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Employing an iterative gradient-based approach to solve generalized coupled Sylvester matrix equations over generalized centro-symmetric matrices

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Abstract. In this research, we introduce a novel approach to solving coupled generalized Sylvester matrix (**GSM**) equations $MV + NW = GVH + D$, $AV + BW = EVF + C$ by leveraging the structural properties of generalized centro-symmetric matrices (***GCSMs***). Specifically, we propose a gradient-based iterative technique that exploits the symmetry and pattern inherent in ***GCSMs*** to facilitate the solution process. The proposed method offers an efficient computational framework, particularly well-suited for problems where the coefficient matrices and initial conditions are generalized centro-symmetric. Our approach focuses on iteratively refining an initial guess—chosen to be a ***GCSM***—toward a generalized centro-symmetric solution (***GCSS***) of the coupled **GSM** equations. A key advantage of this method is its rapid convergence in the idealized setting free of round-off errors. When the matrix equations under consideration are compatible with the structure of the initial ***GCSM***, the method can achieve accurate results in a relatively small number of iterations.

1. Introduction

In this paper, we indicate the set of all real $n \times n$ matrices by $R^{n\times n}$. $\mathcal{S}R^{n\times n}$ and $\mathcal{S}OR^{n\times n}$ represent the set of whole symmetric and symmetric orthogonal matrices in $R^{n\times n}$, consecutively. For $A \in R^{n\times n}$, the symbols A^T, $tr(A)$, $R(A)$, $rank(A)$, $\|A\| = \sqrt{tr(A^TA)}$ refer to the transpose, the trace, the column space, the rank and the Frobenius norm of matrix A, consecutively. The inner product of the two matrices A and B is realized as $\langle A, B\rangle = tr\left(B^TA\right)$. I_n denote the identity matrix with order n. P is called real symmetric orthogonal $n \times n$ matrix if $P = P^T = P^{-1}$.

For the matrix $A \in R^{mxn}$, The vec operator is a mathematical operator that transforms a matrix into a column vector by vertically stacking the columns of the matrix on top of one another, the vec operator is denoted as $vec(A) = ({a_1}^T, {a_2}^T, \ldots\ldots {a_n}^T)^T$. $A \bigotimes B$ represents the Kronecker product of the matrices A and B.

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Definition 1.1. *[22] Generalized centro-symmetric (generalized central anti-symmetric)* ***GCS(GCAS)*** *Taking P into account, the matrix* $A \in R^{n\times n}$ *is called* ***GCS (GCAS)*** *if* $PAP = A$ $(PAP = -A)$*, where P is any real symmetric orthogonal* $n \times n$ *matrix. The set of order n* ***GCS (GCAS)*** *matrices with respect to* $P \in \mathcal{SOR}^{n\times n}$ *is denoted by* $\mathcal{CSR}_P^{n\times n}\left(\mathcal{CASR}_P^{n\times n}\right)$.

We will investigate the following problem in our paper:

Problem 1.2. *For the assumed matrices* $A, B, E, C, M, G, N, D, F, H \in R^{n\times n}$, $P \in \mathcal{SOR}^{n\times n}$ *and* $S \in \mathcal{SOR}^{n\times n}$ *, detect matrices* $V \in \mathcal{CSR}_P^{n\times n}$ *and* $W \in \mathcal{CSR}_S^{n\times n}$ *where*

$$\begin{cases} AV + BW = EVF + C, \\ MV + NW = GVH + D. \end{cases} \tag{1}$$

Pattern recognition [8, 9], engineering problems, and the Harmonic differential method [6] all make extensive use of ***GCSM****. Zhou et al. [43] provided certain conditions for addressing the inverse eigenvalue issue for centro-symmetric matrices. The inverse eigenvalue problem of symmetric and* ***GCSMs*** *is solved, according to Xie et al. [39]. Mukherjee and Maiti [23] investigated the properties and applications of Toeplitz matrices, a type of matrices.*

The GSM equation of the type

$$AV + BW = EVF, \tag{2}$$

is essential to several problems in descriptor system theory, for example the problem of observer design (see [4, 5, 34, 37]) and the eigen structure assignment problem (see [15, 16, 24]). Numerous studies have described numerous strategies for solving linear matrix problems; check refs. ([7, 10, 17, 20, 21, 32, 33, 35]) for more information. Iterative strategies for solving the extended Sylvester matrix problem of the kind $AV + BW = EVF + C$ *were presented by Ramadan and Bayoumi in [25]. The coupled example of GSM equations was investigated by Wang and Zhuo [36]. Wu et al. solved the matrix equation* (2) *in [38]. The GSM problem of the type* $AV - EVF = BW$ *was solved by Zhou and Duan [42]. Zhou and Duan [41] solved the matrix equation* $AX - XF = BY$ *in 2005. Hajarian [18] presented an iterative approach for finding the solution of generic matrices in linked form. M. L. Liang, C.H. You, and colleagues developed an iterative technique for solving the extended centro-symmetric problem* $AXB = C$ *in [22].*

Ref. [3] studied the coupled Sylvester-transpose matrix equations $\sum_{j=1}^{q}\left(A_{ij}X_jB_{ij} + C_{ij}X_j^TD_{ij}\right) = F_i$ *through (GCSM). In addition, [11] solved a set of GSM equations in terms of* ***GCSM****. Ding and Liu et al. [14] used the technique of hierarchical identification to solve the GSM equations. In 2005, Ding and Chen ([12, 13]) introduced hierarchical least squares identification techniques to solve multivariable systems. Several recent works have looked into gradient-based iterative (GI) techniques for solving linear matrix problems.*

[19] represented the (GI) algorithm for getting matrix equation solutions to $A_1X_1B_1 + C_1X_1D_1 = E_1$, $A_2X_2B_2 + C_2X_2D_2 = E_2$. *Ramadan and colleagues [28] introduced a relaxed gradient-based (RGI) technique for solving the extended Sylvester conjugate equations. The accelerated gradient-based iterative (AGBI) technique and the modified gradient-based (MGI) approach were presented by Ramadan and Bayoumi ([2, 26]) for solving extended Sylvester–conjugate matrix equations. Ramadan et al. [30] suggested an iterative technique for solving generalized coupled Sylvester equations through* ***GCSMs****. Ramadan et al. [29] offered two techniques for solving the extended Sylvester equation's reflexive and Hermitian reflexive solutions. Bayoumi [1] developed two finite techniques for obtaining reflexive and Hermitian reflexive solutions to linked complex of conjugate and transpose matrix equations. Y. Xie and Y. Ke [40] presented an iterative technique for solving the generalized complex coupled Sylvester-transpose equations through* (R, S)*-conjugate matrix solution. Ramadan and El-Danaf [27] proposed an iterative method for finding the generalized bisymmetric solution to the generalized coupled Sylvester matrix equations. Also, they obtained the generalized bisymmetric solution within finite iterative steps in the absence of round-off errors when the generalized coupled Sylvester matrix equations are consistent, for any generalized bisymmetric initial iterative matrix pair. Ramadan et al. [31], proposed two iterative algorithms for solving the Sylvester- conjugate matrix equations* $AV + BW = E\bar{V}F + C, AV + B\bar{W} = E\bar{V}F + C$.

This paper is organized in the following manner:
In the second section, two lemmas are introduced to make it possible to solve the matrix equations (1) using

GCSM. The gradient-based iterative (GI) approach for solving these equations will also be given.
We demonstrated in Section 3 how our proposed technique converges to **GCSS** with certain starting matrices .
Section 4 includes numerical examples that indicate that our introduced approach is efficient.
Section 5 discusses the paper conclusion.

2. The Proposed Iterative Method for Solving Problem 1.2

In this section, we prove two lemmas for obtaining the solution of the coupled GSM equation (1). Then, to solve problem 1.2, we propose a gradient-based iterative (GI) algorithm. In these lemmas we present the necessary and sufficient condition for obtaining the solution of the coupled GSM equation (1) as well as the uniqueness of this solution.

Lemma 2.1. *The GSM equations* (1) *have GSCS* $V \in \mathcal{CSR}_P^{n\times n}$ *and* $W \in \mathcal{CSR}_S^{n\times n}$ *if and only if the following system of linear equations is consistent*

$$\begin{cases} AV + BW = EVF + C, \\ APVP + BSWS = EPVPF + C, \\ MV + NW = GVH + D, \\ MPVP + NSWS = GPVPH + D. \end{cases} \tag{3}$$

Proof. At first assume that the system (3) is consistent, then there exists matrices $\tilde{V}$, $\tilde{W}$ in which

$$\begin{cases} A\tilde{V} + B\tilde{W} = E\tilde{V}F + C, \\ AP\tilde{V}P + BS\tilde{W}S = EP\tilde{V}PF + C, \\ M\tilde{V} + N\tilde{W} = G\tilde{V}H + D, \\ MP\tilde{V}P + NS\tilde{W}S = GP\tilde{V}PH + D. \end{cases} \tag{4}$$

Set $\hat{V} = \frac{\tilde{V}+P\tilde{V}P}{2}$ and $\hat{W} = \frac{\tilde{W}+S\tilde{W}S}{2}$.
Clearly, we can see that $\hat{V} \in \mathcal{CSR}_p^{n\times n}$, $\hat{W} \in \mathcal{CSR}_s^{n\times n}$ and

$$\begin{aligned} A\hat{V} + B\hat{W} - E\hat{V}F &= A\frac{\tilde{V} + P\tilde{V}P}{2} + B\frac{\tilde{W} + S\tilde{W}S}{2} - E\frac{\tilde{V} + P\tilde{V}P}{2}F \\ &= \frac{1}{2}\left[A\tilde{V} + B\tilde{W} - E\tilde{V}F\right] + \frac{1}{2}[AP\tilde{V}P + BS\tilde{W}S - EP\tilde{V}PF] = \frac{C}{2} + \frac{C}{2} = C, \end{aligned}$$

$$\begin{aligned} M\hat{V} + N\hat{W} - G\hat{V}H &= M\frac{\tilde{V} + P\tilde{V}P}{2} + N\frac{\tilde{W} + S\tilde{W}S}{2} - G\frac{\tilde{V} + P\tilde{V}P}{2}H \\ &= \frac{1}{2}\left[M\tilde{V} + N\tilde{W} - G\tilde{V}H\right] + \frac{1}{2}[MP\tilde{V}P + NS\tilde{W}S - GP\tilde{V}PH] = \frac{D}{2} + \frac{D}{2} = D. \end{aligned}$$

Conversely, suppose that the matrix equations (1) have the **GCSS** $V^* \in \mathcal{CSR}_P^{n\times n}$, $W^* \in \mathcal{CSR}_S^{n\times n}$ where $V^* = PV^*P$, $W^* = SW^*S$ and $AV^* + BW^* = EV^*F + C$, $MV^* + NW^* = GV^*H + D$.
Then we get

$$APV^*P + BSW^*S - EPV^*PF = AV^* + BW^* - EV^*F = C, \tag{5}$$

and

$$MPV^*P + NSW^*S - GPV^*PH = MV^* + NW^* - GV^*H = D. \tag{6}$$

We can conclude that the solutions V^* and W^* are also solutions to the system (3). As a result, linear matrix equations (3) are valid. This brings the proof of Lemma 2.1 to a close. □

Lemma 2.2. *Let*

$$Q = \begin{pmatrix} I \bigotimes A - F^T \bigotimes E & I \bigotimes B \\ P \bigotimes AP - F^T P \bigotimes EP & S \bigotimes BS \\ I \bigotimes M - H^T \bigotimes G & I \bigotimes N \\ P \bigotimes MP - H^T P \bigotimes GP & S \bigotimes NS \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} vec(C) \\ vec(C) \\ vec(D) \\ vec(D) \end{pmatrix}.$$

Problem 1.2 has a unique pair of solution (V, W) *if and only if rank*(Q, y)= *rank*(Q) *and the matrix* Q *is of full column rank. So we can write this **GSC** pair* (V, W) *as follows:*

$\mathrm{V} = \frac{\mathrm{V_1+PV_1P}}{2}$, $\mathrm{W} = \frac{\mathrm{W_1+SW_1S}}{2}$ *where* $\begin{pmatrix} \mathrm{vec(V_1)} \\ \mathrm{vec(W_1)} \end{pmatrix} = (\mathrm{Q^TQ})^{-1}\mathrm{Q^Ty}$, *and the homogeneous systems*

$$\begin{cases} AV + BW - EVF = 0 \\ MV + NW - GVH = 0, \end{cases} \tag{7}$$

and the solution pair $(V, W) = (0, 0)$ *is the unique solution of the later system:*

$$\begin{cases} AV + BW - EVF = 0, \\ APVP + BSWS - EPVPF = 0, \\ MV + NW - GVH = 0, \\ MPVP + NSWS - GPVPH = 0, \end{cases} \tag{8}$$

To solve problem 1.2, we must determine the inverse of the significant matrix (QT Q), and we will present our proposed gradient-based iterative (GI) algorithm in order to obtain the solution of the equations (1) *via **(GCSM)** by stretching the iterations of the Jacobi and Gauss–Seidel methods and employing the principle of hierarchical identification ([12, 13]).*

Algorithm 2.3. *The proposed (GI) algorithm.*
Step1: *Input matrices* $A, B, E, C, M, G, N, D, F, H \in R^{n\times n}, V(0) \in CSR_p^{n\times n}, W(0) \in CSR_s^{n\times n}, P \in SOR^{n\times n}, S \in SOR^{n\times n}$ *and the convergence factor* μ *as:*

$$0 < \mu < \frac{2}{\|A\|^2 + \|B\|^2 + \|EF\|^2 + \|M\|^2 + \|N\|^2 + \|GH\|^2},$$

Step 2: *Compute*

$$R_1(0) = C - AV(0) - BW(0) + EV(0)F,$$

$$R_2(0) = D - MV(0) - NW(0) + GV(0)H,$$

Step 3: *Update the sequences for* $k = 1, 2 \ldots.$

$$\begin{aligned} V(k) =& V(k-1) + \frac{\mu}{2}[A^T R_1(k-1) + PA^T R_1(k-1)P - E^T R_1(k-1)F^T - PE^T R_1(k-1)F^T P + M^T R_2(k-1) \\ &+ PM^T R_2(k-1)P - G^T R_2(k-1)H^T - PG^T R_2(k-1)H^T P], \end{aligned}$$

$$W(k) = W(k-1) + \frac{\mu}{2}\left[B^T R_1(k-1) + SB^T R_1(k-1)S + N^T R_2(k-1) + SN^T R_2(k-1)S\right],$$

and

$$R_1(k) = C - AV(k) - BW(k) + EV(k)F,$$

$$R_2(k) = D - MV(k) - NW(k) + GV(k)H.$$

Step 4: *Set* $k := k + 1$ *; return to step 3.*

Remark 2.4. *From algorithm 2.3, we can observe that* $V(k) \in \mathcal{CSR}_p^{n\times n},\ W(k) \in \mathcal{CSR}_s^{n\times n}$ *for* $k = 1, 2 \cdots$
For $k = 2$

$$\begin{aligned} PV(2)P &= PV(1)P + \frac{\mu}{2}[PA^TR_1(1)P + P^2A^TR_1(1)P^2 - PE^TR_1(1)F^TP - P^2E^TR_1(1)F^TP^2 + PM^TR_2(1)P \\ &\quad + P^2M^TR_2(1)P^2 - PG^TR_2(1)H^TP - P^2G^TR_2(1)H^TP^2] \\ &= V(1) + \frac{\mu}{2}[PA^TR_1(1)P + A^TR_1(1) - PE^TR_1(1)F^TP - E^TR_1(1)F^T + PM^TR_2(1)P \\ &\quad + M^TR_2(1) - PG^TR_2(1)H^TP - G^TR_2(1)H^T] \\ &= V(2). \end{aligned}$$

And

$$\begin{aligned} SW(2)S &= SW(1)S + \frac{\mu}{2}\left[SB^TR_1(1)S + S^2B^TR_1(1)S^2 + SN^TR_2(1)S + S^2N^TR_2(1)S^2\right] \\ &= W(1) + \frac{\mu}{2}\left[SB^TR_1(1)S + B^TR_1(1) + SN^TR_2(1)S + N^TR_2(1)\right] \\ &= W(2). \end{aligned}$$

3. Main Results

This section examines the convergence analysis of the sequences $\{V(k)\}, \{W(k)\}$ produced by our proposed Algorithm 2.3.

Theorem 3.1. *Presume that the unique* ***GSCS*** *of the coupled equations* (1) *are* V^* *and* W^* *. If the convergence factor* μ *is taken to satisfy*

$$0 < \mu < \frac{2}{\|A\|^2 + \|B\|^2 + \|EF\|^2 + \|M\|^2 + \|N\|^2 + \|GH\|^2}.$$

Then the sequences $\{V(k)\}$ *and* $\{W(k)\}$ *constructed by our suggested Algorithm 2.3 converge to* V^* *and* W^* *with the chosen initial (GCSM)* $V(0)$ *and* $W(0)$ *, that is* $\lim_{k\to\infty} V(k) = V^*$ *and* $\lim_{k\to\infty} W(k) = W^*$ *.*

Proof. Define the error matrix

$$\tilde{V}(k) = V(k) - V^* \in \mathcal{CSR}_p^{n\times n}, \tag{9}$$

Where

$$P\tilde{V}(k)P = PV(k)P - PV^*P = V(k) - V^* = \tilde{V}(k). \tag{10}$$

and

$$\tilde{W}(k) = W(k) - W^* \in \mathcal{CSR}_S^{n\times n}, \tag{11}$$

where

$$S\tilde{W}(k)S = SW(k)S - SW^*S = W(k) - W^* = \tilde{W}(k). \tag{12}$$

Also we have

$$\begin{aligned} R_1(k) &= C - AV(k) - BW(k) + EV(k)F \\ &= -A\tilde{V}(k) - B\tilde{W}(k) + E\tilde{V}(k)F \end{aligned} \tag{13}$$

$$\begin{aligned} R_2(k) &= D - MV(k) - NW(k) + GV(k)H \\ &= -M\tilde{V}(k) - N\tilde{W}(k) + G\tilde{V}(k)H \end{aligned} \tag{14}$$

Let

$$\begin{aligned}&\Delta_1(k) = -R_1(k) = A\tilde{V}(k) + B\tilde{W}(k) - E\tilde{V}(k)F\,,\\&\text{and}\\&\Delta_2(k) = -R_2(k) = M\tilde{V}(k) + N\tilde{W}(k) - G\tilde{V}(k)H.\end{aligned} \tag{15}$$

Now, it is clear that

$$\begin{aligned}\tilde{V}(k) &= V(k) - V^*\\&= V(k-1) + \frac{\mu}{2}[A^TR_1(k-1) + PA^TR_1(k-1)P - E^TR_1(k-1)F^T - PE^TR_1(k-1)F^TP\\&\quad + M^TR_2(k-1) + PM^TR_2(k-1)P - G^TR_2(k-1)H^T - PG^TR_2(k-1)H^TP] - V^*\,,\end{aligned} \tag{16}$$

$$\begin{aligned}&= \widetilde{V}(k-1) - \frac{\mu}{2}[A^T\Delta_1(k-1) + PA^T\Delta_1(k-1)P - E^T\Delta_1(k-1)F^T - PE^T\Delta_1(k-1)F^TP\\&\quad + M^T\Delta_2(k-1) + PM^T\Delta_2(k-1)P - G^T\Delta_2(k-1)H^T - PG^T\Delta_2(k-1)H^TP].\end{aligned} \tag{17}$$

Similarly for (17), we can obtain $\tilde{W}(k)$ as:

$$\begin{aligned}\tilde{W}(k) &= W(k) - W^*\\&= \tilde{W}(k-1) - \frac{\mu}{2}\left[B^T\Delta_1(k-1) + SB^T\Delta_1(k-1)S + \Delta_2{}^TR_2(k-1) + S\Delta_2{}^TR_2(k-1)S\right].\end{aligned} \tag{18}$$

Taking the Frobenius norm of $\tilde{V}(k)$, it follows that

$$\begin{aligned}\left\|\tilde{V}(k)\right\|^2 &= tr(\tilde{V}(k)^T\tilde{V}(k))\\&= \Big\|\tilde{V}(k-1) - \frac{\mu}{2}\Big[A^T\Delta_1(k-1) + PA^T\Delta_1(k-1)P - E^T\Delta_1(k-1)F^T - PE^T\Delta_1(k-1)F^TP\\&\quad + M^T\Delta_2(k-1) + PM^T\Delta_2(k-1)P - G^T\Delta_2(k-1)H^T - PG^T\Delta_2(k-1)H^TP\Big]\Big\|^2\end{aligned} \tag{19}$$

$$\begin{aligned}&= \left\|\tilde{V}(k-1)\right\|^2 - \mu tr\Big(\tilde{V}(k-1)^T\Big[A^T\Delta_1(k-1) + PA^T\Delta_1(k-1)P - E^T\Delta_1(k-1)F^T\\&\quad - PE^T\Delta_1(k-1)F^TP + M^T\Delta_2(k-1) + PM^T\Delta_2(k-1)P - G^T\Delta_2(k-1)H^T - PG^T\Delta_2(k-1)H^TP\Big]\Big)\\&\quad + \frac{\mu^2}{4}\Big\|\Big[A^T\Delta_1(k-1) + PA^T\Delta_1(k-1)P - E^T\Delta_1(k-1)F^T - PE^T\Delta_1(k-1)F^TP + M^T\Delta_2(k-1)\\&\quad + PM^T\Delta_2(k-1)P - G^T\Delta_2(k-1)H^T - PG^T\Delta_2(k-1)H^TP\Big]\Big\|^2\end{aligned} \tag{20}$$

$$\begin{aligned}&\le \left\|\tilde{V}(k-1)\right\|^2 - \mu tr\Big(\Big[A\tilde{V}(k-1) + PA\tilde{V}(k-1)P - E\tilde{V}(k-1)F - PE\tilde{V}(k-1)FP\Big]\Delta_1(k-1)^T\\&\quad + \Big[M\tilde{V}(k-1) + PM\tilde{V}(k-1)P - G\tilde{V}(k-1)H - PG\tilde{V}(k-1)HP\Big]\Delta_2(k-1)^T\Big)\\&\quad + \frac{\mu^2}{2}(\left\|A^T\Delta_1(k-1) - E^T\Delta_1(k-1)F^T + M^T\Delta_2(k-1) - G^T\Delta_2(k-1)H^T\right\|^2\\&\quad + \left\|PA^T\Delta_1(k-1)P - PE^T\Delta_1(k-1)F^TP + PM^T\Delta_2(k-1)P - PG^T\Delta_2(k-1)H^TP\right\|^2)\end{aligned} \tag{21}$$

$$\begin{aligned}&= \left\|\tilde{V}(k-1)\right\|^2 - \mu tr(\tilde{V}(k-1)^T\Big[A^T\Delta_1(k-1) + PA^T\Delta_1(k-1)P - E^T\Delta_1(k-1)F^T - PE^T\Delta_1(k-1)F^TP\\&\quad + M^T\Delta_2(k-1) + PM^T\Delta_2(k-1)P - G^T\Delta_2(k-1)H^T - PG^T\Delta_2(k-1)H^TP\Big]\end{aligned} \tag{22}$$

$$\leq \left\|\tilde{V}(k-1)\right\|^2 - 2\mu tr\left(\left[A\tilde{V}(k-1) - E\tilde{V}(k-1)F\right]\Delta_1(k-1)^T + \left[M\tilde{V}(k-1) - G\tilde{V}(k-1)H\right]\Delta_2(k-1)^T\right) + \mu^2(\left\|A^T\Delta_1(k-1) - E^T\Delta_1(k-1)F^T + M^T\Delta_2(k-1) - G^T\Delta_2(k-1)H^T\right\|^2) \tag{23}$$

$$\leq \left\|\tilde{V}(k-1)\right\|^2 - 2\mu tr\left(\left[A\tilde{V}(k-1) - E\tilde{V}(k-1)F\right]\Delta_1(k-1)^T + \left[M\tilde{V}(k-1) - G\tilde{V}(k-1)H\right]\Delta_2(k-1)^T\right) + \mu^2\left[(||A||^2 + ||EF||^2\right)||\Delta_1(k-1)||^2 + (||M||^2 + ||GH||^2)||\Delta_2(k-1)||^2]. \tag{24}$$

Similarly for (24),

$$\left\|\tilde{W}(k)\right\|^2 \leq \left\|\tilde{W}(k-1)\right\|^2 - 2\mu tr\left([B\tilde{W}(k-1)]\Delta_1(k-1)^T + [N\tilde{W}(k-1)]\Delta_2(k-1)^T\right) + \mu^2\left[||B||^2||\Delta_1(k-1)||^2 + ||N||^2||\Delta_2(k-1)||^2\right] \tag{25}$$

Define the non-negative definite function $\tilde{Q}(k) = \left\|\tilde{V}(k)\right\|^2 + \left\|\tilde{W}(k)\right\|^2$, it follows that:

$$\begin{aligned}
\tilde{Q}(k) \leq{}& \left\|\tilde{V}(k-1)\right\|^2 + \left\|\tilde{W}(k-1)\right\|^2 - 2\mu tr\left(\left[A\tilde{V}(k-1) + B\tilde{W}(k-1) - E\tilde{V}(k-1)F\right]\Delta_1(k-1)^T\right.\\
&\left.+[M\tilde{V}(k-1) + N\tilde{W}(k-1) - G\tilde{V}(k-1)H]\Delta_2(k-1)^T\right)\\
&+\mu^2\left[(||A||^2 + ||B||^2 + ||EF||^2\right)||\Delta_1(k-1)||^2 + (||M||^2 + ||N||^2 + ||GH||^2)||\Delta_2(k-1)||^2]\\
={}& \tilde{Q}(k-1) - 2\mu\left(||\Delta_1(k-1)||^2 + ||\Delta_2(k-1)||^2\right)\\
&+\mu^2(\left[||A||^2 + ||B||^2 + ||EF||^2 + ||M||^2 + ||N||^2 + ||GH||^2\right] \times \left[||\Delta_1(k-1)||^2 + ||\Delta_2(k-1)||^2\right])\\
={}& \tilde{Q}(k-1) - 2\mu(1 - \frac{\mu}{2}\left[||A||^2 + ||B||^2 + ||EF||^2 + ||M||^2 + ||N||^2 + ||GH||^2\right]\left[||\Delta_1(k-1)||^2 + ||\Delta_2(k-1)||^2\right])\\
\leq{}& \tilde{Q}(0) - 2\mu(1 - \frac{\mu}{2}\left[||A||^2 + ||B||^2 + ||EF||^2 + ||M||^2 + ||N||^2 + ||GH||^2\right]\sum_{m=1}^{k}[||\Delta_1(m-1)||^2 + ||\Delta_2(m-1)||^2]).
\end{aligned} \tag{26}$$

If we chose $0 < \mu < \frac{2}{||A||^2+||B||^2+||EF||^2+||M||^2+||N||^2+||GH||^2}$, we can obtain

$$\sum_{m=1}^{k}[||\Delta_1(m-1)||^2 + ||\Delta_2(m-1)||^2]) < \infty. \tag{27}$$

It follows that $\lim_{k\to\infty}\Delta_1(k-1) = 0$, $\lim_{k\to\infty}\Delta_2(k-1) = 0$, and $\lim_{k\to\infty} A\tilde{V}(k) + B\tilde{W}(k) - E\tilde{V}(k)F = 0$,

$$\lim_{k\to\infty} M\tilde{V}(k) + N\tilde{W}(k) - G\tilde{V}(k)H = 0\,. \tag{28}$$

This implies that $\lim_{k\to\infty}\tilde{V}(k) = 0$, $\lim_{k\to\infty}\tilde{W}(k) = 0$ and we get $\lim_{k\to\infty} V(k) = V^*$,

$$\lim_{k\to\infty} W(k) = W^*\,. \tag{29}$$

The proof is complete. □

4. Numerical Examples

We present two test cases in this section to show the effectiveness of our proposed algorithm. All of the reported numerical results were gained using MATLAB (7.1) on a PC Intel ® CoreTMi5-2520M CPU@2.50GHz with 4GB of RAM.

Example 4.1. *In the first example, we deem the matrix equations* (1) *with the following input matrices*

$$A=\begin{pmatrix}-1&1&5&3\\-2&1&7&1\\2&8&6&1\\3&5&2&1\end{pmatrix}\in R^{4\times4},\quad B=\begin{pmatrix}5&1&3&-2\\2&4&5&7\\4&2&1&6\\4&6&5&3\end{pmatrix}\in R^{4\times4},\quad E=\begin{pmatrix}3&-2&5&-2\\-1&1&-2&3\\5&3&6&1\\3&-1&2&5\end{pmatrix}\in R^{4\times4},$$

$$F=\begin{pmatrix}1&2&6&1\\3&-1&-6&1\\2&-5&4&-1\\8&-1&3&4\end{pmatrix}\in R^{4\times4},\quad M=\begin{pmatrix}2&1&5&3\\3&1&6&-4\\2&-1&3&6\\-1&3&4&2\end{pmatrix}\in R^{4\times4},\quad N=\begin{pmatrix}4&5&-2&6\\6&2&-1&4\\2&-3&4&8\\7&-1&5&-3\end{pmatrix}\in R^{4\times4},$$

$$G=\begin{pmatrix}9&2&-1&5\\3&-5&2&4\\6&-4&7&1\\2&-3&1&4\end{pmatrix}\in R^{4\times4},\qquad H=\begin{pmatrix}3&-5&6&1\\5&-1&-4&2\\6&-5&4&3\\9&1&-2&4\end{pmatrix}\in R^{4\times4},$$

$$C=\begin{pmatrix}37&124&-201&72\\-44&-79&64&-10\\-151&173&-359&31\\-174&63&-279&-26\end{pmatrix}\in R^{4\times4},\qquad D=\begin{pmatrix}-418&197&-191&-144\\-185&146&-127&-108\\-419&458&-463&-162\\-147&94&-70&-98\end{pmatrix}\in R^{4\times4}.$$

Then, we can check that this matrix equation is consistent using centro-symmetric matrices and that its GSCSs are correct.

$$Vexact=\begin{pmatrix}3&0&4&0\\0&1&0&1\\3&0&5&0\\0&-2&0&5\end{pmatrix}\in CSR_P^{4\times4},\qquad \textit{where}\qquad P=\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}\in SOR^{4\times4},$$

and

$$Wexact=\begin{pmatrix}6&0&5&0\\0&-4&0&5\\1&0&-1&0\\0&-3&0&4\end{pmatrix}\in CSR_S^{4\times4},\qquad \textit{where}\qquad S=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}\in SOR^{4\times4}$$

Using arbitrary initial matrices $V(0)=0$, $W(0)=0$ and our proposed algorithm, we obtain a sequence of ***GCSSs*** *$V(k)$ and $W(k)$.*

In the Fig. 1 given below, we notify the gained results for different amounts of μ where $r(k) = \log_{10}(\,\|R_1(k)\|^2 + \|R_2(k)\|^2)$. It is obviously easy to see that the initial matrices $V(0)=0$, $W(0) = 0$, $r(k)$ decreases and converges to zero as k increases.

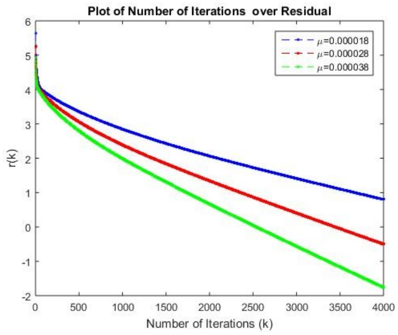


Figure 1: The gained outcomes for Example 4.1 with the initial matrices $V(0) = 0$, $W(0) = 0$

The graph clearly shows that our proposed Algorithm 2.3 is efficient. As we can see, as the convergence factor increases, so does the algorithm's convergence rate.

Example 4.2. *In the second example, we deem the equations* (1) *with the following input matrices*

$$A = \begin{pmatrix} -1 & 1 & 5 & 3 \\ 0 & 1 & 7 & 1 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in R^{4\times4}, \quad B = \begin{pmatrix} 5 & 1 & 3 & -2 \\ 0 & 4 & 5 & 7 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 3 \end{pmatrix} \in R^{4\times4}, \quad E = \begin{pmatrix} 3 & -2 & 5 & -2 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix} \in R^{4\times4},$$

$$F = \begin{pmatrix} 1 & 2 & 6 & 1 \\ 0 & -1 & -6 & 1 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 4 \end{pmatrix} \in R^{4\times4}, \quad M = \begin{pmatrix} 2 & 1 & 5 & 3 \\ 0 & 1 & 6 & -4 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 2 \end{pmatrix} \in R^{4\times4}, \quad N = \begin{pmatrix} 4 & 5 & -2 & 6 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & -3 \end{pmatrix} \in R^{4\times4},$$

$$G = \begin{pmatrix} 9 & 2 & -1 & 5 \\ 0 & -5 & 2 & 4 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix} \in R^{4\times4}, \quad H = \begin{pmatrix} 3 & -5 & 6 & 1 \\ 0 & -1 & -4 & 2 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix} \in R^{4\times4},$$

$$C = \begin{pmatrix} 18 & -23 & -171 & 91 \\ 0 & -14 & 76 & -21 \\ 0 & 0 & -91 & 39 \\ 0 & 0 & 0 & -83 \end{pmatrix} \in R^{4\times4}, \quad D = \begin{pmatrix} -51 & 118 & -223 & -167 \\ 0 & -12 & -29 & -73 \\ 0 & 0 & -129 & -63 \\ 0 & 0 & 0 & -82 \end{pmatrix} \in R^{4\times4}.$$

Then, using centro-symmetric matrices, we can verify that this matrix equation is consistent and that its ***GCSs*** *are correct.*

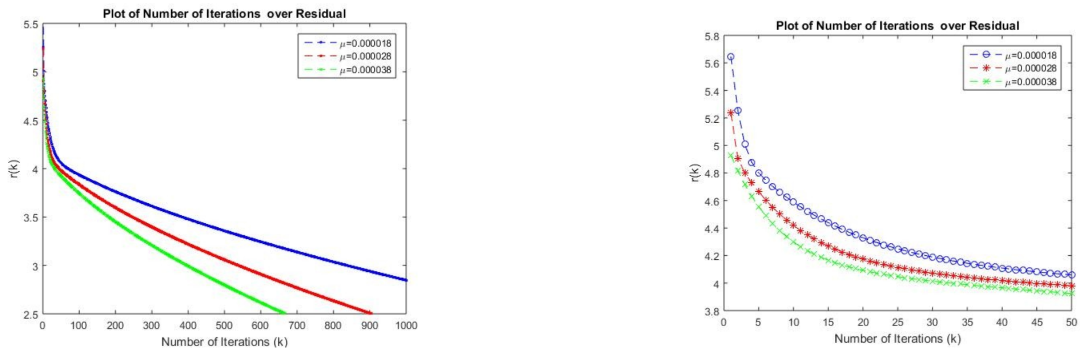
$$Vexact = \begin{pmatrix} 3 & 0 & 4 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix} \in CSR_P^{4\times4}, \quad \textit{where} \quad P = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in SOR^{4\times4},$$

and

$$Wexact = \begin{pmatrix} 6 & 0 & 5 & 0 \\ 0 & -4 & 0 & 5 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \in CSR_S^{4\times4}, \quad \textit{where} \quad S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in SOR^{4\times4}$$

Choosing the arbitrary initial matrices as $V(0) = 0$, $W(0) = 0$ *and applying our suggested algorithm we get a sequence of GCS solutions* $V(k)$ *and* $W(k)$.

In the Fig. 2 given below, we notify the gained results for different amounts of μ *where* $r(k) = \log_{10}(\|R_1(k)\|^2 + \|R_2(k)\|^2)$. *It is obviously easy to see that the initial matrices* $V(0) = 0$, $W(0) = 0$, $r(k)$ *decreases and converges to zero as k increases.*



(a) (b)

Figure 2: The gained outcomes for Example 4.2 with initial matrices $V(0) = 0$, $W(0) = 0$

It is clear from Fig.2 (a) and 2 (b) that our proposed Algorithm 2.3 is efficient. We can recognize that as the convergence factor increases, the algorithm's convergence rate increases.

5. Conclusion

This paper presented the Gradient-based Iterative (***GI***) Algorithm 2.3, a computational procedure developed to obtain solutions to a system of coupled generalized Sylvester matrix (GSM) equations of the form $AV+BW=EVF+C$, $MV+NW=GVH+D$ where the unknown matrices $\boldsymbol{V}$ and $\boldsymbol{W}$ were constrained to belong to the class of generalized centro-symmetric matrices (***GCSMs***). The algorithm was specifically designed to preserve the structural properties of ***GCSMs*** throughout the iterative process, ensuring that the symmetry inherent in the solution space was maintained.We investigated the convergence behavior of the matrix sequences $\{V(k)\}$ and $\{W(k)\}$ generated by the proposed ***GI*** Algorithm 2.3. We discussed sufficient conditions for convergence and provided theoretical justifications to support the stability and consistency of the method. To demonstrate the practical applicability and performance of the algorithm, we presented two numerical experiments. These experiments verified both the accuracy and computational efficiency of the proposed approach in solving ***GSM*** equations under the generalized centro-symmetric constraint. The results confirmed that the ***GI*** Algorithm 2.3 converged reliably and produced high-precision solutions in a relatively small number of iterations. Overall, this work contributed an effective and symmetry-aware numerical scheme for solving structured matrix equations, with potential applications in areas where such symmetries naturally arose, including systems theory, control engineering, and signal processing.

Conflict of Interest

There are no conflicts of interest declared by the authors.

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