.7 it follows that the condition that $0 \notin \operatorname{int}\sigma_l(A)$ ($0 \notin \operatorname{int}\sigma_r(A)$) in the statement (viii) in Theorem 5.5 (Theorem 5.6) can not be replaced with the stronger condition that $0 \notin \operatorname{acc}\sigma_l(A)$ ($0 \notin \operatorname{acc}\sigma_r(A)$). Also, the condition that $0 \notin \operatorname{int}\sigma_p(A)$ ($0 \notin \operatorname{int}\sigma_{cp}(A)$) in the statement (xiv) in Theorem 5.5 (Theorem 5.6), can not be replaced with the stronger condition that $0 \notin \operatorname{acc}\sigma_p(A)$ ($0 \notin \operatorname{acc}\sigma_{cp}(A)$).

The following example shows that the class of left (right) generalized Drazin-Riesz invertible operators is strictly contained in the class of left (right) generalized Drazin-meromorphic invertible operators.

Example 5.7. Let X be an infinite dimensional Banach space and let $\tilde{X} = \oplus_{n=1}^{\infty} X_i$ where $X_i = X$, $i \in \mathbb{N}$. Let $A = \oplus_{n=1}^{\infty}(\frac{1}{n}I)$ where I is the identity on X. In [27, Example 3.9] it is shown that $A \in L(\tilde{X})$ is a meromorphic operator which does not admit a GKRD. Hence A is both left and right generalized Drazin-meromorphic invertible, and A does not admit a GSRD. From Theorems 4.5 and 4.6 it follows that A is neither left generalized Drazin-Riesz invertible nor right generalized Drazin-Riesz invertible.

Theorem 5.8. *Let $A \in L(X)$ be left (resp., right) generalized Drazin-meromorphic invertible. Then there exists an operator $B \in L(X)$ commuting with A such that B is Drazin invertible, $A+B$ is left (resp., right) invertible and AB is meromorphic.*

Proof. Suppose that A is left generalized Drazin-meromorphic invertible. According to Theorem 5.5 there exists a pair $(M, N) \in Red(A)$ such that A_M is left invertible and A_N is meromorphic. Then $I_N - A_N$ is Drazin invertible and hence there exists a pair $(N_1, N_2) \in Red(I_N - A_N)$ such that $I_{N_1} - A_{N_1}$ is nilpotent and $I_{N_2} - A_{N_2}$ is invertible [17, Theorem 4], [8, Lemma 3.4.2]. Let $B = 0_M \oplus (I_N - A_N)$. According Lemma 3.1 we have that $M \oplus N_1$ is closed, and so $(M \oplus N_1, N_2) \in Red(B)$. As $B_{M\oplus N_1} = 0_M \oplus (I_{N_1} - A_{N_1})$ is nilpotent and $B_{N_2} = I_{N_2} - A_{N_2}$ is invertible, it follows that B is Drazin invertible. Since $A + B = A_M \oplus I_N$, according Lemma 3.5 we get that $A + B$ is left invertible. Further we have that

$$\begin{aligned} AB &= (A_M \oplus A_N)(0_M \oplus (I_N - A_N)) = 0_M \oplus A_N(I_N - A_N) \\ &= 0_M \oplus (I_N - A_N)A_N = (0_M \oplus (I_N - A_N))(A_M \oplus A_N) = BA. \end{aligned} \tag{5.1}$$

Since A_N is meromorphic, $\sigma_D(A_N) \subset \{0\}$. Using the spectral mapping theorem [5, Corollary 2.4] for $f(\lambda) = \lambda - \lambda^2$, $\lambda \in \mathbb{C}$, we conclude that $\sigma_D(A_N(I_N - A_N)) = \sigma_D(f(A_N)) = f(\sigma_D(A_N)) \subset \{0\}$. Consequently, $A_N(I_N - A_N)$ is meromorphic, and from (5.1) according Lemma 3.6 (ii) we obtain that AB is meromorphic.

The rest of assertion can be proved similarly. □

Corollary 5.9. *Let $A \in L(X)$. Then A is generalized Drazin-meromorphic invertible if and only if A is both left and right generalized Drazin-meromorphic invertible.*

Proof. ($\Longrightarrow$): It is obvious.

($\Longleftarrow$): Let A be both left and right generalized Drazin-meromorphic invertible. Then from Theorems 5.5 and 5.6 it follows that A admits a GS($\mathcal{M}$)D, and $0 \notin \operatorname{acc} \sigma_D^l(A)$ and $0 \notin \operatorname{acc} \sigma_D^r(A)$. Hence $0 \notin \operatorname{acc}(\sigma_D^l(A) \cup \sigma_D^r(A)) = \operatorname{acc} \sigma_D(A)$, and according to [28, Theorem 5] we get that A is generalized Drazin-meromorphic invertible. □

Similarly as in Remark 4.9 we conclude that every left (right) invertible operator which is not right (left) invertible is left (right) generalized Drazin-meromorphic invertible, but it is not right (left) generalized Drazin-meromorphic invertible.

Corollary 5.10. *For $A \in L(X)$ the following conditions are equivalent:*

(i) *A is generalized Drazin-meromorphic invertible;*

(ii) *A is either left generalized Drazin-meromorphic invertible or right generalized Drazin-meromorphic invertible, and $0 \notin \operatorname{int}\sigma(A)$;*

(iii) *A admits a GS($\mathcal{M}$)D and $0 \notin \operatorname{int}\sigma(A)$.*

Proof. It follows from Theorems 5.5 and 5.6, and [28, Theorem 5]. □

Corollary 5.11. *Let $A \in L(X)$. If $0 \notin \operatorname{int}\sigma(A)$, then A admits a GK($\mathcal{M}$)D if and only if A admits a GS($\mathcal{M}$)D.*

Proof. It follows from Corollary 5.10 and [28, Theorem 5]. □

Since the adjoint of a meromorphic operator is also meromorphic the following assertions are proved in a similar way as appropriate assertions in Section 4.

Proposition 5.12. *Let $A \in L(X)$. Then:*

(i) *If there exists $(M, N) \in Red(A)$ such that A_M is left invertible and A_N is meromorphic, then $(N^\perp, M^\perp) \in Red(A')$, $A'_{N^\perp}$ is right invertible and $A'_{M^\perp}$ is meromorphic.*

(ii) *If there exists $(M, N) \in Red(A)$ such that A_M is right invertible and A_N is meromorphic, then $(N^\perp, M^\perp) \in Red(A')$, $A'_{N^\perp}$ is left invertible and $A'_{M^\perp}$ is meromorphic.*

Corollary 5.13. *Let $A \in L(X)$. Then:*

(i) *If A is left generalized Drazin-meromorphic invertible, then A' is right generalized Drazin-meromorphic invertible.*

(ii) *If A is right generalized Drazin-meromorphic invertible, then A' is left generalized Drazin-meromorphic invertible.*

Proposition 5.14. *Let $A \in L(X)$. If A admits a $GS(\mathcal{M})D(M,N)$, then A' admits a $GS(\mathcal{M})D(N^{\perp}, M^{\perp})$.*

If for an operator $A \in L(X)$ there exists a pair $(M,N) \in Red(A)$ such that A_M is bounded bellow (resp. surjective) and A_N is Riesz, we say that A is *upper generalized Drazin-Riesz invertible* (resp., *lower generalized Drazin-Riesz invertible*). Such kind of operators was considered in [27, Theorems 2.4 and 2.5]. Furthermore, if there exists a pair $(M,N) \in Red(A)$ such that A_M is bounded bellow (resp. surjective) and A_N is meromorphic, then we say that A is *upper generalized Drazin-meromorphic invertible* (resp., *lower generalized Drazin-meromorphic invertible*) (see [28, Theorems 6 and 7]). On a Hilbert space the class of upper (resp., lower) generalized Drazin-Riesz invertible operators coincides with the class of left (resp., right) generalized Drazin-Riesz invertible operators, and the class of upper (resp., lower) generalized Drazin-meromorphic invertible operators coincides with the class of left (resp., right) generalized Drazin-meromorphic invertible operators.

In order to prove that it does not hold on an arbitrary Banach space we need the following theorem that follows immediately from [13, Theorem 3.1].

Theorem 5.15. *Let $A \in L(X)$ is an upper (resp. lower) Drazin invertible operator. Then the following conditions are equivalent:*

(i) *A is of Saphar type;*

(ii) *A admits a GSRD;*

(iii) *A admits a $GS\mathcal{M}D$.*

Remark 5.16. L. Burlando proved that on a Banach space X there exists an operator $A \in L(X)$ which is bounded below (resp., surjective) but its range (resp., null-space) is not complemented [7, Example 1.1]. Clearly this operator is upper generalized Drazin-Riesz invertible and upper generalized Drazin-meromorphic invertible (resp., lower generalized Drazin-Riesz invertible and lower generalized Drazin-meromorphic invertible), but it is not of Saphar type (see [30, p. 170]). Hence from Theorem 5.15 it follows that A admits neither a generalized Saphar-Riesz decomposition nor a generalized Saphar-meromorphic decomposition. Now according to Theorems 4.5 and 5.5 (resp., Theorems 4.6 and 5.6) it follows that A is neither left generalized Drazin-Riesz invertible nor left generalized Drazin-meromorphic invertible (resp., neither right generalized Drazin-Riesz invertible nor right generalized Drazin-meromorphic invertible).

6. Spectra

In this section we introduce and study spectra corresponding to operators and decompositions introduced in the previous two sections.

For $A \in L(X)$, set

$$\begin{aligned} \sigma_{gSR}(A) &= \{\lambda \in \mathbb{C} : A - \lambda I \text{ does not admit a } GSRD\}, \\ \sigma_{gS\mathcal{M}}(A) &= \{\lambda \in \mathbb{C} : A - \lambda I \text{ does not admit a } GS(\mathcal{M})D\}. \end{aligned}$$

The left generalized Drazin spectrum, the right generalized Drazin spectrum, the left generalized Drazin-Riesz spectrum, the right generalized Drazin-Riesz spectrum, the left generalized Drazin-meromorphic spectrum and the right generalized Drazin-meromorphic spectrum of $A \in L(X)$ are denoted by $\sigma^l_{gD}(A)$, $\sigma^r_{gD}(A)$, $\sigma^l_{gDR}(A)$, $\sigma^r_{gDR}(A)$, $\sigma^l_{gD\mathcal{M}}(A)$ and $\sigma^r_{gD\mathcal{M}}(A)$, respectively.

Theorem 6.1. *Let $A \in L(X)$. Then:*

(i) *If A admits a GSRD, then there exists $\epsilon > 0$ such that $A - \lambda I$ is essentially Saphar for each λ such that $0 < |\lambda| < \epsilon$;*

(ii) *If A admits a $GS(\mathcal{M})D$, then there exists $\epsilon > 0$ such that $A - \lambda I$ is of Saphar type for each λ such that $0 < |\lambda| < \epsilon$;*

Proof. There exists $(M, N) \in Red(A)$, such that $A = A_M \oplus A_N$, A_M is Saphar and A_N is Riesz. If $M = \{0\}$, then A is Riesz. According to [29, Theorem 3.2] we have that $A - \lambda I$ is Fredholm for all $\lambda \neq 0$. It follows that $\mathcal{R}(A - \lambda I)$ and $\mathcal{N}(A - \lambda I)$ are complemented subspaces of X. According to [20, Theorem 16.12] we have that $A - \lambda I$ is essentially Kato operator for all $\lambda \neq 0$. Hence $A - \lambda I$ is an essentially Saphar operator for all $\lambda \neq 0$.

Let $M \neq \{0\}$. According to [20, Corollary 12.4 and Lemma 13.6] we have that there exists an $\epsilon > 0$ such that for $|\lambda| < \epsilon$, $A_M - \lambda I_M$ is Saphar. As A_N is Riesz, $A_N - \lambda I_N$ is Fredholm, and hence it is essentially Saphar for all $\lambda \neq 0$. From Lemma 3.6 (iii) and Lemma 3.1 we obtain that $A - \lambda I$ is essentially Saphar for each λ such that $0 < |\lambda| < \epsilon$.

(ii) It can be proved in a similar way as the part (i), by using the fact that if A (A_N) is meromorphic, then $A - \lambda I$ ($A_N - \lambda I_N$) is Drazin invertible for every $\lambda \neq 0$. □

Corollary 6.2. *Let $A \in L(X)$. Then*

(i) *$\sigma_{gSR}(A)$ and $\sigma_{gS\mathcal{M}}(A)$ are compact;*

(ii) *$\sigma_{eS}(A) \setminus \sigma_{gSR}(A) \subset \text{iso}\, \sigma_{eS}(A)$,*
$\sigma_{St}(A) \setminus \sigma_{gS\mathcal{M}}(A) \subset \text{iso}\, \sigma_{St}(A)$.

Proof. It follows from Theorem 6.1. □

Corollary 6.3. *For $A \in L(X)$ and each $* = l, r$ the following holds:*

(i)

$$\sigma^*_{gDR}(A) = \sigma_{gSR}(A) \cup \text{int}\, \sigma_*(A) \tag{6.1}$$
$$= \sigma_{gSR}(A) \cup \text{acc}\, \sigma^*_{\mathcal{B}}(A) \tag{6.2}$$

and

$$\sigma^*_{gD\mathcal{M}}(A) = \sigma_{gS\mathcal{M}}(A) \cup \text{int}\, \sigma_*(A) \tag{6.3}$$
$$= \sigma_{gS\mathcal{M}}(A) \cup \text{acc}\, \sigma^*_{D}(A) \tag{6.4}$$

(ii) $\sigma^*_{gD\mathcal{M}}(A) \subset \sigma^*_{gDR}(A) \subset \sigma_*(A)$;

(iii) *$\sigma^*_{gDR}(A)$ and $\sigma^*_{gD\mathcal{M}}(A)$ are compact;*

(iv) $\text{int}\, \sigma^*_{gD\mathcal{M}}(A) = \text{int}\, \sigma^*_{gDR}(A) = \text{int}\, \sigma^*_{gD}(A) = \text{int}\, \sigma^*_{D}(A) = \text{int}\, \sigma^*_{\mathcal{B}}(A) = \text{int}\, \sigma_*(A)$;

(v) $\partial \sigma^*_{gD\mathcal{M}}(A) \subset \partial\, \sigma^*_{gDR}(A) \subset \partial \sigma^*_{gD}(A) \subset \partial \sigma^*_{D}(A) \subset \partial \sigma^*_{\mathcal{B}}(A) \subset \partial\, \sigma_*(A)$,

(vi) $\sigma^*_{\mathcal{B}}(A) \setminus \sigma^*_{gDR}(A) = (\text{iso}\, \sigma^*_{\mathcal{B}}(A)) \setminus \sigma_{gSR}(A)$,
$\sigma^*_{D}(A) \setminus \sigma^*_{gD\mathcal{M}}(A) = (\text{iso}\, \sigma^*_{D}(A)) \setminus \sigma_{gS\mathcal{M}}(A)$;

(vii) *$\sigma^*_{\mathcal{B}}(A) \setminus \sigma^*_{gDR}(A)$ and $\sigma^*_{D}(A) \setminus \sigma^*_{gD\mathcal{M}}(A)$ are at most countable.*

Proof. (i): It follows from Theorem 4.5 and Theorem 4.6.

(ii): It is clear.

(iii): It follows from (6.2), (6.4) and Corollary 6.2 (i).

(iv): From (6.3) we have that $\text{int}\, \sigma_*(A) \subset \sigma^*_{gD\mathcal{M}}(A)$, and hence $\text{int}\, \sigma_*(A) \subset \text{int}\, \sigma^*_{gD\mathcal{M}}(A)$. From (ii) it follows that $\text{int}\, \sigma^*_{gD\mathcal{M}}(A) \subset \text{int}\, \sigma_*(A)$. Hence

$$\text{int}\, \sigma^*_{gD\mathcal{M}}(A) = \text{int}\, \sigma_*(A),$$

which together with the inclusions

$$\sigma^*_{gD\mathcal{M}}(A) \subset \sigma^*_{gDR}(A) \subset \sigma^*_{gD}(A) \subset \sigma^*_{D}(A) \subset \sigma^*_{\mathcal{B}}(A) \subset \sigma_*(A)$$

gives the desired equalities.

(v): From (ii) and (iv) it follows that

$$\partial\sigma^*_{gD\mathcal{M}}(A) \subset \sigma^*_{gD\mathcal{M}}(A) \setminus \operatorname{int}\sigma^*_{gD\mathcal{M}}(A) \subset \sigma^*_{gDR}(A) \setminus \operatorname{int}\sigma^*_{gDR}(A) = \partial\sigma^*_{gDR}(A).$$

The rest of inclusions can be proved analogously.

(vi): It follows from (6.2) and (6.4) .

(vii): It follows from (vi). □

Corollary 6.4. (i) *Let $A \in L(X)$ have the SVEP. Then* $\operatorname{acc}\sigma^l_{\mathcal{B}}(A) \subset \sigma_{gSR}(A)$ *and* $\operatorname{acc}\sigma^l_D(A) \subset \sigma_{gS\mathcal{M}}(A)$.
(ii) *For $A \in L(X)$ let A' have the SVEP. Then* $\operatorname{acc}\sigma^r_{\mathcal{B}}(A) \subset \sigma_{gSR}(A)$ *and* $\operatorname{acc}\sigma^r_D(A) \subset \sigma_{gS\mathcal{M}}(A)$.

Proof. (i): It follows from the equivalence (ix)⟺(x) in Theorem 4.5 and Theorem 5.5.

(ii): It follows from the equivalence (ix)⟺(x) in Theorem 4.6 and Theorem 5.6. □

Theorem 6.5. *For $A \in L(X)$ and each $* = l, r$ there are inclusions*

$$\partial\sigma_*(A) \cap \operatorname{acc}\sigma^*_{\mathcal{B}}(A) \subset \partial\sigma_{gSR}(A) \tag{6.5}$$

and

$$\partial\sigma_*(A) \cap \operatorname{acc}\sigma^*_D(A) \subset \partial\sigma_{gS\mathcal{M}}(A). \tag{6.6}$$

Proof. Suppose that $A - \lambda I$ admit a GSRD and $\lambda \in \partial\,\sigma_l(A)$. Then $\lambda \notin \operatorname{int}\sigma_l(A)$ and from Theorem 4.5 it follows that $\lambda \notin \operatorname{acc}\sigma^l_{\mathcal{B}}(A)$. Hence

$$\partial\,\sigma_l(A) \cap \operatorname{acc}\sigma^l_{\mathcal{B}}(A) \subset \sigma_{gSR}(A). \tag{6.7}$$

Let $\lambda \in \partial\,\sigma_l(A) \cap \operatorname{acc}\sigma^l_{\mathcal{B}}(A)$. Then there exists a sequence (λ_n) which converges to λ and such that $A - \lambda_n$ is left inverible for every $n \in \mathbb{N}$. Hence $A - \lambda_n$ admits a GSRD, and so $\lambda_n \notin \sigma_{gSR}(A)$ for every $n \in \mathbb{N}$. As from (6.7) we have that $\lambda \in \sigma_{gSR}(A)$, it follows that $\lambda \in \partial\,\sigma_{gSR}(A)$. This proves the inclusion (6.5) for the case $* = l$. The rest of inclusions can be proved similarly by using Theorems 4.6, 5.5 and 5.6. □

If $K \subset \mathbb{C}$ is compact, then for $\lambda \in \partial K$ the following equivalence holds:

$$\lambda \in \operatorname{acc}K \Longleftrightarrow \lambda \in \operatorname{acc}\partial K. \tag{6.8}$$

Corollary 6.6. *For $A \in L(X)$ and each $* = l, r$, if*

$$\sigma_*(A) = \partial\sigma(A) \subset \operatorname{acc}\sigma(A),$$

then

$$\sigma_{gS\mathcal{M}}(A) = \sigma_{gSR}(A) = \sigma_{gS}(A) = \sigma_{St} = \sigma_{eS}(A) = \sigma_S(A) = \sigma_*(A) \tag{6.9}$$

and

$$\sigma^*_{gD\mathcal{M}}(A) = \sigma^*_{gDR}(A) = \sigma_*(A). \tag{6.10}$$

Proof. From $\sigma_*(A) = \partial\sigma(A)$ we get that $\sigma_*(A) = \partial\sigma(A) \subset \partial\sigma_*(A) \subset \sigma_*(A)$, and so $\sigma_*(A) = \partial\sigma^*(A)$. According to (6.8) we conclude that every $\lambda \in \partial\sigma(A)$ is an accumulation point of $\partial\sigma(A) = \sigma_*(A)$. Thus $\sigma_*(A) = \partial\sigma_*(A) = \operatorname{acc}\sigma_*(A)$. From [30, Corollary 5.5 (i)] it follows that $\operatorname{acc}\sigma_*(A) \subset \sigma^*_D(A) \subset \sigma_*(A)$, and so $\sigma_*(A) = \partial\sigma_*(A) = \operatorname{acc}\sigma^*_D(A)$. Using Theorem 6.5 we conclude that

$$\sigma_*(A) = \partial\sigma_*(A) \cap \operatorname{acc}\sigma^*_D(A) \subset \sigma_{gS\mathcal{M}}(A).$$

As $\sigma_{gS\mathcal{M}}(A) \subset \sigma_*(A)$, we obtain that $\sigma_{gS\mathcal{M}}(A) = \sigma_*(A)$, which together with the inclusions

$$\sigma_{gS\mathcal{M}}(A) \subset \sigma_{gSR}(A) \subset \sigma_{gS}(A) \subset \sigma_{St} \subset \sigma_{eS}(A) \subset \sigma_S(A) \subset \sigma_*(A)$$

gives (6.9). Corollary 6.3 (i) and (6.9) imply (6.10). □

Theorem 6.7. *Let $A \in L(X)$. Then the following inclusions hold:*
(i)

$$\begin{array}{ccccc} & & \partial\sigma^l_{gDR}(A) & & \\ & \subset & & \subset & \\ \partial\sigma_{gDR}(A) & & & & \partial\sigma_{gSR}(A) \\ & \subset & & \subset & \\ & & \partial\sigma^r_{gDR}(A) & & \end{array}$$

and

$$\begin{array}{ccccc} & & \partial\sigma^l_{gD\mathcal{M}}(A) & & \\ & \subset & & \subset & \\ \partial\sigma_{gD\mathcal{M}}(A) & & & & \partial\sigma_{gS\mathcal{M}}(A); \\ & \subset & & \subset & \\ & & \partial\sigma^r_{gD\mathcal{M}}(A) & & \end{array}$$

(ii) $\eta\sigma_{gSR}(A) = \eta\sigma^l_{gDR}(A) = \eta\sigma^r_{gDR}(A) = \eta\sigma_{gDR}(A),$
$\eta\sigma_{gS\mathcal{M}}(A) = \eta\sigma^l_{gD\mathcal{M}}(A) = \eta\sigma^r_{gD\mathcal{M}}(A) = \eta\sigma_{gD\mathcal{M}}(A);$

(iii) *Let the complement of $\sigma_*(A)$ be connected, where $\sigma_* \in \{\sigma_{gSR}, \sigma^l_{gDR}, \sigma^r_{gDR}\}$. Then $\sigma_*(A) = \sigma_{gRD}(A)$.*
Let the complement of $\sigma_*(A)$ be connected, where $\sigma_* \in \{\sigma_{gS\mathcal{M}}, \sigma^l_{gD\mathcal{M}}, \sigma^r_{gD\mathcal{M}}\}$. Then $\sigma_*(A) = \sigma_{gD\mathcal{M}}(A)$.

Proof. (i): As

$$\begin{array}{ccccc} & & \sigma^l_{gDR}(A) & & \\ & \subset & & \subset & \\ \sigma_{gSR}(A) & & & & \sigma_{gDR}(A); \\ & \subset & & \subset & \\ & & \sigma^r_{gDR}(A) & & \end{array}$$

according to Corollary 6.2 (i), Corollary 6.3 (iii) and (2.1) it is enough to prove that $\partial\sigma_*(A) \subset \sigma_{gSR}(A)$, for $\sigma_* \in \{\sigma^l_{gDR}, \sigma^r_{gDR}, \sigma_{gDR}\}$. From (6.1) and Corollary 6.3 (iv) it follows that

$$\begin{aligned} \partial\sigma^l_{gDR}(A) &= \sigma^l_{gDR}(A) \setminus \operatorname{int}\sigma^l_{gDR}(A) = (\sigma_{gSR}(A) \cup \operatorname{int}\sigma_l(A)) \setminus \operatorname{int}\sigma_l(A) \\ &= \sigma_{gSR}(A) \setminus \operatorname{int}\sigma_l(A) \subset \sigma_{gSR}(A). \end{aligned}$$

Similarly, $\partial\sigma^r_{gDR}(A) \subset \sigma_{gSR}(A)$.

From Corollary 4.10 we conclude that $\sigma_{gDR}(A) = \sigma_{gSR}(A) \cup \operatorname{int}\sigma(A)$, which implies that $\operatorname{int}\sigma_{gDR}(A) = \operatorname{int}\sigma(A)$. Applying the same method as above we conclude that $\partial\sigma_{gDR}(A) \subset \sigma_{gSR}(A)$.

The rest of the inclusions can be proved similarly.

(ii): It follows from (2.1), (i) and (ii).

(iii): Let the complement of $\sigma_*(A)$ be connected, where $\sigma_* \in \{\sigma_{gSR}, \sigma^l_{gDR}, \sigma^r_{gDR}\}$. Then by using (iii) we get $\sigma_{gDR}(A) \supset \sigma_*(A) = \eta\sigma_*(A) = \eta\sigma_{gDR}(A) \supset \sigma_{gDR}(A)$, and so $\sigma_*(A) = \sigma_{gDR}(A)$.

The second assertion can be proved similarly. □

From Theorem 6.7 (ii) it follows that if one of the spectra $\sigma_{gSR}(A), \sigma^l_{gDR}(A), \sigma^r_{gDR}(A)$, $\sigma_{gDR}(A)$ is at most countable, then they are equal. Also, if one of the spectra $\sigma_{gS\mathcal{M}}(A)$, $\sigma^l_{gD\mathcal{M}}(A), \sigma^r_{gD\mathcal{M}}(A)$, $\sigma_{gD\mathcal{M}}(A)$ is at most countable, then they are equal. Hence using [27, Theorem 3.11] and [28, Theorem 12] respectively, we get the following corollaries.

Corollary 6.8. *For $A \in L(X)$ the following statements are equivalent:*

(i) $\sigma_{gSR}(A) = \emptyset$;

(ii) $\sigma^l_{gDR}(A) = \emptyset$;

(iii) $\sigma^r_{gDR}(A) = \emptyset$;

(iv) $\sigma_{gDR}(A) = \emptyset$;
(v) $\sigma_{gKR}(A) = \emptyset$;
(vi) *A is polynomially Riesz;*
(vii) $\sigma_{\mathcal{B}}(A)$ *is a finite set.*

Corollary 6.9. *For $A \in L(X)$ the following statements are equivalent:*
(i) $\sigma_{gS\mathcal{M}}(A) = \emptyset$;
(ii) $\sigma^l_{gD\mathcal{M}}(A) = \emptyset$;
(iii) $\sigma^r_{gD\mathcal{M}}(A) = \emptyset$;
(iv) $\sigma_{gD\mathcal{M}}(A) = \emptyset$;
(v) $\sigma_{gK\mathcal{M}}(A) = \emptyset$;
(vi) *A is polynomially meromorphic;*
(vii) $\sigma_D(A)$ *is a finite set.*

Example 6.10. The *Cesáro operator* C_p on the classical Hardy space $H_p(\mathbf{D})$, $\mathbf{D}$ is the open unit disc and $1 < p < \infty$, is defined by

$$(C_p f)(\lambda) = \frac{1}{\lambda}\int_0^\lambda \frac{f(\mu)}{1-\mu}\,d\mu, \text{ for all } f \in H_p(\mathbf{D}) \text{ and } \lambda \in \mathbf{D}.$$

We recall that its spectrum is the closed disc Γ_p centered at $p/2$ with radius $p/2$ and $\sigma_{Kt}(C_p) = \partial\Gamma_p$ [3]. Since for $\lambda \in \text{int}\Gamma_p$ we have that $C_p - \lambda I$ is Fredholm, bounded below and not surjective [30, Example 5.13], we have that $\sigma_r(C_p) = \Gamma_p$ and $\sigma_l(C_p) = \partial\Gamma_p$. From Corollary 6.6 it follows that $\sigma_{gS\mathcal{M}}(C_p) = \sigma_{gSR}(C_p) = \partial\Gamma_p$ and $\sigma^l_{gD\mathcal{M}}(C_p) = \sigma^l_{gDR}(C_p) = \partial\Gamma_p$, while from Corollary 6.3 (i) we have $\sigma^r_{gDR}(C_p) = \sigma_{gSR}(C_p) \cup \text{int}\,\sigma_r(C_p) = \Gamma_p$ and $\sigma^r_{gD\mathcal{M}}(C_p) = \sigma_{gS\mathcal{M}}(C_p) \cup \text{int}\,\sigma_r(C_p) = \Gamma_p$.

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