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Geometry of orthogonality using a new angular distance function in normed spaces

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Abstract. This work introduces an angular function orthogonality based on the angular distance function in normed spaces. We examine the geometrical features of the aforementioned orthogonality by discussing its homogeneity, alpha-existence, and the conditions under which it may exhibit symmetry. Motivated by the stated orthogonality, we introduce a well-defined angle using the angular distance function and discuss its geometrical properties by defining acute, obtuse, and right angles in normed spaces. Some non-trivial examples are also provided to support the results. Furthermore, we discuss the relationship of the angle with the Euclidean, isosceles, and Thy angles.

1. Introduction and Preliminaries

The geometry of normed spaces helps in characterizing feasible regions, defining convex sets, and understanding optimality conditions. It can also be used to determine how far apart or similar two data points are. Normed space metrics are useful for regression, grouping, and classification. Within the realm of graphics, normed spaces assist in the processes of transformation, rendering, and modeling geometric shapes. For realistic depiction, it is essential to have a solid understanding of the distances and angles present in these regions. There is a particular kind of normed space known as Hilbert space, and it is used in quantum mechanics to represent the states of a system. It is essential to have a solid understanding of the geometry of these spaces to comprehend quantum states and processes.

The influence of orthogonality and angular relations on normed vector spaces has been observed from Euclidean geometry to contemporary functional analysis. Alternative definitions of angle functions and angular measurements can be explored utilizing a basis in these spaces. This technique is especially fascinating in actual Banach spaces, where geometric characteristics are crucial. Orthogonality is a fundamental concept that enables the examination of these geometric relationships and denotes perpendicularity. By investigating these diverse definitions, we can better understand the structural intricacies of normed vector spaces.

Roberts [11] developed the concept of orthogonality relation in a normed space. In 1935, Birkhoff proposed Birkhoff orthogonality, which is considered one of the fundamental types of orthogonality in a

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normed space [4]. James provided an extensive analysis of the characteristics of Birkhoff orthogonality [9]. Due to this, Birkhoff orthogonality is commonly known as Birkhoff-James orthogonality. Subsequently, James [8] presented the concept of isosceles orthogonality. In addition, James [8] proposed the concept of Pythagorean orthogonality in normed space. This concept extends the idea in Euclidean space that two vectors are perpendicular if and only if a right triangle exists with the two vectors as its sides. Singer [13] introduced the Singer orthogonality that is closely associated with isosceles orthogonality. Later, Dadipour et al. [5] introduced a new orthogonality based on an angular distance inequality and discussed some properties of this orthogonality in the setting of normed spaces.

In a real inner product space X, the orthogonality relation has the following well-known properties for any $\varpi_1, \varpi_2 \in X$,

1. Homogeneity: If $\varpi_1 \perp \varpi_2$, then $\mu\varpi_1 \perp \lambda\varpi_2$ for any λ, μ in $\mathbb{R}$.
2. Symmetry: If $\varpi_1 \perp \varpi_2$, then $\varpi_2 \perp \varpi_1$.
3. α-existence: there exists $\alpha \in \mathbb{R}$ such that $\varpi_1 \perp \alpha\varpi_1 + \varpi_2$.

However, these well-known properties of orthogonality in inner product space, such as symmetry, homogeneity, and α-existence, need not hold for other well-known orthogonalities in normed spaces. For instance, Birkhoff–James orthogonality is homogeneous but fails to be symmetric, whereas both isosceles and Pythagorean orthogonality are symmetric but lack homogeneity. These discrepancies highlight the fact that generalized orthogonalities can demonstrate fundamentally divergent behaviors. Also, the isosceles and Singer orthogonalities are symmetric and satisfy the α-existence property (see [1]). The Singer orthogonality also has the α-uniqueness feature [15], but the isosceles orthogonality doesn't usually have this property. A classification of several types of orthogonality in normed linear spaces, together with their fundamental features and interrelations, is available in the study [2, 14].

Gunawan et al. [6] introduced two angle functions, namely P-angle and I-angle, as a result of the Pythagorean and isosceles orthogonalities concepts. The P-angle preserves Pythagorean orthogonality, while the I-angle preserves isosceles orthogonality. The Cosine Law and Polarization Identity show that the definitions of P-angle and I-angle correspond to the standard angles in inner product space. In a normed space, both the P-angle and the I-angle have identical properties: partial homogeneity, continuity, and symmetry. However, inner product spaces lack numerous parallelism and homogeneity qualities in comparison to the standard angle. Subsequently, Thürey [12] proposed the notion of the Thy-angle, which preserves Singer orthogonality. The Thy-angle possesses the properties of continuity, symmetry, homogeneity, and non-degeneracy. Thy-angle clearly coincides with the standard angle function in Euclidean space. Thy-angle obviously corresponds to the standard angle function in Euclidean space. Many writers have examined the concept of angles in normed spaces (see [3] and [10]).

So motivated by the geometry of angles in normed linear spaces, the main aim of this paper is to introduce a new angular function orthogonality and to address the following crucial question for its existence:

- Does there exist a non-zero vector that is an angular function orthogonal to some given non-zero vectors?

The article's structure is as follows: After the introductory Section 1, we move on to Section 2, where we present some fundamental definitions, notations, and results pertaining to the orthogonality problem. Section 3 defines a well-defined notion of orthogonality termed angular function orthogonality using the concept of the angular distance function. Additionally, we give some examples to discuss the existence of a non-zero vector that is an angular function orthogonal to a given vector. To explore the geometrical properties of the aforementioned orthogonality, we discuss its homogeneity, alpha-existence, and the conditions for which it is symmetric. In addition, we show that the orthogonality of the angular distance is equivalent to the standard concept of orthogonality for inner product spaces. Inspired by the aforementioned orthogonality, we establish a well-defined angle through the angular distance function and examine its geometric features by defining acute, obtuse, and right angles in normed spaces. Several significant examples are also presented to substantiate the findings. Additionally, we examine the relationship of the angle with the Euclidean angle, the isosceles angle, and the Thy angle.

The following are some notations and definitions that will be utilized in the subsequent sections.

Definition 1.1. *The vectors ϖ_1, ϖ_2 in a normed space X are Singer-type orthogonal [7] ($\varpi_1 \perp_{s'} \varpi_2$) if*

$$\left\| \frac{\varpi_1}{1+\|\varpi_1\|} + \frac{\varpi_2}{1+\|\varpi_2\|} \right\| = \left\| \frac{\varpi_1}{1+\|\varpi_1\|} - \frac{\varpi_2}{1+\|\varpi_2\|} \right\|.$$

Definition 1.2. *Let X be an inner product space. The number*

$$\angle_{Euclid}(\varpi_1, \varpi_2) = \arccos \frac{\langle \varpi_1, \varpi_2 \rangle}{\|\varpi_1\|\|\varpi_2\|}$$

is called the Euclidean-angle between two non-zero vectors ϖ_1 and ϖ_2.

Definition 1.3. *The angle between two non-zero vectors ϖ_1, ϖ_2 in a normed space X is called*

- *I-angle [6], if $\angle_I(\varpi_1, \varpi_2) = \arccos \frac{\|\varpi_1+\varpi_2\|^2 - \|\varpi_1-\varpi_2\|^2}{4\|\varpi_1\|\|\varpi_2\|}$.*
- *P-angle [6], if $\angle_P(\varpi_1, \varpi_2) = \arccos \left(\frac{\|\varpi_1\|^2 + \|\varpi_2\|^2 - \|\varpi_1-\varpi_2\|^2}{2\|\varpi_1\|\|\varpi_2\|} \right)$.*
- *Thy-angle [12], if*

$$\angle_{Thy}(\varpi_1, \varpi_2) = \arccos \left(\frac{1}{4} \left(\left\| \frac{\varpi_1}{\|\varpi_1\|} + \frac{\varpi_2}{\|\varpi_2\|} \right\|^2 - \left\| \frac{\varpi_1}{\|\varpi_1\|} - \frac{\varpi_2}{\|\varpi_2\|} \right\|^2 \right) \right).$$

2. Main Results

2.1. Orthogonality and angular distance

We introduce a new orthogonality in a normed space using the angular distance function and the Singer-type orthogonality as inspiration in this section.

First, we define an angular distance function $\phi : X \times X \to \mathbb{R}$ as

$$\phi(\varpi_1, \varpi_2) = \frac{(1+\|\varpi_1\|)(1+\|\varpi_2\|)}{\|\varpi_1\|\|\varpi_2\|} \left\| \frac{\varpi_1}{1+\|\varpi_1\|} - \frac{\varpi_2}{1+\|\varpi_2\|} \right\|^2$$

for two non-zero vectors $\varpi_1, \varpi_2 \in X$.
Assume that Φ is the collection of all angular distance functions.

Using the above angular distance function and Singer-type orthogonality, we introduce an orthogonality between two vectors in a normed space.

Definition 2.1. *Let X be a normed space. ϖ_1 is angular function orthogonal to ϖ_2 ($\varpi_1 \perp_* \varpi_2$) if either of the following hold for all $\varpi_1, \varpi_2 \in X$:*

(a) $\|\varpi_1\|\,\|\varpi_2\| = 0$;
or

(b) for all non-zero vectors ϖ_1, ϖ_2:
(i) $\{\varpi_1, \varpi_2\}$ is linearly independent (L.I.),
(ii) $|\Upsilon(t)| \geq |\Upsilon(0)|$, where

$$\Upsilon(t) = \Upsilon(\varpi_1, \varpi_2; t) := \phi(\varpi_1 + t\varpi_2, -\varpi_2) - \phi(\varpi_1 + t\varpi_2, \varpi_2),$$

for all $t \in \mathbb{R}$ and $\phi \in \Phi$.

It is interesting to note that $\Upsilon(\varpi_1, -\varpi_2; -t) = -\Upsilon(\varpi_1, \varpi_2; t)$.

In the following result, we show that $\Upsilon(t)$ is well-defined.

Proposition 2.2. *The function*

$$\Upsilon(t) := \phi(\varpi_1 + t\varpi_2, -\varpi_2) - \phi(\varpi_1 + t\varpi_2, \varpi_2), \tag{1}$$

is well-defined if $\phi \in \Phi$, the non-zero vectors ϖ_1 and ϖ_2 are linearly independent in a normed space X, and $t \in \mathbb{R}$.

Proof. Clearly if $\{\varpi_1, \varpi_2\}$ is a linearly independent set, then $\varpi_1 + t\varpi_2 \neq 0$ for all $t \in \mathbb{R}$ and so $\phi(\varpi_1 + t\varpi_2, \varpi_2)$ and $\phi(\varpi_1 + t\varpi_2, -\varpi_2)$ are well defined. Therefore, the function $\Upsilon(t)$ is well-defined where ϖ_1 and ϖ_2 are linearly independent. □

Now, we are presenting some examples of non-zero independent vectors, which are angular functional orthogonal.

Example 2.3. *Consider a two-dimensional real sequence space $X = \ell_1$ equipped with the Holder weights $\|.\|$, defined as $\|\varpi_1 = (p, q)\| = |p| + |q|$. Consider the vectors $\varpi_1 = (2, 3)$ and $\varpi_2 = (-1, 1)$ in X. Then, $\|\varpi_1\| = 5$ and $\|\varpi_2\| = 2$. Here ϖ_1 and ϖ_2 are L.I. with $\|\varpi_1\|\,\|\varpi_2\| \neq 0$. Now, consider*

$$\begin{aligned}
&\Upsilon(0)\\
&= \phi(\varpi_1, -\varpi_2) - \phi(\varpi_1, \varpi_2)\\
&= \frac{(1 + \|\varpi_1\|)(1 + \|\varpi_2\|)}{\|\varpi_1\|\|\varpi_2\|}\left[\left\|\frac{\varpi_1}{1 + \|\varpi_1\|} + \frac{\varpi_2}{1 + \|\varpi_2\|}\right\|^2 - \left\|\frac{\varpi_1}{1 + \|\varpi_1\|} - \frac{\varpi_2}{1 + \|\varpi_2\|}\right\|^2\right]\\
&= \frac{6 \times 3}{5 \times 2}\left[\left(\frac{5}{6}\right)^2 - \left(\frac{5}{6}\right)^2\right]\\
&= 0.
\end{aligned}$$

So, $|\Upsilon(t)| \geq |\Upsilon(0)|$ holds clearly for all $t \in \mathbb{R}$. Hence $\varpi_1 \perp_* \varpi_2$.

Example 2.4. *Consider a two-dimensional real sequence space $X = \ell_\infty$ equipped with a norm $\|\varpi_1 = (p, q)\| = \max\{|p|, |q|\}$. Consider the vectors $\varpi_1 = (2, -1)$ and $\varpi_2 = (1, 2)$ in X. Then, $\|\varpi_1\| = 2$ and $\|\varpi_2\| = 2$. Here ϖ_1 and ϖ_2 are L.I. with $\|\varpi_1\|\|\varpi_2\| \neq 0$. Now, consider*

$$\begin{aligned}
&\Upsilon(0)\\
&= \phi(\varpi_1, -\varpi_2) - \phi(\varpi_1, \varpi_2)\\
&= \frac{(1 + \|\varpi_1\|)(1 + \|\varpi_2\|)}{\|\varpi_1\|\|\varpi_2\|}\left[\left\|\frac{\varpi_1}{1 + \|\varpi_1\|} + \frac{\varpi_2}{1 + \|\varpi_2\|}\right\|^2 - \left\|\frac{\varpi_1}{1 + \|\varpi_1\|} - \frac{\varpi_2}{1 + \|\varpi_2\|}\right\|^2\right]\\
&= \frac{3 \times 3}{2 \times 2}\left[\left(\frac{3}{3}\right)^2 - \left(\frac{3}{3}\right)^2\right]\\
&= 0.
\end{aligned}$$

So, $|\Upsilon(t)| \geq |\Upsilon(0)|$ holds clearly for all $t \in \mathbb{R}$. Hence $\varpi_1 \perp_* \varpi_2$.

Remark 2.5. *(a) The two independent non-zero vectors ϖ_1 and ϖ_2 are orthogonal if and only if $|\Upsilon(\varpi_1, \varpi_2; t)| \geq |\Upsilon(\varpi_1, \varpi_2; 0)|$, for all $t \in \mathbb{R}$.*

(b) It is clear that the above notion of orthogonality is not symmetry, in general.

Now, we give an example to show that the angular function orthogonality is not homogeneous.

Example 2.6. *Consider a two-dimensional real sequence space $X = \ell_{1,\infty}$ equipped with a norm*

$$\|\varpi_1 = (p,q)\| = \begin{cases} |p| + |q|, & \text{if } pq \geq 0 \\ \max\{|p|,|q|\}, & \text{if } pq \leq 0. \end{cases}$$

Consider the vectors $\varpi_1 = (\frac{1}{5}, 0)$ and $\varpi_2 = (4, -5)$ in X. Then, $\|\varpi_1\| = \frac{1}{5}$ and $\|\varpi_2\| = 5$. Here ϖ_1 and ϖ_2 are L.I. with $\|\varpi_1\|\,\|\varpi_2\| \neq 0$. Here it is easy to see that $\Upsilon(0) = 0$. So, $|\Upsilon(t)| \geq |\Upsilon(0)|$ holds clearly for all $t \in \mathbb{R}$. Hence $\varpi_1 \perp_* \varpi_2$. Now, $5\varpi_1 = (1, 0)$ and it is easy to see that $|\Upsilon(5\varpi_1, \varpi_2; 0)| = \frac{24}{15} = 1.6$. Choosing $t = -\frac{1}{10}$, we have $5\varpi_1 + t\varpi_2 = 5\varpi_1 - \frac{1}{10}\varpi_2 = (1, 0) - \frac{1}{10}(4, -5) = (\frac{3}{5}, \frac{1}{2})$. So, $1 + \|5\varpi_1 + t\varpi_2\| = \frac{21}{10}$. Now, we have

$$\begin{aligned}\Upsilon(5\varpi_1, \varpi_2; t = -\frac{1}{10}) &= \phi(5\varpi_1 + t\varpi_2, -\varpi_2) - \phi(5\varpi_1 + t\varpi_2, \varpi_2) \\ &= \frac{21 \times 6}{11 \times 5}\left[\left(\frac{20}{21}\right)^2 - \left(\frac{15}{14}\right)^2\right] \\ &\approx -0.55.\end{aligned}$$

Therefore, $|\Upsilon(5\varpi_1, \varpi_2; t = -\frac{1}{10})| = \left|\frac{21\times6}{11\times5}\left[\left(\frac{20}{21}\right)^2 - \left(\frac{15}{14}\right)^2\right]\right| \approx 0.55$. So, $5\varpi_1 \not\perp_* \varpi_2$.
Hence $\varpi_1 \perp_* \varpi_2$ but $5\varpi_1 \not\perp_* \varpi_2$. It implies that the angular function orthogonality is not homogeneous.

Now, in an inner product space to show that the angular function orthogonality is equivalent to the standard orthogonality, we need the following result:

Lemma 2.7. *For an inner product space X, $\Upsilon(t) = \frac{\langle \varpi_1 + t\varpi_2, \varpi_2 \rangle}{\|\varpi_1 + t\varpi_2\|\|\varpi_2\|}$ for all linearly independent vectors $\varpi_1, \varpi_2 \in X$.*

Proof. For all linearly independent vectors $\varpi_1, \varpi_2 \in X$ we have,

$$\begin{aligned}\Upsilon(t) &= \phi(\varpi_1 + t\varpi_2, -\varpi_2) - \phi(\varpi_1 + t\varpi_2, \varpi_2) \\ &= \frac{(1 + \|\varpi_1 + t\varpi_2\|)(1 + \|\varpi_2\|)}{\|\varpi_1 + t\varpi_2\|\|\varpi_2\|}\Bigg[\left\|\frac{(\varpi_1 + t\varpi_2)}{1 + \|\varpi_1 + t\varpi_2\|} + \frac{\varpi_2}{1 + \|\varpi_2\|}\right\|^2 \\ &\qquad - \left\|\frac{(\varpi_1 + t\varpi_2)}{1 + \|\varpi_1 + t\varpi_2\|} - \frac{\varpi_2}{1 + \|\varpi_2\|}\right\|^2\Bigg] \\ &= \frac{(1 + \|\varpi_1 + t\varpi_2\|)(1 + \|\varpi_2\|)}{\|\varpi_1 + t\varpi_2\|\|\varpi_2\|} \times 4\left\langle \frac{\varpi_1 + t\varpi_2}{1 + \|\varpi_1 + t\varpi_2\|}, \frac{\varpi_2}{1 + \|\varpi_2\|}\right\rangle \\ &= \frac{(1 + \|\varpi_1 + t\varpi_2\|)(1 + \|\varpi_2\|)}{\|\varpi_1 + t\varpi_2\|\|\varpi_2\|} \times \frac{4\langle \varpi_1 + t\varpi_2, \varpi_2\rangle}{(1 + \|\varpi_1 + t\varpi_2\|)(1 + \|\varpi_2\|)} \\ &= \frac{4\langle \varpi_1 + t\varpi_2, \varpi_2\rangle}{\|\varpi_1 + t\varpi_2\|\,\|\varpi_2\|}.\end{aligned}$$

Hence the result. □

Theorem 2.8. *Let ϖ_1 and ϖ_2 be two vectors in an inner product space X. Then $\varpi_1 \perp_* \varpi_2$ if and only if $\langle \varpi_1, \varpi_2 \rangle = 0$.*

Proof. Let $\varpi_1 \perp_* \varpi_2$ holds for $\varpi_1, \varpi_2 \in X$. So, $|\Upsilon(t)| \geq |\Upsilon(0)|$ holds. Using the Lemma 2.7, we get

$$\begin{aligned}&|\phi_2(\varpi_1, -\varpi_2) - \phi_2(\varpi_1, \varpi_2)| \leq |\phi_2(\varpi_1 + t\varpi_2, -\varpi_2) - \phi_2(\varpi_1 + t\varpi_2, \varpi_2)| \\ \iff &\left|\frac{4\langle \varpi_1, \varpi_2\rangle}{\|\varpi_1\|\|\varpi_2\|}\right| \leq \left|\frac{4\langle \varpi_1 + t\varpi_2, \varpi_2\rangle}{\|\varpi_1 + t\varpi_2\|\|\varpi_2\|}\right| \\ \iff &\|\varpi_1 + t\varpi_2\||\langle \varpi_1, \varpi_2\rangle| \leq \|\varpi_1\||\langle \varpi_1 + t\varpi_2, \varpi_2\rangle|. \qquad (2)\end{aligned}$$

Putting $t = -\frac{\langle \varpi_1, \varpi_2\rangle}{\|\varpi_2\|^2}$ in (2), we get $\langle \varpi_1, \varpi_2 \rangle = 0$.
Conversely, if $\langle \varpi_1, \varpi_2 \rangle = 0$, then (2) holds and so $|\Upsilon(t)| \geq 0$ for all $t \in \mathbb{R}$ and hence $\varpi_1 \perp_* \varpi_2$. □

Remark 2.9. *If the norm comes from an inner product space, then angular function orthogonality is symmetry.*

Now, to discuss the α-existence property of the angular function orthogonality, we need the following results.

Lemma 2.10. *Let ϖ_1, ϖ_2 be two independent vectors in a normed space X. Then, for $\phi \in \Phi$, $\lim_{t\to\pm\infty} \Upsilon(t) = \pm 4$, that is,*

$$\lim_{t\to\pm\infty} \phi(\varpi_1 + t\varpi_2, -\varpi_2) - \phi(\varpi_1 + t\varpi_2, \varpi_2) = \pm 4.$$

Proof. Consider

$$\begin{aligned}
&\lim_{t\to+\infty} \phi(\varpi_1 + t\varpi_2, -\varpi_2)\\
&= \lim_{t\to+\infty} \frac{(1+\|\varpi_1 + t\varpi_2\|)(1+\|\varpi_2\|)}{\|\varpi_1 + t\varpi_2\|\|\varpi_2\|} \left\| \frac{\varpi_1 + t\varpi_2}{1+\|\varpi_1 + t\varpi_2\|} + \frac{\varpi_2}{1+\|\varpi_2\|} \right\|^2\\
&= \frac{(\|\varpi_2\|)(1+\|\varpi_2\|)}{\|\varpi_2\|\|\varpi_2\|} \left\| \frac{\varpi_2}{\|\varpi_2\|} + \frac{\varpi_2}{1+\|\varpi_2\|} \right\|^2\\
&= \frac{(1+\|\varpi_2\|)}{\|\varpi_2\|} \times \frac{(1+2\|\varpi_2\|)^2}{(1+\|\varpi_2\|)^2}\\
&= \frac{(1+2\|\varpi_2\|)^2}{\|\varpi_2\|(1+\|\varpi_2\|)}.
\end{aligned}$$

Also,

$$\begin{aligned}
&\lim_{t\to+\infty} \phi(\varpi_1 + t\varpi_2, \varpi_2)\\
&= \lim_{t\to+\infty} \frac{(1+\|\varpi_1 + t\varpi_2\|)(1+\|\varpi_2\|)}{\|\varpi_1 + t\varpi_2\|\|\varpi_2\|} \left\| \frac{\varpi_1 + t\varpi_2}{1+\|\varpi_1 + t\varpi_2\|} - \frac{\varpi_2}{1+\|\varpi_2\|} \right\|^2\\
&= \frac{(\|\varpi_2\|)(1+\|\varpi_2\|)}{\|\varpi_2\|\|\varpi_2\|} \left\| \frac{\varpi_2}{\|\varpi_2\|} - \frac{\varpi_2}{1+\|\varpi_2\|} \right\|^2\\
&= \frac{1}{\|\varpi_2\|(1+\|\varpi_2\|)}.
\end{aligned}$$

Therefore, $\lim_{t\to+\infty} \phi(\varpi_1 + t\varpi_2, -\varpi_2) - \phi(\varpi_1 + t\varpi_2, \varpi_2) = 4$.

On calculating, we obtain

$$\begin{aligned}
\lim_{t\to-\infty} \phi(\varpi_1 + t\varpi_2, -\varpi_2) &= \frac{(\|\varpi_2\|)(1+\|\varpi_2\|)}{\|\varpi_2\|\|\varpi_2\|} \left\| -\frac{\varpi_2}{\|\varpi_2\|} + \frac{\varpi_2}{1+\|\varpi_2\|} \right\|^2\\
&= \frac{1}{\|\varpi_2\|(1+\|\varpi_2\|)}.
\end{aligned}$$

and

$$\begin{aligned}
\lim_{t\to-\infty} \phi(\varpi_1 + t\varpi_2, \varpi_2) &= \frac{(\|\varpi_2\|)(1+\|\varpi_2\|)}{\|\varpi_2\|\|\varpi_2\|} \left\| -\frac{\varpi_2}{\|\varpi_2\|} - \frac{\varpi_2}{1+\|\varpi_2\|} \right\|^2\\
&= \frac{(1+2\|\varpi_2\|)^2}{\|\varpi_2\|(1+\|\varpi_2\|)}.
\end{aligned}$$

Therefore,

$$\lim_{t\to-\infty} \phi(\varpi_1 + t\varpi_2, -\varpi_2) - \phi(\varpi_1 + t\varpi_2, \varpi_2) = -4.$$

Hence the result. □

Lemma 2.11. *Suppose that ϖ_1, ϖ_2 are two independent vectors in a normed space X. Then* $|\Upsilon(t)| < \frac{16(1+\|\varpi_1\|)(1+\|\varpi_2\|)}{\|\varpi_1\|\|\varpi_2\|}$.

Proof. For two independent vectors $\varpi_1, \varpi_2 \in X$, we have

$$\begin{aligned}
&|\Upsilon(\varpi_1, \varpi_2; t)| \\
&= |\phi(\varpi_1 + t\varpi_2, -\varpi_2) - \phi(\varpi_1 + t\varpi_2, \varpi_2)| \\
&= \left| \frac{(1+\|\varpi_1\|)(1+\|\varpi_2\|)}{\|\varpi_1\|\|\varpi_2\|} \left[\left\| \frac{\varpi_1}{1+\|\varpi_1\|} + \frac{\varpi_2}{1+\|\varpi_2\|} \right\|^2 - \left\| \frac{\varpi_1}{1+\|\varpi_1\|} - \frac{\varpi_2}{1+\|\varpi_2\|} \right\|^2 \right] \right| \\
&= \frac{(1+\|\varpi_1\|)(1+\|\varpi_2\|)}{\|\varpi_1\|\|\varpi_2\|} \left| \left[\left\| \frac{\varpi_1}{1+\|\varpi_1\|} + \frac{\varpi_2}{1+\|\varpi_2\|} \right\|^2 - \left\| \frac{\varpi_1}{1+\|\varpi_1\|} - \frac{\varpi_2}{1+\|\varpi_2\|} \right\|^2 \right] \right|.
\end{aligned}$$

Taking $A = \frac{\varpi_1}{1+\|\varpi_1\|}$ with $\|A\| < 1$ and $B = \frac{\varpi_2}{1+\|\varpi_2\|}$ with $\|B\| < 1$, we have

$$\begin{aligned}
&\left| \left[\left\| \frac{\varpi_1}{1+\|\varpi_1\|} + \frac{\varpi_2}{1+\|\varpi_2\|} \right\|^2 - \left\| \frac{\varpi_1}{1+\|\varpi_1\|} - \frac{\varpi_2}{1+\|\varpi_2\|} \right\|^2 \right] \right| \\
&= |\|A+B\|^2 - \|A-B\|^2| \\
&\le |\|A+B\| + \|A-B\||^2 \\
&\le |\|A\| + \|B\| + \|A\| + \|B\||^2 \\
&< 16.
\end{aligned}$$

Therefore, $|\Upsilon(\varpi_1, \varpi_2; t)| < \frac{16(1+\|\varpi_1\|)(1+\|\varpi_2\|)}{\|\varpi_1\|\|\varpi_2\|}$, for two independent vectors $\varpi_1, \varpi_2 \in X$. □

In the following result, we show the α-existence property for the angular function orthogonality.

Theorem 2.12. *The angular function orthogonality has an α-existence property.*

Proof. Let ϖ_1, ϖ_2 be two independent vectors in a normed space X. We'll show that there exists $t_0 \in \mathbb{R}$ such that $\varpi_1 + t_0\varpi_2 \perp_* \varpi_2$. Since $\Upsilon(\varpi_1, \varpi_2; t) \ge \Upsilon(\varpi_1, \varpi_2; 0)$, it is sufficient to prove that there exists $t_0 \in \mathbb{R}$ such that the function $\Upsilon(\varpi_1 + t_0\varpi_2, \varpi_2; t)$ takes a minimum at t_0. Also, we know that

$$\Upsilon(\varpi_1 + t_0\varpi_2, \varpi_2; t) \ge \Upsilon(\varpi_1 + t_0\varpi_2, \varpi_2; 0) \quad \text{if and only if} \quad \Upsilon(\varpi_1, \varpi_2; t + t_0) \ge \Upsilon(\varpi_1, \varpi_2; t_0). \tag{3}$$

Also, $\lim_{t\to\pm\infty} \Upsilon(\varpi_1, \varpi_2; t) = \pm 4$ (by Lemma 2.10) and $-\frac{16(1+\|\varpi_1\|)(1+\|\varpi_2\|)}{\|\varpi_1\|\|\varpi_2\|} < \Upsilon(\varpi_1, \varpi_2; t) < \frac{16(1+\|\varpi_1\|)(1+\|\varpi_2\|)}{\|\varpi_1\|\|\varpi_2\|}$ (by Lemma 2.11). Thus by using the continuity of Υ, $\Upsilon(\varpi_1, \varpi_2; t)$ takes a minimum at some $t_0 \in \mathbb{R}$, that is, $\Upsilon(\varpi_1, \varpi_2; t) \ge \Upsilon(\varpi_1, \varpi_2; t_0)$. Hence, the result follows from (3). □

2.2. S_*-type angle

Angles between vectors in normed spaces have been defined in several ways, all of them are equivalent to the Euclidean angle in inner product spaces. Inspired by this, we provide a new perspective utilizing the angular distance function and derive several of its geometric features in this subsection. Also, we define the acute, obtuse, and right angles using it.

Suppose that X is a real inner product space with an inner product $\langle ., . \rangle$, and induced norm $\|.\|$. Then for any $\varpi_1, \varpi_2 \in X$, we have

$$\|\varpi_1 + \varpi_2\|^2 - \|\varpi_1 - \varpi_2\|^2 = 4\langle \varpi_1, \varpi_2 \rangle. \tag{4}$$

For any two non-zero vectors $\varpi_1, \varpi_2 \in X$, replacing ϖ_1 with $\frac{\varpi_1}{1+\|\varpi_1\|}$ and ϖ_2 with $\frac{\varpi_2}{1+\|\varpi_2\|}$ in (4), we get

$$\left\| \frac{\varpi_1}{1+\varpi_1} + \frac{\varpi_2}{1+\|\varpi_2\|} \right\|^2 - \left\| \frac{\varpi_1}{1+\|\varpi_1\|} - \frac{\varpi_2}{1+\|\varpi_2\|} \right\|^2$$

$$= 4\left\langle \frac{\varpi_1}{1+\|\varpi_1\|}, \frac{\varpi_2}{1+\|\varpi_2\|} \right\rangle$$
$$= \frac{4\langle \varpi_1, \varpi_2 \rangle}{(1+\|\varpi_1\|)(1+\|\varpi_2\|)}.$$

Taking modulus on both sides, we have

$$\left| \left\| \frac{\varpi_1}{1+\|\varpi_1\|} + \frac{\varpi_2}{1+\|\varpi_2\|} \right\|^2 - \left\| \frac{\varpi_1}{1+\|\varpi_1\|} - \frac{\varpi_2}{1+\|\varpi_2\|} \right\|^2 \right|$$
$$= \frac{4|\langle \varpi_1, \varpi_2 \rangle|}{(1+\|\varpi_1\|)(1+\|\varpi_2\|)}$$
$$\leq \frac{4\|\varpi_1\|\|\varpi_2\|}{(1+\|\varpi_1\|)(1+\|\varpi_2\|)}.$$

Therefore,

$$\left| \frac{(1+\|\varpi_1\|)(1+\|\varpi_2\|)}{4\|\varpi_1\|\|\varpi_2\|} \left[\left\| \frac{\varpi_1}{1+\|\varpi_1\|} + \frac{\varpi_2}{1+\|\varpi_2\|} \right\|^2 - \left\| \frac{\varpi_1}{1+\|\varpi_1\|} - \frac{\varpi_2}{1+\|\varpi_2\|} \right\|^2 \right] \right| \leq 1. \tag{5}$$

Inspired by this, we introduce the following notion of angle using the angular distance function in a real normed space X.

Definition 2.13. *Let $(X, \|.\|)$ be a real normed space. We define S_*-type angle function $\angle_{S_*} : X \times X \mapsto [0, \pi]$ as*

$$\angle_{S_*}(\varpi_1, \varpi_2) = \arccos \begin{cases} 0, & \text{for } \|\varpi_1\|\|\varpi_2\| = 0, \\ \{1, -1\}, & \text{for } \varpi_1, \varpi_2 \text{ are L.D.}, \\ \frac{1}{4}\left(\phi(\varpi_1, -\varpi_2) - \phi(\varpi_1, \varpi_2)\right), & \text{for } \|\varpi_1\|\|\varpi_2\| \neq 0, \text{and } \varpi_1, \varpi_2 \text{ are L.I.} \end{cases}$$

where $\phi \in \Phi$ and $\varpi_1, \varpi_2 \in X$.

Remark 2.14. *(i) When $\|\varpi_1\|\,\|\varpi_2\| \neq 0$ and ϖ_1, ϖ_2 are L.I., we can also write $\angle_{S_*}(\varpi_1, \varpi_2) = \frac{1}{4}\Upsilon(\varpi_1, \varpi_2; 0)$.*

(ii) From (5), it is easy to see that in an inner product space, $\angle_{S_*}$ is less than or equal to unity.

(iii) Suppose that $\varpi_1, \varpi_2 \in X$ are L.I. with $\|\varpi_1\|\|\varpi_2\| \neq 0$. If $\angle_{S_*}(\varpi_1, \varpi_2) = \frac{\pi}{2}$, then $\varpi_1 \perp_* \varpi_2$ because $|\phi(\varpi_1 + t\varpi_2, -\varpi_2) - \phi(\varpi_1 + t\varpi_2, \varpi_2)| \geq |\phi(\varpi_1, -\varpi_2) - \phi(\varpi_1, \varpi_2)| = 0$ that is, $|\Upsilon(t)| \geq |\Upsilon(0)|$ for all $t \in \mathbb{R}$.

We discuss the geometrical properties of the $\mathcal{S}_*$-type angle ($\angle_{\mathcal{S}_*}$).

Proposition 2.15. *The angle function $\angle_{\mathcal{S}_*}$ satisfies the following properties for all non-zero $\varpi_1, \varpi_2 \in X$:*

(i) Parallelism Property: For $\alpha \in \mathbb{R}$, we have

$$\angle_{\mathcal{S}_*}(\varpi_1, \alpha\varpi_1) = \begin{cases} 0, & \text{if } \alpha > 0 \\ \pi, & \text{if } \alpha < 0. \end{cases}$$

(ii) Symmetry Property: $\angle_{\mathcal{S}_*}(\varpi_1, \varpi_2) = \angle_{\mathcal{S}_*}(\varpi_2, \varpi_1)$,

(iii) Part of Homogeneity Property: For $a, b \in \{-1, 1\}$ we have,

$$\angle_{\mathcal{S}_*}(a\varpi_1, b\varpi_2) = \begin{cases} \angle_{\mathcal{S}_*}(\varpi_1, \varpi_2), & \text{if } ab > 0 \\ \pi - \angle_{\mathcal{S}_*}(\varpi_1, \varpi_2), & \text{if } ab < 0. \end{cases}$$

(iv) Continuity Property: If $\varpi_{1n} \to \varpi_1$ and $\varpi_{2n} \to \varpi_2$ (in norm) then $\angle_{\mathcal{S}_*}(\varpi_{1n}, \varpi_{2n}) \to \angle_{\mathcal{S}_*}(\varpi_1, \varpi_2)$.

Proof. (*i*) From the definition of angle, the parallelogram property is obvious.

(*ii*) When ϖ_1 and ϖ_2 are L.I., we have

$$\begin{aligned}
&\angle_{\mathcal{S}_*}(\varpi_2, \varpi_1)\\
&= \arccos\left[\frac{1}{4}\left(\phi(\varpi_2, -\varpi_1) - \phi(\varpi_2, \varpi_1)\right)\right]\\
&= \arccos \frac{(1+\|\varpi_2\|)(1+\|\varpi_1\|)}{4\|\varpi_2\|\|\varpi_1\|}\left[\left\|\frac{\varpi_2}{1+\|\varpi_2\|} + \frac{\varpi_1}{1+\|\varpi_1\|}\right\|^2 - \left\|\frac{\varpi_2}{1+\|\varpi_2\|} - \frac{\varpi_1}{1+\|\varpi_1\|}\right\|^2\right]\\
&= \arccos \frac{(1+\|\varpi_1\|)(1+\|\varpi_2\|)}{4\|\varpi_1\|\|\varpi_2\|}\left[\left\|\frac{\varpi_1}{1+\|\varpi_1\|} + \frac{\varpi_2}{1+\|\varpi_2\|}\right\|^2 - \left\|\frac{\varpi_1}{1+\|\varpi_1\|} - \frac{\varpi_2}{1+\|\varpi_2\|}\right\|^2\right]\\
&= \angle_{\mathcal{S}_*}(\varpi_1, \varpi_2).
\end{aligned}$$

(*iii*) For the homogeneity property, we take only the non-zero L.I. vectors; otherwise, it is obvious.

(a) If $a, b \in \{-1, 1\}$ and $ab > 0$ then we get two cases.

Case-I: For $a = 1$ and $b = 1$, $\angle_{\mathcal{S}_*}(a\varpi_1, b\varpi_2) = \angle_{\mathcal{S}_*}(\varpi_1, \varpi_2)$ is obvious.

Case-II: For $a = -1$ and $b = -1$, we get

$$\begin{aligned}
\angle_{\mathcal{S}_*}(a\varpi_1, b\varpi_2) &= \angle_{\mathcal{S}_*}(-\varpi_1, -\varpi_2)\\
&= \arccos\left[\frac{1}{4}\left(\phi(-\varpi_1, \varpi_2) - \phi(-\varpi_1, -\varpi_2)\right)\right]\\
&= \arccos \frac{(1+\|\varpi_1\|)(1+\|\varpi_2\|)}{4\|\varpi_1\|\|\varpi_2\|}\left[\left\|\frac{(-\varpi_1)}{1+\|-\varpi_1\|} - \frac{\varpi_2}{1+\|-\varpi_2\|}\right\|^2 - \left\|\frac{(-\varpi_1)}{1+\|-\varpi_1\|} - \frac{(-\varpi_2)}{1+\|-\varpi_2\|}\right\|^2\right]\\
&= \arccos \frac{(1+\|\varpi_1\|)(1+\|\varpi_2\|)}{4\|\varpi_1\|\|\varpi_2\|}\left[\left\|\frac{\varpi_1}{1+\|\varpi_1\|} + \frac{\varpi_2}{1+\|\varpi_2\|}\right\|^2 - \left\|\frac{\varpi_1}{1+\|\varpi_1\|} - \frac{\varpi_2}{1+\|\varpi_2\|}\right\|^2\right]\\
&= \angle_{\mathcal{S}_*}(\varpi_1, \varpi_2).
\end{aligned}$$

(b) If $a, b \in \{-1, 1\}$ and $ab < 0$ then we get two cases.
Case-I: For $a = 1$ and $b = -1$, we get

$$\begin{aligned}
\angle_{\mathcal{S}_*}(a\varpi_1, b\varpi_2) &= \angle_{\mathcal{S}_*}(\varpi_1, -\varpi_2)\\
&= \arccos\left[\frac{1}{4}\left(\phi(\varpi_1, \varpi_2) - \phi(\varpi_1, -\varpi_2)\right)\right]\\
&= \arccos \frac{(1+\|\varpi_1\|)(1+\|\varpi_2\|)}{4\|\varpi_1\|\|\varpi_2\|}\left[\left\|\frac{\varpi_1}{1+\|\varpi_2\|} - \frac{\varpi_2}{1+\|\varpi_2\|}\right\|^2 - \left\|\frac{\varpi_1}{1+\|\varpi_1\|} - \frac{(-\varpi_2)}{1+\|-\varpi_2\|}\right\|^2\right]\\
&= \arccos \frac{(1+\|\varpi_1\|)(1+\|\varpi_2\|)}{4\|\varpi_1\|\|\varpi_2\|}\left[\left\|-\frac{\varpi_1}{1+\|\varpi_1\|} - \frac{\varpi_2}{1+\|\varpi_2\|}\right\|^2 - \left\|\frac{\varpi_1}{1+\|\varpi_1\|} + \frac{\varpi_2}{1+\|\varpi_2\|}\right\|^2\right]\\
&= \arccos(-1)\frac{(1+\|\varpi_1\|)(1+\|\varpi_2\|)}{4\|\varpi_1\|\|\varpi_2\|}\left[\left\|\frac{\varpi_1}{1+\|\varpi_1\|} + \frac{\varpi_2}{1+\|\varpi_2\|}\right\|^2 - \left\|\frac{\varpi_1}{1+\|\varpi_1\|} - \frac{\varpi_2}{1+\|\varpi_2\|}\right\|^2\right]\\
&= \arccos\left[-\frac{1}{4}\left(\phi(\varpi_1, -\varpi_2) - \phi(\varpi_1, \varpi_2)\right)\right]\\
&= \pi - \arccos\left[\frac{1}{4}\left(\phi(\varpi_1, -\varpi_2) - \phi(\varpi_1, \varpi_2)\right)\right]
\end{aligned}$$

$$= \pi - \angle_{\mathcal{S}_*}(\varpi_1, \varpi_2).$$

Case-II: For $a = -1$ and $b = 1$, in a similar manner, we have $\angle_{\mathcal{S}_*}(a\varpi_1, b\varpi_2) = \pi - \angle_{\mathcal{S}_*}(\varpi_1, \varpi_2)$.

(iv) We know that the norm and cosine inverse functions are continuous. This continuity implies that the angle function $\angle_{\mathcal{S}_*}$ has a continuity property, that is if $\varpi_{1n} \to \varpi_1$ and $\varpi_{2n} \to \varpi_2$ (in norm) then $\angle_{\mathcal{S}_*}(\varpi_{1n}, \varpi_{2n}) = \angle_{\mathcal{S}_*}(\varpi_1, \varpi_2)$. □

Remark 2.16. *Using Proposition 2.15 (iii), it is easy to see that the angle function $\angle_{\mathcal{S}_*}$ satisfies the supplementary angle property, that is, $\angle_{\mathcal{S}_*}(\varpi_1, \varpi_2) + \angle_{\mathcal{S}_*}(\varpi_2, -\varpi_1) = \pi$.*

In the following example, we show that the angle function $\angle_{\mathcal{S}_*}$ does not satisfy the additivity property in general, that is, $\angle(\varpi_1, \alpha\varpi_1 + \beta\varpi_2) + \angle(\alpha\varpi_1 + \beta\varpi_2, \varpi_2) \neq \angle(\varpi_1, \varpi_2)$ for every $\varpi_1, \varpi_2 \in X\backslash\{0\}$ and $\alpha, \beta > 0$.

Example 2.17. *Consider a two-dimensional real sequence space $X = \ell_1$ equipped with the Holder weights $\|.\|$, where $\|\varpi_1 = (p, q)\| = |p| + |q|$. Consider the vectors $\varpi_1 = (2, 3)$ and $\varpi_2 = (-1, 1)$ in a normed space X. Then, $\|\varpi_1\| = 5$ and $\|\varpi_2\| = 2$. Here ϖ_1 and ϖ_2 are L.I. with $\|\varpi_1\|\|\varpi_2\| \neq 0$. From the definition of the angle function, we have*

$$\angle_{\mathcal{S}_*}(\varpi_1, \varpi_2) = \arccos\left[\frac{1}{4}\left(\phi(\varpi_1, -\varpi_2) - \phi(\varpi_1, \varpi_2)\right)\right]$$
$$= \arccos(0)$$
$$= \frac{\pi}{2}.$$

Also, we have

$$\angle_{\mathcal{S}_*}(\varpi_1, \varpi_1 + \varpi_2) = \angle_{\mathcal{S}_*}((2, 3), (1, 4))$$
$$= \arccos\frac{48}{50}$$
$$= \frac{9\pi}{100},$$

and

$$\angle_{\mathcal{S}_*}(\varpi_1 + \varpi_2, \varpi_2) = \angle_{\mathcal{S}_*}((1, 4), (2, 3))$$
$$= \arccos 1$$
$$= 0.$$

Therefore $\angle_{\mathcal{S}_*}(\varpi_1, \varpi_1 + \varpi_2) + \angle_{\mathcal{S}_*}(\varpi_1 + \varpi_2, \varpi_2) \neq \angle_{\mathcal{S}_*}(\varpi_1, \varpi_2)$.

In the following example, we show that the angle function $\angle_{\mathcal{S}_*}$ does not satisfy the homogeneity property in general, that is, $\angle_{\mathcal{S}_*}(\alpha\varpi_1, \beta\varpi_2) \neq \angle_{\mathcal{S}_*}(\varpi_1, \varpi_2)$ for every linearly independent vector $\varpi_1, \varpi_2 \in X\backslash\{0\}$.

Example 2.18. *Consider a two-dimensional real sequence space $X = \ell_1$ equipped with a Holder weight $\|.\|$, where $\|(p, q)\| = |p| + |q|$. Consider the vectors $\varpi_1 = (2, 3)$ and $\varpi_2 = (-1, 1)$ in a normed space X. From Example 2.17, we have*

$$\angle_{\mathcal{S}_*}(\varpi_1, \varpi_2) = \arccos 0.$$

Choosing $\alpha = 2$ and $\beta = 3$, we have

$$\angle_{\mathcal{S}_*}(2\varpi_1, 3\varpi_2) = \arccos\left[\frac{1}{4}\left(\phi(2\varpi_1, -3\varpi_2) - \phi(2\varpi_1, 3\varpi_2)\right)\right]$$
$$= \arccos\left(\frac{25}{308}\right).$$

Therefore $\angle_{\mathcal{S}_*}(2\varpi_1, 3\varpi_2) \neq \angle_{\mathcal{S}_*}(\varpi_1, \varpi_2)$.

In the following example, we discuss the geometry of acute, obtuse, and right angles.

Definition 2.19. *Let ϖ_1, ϖ_2 be two non-zero independent vectors in a normed space X. The angle between two vectors ϖ_1 and ϖ_2 is called an obtuse (respectively, acute) angle if there exists a unique number $t_0 \in (0, \infty)$ (respectively, $t_0 \in (-\infty, 0)$) such that*

$$|\Upsilon(t_0)| = 0,$$

where Υ function is same as in Proposition 2.2.
Moreover, the angle between ϖ_1 and ϖ_2 is right if $|\Upsilon(0)| = 0$.

Theorem 2.20. *Let ϖ_1, ϖ_2 be two non-zero independent vectors in an inner product space X. Then the angle between ϖ_1 and ϖ_2 is*

(i) acute angle if and only if $0 < \arccos \frac{\langle \varpi_1, \varpi_2 \rangle}{\|\varpi_1\|\|\varpi_2\|} < \frac{\pi}{2}$;

(ii) obtuse angle if and only if $\frac{\pi}{2} < \arccos \frac{\langle \varpi_1, \varpi_2 \rangle}{\|\varpi_1\|\|\varpi_2\|} < \pi$;

(iii) right angle if and only if $\arccos \frac{\langle \varpi_1, \varpi_2 \rangle}{\|\varpi_1\|\|\varpi_2\|} = \frac{\pi}{2}$.

Proof. In an inner product space X, using Lemma 2.7 we have

$$\begin{aligned} \Upsilon(t) &= \frac{4\langle \varpi_1 + t\varpi_2, \varpi_2 \rangle}{\|\varpi_1 + t\varpi_2\|\|\varpi_2\|} \\ &= 4 \times \frac{\langle \varpi_1, \varpi_2 \rangle + t\|\varpi_2\|^2}{\|\varpi_1 + t\varpi_2\|\|\varpi_2\|}, \end{aligned} \tag{6}$$

where ϖ_1 and ϖ_2 are non-zero L.I. vector.

(i) Let the angle between ϖ_1 and ϖ_2 be an acute angle. Then there exists $t_0 \in (-\infty, 0)$ such that $|\Upsilon(t_0)| = 0$. From (6) we have $\langle \varpi_1, \varpi_2 \rangle > 0$ which implies that $0 < \arccos \frac{\langle \varpi_1, \varpi_2 \rangle}{\|\varpi_1\|\|\varpi_2\|} < \frac{\pi}{2}$.
Conversely, assume that $0 < \arccos \frac{\langle \varpi_1, \varpi_2 \rangle}{\|\varpi_1\|\|\varpi_2\|} < \frac{\pi}{2}$ that is, $\langle \varpi_1, \varpi_2 \rangle > 0$. Using (6), $\Upsilon(t) = 0$ implies that $t < 0$, that is, the angle between ϖ_1 and ϖ_2 is an acute angle.

(ii) Let the angle between ϖ_1 and ϖ_2 be obtuse. Then there exists $t_0 \in (0, \infty)$ such that $|\Upsilon(t_0)| = 0$. From (6), we have $\langle \varpi_1, \varpi_2 \rangle \leq 0$ which implies that $\frac{\pi}{2} < \arccos \frac{\langle \varpi_1, \varpi_2 \rangle}{\|\varpi_1\|\|\varpi_2\|} < \pi$.
Conversely, assume that $\frac{\pi}{2} < \arccos \frac{\langle \varpi_1, \varpi_2 \rangle}{\|\varpi_1\|\|\varpi_2\|} < \pi$, that is, $\langle \varpi_1, \varpi_2 \rangle < 0$. Using (6), $\Upsilon(t) = 0$ implies that $t > 0$ that is, the angle between ϖ_1 and ϖ_2 is an obtuse angle.

(iii) Let the angle between ϖ_1 and ϖ_2 be a right angle. Then there exists $t_0 = 0$ such that $|\Upsilon(t_0)| = 0$. From (6) we have $\langle \varpi_1, \varpi_2 \rangle = 0$.
Conversely, assume that $\arccos \frac{\langle \varpi_1, \varpi_2 \rangle}{\|\varpi_1\|\|\varpi_2\|} = \frac{\pi}{2}$ that is $\langle \varpi_1, \varpi_2 \rangle = 0$. Using (6) we have $\Upsilon(t) = 0$ implies $t = 0$. Therefore, the angle between ϖ_1 and ϖ_2 is a right angle.

□

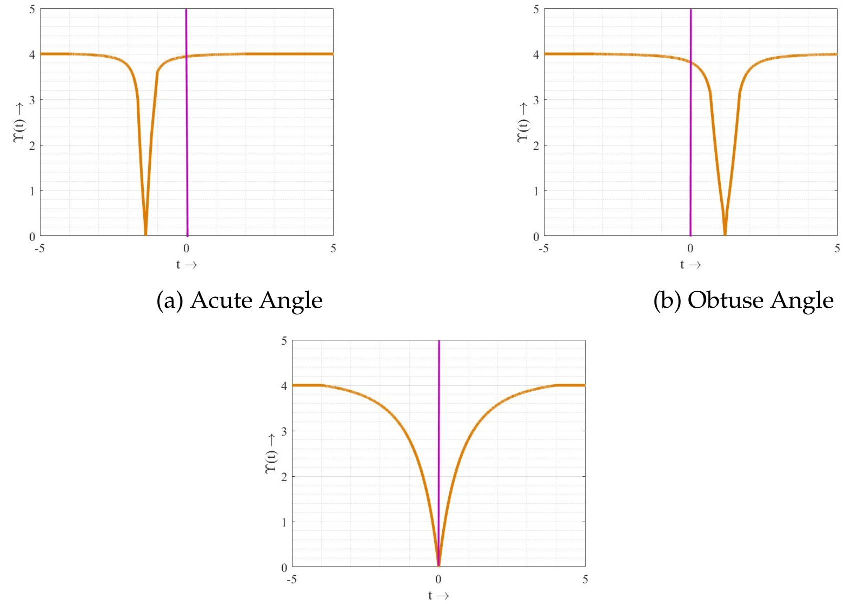
In the following example, we discuss the geometry of acute, obtuse, and right angles.

Example 2.21. *Consider a two-dimensional real sequence space with a ℓ_1 norm.*

(a) Taking $\varpi_1 = (5, 2)$ and $\varpi_2 = (3, 2)$ we have the angle between ϖ_1 and ϖ_2 is an acute angle. In Figure 1(a), we see that $\Upsilon(t) = 0$ when $t = -1.4 < 0$ for $\varpi_1 = (5, 2)$ and $\varpi_2 = (3, 2)$. This indicates that the angle between ϖ_1 and ϖ_2 is an acute angle, and the value of the angle is approximately $10°$.

(b) Taking $\varpi_1 = (5, -2)$ and $\varpi_2 = (-3, 3)$ we have the angle between ϖ_1 and ϖ_2 is obtuse angle. In Figure 1(b), we see that $\Upsilon(t) = 0$ when $t = 0.5 > 0$ for $\varpi_1 = (5, -2)$ and $\varpi_2 = (-3, 3)$. This indicates that the angle between ϖ_1 and ϖ_2 is an obtuse angle, and the value of the angle is approximately $162°$.

(c) *Taking $\varpi_1 = (3, 0)$ and $\varpi_2 = (0, 2)$ we have the angle between ϖ_1 and ϖ_2 is a right angle. In Figure 1(c), we see that $\Upsilon(t) = 0$ when $t = 0$ for $\varpi_1 = (3, 0)$ and $\varpi_2 = (0, 2)$. This indicates that the angle between ϖ_1 and ϖ_2 is a right angle.*



(a) Acute Angle

(b) Obtuse Angle

(c) Right Angle

Figure 1: Plot of $\Upsilon(t)$ vs t of Example 2.21 (a), (b), (c)

Now, we discuss the relationship between the angle function $\angle_{\mathcal{S}_*}$ and other renowned angles present in the literature.

First, we obtain that in an inner product space, the angle function $\angle_{\mathcal{S}_*}$ corresponds to the Euclidean angle.

Theorem 2.22. *Assume that $(X, \langle ., . \rangle)$ is an inner product space with $\langle ., . \rangle$ and $\|.\|$ is the generated norm. Then $\angle_{\mathcal{S}_*}(\varpi_1, \varpi_2) = \angle_{Euclid}(\varpi_1, \varpi_2)$, for all non-zero linearly independent vectors $\varpi_1, \varpi_2 \in X$.*

Proof. For all non-zero linearly independent vectors $\varpi_1, \varpi_2 \in X$, we have

$$\begin{aligned}
&\frac{(1 + \|\varpi_1\|)(1 + \|\varpi_2\|)}{4\|\varpi_1\|\|\varpi_2\|}\left[\left\|\frac{\varpi_1}{1 + \|\varpi_1\|} + \frac{\varpi_2}{1 + \|\varpi_2\|}\right\|^2 - \left\|\frac{\varpi_1}{1 + \|\varpi_1\|} - \frac{\varpi_2}{1 + \|\varpi_2\|}\right\|^2\right] \\
&= \frac{(1 + \|\varpi_1\|)(1 + \|\varpi_2\|)}{4\|\varpi_1\|\|\varpi_2\|} \times 4\left\langle \frac{\varpi_1}{1 + \|\varpi_1\|}, \frac{\varpi_2}{1 + \|\varpi_2\|}\right\rangle \\
&= \frac{(1 + \|\varpi_1\|)(1 + \|\varpi_2\|)}{4\|\varpi_1\|\|\varpi_2\|} \times \frac{4\langle \varpi_1, \varpi_2\rangle}{(1 + \|\varpi_1\|)(1 + \|\varpi_2\|)} \\
&= \frac{\langle \varpi_1, \varpi_2\rangle}{\|\varpi_1\|\|\varpi_2\|}.
\end{aligned}$$

Therefore, $\angle_{\mathcal{S}_*}(\varpi_1, \varpi_2) = \angle_{Euclid}(\varpi_1, \varpi_2)$ for all linearly independent vectors $\varpi_1, \varpi_2 \neq 0$. □

Now, we discuss the relationship between angle ($\angle_{\mathcal{S}_*}$), isosceles angle ($\angle_I$) and Thy angle ($\angle_{Thy}$).

It is interesting to note that for the L.I. unit vectors $\varpi_1, \varpi_2 \in X$, $\angle_{\mathcal{S}_*}(\varpi_1, \varpi_2) = \angle_I(\varpi_1, \varpi_2) = \angle_{Thy}(\varpi_1, \varpi_2)$.

Now, we give an example of non-zero linearly independent vectors ϖ_1, ϖ_2 such that $\angle_{\mathcal{S}_*}(\varpi_1, \varpi_2) \neq \angle_I(\varpi_1, \varpi_2) \neq \angle_{Thy}(\varpi_1, \varpi_2)$.

Example 2.23. *We consider $X = \mathbb{R}^2$ equipped with a max norm. Choose $\varpi_1 = (1,2) \in X$ and $\varpi_2 = (3,-1) \in X$ such that $\|\varpi_1\| = 2$ and $\|\varpi_2\| = 3$. Consider*

$$\begin{aligned}
&\angle_{\mathcal{S}_*}(\varpi_1, \varpi_2)\\
&= \arccos \frac{(1+\|\varpi_1\|)(1+\|\varpi_2\|)}{4\|\varpi_1\|\|\varpi_2\|}\left[\left\|\frac{\varpi_1}{1+\|\varpi_1\|}+\frac{\varpi_2}{1+\|\varpi_2\|}\right\|^2-\left\|\frac{\varpi_1}{1+\|\varpi_1\|}-\frac{\varpi_2}{1+\|\varpi_2\|}\right\|^2\right]\\
&= \arccos \frac{(1+2)(1+3)}{4\times 2\times 3}\left[\left\|\frac{1}{3}(1,2)+\frac{1}{4}(3,-1)\right\|^2-\left\|\frac{1}{3}(1,2)-\frac{1}{4}(3,-1)\right\|^2\right]\\
&= \arccos \frac{1}{2}\times\frac{1}{12\times 12}\left[\|(13,5)\|^2-\|(-5,11)\|^2\right]\\
&= \arccos \frac{13^2-11^2}{2\times 144}\\
&= \arccos\left(\frac{1}{6}\right)\\
&= \frac{4\pi}{9}.
\end{aligned}$$

$$\begin{aligned}
\angle_I(\varpi_1, \varpi_2) &= \arccos \frac{\|\varpi_1+\varpi_2\|^2-\|\varpi_1-\varpi_2\|^2}{4\|\varpi_1\|\|\varpi_2\|}\\
&= \arccos \frac{\|(1,2)+(3,-1)\|^2-\|(1,2)-(3,-1)\|^2}{4\times 2\times 3}\\
&= \arccos \frac{\|(4,1)\|^2-\|(-2,3)\|^2}{4\times 2\times 3}\\
&= \arccos \frac{16-9}{24}\\
&= \arccos\left(\frac{7}{24}\right)\\
&= \frac{73\pi}{180}.
\end{aligned}$$

$$\begin{aligned}
\angle_{Thy}(\varpi_1, \varpi_2) &= \arccos \frac{1}{4}\left[\left\|\frac{\varpi_1}{\|\varpi_1\|}+\frac{\varpi_2}{\|\varpi_2\|}\right\|^2-\left\|\frac{\varpi_1}{\|\varpi_1\|}-\frac{\varpi_2}{\|\varpi_2\|}\right\|^2\right]\\
&= \arccos \frac{1}{4}\left[\left\|\frac{1}{2}(1,2)+\frac{1}{3}(3,-1)\right\|^2-\left\|\frac{1}{2}(1,2)-\frac{1}{3}(3,-1)\right\|^2\right]\\
&= \arccos \frac{1}{4\times 36}\left[\|(9,4)\|^2-\|(-3,8)\|^2\right]\\
&= \arccos \frac{81-64}{4\times 36}\\
&= \arccos\left(\frac{17}{4\times 36}\right)\\
&= \frac{83\pi}{180}.
\end{aligned}$$

$$
\begin{aligned}
\angle_P(\varpi_1, \varpi_2) &= \arccos\left(\frac{\|\varpi_1\|^2 + \|\varpi_2\|^2 - \|\varpi_1 - \varpi_2\|^2}{2\|\varpi_1\|\|\varpi_2\|}\right) \\
&= \arccos\left(\frac{\|(1,2)\|^2 + \|(3,-1)\|^2 - \|(1,2)-(3,1)\|^2}{2\|(1,2)\|\|(3,-1)\|}\right) \\
&= \arccos\left(\frac{4+9-4}{2\times 2\times 3}\right) \\
&= \arccos\left(\frac{3}{4}\right) \\
&= \frac{41\pi}{180}.
\end{aligned}
$$

Therefore, $\angle_{\mathcal{S}_*}(\varpi_1, \varpi_2) \neq \angle_I(\varpi_1, \varpi_2) \neq \angle_{Thy}(\varpi_1, \varpi_2) \neq \angle_P(\varpi_1, \varpi_2)$ *non-zero linearly independent vectors* ϖ_1, ϖ_2.

3. Conclusion

In conclusion, the paper presents a novel concept of angular function orthogonality in normed spaces, exploring its properties and geometrical implications. It establishes a connection between this new orthogonality and standard orthogonality in inner product spaces. We present a well-defined angle using the angular distance function and analyze its geometric features, classifying angles as acute, obtuse, or right in normed spaces. We demonstrate our conclusions with several non-trivial cases and describe how this angle relates to other angles.

We have the following open question from our observation:
" Is it possible to connect two non-zero angular function orthogonal vectors through a third non-zero vector, which is an angular function orthogonal to both?"

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