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Stability of variable coefficients Rayleigh beam with indefinite damping under a force control in position and velocity

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Abstract. In this paper we study the exponential stability of a damped variable coefficients Rayleigh beam, that is clamped at one end and is free at the other. This is a continuation of the work of Wang et al in [17]. To stabilize the system, we apply a leaner boundary control in position and velocity at the free end of the beam. Using the modern spectral analysis approach, we obtain the Riesz basis property and the exponential stability.

1. Introduction

We study the nonuniform Rayleigh beam system with a indefinite damping term under boundary controls feedbacks given by:

$$\begin{cases} \rho(x)\dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial}{\partial x}\left(I_\rho(x)\dfrac{\partial^3 u}{\partial t^2 \partial x}\right) + \dfrac{\partial^2}{\partial x^2}\left(EI(x)\dfrac{\partial^2 u}{\partial x^2}\right) - \dfrac{\partial}{\partial x}\left(a(x)\dfrac{\partial^2 u}{\partial x \partial t}\right) = 0, \ x \in (0,1), \ t > 0 \\ u(0,t) = \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial^2 u}{\partial x^2}(1,t) = 0, \ t > 0 \\ \dfrac{\partial}{\partial x}\left(EI(\cdot)\dfrac{\partial^2 u}{\partial x^2}\right)(1,t) - I_\rho(1)\dfrac{\partial^3 u}{\partial t^2 \partial x}(1,t) = \alpha u(1,t) + \beta \dfrac{\partial u}{\partial t}(1,t), \ t > 0 \\ u(x,0) = u_0(x), \ \dfrac{\partial u}{\partial t}(x,0) = u_1(x), \ x \in (0,1). \end{cases} \tag{1}$$

Here $u(x,t)$ is the transversal deviation of the beam, x and t stand respectively for the position and time, $\rho(x) > 0$ is the mass density, $EI(x) > 0$ is the stiffness of the beam, $I_\rho(x) > 0$ is the mass moment of inertia. The length of the beam is chosen to be unity, α and β are two given positive constants. The damping $a(x)$ is indefinite and continuously differentiable coefficient function such that

$$a(1) = 0, \tag{2}$$

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$$\int_0^1 \frac{a(x)}{I_\rho(x)} dx > 0. \tag{3}$$

For convenience, we assume that

$$\rho(x),\ I_\rho(x),\ EI(x) \in C^4[0,1].$$

The nonuniform Rayleigh problem was investigate in [16] except that there was no damping term. In [16], the authors use the Riesz basis approach to stabilize the system. We refer the readers to [1, 15–17] for further details of this model. The uniform and nonuniform Rayleigh problem with boundary feedbacks controls were discussed respectably in [1, 7, 16, 17], where a conjecture on how the control parameters affect decay rate was proposed and then answered, respectively. There are several approaches to study these perturbed systems.

In this paper we use the Riesz basis approach, which shows that the generalized eigenfunctions of the system form a Riesz basis, and then deduces the spectrum determined growth condition and various stability results from the eigenvalue distribution of the system.This approach was used in [1, 16, 17]. The system (1) was studied in [17] with zero boundary conditions where the authors obtained a necessary condition for the stability and establish the Riesz basis property as well as the spectrum determined growth condition for the system.

The rest of this paper is organized as follows. In Section 2, we will find a suitable Hilbert space framework for system (1), and then the system will be shown to generate a C_0-semigroup. We then study the eigenvalue problem. The main trick is to use a space-scaling transformation to derive an equivalent eigenvalue boundary problem and this leads to much simpler asymptotic expansions. In Section 3, we shall apply techniques in [12–14] to the fundamental solutions of the eigenvalue boundary problem, and then use the results to expand the characteristic determinant in deducing the asymptotic behavior of the eigenvalues. In the last Section, we shall reveal how the damping term affects the decay rate asymptotically, and then discuss how negative the damping could be so that the system is still exponentially stable.

2. Formulation of the problem and well-posedness

We start our investigation by formulating the system (1) in the suitable functional setting. We first introduce the following spaces

$$V = \left\{ u \in H^1(0,1);\ u(0) = 0 \right\},$$

with the inner product

$$\langle u, v\rangle_V = \int_0^1 \left(\rho(x) u(x)\overline{v(x)} + I_\rho(x) \frac{du}{dx}(x)\overline{\frac{dv}{dx}(x)} \right) dx,$$

$$W = \left\{ u \in H^2(0,1);\ u(0) = 0,\ \frac{du}{dx}(0) = 0 \right\},$$

with the inner product

$$\langle u, v\rangle_W = \int_0^1 EI(x) \frac{d^2u}{dx^2}(x)\overline{\frac{d^2v}{dx^2}(x)} dx + \alpha u(1)\overline{v(1)}.$$

Also, the energy space $\mathcal{H} = W \times V$ which is endowed with the inner product

$$\langle (u,v),(f,g)\rangle_{\mathcal{H}} = \langle u, f\rangle_W + \langle v, g\rangle_V,\ (u,v),\ (f,g) \in \mathcal{H}.$$

Easy to see that

$$W \subset V \subset L^2(0,1) \subset V' \subset W', \tag{4}$$

where W' and V' are the dual spaces of W and V, respectively.

Let u be a smooth solution of system (1) and $\phi \in W$. Multiplying the first equation of system (1) and integrating on $[0,1]$. Then integrating by parts, we get

$$\int_0^1 \left(\rho(x)\frac{\partial^2 u}{\partial t^2}(x,t)\overline{\phi(x)} + I_\rho(x)\frac{\partial^3 u}{\partial x \partial t^2}(x,t)\overline{\frac{\partial \phi}{\partial x}(x)}\right)dx + \int_0^1 EI(x)\frac{\partial^2 u}{\partial x^2}(x,t)\overline{\frac{\partial^2 \phi}{\partial x^2}(x)}dx$$

$$+ \int_0^1 a(x)\frac{\partial^2 u}{\partial x \partial t}(x,t)\overline{\frac{\partial \phi}{\partial x}(x)}dx + \alpha u(1,t)\overline{\phi(1)} + \beta \frac{\partial u}{\partial t}(1,t)\overline{\phi(1)} = 0 \tag{5}$$

Now we define the continuous linear operators $A \in \mathcal{L}(W, W')$ and $B,\ C,\ D \in \mathcal{L}(V, V')$ by

$$\begin{array}{llllll} A: & W & \longrightarrow & W' & & \\ & \phi & \longmapsto & A\phi: & W \longrightarrow \mathbb{C} & \\ & & & & \psi \longmapsto \left\langle A\phi, \psi \right\rangle = \displaystyle\int_0^1 EI(x)\frac{d^2\phi}{dx^2}(x)\overline{\frac{d^2\psi}{dx^2}(x)}dx + \alpha\phi(1)\overline{\psi(1)}, & \end{array} \tag{6}$$

$$\begin{array}{llllll} B: & V & \longrightarrow & V' & & \\ & \phi & \longmapsto & B\phi: & V \longrightarrow \mathbb{C} & \\ & & & & \psi \longmapsto \left\langle B\phi, \psi \right\rangle = \displaystyle\int_0^1 a(x)\frac{d\phi}{dx}(x)\overline{\frac{d\psi}{dx}(x)}dx, & \end{array} \tag{7}$$

$$\begin{array}{llllll} C: & V & \longrightarrow & V' & & \\ & \phi & \longmapsto & C\phi: & V \longrightarrow \mathbb{C} & \\ & & & & \psi \longmapsto \left\langle C\phi, \psi \right\rangle = \displaystyle\int_0^1 \left(\rho(x)\phi(x)\overline{\psi(x)} + I_\rho(x)\frac{d\phi}{dx}(x)\overline{\frac{d\psi}{dx}(x)}\right)dx. & \end{array} \tag{8}$$

$$\begin{array}{llllll} D: & V & \longrightarrow & V' & & \\ & \phi & \longmapsto & D\phi: & V \longrightarrow \mathbb{C} & \\ & & & & \psi \longmapsto \left\langle D\phi, \psi \right\rangle = \phi(1)\overline{\psi(1)}, & \end{array} \tag{9}$$

Then by the Lax-Milgram Theorem (see [3, 18]), the operator A (resp. C) is a canonical isomorphism of W (resp. V) onto W' (resp. V'). With these operators, the equation (5) becomes

$$\left\langle C\frac{\partial^2 u}{\partial t^2}, \phi \right\rangle + \left\langle Au, \phi \right\rangle + \left\langle B\frac{\partial u}{\partial t}, \phi \right\rangle + \beta \left\langle D\frac{\partial u}{\partial t}, \phi \right\rangle = 0,\ \forall \phi \in W. \tag{10}$$

Assume that $Au \in V'$, since $C: V \longrightarrow V'$ is isomorphism, we get:

$$\frac{\partial^2 u}{\partial t^2} = -C^{-1}\left(Au + B\frac{\partial u}{\partial t} + \beta D\frac{\partial u}{\partial t}\right) \quad \text{in } V.$$

Therefore, we define the linear unbounded operator $\mathscr{A}$ by

$$\mathscr{D}(\mathscr{A}) = \{(u,v) \in \mathcal{H};\ v \in W \text{ and } Au \in V'\}, \tag{11}$$

$$\mathscr{A}(u,v) = \left(v, -C^{-1}Au\right),\ \forall\ (u,v) \in \mathscr{D}(\mathscr{A}), \tag{12}$$

and the linear bounded operator $\mathscr{B}$ as follows

$$\mathscr{B}(u,v) = \left(0, -C^{-1}(Bv + \beta Dv)\right),\ \forall\ (u,v) \in \mathcal{H}. \tag{13}$$

Thus, (5) can be formulated into an evolution equation in $\mathcal{H}$ as

$$\begin{cases} \dfrac{dU}{dt}(t) = (\mathscr{A} + \mathscr{B})\, U(t) \\ U(0) = U_0, \end{cases} \tag{14}$$

where $U = \left(u, \dfrac{\partial u}{\partial t}\right)$ and $U_0 = (u_0, u_1)$.

Lemma 2.1. *$\mathscr{A}$ is a skew-adjoint operator on the energy space $\mathcal{H}$ and $0 \in \rho(\mathscr{A})$.*

Proof. Let $(u, v),\ (f, g) \in \mathscr{D}(\mathscr{A})$, using the fact that

$$\langle Au, v\rangle = \langle u, v\rangle_W\,,\ \forall u, v \in W,$$

$$\langle Cu, v\rangle = \langle u, v\rangle_V\,,\ \forall u, v \in V,$$

we get

$$\begin{aligned}
\langle \mathscr{A}(u, v), (f, g)\rangle_{\mathcal{H}} &= \left\langle (v, -C^{-1}Au), (f, g)\right\rangle_{\mathcal{H}} \\
&= \langle v, f\rangle_W - \left\langle C^{-1}Au, g\right\rangle_V \\
&= -\left\langle CC^{-1}Au, g\right\rangle + \overline{\langle f, v\rangle_W} \\
&= -\langle Au, g\rangle + \overline{\langle Af, v\rangle} \\
&= -\langle u, g\rangle_W + \overline{\langle CC^{-1}Af, v\rangle} \\
&= -\langle u, g\rangle_W + \overline{\langle C^{-1}Af, v\rangle_V} \\
&= -\langle u, g\rangle_W + \left\langle v, C^{-1}Af\right\rangle_V \\
&= -\langle (u, v), \mathscr{A}(f, g)\rangle_{\mathcal{H}}\,.
\end{aligned}$$

To show that $0 \in \rho(\mathscr{A})$, let $(f, g) \in \mathcal{H}$, we look for a unique element $(u, v) \in \mathscr{D}(\mathscr{A})$ such that

$$\mathscr{A}\,(u, v) = (f, g)\,,$$

then $v = f$ and

$$-C^{-1}Au = g.$$

Since $C : V \longrightarrow V'$ is isomorphism,

$$Au = -Cg.$$

So, for any $\phi \in W$, we get

$$\left\langle Au, \phi\right\rangle = -\left\langle Cg, \phi\right\rangle. \tag{15}$$

For any $\phi \in W$, we have $\left\langle A\phi, \phi\right\rangle = \|\phi\|_W^2$.
The left-hand side of (15) is a continuous coercive bilinear form Ψ of u and ϕ. Moreover, the right-hand side of (15) is a continuous linear form L of ϕ on W.
Using the well-known Lax-Milgram Theorem, there exists a unique $u \in W$ so that

$$\Psi(u, \phi) = L(\phi),\quad \forall \phi \in W,$$

holds and $0 \in \rho(\mathcal{A})$. □

Furthermore we have the following characterization of the space $\mathscr{D}(\mathscr{A})$ (see [16]).

Lemma 2.2. *Let $(u,v) \in \mathcal{H}$. Then $(u,v) \in \mathscr{D}(\mathscr{A})$ if and only if $u \in W \cap H^3(0,1)$ and $v \in W$ such that*

$$\frac{d^2u}{dx^2}(1) = 0. \tag{16}$$

In particular, $\mathscr{A}^{-1}$ is compact on the energy space $\mathcal{H}$.

Proof. The sufficiency is obvious. To prove the necessity, let $(u,v) \in \mathscr{D}(\mathscr{A})$ such that $\mathscr{A}(u,v) = (f,g) \in \mathcal{H}$. Then we have $v = f$ and

$$-C^{-1}Au = g.$$

Since $C : V \longrightarrow V'$ is isomorphism,

$$Au + Cg = 0,$$

hence

$$\int_0^1 EI(x)\frac{d^2u}{dx^2}(x)\overline{\frac{d^2\psi}{dx^2}(x)}dx + \alpha u(1)\overline{\psi(1)} + \int_0^1 \left(\rho(x)g(x)\overline{\psi(x)} + I_\rho(x)\frac{dg}{dx}(x)\overline{\frac{d\psi}{dx}(x)}\right)dx = 0, \forall \psi \in W. \tag{17}$$

Let $\phi \in C_0^\infty(0,1)$, taking $\psi(x) = \int_0^x \phi(t)dt$ and substitute it into (17) yields

$$\int_0^1 EI(x)\frac{d^2u}{dx^2}(x)\overline{\frac{d\phi}{dx}(x)}dx + \alpha u(1)\int_0^1 \overline{\phi(x)}dx + \int_0^1 \left(\int_x^1 \rho(t)g(t)dt\right)\overline{\phi(x)}dx + \int_0^1 I_\rho(x)\frac{dg}{dx}(x)\overline{\phi(x)}dx = 0, \tag{18}$$

then

$$\int_0^1 EI(x)\frac{d^2u}{dx^2}(x)\overline{\frac{d\phi}{dx}(x)}dx = -\int_0^1 \left[\alpha u(1) + \int_x^1 \rho(t)g(t)dt + I_\rho(x)\frac{dg}{dx}(x)\right]\overline{\phi(x)}dx.$$

Thus

$$\frac{d}{dx}\left(EI(\cdot)\frac{d^2u}{dx^2}\right) = \alpha u(1) + \int_x^1 \rho(t)g(t)dt + I_\rho\frac{dg}{dx} \in L^2(0,1). \tag{19}$$

Since $EI \in C^4(0,1)$, so $u \in W \cap H^3(0,1)$. Let $\phi \in V$ such that $\phi(1) = 1$ and $\psi(x) = \int_0^x \phi(t)dt$, substitute it into (17). Then integrating by parts and using (19), we get (16) holds. By Lemma 2.1, $\mathscr{A}^{-1}$ exists and is bounded on $\mathcal{H}$. From The Sobolev Embedding Theorem, $\mathscr{A}^{-1}$ is compact. □

Therefore, since $\mathscr{A}$ is a skew-adjoint operator with compact resolvent and $\mathscr{B}$ is bounded, a standard perturbation result (see Pazy[10]) will then imply that $\mathscr{A} + \mathscr{B}$ generates a C_0-group on $\mathcal{H}$ and has compact resolvent. Thus, the spectrum $\sigma(\mathscr{A} + \mathscr{B})$ consists of only isolated eigenvalues that have a finite algebraic multiplicity.

Using standard semigroup theory, we get the following theorem on the existence, uniqueness, and regularity of the equation (14) solution.

Theorem 2.3. *(Existence and uniqueness)*

1. *If $U_0 \in D(\mathscr{A})$, then system* (14) *has a unique strong solution*

$$U \in C\left([0,+\infty[\,,\mathscr{D}(\mathscr{A})\right) \cap C^1\left([0,+\infty[\,,\mathcal{H}\right).$$

2. *If $U_0 \in \mathcal{H}$, then system* (14) *has a unique weak solution*

$$U \in C\left([0,+\infty[\,,\mathcal{H}\right).$$

3. High frequencies

In this section, we obtain asymptotic expansions for the eigenvalues of $\mathscr{A}+\mathscr{B}$. Let $\lambda \in \sigma(\mathscr{A}+\mathscr{B})$ and $U=(u,v)\in \mathscr{D}(\mathscr{A})$ such that

$$(\mathscr{A}+\mathscr{B})\,U=\lambda U.$$

Equivalently

$$\begin{cases} v=\lambda u \\ -C^{-1}(Au+Bv+\beta Dv)=\lambda v. \end{cases}$$

Since $C: V \longrightarrow V'$ is an isomorphism, we have $v=\lambda u$ and

$$Au+\lambda Bu+\lambda\beta Du+\lambda^2 Cu=0. \tag{20}$$

Using (6)-(9), we get

$$\begin{aligned} &\int_0^1 EI(x)\frac{d^2u}{dx^2}(x)\overline{\frac{d^2\psi}{dx^2}(x)}dx+\alpha u(1)\overline{\psi(1)}+\lambda\int_0^1 a(x)\frac{du}{dx}(x)\overline{\frac{d\psi}{dx}(x)}dx+\lambda\beta u(1)\overline{\psi(1)}\\ &+\lambda^2\int_0^1\left(\rho(x)u(x)\overline{\psi(x)}+I_\rho(x)\frac{du}{dx}(x)\overline{\frac{d\psi}{dx}(x)}\right)dx=0,\ \forall\psi\in W. \end{aligned} \tag{21}$$

Let $\psi\in C_0^\infty(0,1)$ and substitute it into (21), we get

$$\left\langle \frac{d^2}{dx^2}\left(EI\frac{d^2u}{dx^2}\right)-\lambda\frac{d}{dx}\left(a\frac{du}{dx}\right)+\lambda^2\left(\rho u-\frac{d}{dx}\left(I_\rho\frac{du}{dx}\right)\right),\psi\right\rangle=0.$$

Then, we have

$$\frac{d^2}{dx^2}\left(EI\frac{d^2u}{dx^2}\right)=\lambda\frac{d}{dx}\left(a\frac{du}{dx}\right)-\lambda^2\left(\rho u-\frac{d}{dx}\left(I_\rho\frac{du}{dx}\right)\right)\in L^2(0,1). \tag{22}$$

Integrating by parts (21) and using (22), we get

$$\frac{d}{dx}\left(EI(\cdot)\frac{d^2u}{dx^2}\right)(1)-\lambda^2 I_\rho(1)\frac{du}{dx}(1)=\alpha u(1)+\lambda\beta u(1).$$

Consequently, the function u is determined by the following system:

$$\begin{cases} \dfrac{d^2}{dx^2}\left(EI(x)\dfrac{d^2u}{dx^2}\right)-\lambda\dfrac{d}{dx}\left(a(x)\dfrac{du}{dx}\right)+\lambda^2\left(\rho(x)u-\dfrac{d}{dx}\left(I_\rho(x)\dfrac{du}{dx}\right)\right)=0\\ u(0)=\dfrac{du}{dx}(0)=\dfrac{d^2u}{dx^2}(1)=0\\ \dfrac{d}{dx}\left(EI(\cdot)\dfrac{d^2u}{dx^2}\right)(1)-\lambda^2 I_\rho(1)\dfrac{du}{dx}(1)=\alpha u(1)+\lambda\beta u(1). \end{cases} \tag{23}$$

We showed that (20) implies (23). Now, we will prove the converse : that (23) implies (20). From (22), we multiply the first equation of (23) by $\overline{\psi}$ with $\psi\in W$ and integrate over $[0,1]$. By integrating by parts, the second equation of (23) and the fact that $\psi\in W$, we obtain

$$\begin{aligned} &\frac{d}{dx}\left(EI(\cdot)\frac{d^2u}{dx^2}\right)(1)\overline{\psi(1)}+\int_0^1 EI(x)\frac{d^2u}{dx^2}(x)\overline{\frac{d^2\psi}{dx^2}(x)}dx+\lambda\int_0^1 a(x)\frac{du}{dx}(x)\overline{\frac{d\psi}{dx}(x)}dx\\ &-\lambda^2 I_\rho(1)\frac{du}{dx}(1)\overline{\psi(1)}+\lambda^2\int_0^1\left(\rho(x)u(x)\overline{\psi(x)}+I_\rho(x)\frac{du}{dx}(x)\overline{\frac{d\psi}{dx}(x)}\right)dx=0. \end{aligned}$$

And by the third equation of (23), we get (21):

$$\int_0^1 EI(x)\frac{d^2u}{dx^2}(x)\overline{\frac{d^2\psi}{dx^2}(x)}dx + \alpha u(1)\overline{\psi(1)} + \lambda\int_0^1 a(x)\frac{du}{dx}(x)\overline{\frac{d\psi}{dx}(x)}dx + \lambda\beta u(1)\overline{\psi(1)}$$
$$+\lambda^2\int_0^1\left(\rho(x)u(x)\overline{\psi(x)} + I_\rho(x)\frac{du}{dx}(x)\overline{\frac{d\psi}{dx}(x)}\right)dx = 0,\ \forall\psi\in W.$$

From the operators (6)-(9), the preceding equation is written as

$$\left\langle Au + \lambda Bu + \lambda\beta Du + \lambda^2 Cu, \psi\right\rangle = 0,\ \forall\psi\in W.$$

What implies (20). Consequently, any solution to equation (23) is a solution to (20) and vice versa. For this, we focus on solving equation (23).

We need the following lemma (see [16, 17]).

Lemma 3.1. *Let $h_1(x)$ and $h_2(x)$ be two linearly independent solutions for the second order linear homogeneous differential equation*

$$\left(I_\rho(x)u'(x)\right)' - \rho(x)u(x) = 0.$$

Then we have

$$D := h_1(0)h_2'(1) - h_1'(1)h_2(0) \neq 0. \tag{24}$$

Expanding (23) into the following form:

$$\begin{cases} u^{(4)}(x) + 2\dfrac{EI'(x)}{EI(x)}u'''(x) + \dfrac{EI''(x)}{EI(x)}u''(x) - \lambda^2\left(\dfrac{I_\rho(x)}{EI(x)}u''(x) + \dfrac{I'_\rho(x)}{EI(x)}u'(x) - \dfrac{\rho(x)}{EI(x)}u(x)\right) \\ -\lambda\left(\dfrac{a(x)}{EI(x)}u''(x) + \dfrac{a'(x)}{EI(x)}u'(x)\right) = 0 \\ u(0) = u'(0) = u''(1) = 0 \\ (EI\ u'')'(1) - \lambda^2 I_\rho(1)u'(1) = \alpha u(1) + \lambda\beta u(1). \end{cases} \tag{25}$$

Now we convert the eigenvalue problem (25) into a more convenient form by a space-scaling transformation. For this, let

$$u(x) = f(z),\ z = \frac{1}{h}\int_0^x\left(\frac{I_\rho(\xi)}{EI(\xi)}\right)^{\frac{1}{2}}d\xi,\ h = \int_0^1\left(\frac{I_\rho(\xi)}{EI(\xi)}\right)^{\frac{1}{2}}d\xi, \tag{26}$$

then (25) can be rewritten as

$$\begin{cases} f^{(4)}(z) + a_1(z)f'''(z) + a_2(z)f''(z) + a_3(z)f'(z) - \lambda^2h^2\left[f''(z) + b_1(z)f'(z) - b_2(z)f(z)\right] \\ -\lambda\left[c_0(z)f''(z) + c_1(z)f'(z)\right] = 0 \\ f(0) = f'(0) = 0 \\ b_{21}f''(1) + b_{22}f'(1) = 0 \\ b_{11}f'''(1) + b_{12}f''(1) + b_{13}f'(1) - \lambda^2 b_{14}f'(1) - \alpha f(1) - \lambda\beta f(1) = 0, \end{cases} \tag{27}$$

where

$$a_1(z) = 6\frac{z_{xx}}{z_x^2} + 2\frac{EI'(x)}{z_x EI(x)},\ z_x = \frac{1}{h}\left(\frac{I_\rho(x)}{EI(x)}\right)^{\frac{1}{2}}, \tag{28}$$

$$a_2(z) = 3\frac{z_{xx}^2}{z_x^4} + 4\frac{z_{xxx}}{z_x^3} + 6\frac{z_{xx}EI'(x)}{z_x^3 EI(x)} + \frac{EI''(x)}{z_x^2 EI(x)}, \tag{29}$$

$$a_3(z) = \frac{z_{xxxx}}{z_x^4} + 2\frac{z_{xxx}EI'(x)}{z_x^4 EI(x)} + \frac{z_{xx}EI''(x)}{z_x^4 EI(x)}, \tag{30}$$

$$b_1(z) = \frac{z_{xx}}{z_x^2} + \frac{I'_\rho(x)}{h^2 z_x^3 EI(x)},\ b_2(z) = \frac{\rho(x)}{h^2 z_x^4 EI(x)}, \tag{31}$$

$$c_0(z) = \frac{a(x)}{z_x^2 EI(x)},\ c_1(z) = \frac{z_{xx}a(x)}{z_x^4 EI(x)} + \frac{a'(x)}{z_x^3 EI(x)}, \tag{32}$$

$$b_{11} = z_x^3(1)EI(1),\ b_{12} = 3z_x(1)z_{xx}(1)EI(1) + z_x^2(1)EI'(1), \tag{33}$$

$$b_{13} = z_{xxx}(1)EI(1) + z_{xx}(1)EI'(1),\ b_{14} = I_\rho(1)z_x(1), \tag{34}$$

$$b_{21} = z_x^2(1),\ b_{22} = z_{xx}(1). \tag{35}$$

If we replace λ by $\mu := h\lambda$, then (27) becomes

$$\begin{cases} f^{(4)}(z) + a_1(z)f'''(z) + a_2(z)f''(z) + a_3(z)f'(z) - \mu^2\left[f''(z) + b_1(z)f'(z) - b_2(z)f(z)\right] \\ -\mu\left[\frac{1}{h}c_0(z)f''(z) + \frac{1}{h}c_1(z)f'(z)\right] = 0 \\ f(0) = f'(0) = 0 \\ b_{21}f''(1) + b_{22}f'(1) = 0 \\ b_{11}f'''(1) + b_{12}f''(1) + b_{13}f'(1) - h^{-2}\mu^2 b_{14}f'(1) - \alpha f(1) - h^{-1}\mu\beta f(1) = 0, \end{cases} \tag{36}$$

In summary, we have the following theorem:

Theorem 3.2. *λ is an eigenvalue of $\mathscr{A} + \mathscr{B}$ if and only if (36) has a nonzero solution $f(z)$ for $\mu = h\lambda$. Moreover, the function u in the corresponding eigenfunction $(\phi, \lambda\phi)$ of $\mathscr{A} + \mathscr{B}$ is given by (26).*

First, we start to treat the fundamental solutions of equation (36). We notice that this equation is not in the classical form that is to say the highest power of the eigenvalue parameter μ is not equal to the order of the highest derivative of the equation. For this, the results of [2, 9] is not applicable. Its results are modified in the papers of Tretter [12–14] that we can apply in our case. To begin, we divide the complex plan into four sectors:

$$\mathscr{S}_k = \left\{z \in \mathbb{C}\ :\ \frac{k\pi}{2} \le \arg z \le \frac{(k+1)\pi}{2}\right\},\ k = 0,\ 1,\ 2,\ 3 \tag{37}$$

and for each $\mathscr{S}_k$, we will pick ω_1 and ω_2 to be the square roots of -1 so that

$$\mathrm{Re}\,(\rho\omega_1) \le \mathrm{Re}\,(\rho\omega_2),\ \forall\rho \in \mathscr{S}_k. \tag{38}$$

In particular, we will choose $\omega_1 = e^{i\pi/2}$, $\omega_2 = e^{i3\pi/2}$ in sector $\mathscr{S}_0$ and re-shuffle them in each the remaining sectors so that (38) holds. Writing $\mu = \rho\omega_1$ for ρ in each sector $\mathscr{S}_k$, we have the following result on the fundamental solutions of (36) from [1, 17] (see also [15]).

Lemma 3.3. *In each sector $\mathscr{S}_k$, for $\rho \in \mathscr{S}_k$, with $|\rho|$ sufficiently large, the equation*

$$f^{(4)}(z) + a_1(z)f'''(z) + a_2(z)f''(z) + a_3(z)f'(z) + \rho^2\left[f''(z) + b_1(z)f'(z) - b_2(z)f(z)\right]$$

$$-\mathrm{i}\rho\left[\frac{1}{h}c_0(z)f''(z)+\frac{1}{h}c_1(z)f'(z)\right]=0, \tag{39}$$

has four linearly independent fundamental solutions $y_s(z,\rho)(s=1,2,3,4)$ and they possess the following asymptotic expressions (for $j=0,1,2,3$)

$$y_s^{(j)}(z,\rho)=h_s^{(j)}(z)+O\left(\rho^{-1}\right),\ s=1,2, \tag{40}$$
$$y_s^{(j)}(z,\rho)=(\rho\omega_{s-2})^j\,e^{\rho\omega_{s-2}x}\left[y_s(z)+O\left(\rho^{-1}\right)\right],\ s=3,4, \tag{41}$$

where for $s=3,4$,

$$y_s(z)=\exp\left(\int_0^z\frac{1}{2}\left[b_1(t)-a_1(t)-\frac{\mathrm{i}}{h}\omega_{s-2}c_0(t)\right]dt\right), \tag{42}$$

$$y_s(0)=1,\ y_s(1)=\exp\left(D_1+\mathrm{i}\omega_{s-2}D_2\right), \tag{43}$$

with

$$D_1=\frac{1}{2}\int_0^1\left[b_1(t)-a_1(t)\right]dt,\ D_2=-\frac{1}{2h}\int_0^1c_0(t)dt \tag{44}$$

and $h_1(z)=h_1(x(z))$, $h_2(z)=h_2(x(z))$ are two linearly independent solutions to the following equation:

$$f''(z)+b_1(z)f'(z)-b_2(z)f(z)=0.$$

From (40) and (41), we can obtain asymptotic expansions for the boundary conditions of system (36). In the following, we introduce the notation

$$[a]_1=a+O\left(\rho^{-1}\right).$$

Theorem 3.4. *Denote the boundary conditions of system (36) by U_1, U_2, U_3 and U_4, then for $\rho\in\mathscr{S}_0$, with $|\rho|$ sufficiently large, we have the following asymptotic expansions:*

$$U_4(y_s,\rho)=y_s(0,\rho)=\begin{cases}h_s(0)+O\left(\rho^{-1}\right)=[h_s(0)]_1, & s=1,2,\\ 1+O\left(\rho^{-1}\right)=[1]_1, & s=3,4,\end{cases} \tag{45}$$

$$U_3(y_s,\rho)=y_s'(0,\rho)=\begin{cases}x_z(0)h_s'(0)+O\left(\rho^{-1}\right)=[x_z(0)h_s'(0)]_1, & s=1,2,\\ \rho\omega_{s-2}\left(1+O\left(\rho^{-1}\right)\right)=\rho\omega_{s-2}[1]_1, & s=3,4,\end{cases} \tag{46}$$

$$\begin{aligned}U_2(y_s,\rho)&=y_s''(1,\rho)+\frac{b_{22}}{b_{21}}y_s'(1,\rho)\\&=\begin{cases}x_z''(1)h_s'(1)+x_z'^2(1)h_s''(1)+\frac{b_{22}}{b_{21}}x_z'(1)h_s'(1)+O\left(\rho^{-1}\right), & s=1,2,\\ \rho^2e^{\rho\omega_{s-2}}\left(y_s(1)\omega_{s-2}^2+O\left(\rho^{-1}\right)\right), & s=3,4,\end{cases}\\&=\begin{cases}\left[x_z''(1)h_s'(1)+x_z'^2(1)h_s''(1)+\frac{b_{22}}{b_{21}}x_z'(1)h_s'(1)\right]_1, & s=1,2,\\ \rho^2e^{\rho\omega_{s-2}}\left[y_s(1)\omega_{s-2}^2\right]_1, & s=3,4,\end{cases}\end{aligned} \tag{47}$$

$$U_1(y_s,\rho) = y_s'''(1,\rho) + \frac{b_{12}}{b_{11}}y_s''(1,\rho) + +\frac{b_{13}}{b_{11}}y_s'(1,\rho) + \rho^2\frac{b_{14}}{h^2 b_{11}}y_s'(1,\rho) - \frac{\alpha}{b_{11}}y_s(1,\rho)$$
$$- \mathrm{i}\frac{\beta}{hb_{11}}\rho y_s(1,\rho)$$
$$= \begin{cases} \rho^2\left(\frac{b_{14}}{h^2 b_{11}}x_z(1)h_s'(1) + \mathrm{O}\left(\rho^{-1}\right)\right), & s=1,2, \\ \rho^3 e^{\rho\omega_{s-2}}\left(y_s(1)\omega_{s-2}^3 + \frac{b_{14}}{h^2 b_{11}}y_s(1)\omega_{s-2} + O\left(\rho^{-1}\right)\right), & s=3,4, \end{cases} \tag{48}$$
$$= \begin{cases} \rho^2\left[\frac{b_{14}}{h^2 b_{11}}x_z(1)h_s'(1)\right]_1, & s=1,2, \\ \rho^3 e^{\rho\omega_{s-2}}\left[y_s(1)\omega_{s-2}^3 + \frac{b_{14}}{h^2 b_{11}}y_s(1)\omega_{s-2}\right]_1, & s=3,4. \end{cases}$$

Proof. The proof is just a direct substitution of the fundamental solutions (40) and (41) into the boundary conditions. □

Note that $\mu = \mathrm{i}\rho$ for any $\rho \in \mathscr{S}_0$, is the eigenvalue in (36) if and only if ρ satisfies the characteristic equation

$$\Delta(\rho) = \begin{vmatrix} U_4(y_1,\rho) & U_4(y_2,\rho) & U_4(y_3,\rho) & U_4(y_4,\rho) \\ U_3(y_1,\rho) & U_3(y_2,\rho) & U_3(y_3,\rho) & U_3(y_4,\rho) \\ U_2 y_1,\rho) & U_2(y_2,\rho) & U_2(y_3,\rho) & U_2(y_4,\rho) \\ U_1(y_1,\rho) & U_1(y_2,\rho) & U_1(y_3,\rho) & U_1(y_4,\rho) \end{vmatrix} = 0. \tag{49}$$

We have the following asymptotic expansion of the characteristic equation $\Delta(\rho)$.

Theorem 3.5. *In sector $\mathscr{S}_0$, the characteristic determinant $\Delta(\rho)$ of the eigenvalue problem (36) has an asymptotic expansion*

$$\Delta(\rho) = -\mathrm{i}\rho^5 x_z(1)D\left\{e^{\mathrm{i}\rho}e^{D_1-D_2} + e^{-\mathrm{i}\rho}e^{D_1+D_2} + O\left(\rho^{-1}\right)\right\}, \tag{50}$$

where D is defined in (24) and D_1, D_2 are defined in (44). The asymptotic expansion (50) also holds in the other sectors as well. Furthermore, the boundary problem (36) is strongly regular in the sense of [13, Definition 2.7*]. Then, the zeros of $\Delta(\rho)$ are simple when their modulus are sufficiently large.*

Proof. In sector $\mathscr{S}_0$, using the fact that in (48),

$$\frac{b_{14}}{b_{11}} = \frac{I_\rho(1)z_x(1)}{z_x^3(1)EI(1)} = h^2,$$

and substituting (45)-(48) into the characteristic determinant $\Delta(\rho)$, we have

$$\Delta(\rho) = \begin{vmatrix} [h_1(0)]_1 \\ \left[x_z(0)h_1'(0)\right]_1 \\ \left[x_z''(1)h_1'(1) + x_z'^2(1)h_1''(1) + \frac{b_{22}}{b_{21}}x_z'(1)h_1'(1)\right]_1 \\ \rho^2\left[x_z(1)h_1'(1)\right]_1 \end{vmatrix}$$

$$\begin{matrix} [h_2(0)]_1 \\ \left[x_z(0)h_2'(0)\right]_1 \\ \left[x_z''(1)h_2'(1) + x_z'^2(1)h_2''(1) + \frac{b_{22}}{b_{21}}x_z'(1)h_2'(1)\right]_1 \\ \rho^2\left[x_z(1)h_2'(1)\right]_1 \end{matrix}$$

$$\left.\begin{matrix} [1]_1 & [1]_1 \\ i\rho[1]_1 & -i\rho[1]_1 \\ -\rho^2 e^{i\rho}e^{D_1-D_2}[1]_1 & -\rho^2 e^{-i\rho}e^{D_1+D_2}[1]_1 \\ \rho^3 e^{i\rho}e^{D_1-D_2}[0]_1 & \rho^3 e^{-i\rho}e^{D_1+D_2}[0]_1 \end{matrix}\right| .$$

Developing the above determinant, we obtain (50). The strong regularity defined in [13, Definition 2.7] together with the simplicity of the zeros with large enough modulus can be verified directly from the fact that $e^{D_1-D_2}$, $e^{D_1+D_2}$, $x_z(1) > 0$ and (24). □

Theorem 3.6. *There exists an integer $N > 0$ such that the eigenvalues λ_k of problem (23) are simple and possess the following asymptotic expressions:*

$$\lambda_k = \frac{1}{h}\left(-\frac{1}{2h}\int_0^1 c_0(t)dt + i\left(\frac{\pi}{2} + k\pi\right)\right) + O\left(k^{-1}\right),\ k > N,\ k \in \mathbb{Z}, \tag{51}$$

where N is a sufficiently large positive integer, $h = \int_0^1 \left(\frac{I_\rho(\xi)}{EI(\xi)}\right)^{\frac{1}{2}} d\xi$ defined in (26) and $c_0(z) = \frac{a(x)}{z_x^2 EI(x)}$ defined in (32). Thus,

$$Re\lambda_k \to -\frac{1}{2h^2}\int_0^1 c_0(t)dt = -\frac{1}{2}\int_0^1 \frac{a(x)}{I_\rho(x)}dx \text{ as } k \to \infty. \tag{52}$$

Proof. In sector $\mathscr{S}_0$, we see from (24) and (50) that equation (49) becomes

$$e^{i\rho}e^{D_1-D_2} + e^{-i\rho}e^{D_1+D_2} + O\left(\rho^{-1}\right) = 0 \tag{53}$$

If we consider the equation

$$e^{i\rho}e^{D_1-D_2} + e^{-i\rho}e^{D_1+D_2} = 0$$

equivalently

$$e^{2i\rho - 2D_2} = -1,$$

with solutions

$$\tilde{\mu}_k = i\rho_k = D_2 + i\left(\frac{\pi}{2} + k\pi\right),\ k = 1, 2, \ldots, \tag{54}$$

where D_2 is defined in (44). Using (54) and the Rouché's theorem, the solutions of (53) will satisfy

$$\mu_k = D_2 + i\left(\frac{\pi}{2} + k\pi\right) + O\left(k^{-1}\right),\ k > N,\ k \in \mathbb{N}, \tag{55}$$

where N is a sufficiently large positive integer. Note that the eigenvalues of $\mathscr{A}+\mathscr{B}$ are distributed symmetrically with respect to the real axis. So the dual eigenvalues are

$$\mu_k = D_2 - \mathrm{i}\left(\frac{\pi}{2}+k\pi\right)+O\left(k^{-1}\right),\ k>N,\ k\in\mathbb{N}. \tag{56}$$

Hence, we can conclude from (55), (56) and $\mu = h\lambda$ that

$$\lambda_k = \frac{1}{h}\mu_k = \frac{1}{h}\left(D_2+\mathrm{i}\left(\frac{\pi}{2}+k\pi\right)\right)+O\left(k^{-1}\right),\ k>N,\ k\in\mathbb{Z}$$

Also, for large enough k, λ_k is simple because problem (36) is strongly regular [9, pp. 64 − 74]. □

Remark 3.7. *The same remark of [17] , we explain the interplay between the strong regularity and the simplicity of the eigenvalues for problem (36) (or (27)). From the characteristic equation (36) (or (27)) and the structure of the corresponding Green's function for the inverse of the associated ordinary differential operator [9, pp. 34 − 37], the multiplicity of each $\lambda\in\sigma(\mathscr{A})$ with large enough modulus, as a pole of the resolvent operator $R(\lambda,\mathscr{A})$, is less than or equal to the order of λ as a zero for the entire function $\Delta(\rho)$ with $\lambda=\frac{\mathrm{i}\rho}{h}$. On the other hand, since problem (36) is strongly regular, it is easy to verify that λ is geometrically simple when modulus of λ is large and the zeros of $\Delta(\rho)=0$ are simple when their modulus are large. So eigenvalues of problem (36) with sufficiently large modulus are algebraically simple because: $m_a\le p.m_g$ [8, pp. 148], where p denotes the order of the pole of the resolvent operator and m_a, m_g denote the algebraic and geometric multiplicities, respectively.*

All the above discussions can be summarized into the following result on the spectrum of $\mathscr{A}+\mathscr{B}$.

Theorem 3.8. *Let $\mathscr{A}+\mathscr{B}$ be defined as in (11), (12) and (13). Then each $\lambda\in\sigma(\mathscr{A}+\mathscr{B})$ is simple when $|\lambda|$ is large enough, and has an asymptotic expression given by (51).*

4. Exponential stability of the system

In this section, we investigate the Riesz basis property and the stability of system (14). We will first establish the completeness of the generalized eigenfunctions of $\mathscr{A}+\mathscr{B}$. For this, we need the following Theorem [4, pp. 170].

Theorem 4.1. *Let K be a compact self-adjoint operator in a Hilbert space H with $\ker K=\{0\}$ and eigenvalues $\lambda_j(K)$, $j=1,2,\dots,\infty$. Assume that*

$$\sum_{j=1}^{\infty}|\lambda_j(K)|^r<\infty,$$

for some $r\ge 1$, and let S be a compact operator such that $I+S$ is invertible. Then the system of generalized eigenfunctions of the operator $A:=K(I+S)$ is complete in H.

Theorem 4.2. *Let $\mathscr{A}+\mathscr{B}$ be defined as in (11),(12) and (13). Then the system of the generalized eigenfunctions of $\mathscr{A}+\mathscr{B}$ are complete in Hilbert space $\mathcal{H}$.*

Proof. Lemma 2.1 ensures that $(i\mathscr{A})^{-1}$ is a compact self-adjoint operator with $\ker(i\mathscr{A})^{-1}=\{0\}$. By Theorem 3.6 (when $c_0(t)\equiv 0$), it is easy to see that $\{\lambda_k((i\mathscr{A})^{-1})\}_{k=1}^{\infty}\in l^2$. We have

$$\begin{aligned}(i(\mathscr{A}+\mathscr{B}))^{-1} &= (i\mathscr{A})^{-1}(I+\mathscr{B}\mathscr{A}^{-1})^{-1}\\ &= (i\mathscr{A})^{-1}(I-\mathscr{B}\mathscr{A}^{-1}(I+\mathscr{B}\mathscr{A}^{-1})^{-1}).\end{aligned}$$

Since $\mathscr{B}\mathscr{A}^{-1}(I+\mathscr{B}\mathscr{A}^{-1})^{-1}$ is compact and $I-\mathscr{B}\mathscr{A}^{-1}(I+\mathscr{B}\mathscr{A}^{-1})^{-1}$ is invertible. So by Theorem 4.1, the proof is complete. □

To show the basis property of generalized eigenvectors of $\mathscr{A}+\mathscr{B}$, we need the following theorem (see [16]).

Theorem 4.3. *Let X be a separable Hilbert space, and A the generator of a C_0-semigroup $T(t)$ on X. Suppose that the following conditions hold:*

(1) *We can decompose $\sigma(A) = \sigma_1(A) \cup \sigma_2(A)$ and $\sigma_2(A) = \{\lambda_k\}_{k=1}^{\infty}$ consists of only isolated eigenvalues of finite algebraic multiplicity.*
(2) *For $m_a(\lambda_k) := \dim E(\lambda_k, A)X$, where $E(\lambda_k, A)$ denotes the Riesz-projection associated with λ_k, we have $\sup_{k\geq 1} m_a(\lambda_k) < \infty$.*
(3) *There is a constant α such that*

$$\sup\{Re\lambda | \lambda \in \sigma_1(A)\} \leq \alpha \leq \inf\{Re\lambda | \lambda \in \sigma_2(A)\}$$

and

$$\inf_{n\neq m} |\lambda_n - \lambda_m| > 0.$$

Then the following assertions are true:

(i) *There exist two $T(t)$-invariant closed subspaces X_1 and X_2 such that $\sigma(A|_{X_1}) = \sigma_1(A)$, $\sigma(A|_{X_2}) = \sigma_2(A)$ and $\{E(\lambda_k, A)X_2\}_{k=1}^{\infty}$ forms a Riesz basis of subspaces for X_2. Furthermore,*

$$X = \overline{X_1 \oplus X_2}.$$

(ii) *If $\sup_{k\geq 1} \|E(\lambda_k, A)\| < \infty$, then $\mathscr{D}(A) \subset X_1 \oplus X_2 \subset X$.*

(iii) *X has the topological direct sum decomposition $X = X_1 \oplus X_2$ if and only if*

$$\sup_{n\geq 1} \left\| \sum_{k=1}^{n} E(\lambda_k, A) \right\| < \infty.$$

Combining Theorems 3.6, 3.8, 4.2 and 4.1, we have the following result.

Theorem 4.4. *System (14) is a Riesz spectral system (in the sense that its generalized eigenfunctions form a Riesz basis in $\mathcal{H}$). Thus, the spectrum determined growth condition holds, i.e.*

$$\omega(\mathscr{A}+\mathscr{B}) = s(\mathscr{A}+\mathscr{B}),$$

with $s(\mathscr{A}+\mathscr{B}) := \sup\{Re\lambda | \lambda \in \sigma(\mathscr{A}+\mathscr{B})\}$ being the spectral bound of $\mathscr{A}+\mathscr{B}$ and $\omega(\mathscr{A}+\mathscr{B})$ being the growth bound of the semigroup $e^{(\mathscr{A}+\mathscr{B})t}$.

Proof. Let $\sigma_2(\mathscr{A}+\mathscr{B}) = \sigma(\mathscr{A}+\mathscr{B})$ and $\sigma_1(\mathscr{A}+\mathscr{B}) = \{-\infty\}$. From Theorem 3.6 and 3.8, all hypotheses in Theorem 4.3 are true. So Theorem 4.2 implies that $X_1 = \{0\}$. Therefore, the first assertion of Theorem 4.3 says that there is a sequence of generalized eigenvectors of $\mathscr{A}+\mathscr{B}$ that forms a Riesz basis in $\mathcal{H}$. Accordingly, the spectrum determined growth condition can be obtained by a direct consequence of the Riesz basis property and the simplicity of the high eigenfrequencies of $\mathscr{A}+\mathscr{B}$. □

Finally, we state the following two results describe how stability depends upon the sign of damping function.

Theorem 4.5. *Suppose that conditions (2) hold, $a(x)$ is continuously differentiable on $[0,1]$ and α, $\beta \geq 0$. If $a(x) \geq 0$, then, (14) is exponentially stable.*

Proof. We have

$$\mathrm{Re}\left\langle (\mathscr{A}+\mathscr{B})(u,v),(u,v)\right\rangle_{\mathcal{H}} = -\int_0^1 a(x)\left|\frac{dv}{dx}(x)\right|^2 dx - \beta\,|v(1)|^2 \leq 0,\ \forall\,(u,v)\in\mathscr{D}(\mathscr{A}),$$

then $\mathscr{A}+\mathscr{B}$ is dissipative in $\mathcal{H}$ and it is easy to see that $0\in\rho(\mathscr{A}+\mathscr{B})$. Hence, $\mathscr{A}+\mathscr{B}$ generates a C_0-semigroup of contractions on $\mathcal{H}$. We show now $\Re e\lambda<0$ for all $\lambda\in\sigma(\mathscr{A}+\mathscr{B})$. Multiplying the first equation of system (23) by $\overline{u}$, the conjugate of u, and integrating on $[0,1]$, we obtain

$$\lambda^2\int_0^1\left(\rho(x)\,|u(x)|^2+I_\rho(x)\left|\frac{du}{dx}(x)\right|^2\right)dx+\alpha\,|u(1)|^2+\lambda\beta\,|u(1)|^2+\int_0^1 EI(x)\left|\frac{d^2u}{dx^2}(x)\right|^2dx+\lambda\int_0^1 a(x)\left|\frac{du}{dx}(x)\right|^2dx=0. \tag{57}$$

Let $\lambda=\mathrm{Re}\lambda+\mathrm{i}\mathrm{Im}\lambda$, where $\mathrm{Re}\lambda,\ \mathrm{Im}\lambda$ are real. Then

$$\left((\mathrm{Re}\lambda)^2-(\mathrm{Im}\lambda)^2\right)\int_0^1\left(\rho(x)\,|u(x)|^2+I_\rho(x)\left|\frac{du}{dx}(x)\right|^2\right)dx+\alpha\,|u(1)|^2+(\mathrm{Re}\lambda)\,\beta\,|u(1)|^2$$

$$+\int_0^1 EI(x)\left|\frac{d^2u}{dx^2}(x)\right|^2dx+(\mathrm{Re}\lambda)\int_0^1 a(x)\left|\frac{du}{dx}(x)\right|^2dx=0, \tag{58}$$

and

$$2\,(\mathrm{Re}\lambda)\,(\mathrm{Im}\lambda)\int_0^1\left(\rho(x)\,|u(x)|^2+I_\rho(x)\left|\frac{du}{dx}(x)\right|^2\right)dx+(\mathrm{Im}\lambda)\int_0^1 a(x)\left|\frac{du}{dx}(x)\right|^2dx+(\mathrm{Im}\lambda)\,\beta\,|u(1)|^2=0. \tag{59}$$

If $\mathrm{Im}\lambda=0$, then by (58), we have

$$(\mathrm{Re}\lambda)^2\int_0^1\left(\rho(x)\,|u(x)|^2+I_\rho(x)\left|\frac{du}{dx}(x)\right|^2\right)dx+\alpha\,|u(1)|^2+(\mathrm{Re}\lambda)\,\beta\,|u(1)|^2$$

$$+\int_0^1 EI(x)\left|\frac{d^2u}{dx^2}(x)\right|^2dx+(\mathrm{Re}\lambda)\int_0^1 a(x)\left|\frac{du}{dx}(x)\right|^2dx=0, \tag{60}$$

thus $\mathrm{Re}\lambda<0$ because $a(x)\geq 0$.
If $\mathrm{Im}\lambda\neq 0$, then by (59), we have

$$(\mathrm{Im}\lambda)\left[2\,(\mathrm{Re}\lambda)\int_0^1\left(\rho(x)\,|u(x)|^2+I_\rho(x)\left|\frac{du}{dx}(x)\right|^2\right)dx+\int_0^1 a(x)\left|\frac{du}{dx}(x)\right|^2dx+\beta\,|u(1)|^2\right]=0. \tag{61}$$

Hence

$$\mathrm{Re}\lambda=-\frac{1}{2}\,\frac{\displaystyle\int_0^1 a(x)\left|\frac{du}{dx}(x)\right|^2dx+\beta\,|u(1)|^2}{\displaystyle\int_0^1\left(\rho(x)\,|u(x)|^2+I_\rho(x)\left|\frac{du}{dx}(x)\right|^2\right)dx}<0, \tag{62}$$

thus $\mathrm{Re}\lambda<0$ because $a(x)\geq 0$. Moreover, Theorem 3.6 ensure that the imaginary axis is not an asymptote of $\sigma(\mathscr{A}+\mathscr{B})$. Therefore, from Theorem 4.4, we get

$$\omega(\mathscr{A}+\mathscr{B})=\sup\{\mathrm{Re}\lambda|\lambda\in\sigma(\mathscr{A}+\mathscr{B})\}<0$$

and the proof is completed. □

Now, we are ready to consider the case that $a(x)$ can change its sign on $[0,1]$. We note

$$a^{+}(x) = \max\{a(x), 0\} \text{ and } a^{-}(x) = \max\{-a(x), 0\}, \text{ for } x \in [0,1], \tag{63}$$

and $\mathscr{B}^{\pm}$ be the corresponding damping operators with respect to $a^{\pm}(x)$, respectively.

Theorem 4.6. *Suppose that conditions (2) and (3) hold, $a(x)$ is indefinite and continuously differentiable on $[0,1]$ and α, $\beta \geq 0$. If*

$$\max\left[\sup_{x\in[0,1]}\left\{\frac{a^{-}(x)}{I_\rho(x)}\right\}, \beta\left(\sup_{x\in[0,1]}\left\{\frac{1}{\sqrt{I_\rho(x)}}\right\}\right)^2\right] < \left|s\left(\mathscr{A}+\mathscr{B}^{+}\right)\right|, \tag{64}$$

where $s\left(\mathscr{A}+\mathscr{B}^{+}\right) = \sup\{Re\lambda | \lambda \in \sigma\left(\mathscr{A}+\mathscr{B}^{+}\right)\}$, then, system (14) is exponentially stable.

Proof. Assume that $a(x)$ is indefinite on $[0,1]$. Then, it is easy to see that $a^{+}(x)$ satisfies the assumptions of Theorem 4.5. Thus, we can decompose $\mathscr{A}+\mathscr{B}$ into

$$\mathscr{A}+\mathscr{B} = \mathscr{A}+\mathscr{B}^{+} - \mathscr{B}^{-}.$$

Let $(u,v) \in \mathcal{H}$, we have

$$\begin{aligned}
&\left\|\mathscr{B}^{-}(u,v)\right\|_{\mathcal{H}}^2 \\
&= \left\|C^{-1}\left(B^{-}v + \beta Dv\right)\right\|_V^2 \\
&= \left\langle C^{-1}\left(B^{-}v + \beta Dv\right), C^{-1}\left(B^{-}v + \beta Dv\right)\right\rangle_V \\
&= \left\langle B^{-}v + \beta Dv, C^{-1}\left(B^{-}v + \beta Dv\right)\right\rangle \\
&= \int_0^1 a^{-}(x)\frac{dv}{dx}(x)\overline{\frac{d}{dx}C^{-1}\left(B^{-}v + \beta Dv\right)(x)}dx + \beta v(1)\overline{C^{-1}\left(B^{-}v + \beta Dv\right)(1)} \\
&\leq \max\left[\sup_{x\in[0,1]}\left\{\frac{a^{-}(x)}{I_\rho(x)}\right\}, \beta\left(\sup_{x\in[0,1]}\left\{\frac{1}{\sqrt{I_\rho(x)}}\right\}\right)^2\right]\|v\|_V \left\|C^{-1}\left(B^{-}v + \beta Dv\right)\right\|_V \\
&\leq \max\left[\sup_{x\in[0,1]}\left\{\frac{a^{-}(x)}{I_\rho(x)}\right\}, \beta\left(\sup_{x\in[0,1]}\left\{\frac{1}{\sqrt{I_\rho(x)}}\right\}\right)^2\right]\|(u,v)\|_{\mathcal{H}} \left\|C^{-1}\left(B^{-}v + \beta Dv\right)\right\|_V,
\end{aligned}$$

where B^{-} is the linear operator corresponding to $a^{-}(x)$ defined in (7). Then

$$\left\|\mathscr{B}^{-}\right\| \leq \max\left[\sup_{x\in[0,1]}\left\{\frac{a^{-}(x)}{I_\rho(x)}\right\}, \beta\left(\sup_{x\in[0,1]}\left\{\frac{1}{\sqrt{I_\rho(x)}}\right\}\right)^2\right]. \tag{65}$$

Applying the perturbation theory of contractive semigroups (see [10]), we have $\lambda \in \rho\left(\mathscr{A}+\mathscr{B}\right)$ whenever

$$\mathrm{Re}\lambda > s\left(\mathscr{A}+\mathscr{B}^{+}\right) + \left\|\mathscr{B}^{-}\right\|.$$

Then

$$s\left(\mathscr{A}+\mathscr{B}\right) \leq s\left(\mathscr{A}+\mathscr{B}^{+}\right) + \left\|\mathscr{B}^{-}\right\|.$$

So from Theorem 4.4 and (64)-(65), we get

$$\omega\left(\mathscr{A}+\mathscr{B}\right) = s\left(\mathscr{A}+\mathscr{B}\right) < 0.$$

Thus system (14) is exponentially stable. □

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