



The inhomogeneous Calderón–Zygmund operators on weak local Hardy spaces associated with ball quasi-Banach function spaces

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Abstract. Let X be a ball quasi-Banach function space on $\mathbb{R}^n$, WX be the weak ball quasi-Banach function space on $\mathbb{R}^n$, h_X be the local Hardy space associated with X . In this paper, we introduce the weak local Hardy-type space Wh_X associated with X via using maximal function characterization. Moreover, we obtain the boundedness of inhomogeneous Calderón–Zygmund operators from h_X to WX or Wh_X . All these results have a wide range of generality and, particularly, to our best knowledge, even when they are applied to the Morrey spaces, Orlicz-slice spaces and mixed-norm Lebesgue spaces, the results in this paper are also new.

1. Introduction

The theory of local Hardy space plays an important role in various fields of analysis and partial differential equations, see [11, 26]. Moreover, various variants of Hardy spaces on $\mathbb{R}^n$ and their real-variable theories have been developed (see, for instance, [1, 22]). Recently, to extend the Banach function space further so that Morrey spaces, Orlicz-slice spaces and mixed-norm Lebesgue spaces are included in a unified framework, Sawano et al. [28] introduced the ball quasi-Banach function space X on $\mathbb{R}^n$ and also investigated the Hardy space H_X via the grand maximal function. Besides, the local Hardy space h_X was also introduced in [28], and the relation between H_X and h_X was established. For more details on Hardy spaces associated with ball quasi-Banach function spaces, we refer to [19, 20, 32, 33, 35].

On the other hand, the theory of singular integral operators began in the 1950s when Calderón and Zygmund studied convolution operators that appeared in elliptic partial differential equations with constant coefficients. In order to find the biggest function space A such that Calderón–Zygmund operators are bounded from A to WL^1 , the weak H^1 space theory was first introduced by Fefferman and Soria in [8]. Then the weak H^p ($0 < p < 1$) space theory was studied by Liu in [18]. Very recently, Zhang et al. [38] introduce the weak Hardy-type space and obtain the boundedness of Calderón–Zygmund operators from the Hardy

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space H_X to WH_X , which includes the critical case. Also see [27] for the case on spaces of homogeneous type. Meanwhile, Tan [29] proved that the inhomogeneous Calderón–Zygmund operator is bounded on local variable Hardy spaces $h^{p(\cdot)}$ via establishing the finite atomic decomposition for the local variable Hardy space. Inspired by these results, we will focus on the boundedness of inhomogeneous Calderón–Zygmund operators from h_X to WX and from h_X to Wh_X , which includes the critical case.

Recall that, the inhomogeneous Calderón–Zygmund operator was considered in [7]. Define $\mathcal{D}$ to be the space of all smooth functions with compact support. The operator T is said to be an inhomogeneous Calderón–Zygmund operator if T is bounded on L^2 , and T is a continuous linear operator from $\mathcal{D}$ to $\mathcal{D}'$ defined

$$\langle T(f), g \rangle = \int \mathcal{K}(x, y) f(y) g(x) dx dy$$

for all $f, g \in \mathcal{D}$ with disjoint supports, where $\mathcal{K}(x, y)$, the kernel of T , satisfies the conditions as follows:

$$|\mathcal{K}(x, y)| \leq C \min \left\{ \frac{1}{|x - y|^n}, \frac{1}{|x - y|^{n+\delta}} \right\}, \text{ for some } \delta > 0 \text{ and } x \neq y$$

and for $\epsilon \in (0, 1)$

$$|\mathcal{K}(x, y) - \mathcal{K}(x, y')| + |\mathcal{K}(y, x) - \mathcal{K}(y', x)| \leq C \frac{|y - y'|^\epsilon}{|x - y|^{n+\epsilon}}$$

when $|y - y'| \leq \frac{1}{2}|x - y|$.

The main purpose of this paper is threefold. The first goal is to introduce the weak local Hardy-type space Wh_X via the radial maximal function. The second goal is to obtain the boundedness of inhomogeneous Calderón–Zygmund operators from h_X to WX or Wh_X . Last but not least, we will apply these results to some concrete function spaces, such as Morrey spaces, Orlicz-slice spaces and mixed-norm Lebesgue spaces.

Now we state the main results of the paper as follows.

Theorem 1.1. *Let $0 < \theta < s \leq 1$, $q \in (1, \infty)$, $\delta > 0$ and $\epsilon \in (0, 1)$. Assume that X is a ball quasi-Banach function space satisfying Assumptions 2.12, 2.13 and 2.14. Assume that $X^{1/s}$ is a ball Banach function space. Let T be an inhomogeneous Calderón–Zygmund operator. If $\theta \in [\max\{\frac{n}{n+\delta}, \frac{n}{n+\epsilon}\}, 1)$ and there exists a positive constant C_0 such that for any $\alpha \in (0, \infty)$ and any sequence $\{f_j\}_{j \in \mathbb{N}} \subset \mathcal{M}$,*

$$\begin{aligned} & \alpha \left\| \chi_{\left\{x \in \mathbb{R}^n : \left\{ \sum_{j \in \mathbb{N}} [M(f_j)(x)]^{\max\{\frac{1}{n+\delta}, \frac{n}{n+\epsilon}\}} \right\}^{\max\{\frac{n}{n+\delta}, \frac{n}{n+\epsilon}\}} > \alpha \right\}} \right\|_{X^{\max\{\frac{1}{n+\delta}, \frac{n}{n+\epsilon}\}}} \\ & \leq C_0 \left\| \left(\sum_{j \in \mathbb{N}} |f_j|^{\max\{\frac{1}{n+\delta}, \frac{n}{n+\epsilon}\}} \right)^{\max\{\frac{n}{n+\delta}, \frac{n}{n+\epsilon}\}} \right\|_{X^{\max\{\frac{1}{n+\delta}, \frac{n}{n+\epsilon}\}}}, \end{aligned} \quad (1)$$

then T has a unique extension on h_X . Moreover, there exists a positive constant C such that for any $f \in h_X$,

$$\|Tf\|_{WX} \leq C \|f\|_{h_X}.$$

To state the (h_X, Wh_X) -boundedness of T , we also need to assume one additional condition on T , $\int_{\mathbb{R}^n} T(a)(x) dx = 0$ for the local (X, q, d) -atom a and $\text{supp } a \subset Q$ with $|Q| < 1$. For convenience, we write $T_{loc}^*(1) = 0$, if T satisfies the above moment condition.

Theorem 1.2. *Let $0 < \theta < s \leq 1$, $q \in (1, \infty)$ and $\epsilon \in (0, 1)$. Assume that X is a ball quasi-Banach function space satisfying Assumptions 2.12, 2.13 and 2.14. Assume that $X^{1/s}$ is a ball Banach function space. Let T be an inhomogeneous Calderón–Zygmund operator and $T_{loc}^*(1) = 0$. If $\theta \in [\max\{\frac{n}{n+\delta}, \frac{n}{n+\epsilon}\}, 1)$ and there exists a positive*

constant C_0 such that for any $\alpha \in (0, \infty)$ and any sequence $\{f_j\}_{j \in \mathbb{N}} \subset \mathcal{M}$, (1) holds true, then T has a unique extension on h_X . Moreover, there exists a positive constant C such that for any $f \in h_X$,

$$\|Tf\|_{Wh_X} \leq C\|f\|_{h_X}.$$

2. Preliminaries

In this section, we present some notions and known results concerning the ball quasi-Banach function spaces and the weak ball quasi-Banach function spaces. Then we give some assumptions on the Hardy–Littlewood maximal operator. Meanwhile, we recall the definition and some results of local Hardy type spaces h_X .

2.1. Weak ball quasi-Banach function spaces

Denote by $\mathcal{M}$ the set of all measurable functions on $\mathbb{R}^n$. Before presenting the notion of weak ball quasi-Banach function spaces, we first recall the concepts of Banach function spaces and ball quasi-Banach function spaces. For the sake of simplicity, $Y(\mathbb{R}^n) =: Y$, where Y is any (quasi-)Banach function spaces or ball (quasi-)Banach function spaces on $\mathbb{R}^n$.

Definition 2.1. [3, Chapter 1, Definitions 1.1 and 1.3] A Banach function space $Y \subset \mathcal{M}$ is called a Banach function space if it satisfies (i) $\|f\|_Y = 0$ if and only if $f = 0$ almost everywhere;

(ii) $|g| \leq |f|$ almost everywhere implies that $\|g\|_Y \leq \|f\|_Y$;

(iii) $0 \leq f_m \uparrow f$ almost everywhere implies that $\|f_m\|_Y \uparrow \|f\|_Y$;

(iv) $\chi_E \in Y$ for any measurable set $E \subset \mathbb{R}^n$ with finite measure;

(v) for any measurable set $E \subset \mathbb{R}^n$ with finite measure, there exists a positive constant $C(E)$, depending on E , such that, for all $f \in Y$,

$$\int_E |f(x)| dx \leq C(E) \|f\|_Y,$$

where f and g are measurable functions. In fact, the above conditions ensure that the norm $\|\cdot\|_Y$ is a Banach function norm. For $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, let $B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$, and

$$\mathbb{B} := \{B(x, r) : x \in \mathbb{R}^n \text{ and } r \in (0, \infty)\}. \quad (2)$$

We now present the notion of ball quasi-Banach function spaces as follows [28].

Definition 2.2. A quasi-Banach space $X \subset \mathcal{M}$ is called a ball quasi-Banach function space if it satisfies

(i) $\|f\|_X = 0$ implies that $f = 0$ almost everywhere;

(ii) $|g| \leq |f|$ almost everywhere implies that $\|g\|_X \leq \|f\|_X$;

(iii) $0 \leq f_m \uparrow f$ almost everywhere implies that $\|f_m\|_X \uparrow \|f\|_X$;

(iv) $B \in \mathbb{B}$ implies that $\chi_B \in X$, where $\mathbb{B}$ is as in (2).

For any ball Banach function space X , the associate space (Köthe dual) X' is defined by setting

$$X' := \{f \in \mathcal{M} : \|f\|_{X'} := \sup \{\|fg\|_{L^1} : g \in X, \|g\|_X = 1\} < \infty\}, \quad (3)$$

where $\|\cdot\|_{X'}$ is called the associate norm of $\|\cdot\|_X$.

Remark 2.3. The second condition of the Banach function space X is called the ideal property, which implies that X is a quasi-Banach lattice [23]. From [28, Proposition 2.3], we know that, if X is a ball Banach function space, then its associate space X' is also a ball Banach function space.

Then we recall the notions of the convexity and the concavity of ball quasi-Banach function spaces.

Definition 2.4. [17, Definition 1.d.3] Let X be a ball quasi-Banach function space and $p \in (0, \infty)$. (i) The p -convexification X^p of X is defined by setting $X^p := \{f \in \mathcal{M} : |f|^p \in X\}$ equipped with the quasi-norm

$$\|f\|_{X^p} := \left\| |f|^p \right\|_X^{1/p}.$$

(ii) The space X is said to be p -concave if there exists a positive constant C such that for any sequence $\{f_j\}_{j \in \mathbb{N}}$ of $X^{1/p}$,

$$\sum_{j \in \mathbb{N}} \|f_j\|_{X^{1/p}} \leq C \left\| \sum_{j \in \mathbb{N}} |f_j| \right\|_{X^{1/p}}.$$

Particularly, X is said to be strictly p -concave when $C = 1$. Next we present the notion of weak ball quasi-Banach function spaces as follows [38, Definition 2.8 and Remark 2.9].

Definition 2.5. Let X be a ball quasi-Banach function space. The weak ball quasi-Banach function space WX is defined to be the set of all measurable functions f satisfying

$$\|f\|_{WX} := \sup_{\alpha \in (0, \infty)} \left[\alpha \left\| \chi_{\{x \in \mathbb{R}^n : |f(x)| > \alpha\}} \right\|_X \right] < \infty. \quad (4)$$

Remark 2.6. (i) Let X be a ball quasi-Banach function space. For any $f \in X$ and $\alpha \in (0, \infty)$, we have $\chi_{\{x \in \mathbb{R}^n : |f(x)| > \alpha\}}(x) \leq |f(x)|/\alpha$ for any $x \in \mathbb{R}^n$, which, together with Definition 2.2 (ii), further implies that

$$\sup_{\alpha \in (0, \infty)} \left[\alpha \left\| \chi_{\{x \in \mathbb{R}^n : |f(x)| > \alpha\}} \right\|_X \right] \leq \|f\|_X.$$

This shows that $X \subset WX$. (ii) Let $f, g \in WX$ with $|f| \leq |g|$. By Definition 2.2 (ii), we conclude that $\|f\|_{WX} \leq \|g\|_{WX}$.

Lemma 2.7. [38, Lemma 2.10] Let X be a ball quasi-Banach function space. Then $\|\cdot\|_{WX}$ is a quasi-norm on WX , namely,

- (i) $\|f\|_{WX} = 0$ if and only if $f = 0$ almost everywhere.
- (ii) For any $\lambda \in \mathbb{C}$ and $f \in WX$,

$$\|\lambda f\|_{WX} = |\lambda| \|f\|_{WX}.$$

- (iii) For any $f, g \in WX$, there exists a positive constant C such that

$$\|f + g\|_{WX} \leq C [\|f\|_{WX} + \|g\|_{WX}].$$

Moreover, if $p \in (0, \infty)$ and $X^{1/p}$ is a ball Banach function space, then

$$\|f + g\|_{WX}^{1/p} \leq 2^{\max\{1/p, 1\}} [\|f\|_{WX}^{1/p} + \|g\|_{WX}^{1/p}].$$

Remark 2.8. Let X be a ball quasi-Banach function space. By [38, Lemma 2.13], we know that WX is also a ball quasi-Banach function space. For any given $s \in (0, \infty)$, it is easy to show that X^s is also a ball quasi-Banach function space. Thus, $(WX)^s$ and $W(X^s)$ make sense and coincide with equal quasi-norms. Indeed, for any $f \in (WX)^s$, by Definitions 2.4 (i) and 2.5, we have

$$\|f\|_{(WX)^s}^s = \| |f|^s \|_{WX} = \|f\|_{W(X^s)}^s.$$

Let X be a ball quasi-Banach function space, then from [38, Lemma 2.13] we know the space WX is also a ball quasi-Banach function space.

Now, we recall the notions of Muckenhoupt weights A_p in [12].

Definition 2.9. An A_p weight ω , with $p \in [1, \infty)$, is a locally integrable and non-negative function on $\mathbb{R}^n$ satisfying that, when $p \in (1, \infty)$,

$$\sup_{B \in \mathbb{B}} \left[\frac{1}{|B|} \int_B \omega(x) dx \right] \left[\frac{1}{|B|} \int_B \{\omega(x)\}^{\frac{1}{1-p}} dx \right]^{p-1} < \infty,$$

and, when $p = 1$,

$$\sup_{B \in \mathbb{B}} \frac{1}{|B|} \int_B \omega(x) dx \left[\|\omega^{-1}\|_{L^\infty(B)} \right] < \infty,$$

where $\mathbb{B}$ is as in (2). Define $A_\infty := \bigcup_{p \in [1, \infty)} A_p$.

Definition 2.10. Let $p \in (0, \infty)$ and $\omega \in A_\infty$. The weighted Lebesgue space L_ω^p is defined to be the set of all measurable functions f such that

$$\|f\|_{L_\omega^p} := \left[\int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx \right]^{\frac{1}{p}} < \infty.$$

Lemma 2.11. [38, Lemma 2.17] Let X be a ball quasi-Banach function space. Assume that there exists an $s \in (0, \infty)$ such that $X^{1/s}$ is a ball Banach function space and M is bounded on $(X^{1/s})'$. Then there exists an $\epsilon \in (0, 1)$ such that X continuously embeds into L_ω^s with $\omega := \left[M \left(\chi_{B(\vec{0}_n, 1)} \right) \right]^\epsilon$, namely, there exists a positive constant C such that for any $f \in X$,

$$\|f\|_{L_\omega^s} \leq C \|f\|_X.$$

2.2. Assumptions on the Hardy–Littlewood maximal operator

Denote by L_{loc}^1 the set of all locally integrable functions on $\mathbb{R}^n$. Recall that the Hardy–Littlewood maximal operator M is defined by setting, for all $f \in L_{\text{loc}}^1$ and $x \in \mathbb{R}^n$,

$$Mf(x) := \sup_{r \in (0, \infty)} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy. \quad (5)$$

For any $\theta \in (0, \infty)$, the powered Hardy–Littlewood maximal operator $M^{(\theta)}$ is defined by setting, for all $f \in L_{\text{loc}}^1$ and $x \in \mathbb{R}^n$,

$$M^{(\theta)}(f)(x) := \left\{ M(|f|^\theta)(x) \right\}^{1/\theta}.$$

In order to prove several theorems in this paper, we need the following assumptions.

Assumption 2.12. Let X be ball quasi-Banach function space. For some $\theta, s \in (0, 1]$ and $\theta < s$, there exists a positive constant C such that, for any $\{f_j\}_{j=1}^\infty \subset \mathcal{M}$,

$$\left\| \left\{ \sum_{j=1}^\infty [M^{(\theta)}(f_j)]^s \right\}^{1/s} \right\|_X \leq C \left\| \left\{ \sum_{j=1}^\infty |f_j|^s \right\}^{1/s} \right\|_X. \quad (6)$$

Assumption 2.13. Let X be a ball quasi-Banach function space satisfying (6) for some $\theta, s \in (0, 1]$. Let $d \geq \lfloor n(1/\theta - 1) \rfloor$ be a fixed integer and $q \in (1, \infty]$. Assume that for any $f \in \mathcal{M}$,

$$\left\| M^{((q/s)')} (f) \right\|_{(X^{1/s})'} \leq C \|f\|_{(X^{1/s})'}, \quad (7)$$

where the implicit positive constant is independent of f .

Assumption 2.14. Let X be a ball quasi-Banach function space. Assume that there exists an $r \in (0, \infty)$ such that M in (5) is bounded on $(WX)^{1/r}$.

Now we recall the definition of the local Hardy type space h_X established in [28].

Definition 2.15. Let X be a ball quasi-Banach function space. Let $\Phi \in \mathcal{S}$ satisfy $\int_{\mathbb{R}^n} \Phi(x) dx \neq 0$ and $b \in (0, \infty)$ sufficiently large. Then the Hardy space h_X associated with X is defined as

$$h_X := \{f \in \mathcal{S}' : \|f\|_{h_X} := \|m_b^{**}(f, \Phi)\|_X < \infty\},$$

where the maximal function $m_b^{**}(f, \Phi)$ of Peetre type is defined by setting, for all $x \in \mathbb{R}^n$,

$$m_b^{**}(f, \Phi)(x) := \sup_{(y,t) \in \mathbb{R}^n \times (0,1)} \frac{|(\Phi_t * f)(x - y)|}{(1 + t^{-1}|y|)^b}.$$

Moreover, the atomic decomposition theory is very useful when we consider the boundedness of operators on Hardy spaces. For instance, see [9, 30, 31, 36]. Then we recall the definition of the local- (X, q, d) -atom in [34]. Denote that

$$d_X := \lceil n(1/\theta - 1) \rceil. \quad (8)$$

Definition 2.16. Let X be a ball quasi-Banach function space and $q \in (1, \infty]$. Assume that $d \in \mathbb{Z}_+$ fulfills $d \geq d_X$. Then a measurable function a is called a local- (X, q, d) -atom if there exists a cube $Q \subset \mathbb{R}^n$ such that $\text{supp}(a) \subset Q$; $\|a\|_{L^q} \leq \frac{|Q|^{1/q}}{\|\chi_Q\|_X}$ and if $|Q| < 1$, then $\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0$ for any multi-index $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq d$.

Lemma 2.17. [34, Theorem 4.8] Let X be a ball quasi-Banach function space, satisfying both Assumption 2.12 with $0 < \theta < s \leq 1$ and Assumption 2.13 with the same $s \in (0, 1]$, and $d \in \mathbb{N}$ such that $d \geq d_X$, where d_X is as in (8). Then $f \in h_X$ if and only if $f \in \mathcal{S}'$ and there exist a sequence $\{a_j\}_{j=1}^\infty$ of local- (X, ∞, d) -atoms supported, respectively, in cubes $\{Q_j\}_{j=1}^\infty$ and a sequence $\{\lambda_j\}_{j=1}^\infty$ of non-negative numbers such that

$$f = \sum_{j=1}^\infty \lambda_j a_j \quad \text{in } \mathcal{S}' \quad (9)$$

and

$$\left\| \left[\sum_{j=1}^\infty \left(\frac{\lambda_j}{\|\chi_{Q_j}\|_X} \right)^s \chi_{Q_j} \right]^{1/s} \right\|_X < \infty.$$

Moreover,

$$\|f\|_{h_X} \sim \inf \left\{ \left\| \left[\sum_{j=1}^\infty \left(\frac{\lambda_j}{\|\chi_{Q_j}\|_X} \right)^s \chi_{Q_j} \right]^{1/s} \right\|_X \right\},$$

where the infimum is taken over all decompositions of f as in (9) and the positive equivalence constants are independent of f but may depend on s .

Lemma 2.18. [34, Lemma 2.21] Assume that X is a ball quasi-Banach function space satisfying Assumption 2.12 with some $s \in (0, 1]$ and Assumption 2.13 for some $q \in (1, \infty]$ and the same $s \in (0, 1]$ as in (6). Let $\{a_j\}_{j=1}^\infty \subset L^q$ be supported, respectively, in cubes $\{Q_j\}_{j=1}^\infty$, and a sequence $\{\lambda_j\}_{j=1}^\infty \subset [0, \infty)$ such that, for any $j \in \mathbb{N}$,

$$\|a_j\|_{L^q} \leq \frac{|Q_j|^{1/q}}{\|\chi_{Q_j}\|_X}$$

and

$$\left\| \left\{ \sum_{j=1}^{\infty} \left(\frac{\lambda_j}{\|\chi_{Q_j}\|_X} \right)^s \chi_{Q_j} \right\}^{1/s} \right\|_X < \infty.$$

Then $f = \sum_{j=1}^{\infty} \lambda_j a_j$ converges in S' and there exists a positive constant C , independent of f , such that

$$\|f\|_X \leq C \left\| \left\{ \sum_{j=1}^{\infty} \left(\frac{\lambda_j}{\|\chi_{Q_j}\|_X} \right)^s \chi_{Q_j} \right\}^{1/s} \right\|_X.$$

3. Weak local Hardy spaces and their maximal characterizations

In this section, we introduce the weak local Hardy type space Wh_X , associated with the ball quasi-Banach space X , and obtain several maximal function characterizations. To state the results, we need some definitions. In what follows, we denote by $\mathcal{S}$ the space of all Schwartz functions, equipped with the well-known topology determined by a countable family of seminorms, and by $\mathcal{S}'$ its topological dual space, equipped with the weak-* topology. For any $N \in \mathbb{N}$, let

$$\mathcal{F}_N := \left\{ \varphi \in \mathcal{S} : \sum_{\beta \in \mathbb{Z}_+^n, |\beta| \leq N} \sup_{x \in \mathbb{R}^n} \left[(1 + |x|)^{N+n} \left| \partial_x^\beta \varphi(x) \right| \right] \leq 1 \right\},$$

here and hereafter, for any $\beta := (\beta_1, \dots, \beta_n) \in \mathbb{Z}_+^n$ and $x \in \mathbb{R}^n$, $|\beta| := \beta_1 + \dots + \beta_n$ and $\partial_x^\beta := \left(\frac{\partial}{\partial x_1} \right)^{\beta_1} \dots \left(\frac{\partial}{\partial x_n} \right)^{\beta_n}$.

First, we recall the local vertical, non-tangential grand maximal functions as follows.

Definition 3.1. Let $f \in \mathcal{S}'$, $N \in \mathbb{N}$, $a, b \in (0, \infty)$ and $\Phi \in \mathcal{S}$.

(i) The local radial maximal function $m(f, \Phi)$ is defined by setting, for all $x \in \mathbb{R}^n$,

$$m(f, \Phi)(x) := \sup_{t \in (0,1)} |(\Phi_t * f)(x)|.$$

(ii) The local grand maximal function $m_N(f)$ is defined by setting, for all $x \in \mathbb{R}^n$,

$$m_N(f)(x) := \sup \left\{ \left| t^{-n} \psi(t^{-1}) * f(y) \right| : t \in (0,1), |x - y| < t, \psi \in \mathcal{F}_N \right\}. \quad (10)$$

(iii) The local non-tangential maximal function $m_a^*(f, \Phi)$, with aperture $a \in (0, \infty)$, is defined by setting, for all $x \in \mathbb{R}^n$,

$$m_a^*(f, \Phi)(x) := \sup_{t \in (0,1)} \left\{ \sup_{y \in \mathbb{R}^n, |y-x| < at} |\Phi_t * f(y)| \right\}.$$

(iv) The local maximal function $m_b^{**}(f, \Phi)$ of Peetre type is defined by setting, for all $x \in \mathbb{R}^n$,

$$m_b^{**}(f, \Phi)(x) := \sup_{(y,t) \in \mathbb{R}^n \times (0,1)} \frac{|(\Phi_t * f)(x - y)|}{(1 + t^{-1}|y|)^b}.$$

(v) The local grand maximal function $m_{b,N}^{**}(f)$ of Peetre type is defined by setting, for all $x \in \mathbb{R}^n$,

$$m_{b,N}^{**}(f)(x) := \sup_{\psi \in \mathcal{F}_N} \left\{ \sup_{(y,t) \in \mathbb{R}^n \times (0,1)} \frac{|(\psi_t * f)(x - y)|}{(1 + t^{-1}|y|)^b} \right\}.$$

Definition 3.2. Let X be a ball quasi-Banach function space. Then the weak Hardy-type space Wh_X associated with X is defined by setting

$$Wh_X := \{f \in \mathcal{S}' : \|f\|_{Wh_X} := \|m_N(f)\|_{WX} < \infty\},$$

where $m_N(f)$ is as in (10) with $N \in \mathbb{N}$ sufficiently large.

Then we have the following estimates, which further give several maximal function characterizations of the space Wh_X .

Theorem 3.3. Let $a, b \in (0, \infty)$ and X be a ball quasi-Banach function space. Let $\Phi \in \mathcal{S}$ satisfy $\int_{\mathbb{R}^n} \Phi(x) dx \neq 0$. (i) Let $N \geq \lfloor b + 2 \rfloor$ be an integer. Then, for any $f \in \mathcal{S}'$,

$$\|m(f, \Phi)\|_{WX} \lesssim \|m_a^*(f, \Phi)\|_{WX} \lesssim \|m_b^{**}(f, \Phi)\|_{WX}, \quad (11)$$

$$\|m(f, \Phi)\|_{WX} \lesssim \|m_N(f)\|_{WX} \lesssim \|m_{\lfloor b+2 \rfloor}(f)\|_{WX} \lesssim \|m_b^{**}(f, \Phi)\|_{WX}, \quad (12)$$

$$\|m_b^{**}(f, \Phi)\|_{WX} \sim \|m_{b,N}^{**}(f)\|_{WX}, \quad (13)$$

where the implicit positive constants are independent of f .

(ii) Let $r, b, A \in (0, \infty)$ satisfy

$$(b - A)r > n. \quad (14)$$

From Remark 2.8, if X is strictly r -convex and for all $f \in WX$,

$$\left\| \left\{ \int_{[0,1]^n} |f(\cdot - z)|^r dz \right\}^{1/r} \right\|_{WX} \lesssim (1 + |z|)^A \|f\|_{WX}, \quad (15)$$

where the implicit positive constant is independent of f , then, for all $f \in \mathcal{S}'$,

$$\|m_{b,N}^{**}(f)\|_{WX} \lesssim \|m(f, \Phi)\|_{WX},$$

where the implicit positive constant is independent of f . In particular, when $N \in \mathbb{N} \cap \lfloor b + 2 \rfloor, \infty)$, and one of the quantities

$$\|m(f, \Phi)\|_{WX}, \|m_a^*(f, \Phi)\|_{WX}, \|m_N(f)\|_{WX}, \|m_b^{**}(f, \Phi)\|_{WX}, \|m_{b,N}^{**}(f)\|_{WX}$$

is finite, then the other quantities are also finite and mutually equivalent with the implicit positive constants independent of f .

Remark 3.4. We point out that (15) holds true for any ball quasi-Banach function space X satisfying Remark 2.6 (ii) and Assumption 2.14. Indeed, for any given $r \in (0, \infty)$ and any $f \in X, x, z \in \mathbb{R}^n$,

$$\left\{ \int_{z+[0,1]^n} |f(x - y)|^r dy \right\}^{1/r} \lesssim (1 + |z|)^{\frac{n}{r}} M^{(r)}(f)(x),$$

where the implicit positive constants are independent of f, x and z , but may depend on n and r . From this and the assumption that X satisfies Remark 2.6 (ii) and Assumption 2.14, we deduce that, for any given $r := \theta$ and $A > \frac{n}{r}$ and any $f \in WX$ and $z \in \mathbb{R}^n$,

$$\left\| \left\{ \int_{z+[0,1]^n} |f(\cdot - y)|^r dy \right\}^{1/r} \right\|_{WX} \lesssim (1 + |z|)^{\frac{n}{r}} \|M^{(r)}(f)\|_{WX} \lesssim (1 + |z|)^A \|f\|_{WX},$$

where the implicit positive constants are independent of f and z , but may depend on n and r .

Proof of Theorem 3.3 The proof of this theorem is similar to that of [28, Theorem 5.3]. For the convenience of the reader, we present some details. Let $f \in \mathcal{S}'$. We first prove (i). From (i), (ii) and (iii) of Definition 3.1, it follows that, for any $x \in \mathbb{R}^n$,

$$m(f, \Phi)(x) \leq m_a^*(f, \Phi)(x) \lesssim m_b^{**}(f, \Phi)(x),$$

which, together with Remark 2.6 (ii), implies (11). Moreover, by (i) and (iv) of Definition 3.1 again, we have, for any $x \in \mathbb{R}^n$,

$$m(f, \Phi)(x) \lesssim m_N(f)(x) \lesssim m_{\lfloor b+1 \rfloor}(f)(x). \quad (16)$$

In addition, from the proof of [13, Theorem 2.1.4(d)], we deduce that, for any $x \in \mathbb{R}^n$,

$$m_{\lfloor b+1 \rfloor}(f)(x) \lesssim m_b^{**}(f, \Phi)(x),$$

which, together with (16) and Remark 2.6 (ii), implies (12). It is easy to see that, for any $x \in \mathbb{R}^n$, $m_b^{**}(f, \Phi)(x) \lesssim m_{b,N}^{**}(f)(x)$, which, combined with [28, Lemma 2.13], implies (13).

Now we prove (ii). Let $b, N \in (0, \infty)$ satisfy

$$Nr > n \quad \text{and} \quad (b - A)r > n. \quad (17)$$

Using [28, Lemma 2.8] with a dilation, we conclude that, for any $m \in \mathbb{N}$, $t \in [1, 2]$ and $x \in \mathbb{R}^n$,

$$\begin{aligned} \sup_{y \in \mathbb{R}^n} \frac{|\Phi_{2^{-m}t} * f(y)|^r}{(1 + 2^m|x - y|)^{br}} &\lesssim r \sum_{k=0}^{\infty} 2^{k(n-Nr)} \int_{\mathbb{R}^n} \frac{|\Phi_{2^{-k-m}t} * f(y)|^r}{(1 + 2^m|x - y|)^{br}} dy \\ &\lesssim r \sum_{k=0}^{\infty} 2^{k(n-Nr)} \int_{\mathbb{R}^n} \frac{|\Phi_{2^{-k-m}t} * f(y)|^r}{(1 + |x - y|)^{br}} dy \end{aligned}$$

which, since $m \in \mathbb{N}$ and $t \in [1, 2]$ are arbitrary, further implies that, for any $x \in \mathbb{R}^n$,

$$\begin{aligned} m_b^{**}(f, \Phi)(x) &\lesssim r \left\{ \sum_{k=0}^{\infty} 2^{k(n-Nr)} \int_{\mathbb{R}^n} \frac{[m(f, \Phi)(x - y)]^r}{(1 + |y|)^{br}} dy \right\}^{1/r} \\ &\sim \left\{ \sum_{k=0}^{\infty} 2^{k(n-Nr)} \sum_{m \in \mathbb{Z}^n} \int_{m+[0,1]^n} \frac{[m(f, \Phi)(x - y)]^r}{(1 + |m|)^{br}} dy \right\}^{1/r}. \end{aligned}$$

From this and the assumption that X is strictly r -convex, it follows that

$$\begin{aligned} \|m_b^{**}(f, \Phi)\|_{WX} &\lesssim_r \left\| \left\{ \sum_{k=0}^{\infty} 2^{k(n-Nr)} \sum_{m \in \mathbb{Z}^n} \int_{m+[0,1]^n} \frac{[m(f, \Phi)(\cdot - y)]^r}{(1 + |m|)^{br}} dy \right\}^{1/r} \right\|_{WX} \\ &\lesssim_r \left\| \sum_{k=0}^{\infty} 2^{k(n-Nr)} \sum_{m \in \mathbb{Z}^n} \int_{m+[0,1]^n} \frac{[m(f, \Phi)(\cdot - y)]^r}{(1 + |m|)^{br}} dy \right\|_{(WX)^{1/r}}^{1/r} \\ &\lesssim_r \left\{ \sum_{k=0}^{\infty} 2^{k(n-Nr)} \sum_{m \in \mathbb{Z}^n} \left\| \int_{m+[0,1]^n} \frac{[m(f, \Phi)(\cdot - y)]^r}{(1 + |m|)^{br}} dy \right\|_{(WX)^{1/r}} \right\}^{1/r} \\ &\sim_r \left\{ \sum_{k=0}^{\infty} 2^{k(n-Nr)} \sum_{m \in \mathbb{Z}^n} \left\| \left[\int_{m+[0,1]^n} \left[\frac{m(f, \Phi)(\cdot - y)}{(1 + |m|)^b} \right]^r dy \right]^{1/r} \right\|_{WX}^r \right\}^{1/r} \end{aligned}$$

which, combined with (14), (15) and (17), further implies that

$$\begin{aligned} \|m_b^{**}(f, \Phi)\|_{WX} &\lesssim \|m(f, \Phi)\|_{WX} \left\{ \sum_{k=0}^{\infty} 2^{k(n-Nr)} \sum_{m \in \mathbb{Z}^n} \frac{1}{(1+|m|)^{(b-A)r}} \right\}^{1/r} \\ &\sim \|m(f, \Phi)\|_{WX}. \end{aligned}$$

This finishes the proof of (ii), and hence of Theorem 3.3. $\square$

4. Proofs of Main Results

Proof of Theorem 1.1 Assume $f \in h_X$ and $d \in [d_X, \infty) \cap \mathbb{Z}_+$ is a fixed integer, where d_X is defined by $d_X := \lceil n(1/\theta - 1) \rceil$. Then from [34], we find that there exists a sequence $\{a_j\}_{j=1}^{\infty}$ of (X, ∞, d) -atoms supported, respectively, in a $\{Q_j\}_{j=1}^{\infty}$ of cubes, and a sequence $\{\lambda_j\}_{j=1}^{\infty}$ of non-negative numbers, independent of f but depending on s , such that

$$f = \sum_{j=1}^{\infty} \lambda_j a_j \quad \text{in } S', \quad (18)$$

and

$$\left\| \left\{ \sum_{j=1}^{\infty} \left(\frac{\lambda_j}{\|\chi_{Q_j}\|_X} \right)^s \chi_{Q_j} \right\} \right\|_X^{1/s} \lesssim \|f\|_{h_X}. \quad (19)$$

In addition, from Lemma 2.11, we find that there exists an $\varepsilon \in (0, 1)$ such that, for any $h \in \mathcal{M}$,

$$\|h\|_{L_v^{s_0}} \lesssim \|h\|_X$$

where $v := [\mathcal{M}(1_{Q(0,1)})]^\varepsilon$. Combining this, (18) and (19), we conclude that

$$f = \sum_{j \in \mathbb{N}} \lambda_j a_j = \sum_{j \in \mathbb{N}} \left[\lambda_j \frac{\|\chi_{Q_j}\|_{L_v^{s_0}}}{\|\chi_{Q_j}\|_X} \right] \left[\frac{\|\chi_{Q_j}\|_X}{\|\chi_{Q_j}\|_{L_v^{s_0}}} a_j \right], \quad (20)$$

in S' and

$$\begin{aligned} &\left\| \left\{ \sum_{j \in \mathbb{N}} \left[\frac{\lambda_j \|\chi_{Q_j}\|_{L_v^{s_0}}}{\|\chi_{Q_j}\|_X} \right]^s \chi_{Q_j} \right\} \right\|_{L_v^{s_0}}^{1/s} \\ &\lesssim \left\| \left\{ \sum_{j \in \mathbb{N}} \left(\frac{\lambda_j}{\|\chi_{Q_j}\|_X} \right)^s \chi_{Q_j} \right\} \right\|_X^{1/s} \lesssim \|f\|_{h_X} < \infty. \end{aligned} \quad (21)$$

Observe that, for any $j \in \mathbb{N}$, $\frac{\|\chi_{Q_j}\|_X}{\|\chi_{Q_j}\|_{L_v^{s_0}}} a_j$ is an local- $(L_v^{s_0}, \infty, d)$ -atom supported in Q_j . This, together with (20), (21) and [5, Theorem 3.6] further implies that $f \in h_v^{s_0}$ and $f = \sum_{j \in \mathbb{N}} \lambda_j a_j$ in $h_v^{s_0}$. Applying this and [6, Theorem 1.9], we find that $T(f) = \sum_{j \in \mathbb{N}} \lambda_j T(a_j)$ holds true in $L_v^{s_0}$ and L^2 . Therefore, we further infer that

$$T(f) = \sum_{j \in \mathbb{N}} \lambda_j T(a_j) \quad \text{in } S'. \quad (22)$$

Then, to prove this theorem, by Definition 2.5, we only need to show that for any $f \in h_X$ and any $\alpha \in (0, \infty)$,

$$\alpha \left\| \chi_{\{x \in \mathbb{R}^n: |T(f)(x)| > \alpha\}} \right\|_X \lesssim \|f\|_{h_X}. \quad (23)$$

Moreover, we denote $Q(c_{Q_j}, l_j)$ the closed cube centered at x and of sidelength l_j . Similarly, given $Q = Q(c_{Q_j}, l_j)$ and $\lambda > 0$, λQ means that the cube with the same center c_{Q_j} and with sidelength λl_j . We denote $\tilde{Q} = 2\sqrt{n}Q$ simply. Then for any $\alpha \in (0, \infty)$, by (22), Lemma 2.7 (iii) and Remark 2.6 (i), we have

$$\begin{aligned} \alpha \left\| \chi_{\{x \in \mathbb{R}^n: |T(f)(x)| > \alpha\}} \right\|_X &= \alpha \left\| \chi_{\{x \in \mathbb{R}^n: |\sum_{j=1}^{\infty} \lambda_j T(a_j)(x)| > \alpha\}} \right\|_X \\ &\lesssim \alpha \left\| \chi_{\{x \in \mathbb{R}^n: |\sum_{j=1}^{\infty} \lambda_j T(a_j)(x) \chi_{2\sqrt{n}Q_j}| > \frac{\alpha}{2}\}} \right\|_X + \alpha \left\| \chi_{\{x \in \mathbb{R}^n: |\sum_{j=1}^{\infty} \lambda_j T(a_j)(x) \chi_{(2\sqrt{n}Q_j)^c}| > \frac{\alpha}{2}\}} \right\|_X \\ &\lesssim \left\| \sum_{j=1}^{\infty} \lambda_j T(a_j) \chi_{2\sqrt{n}Q_j} \right\|_X + \alpha \left\| \chi_{\{x \in \mathbb{R}^n: |\sum_{j=1}^{\infty} \lambda_j T(a_j)(x) \chi_{(2\sqrt{n}Q_j)^c}| > \frac{\alpha}{2}\}} \right\|_X \\ &= I_1 + I_2. \end{aligned} \quad (24)$$

From this, to prove (23), it is only necessary to prove $I_1 \lesssim \|f\|_{h_X}$ and $I_2 \lesssim \|f\|_{h_X}$, respectively. Notice that, for any $j \in \mathbb{N}$, $a_j \in L^2$. Since T is bounded on L^2 , it follows that

$$\|T(a_j) \chi_{2\sqrt{n}Q_j}\|_{L^2} \leq \|T(a_j)\|_{L^2} \lesssim \|a_j\|_{L^2} \lesssim \frac{|Q_j|^{1/2}}{\|\chi_{Q_j}\|_X},$$

which, combined with Lemma 2.18 and (19), implies that

$$I_1 \lesssim \left\| \left\{ \sum_{j=1}^{\infty} \left(\frac{\lambda_j}{\|\chi_{Q_j}\|_X} \right)^s \chi_{Q_j} \right\}^{1/s} \right\|_X \lesssim \|f\|_{h_X}, \quad (25)$$

where the atom a_j in Lemma 2.18 is replaced by $T(a_j) \chi_{2\sqrt{n}Q_j}$.

To estimate the term I_2 , we will divide into the following two cases.

Case 1: $|Q_j| \leq 1$. In this case, a_j satisfies the vanishing moment condition. Notice that l_j is the side length and c_{Q_j} is the center of Q_j . When $x \in \tilde{Q}_j^c$, we have $|x - c_{Q_j}| \geq 2|y - c_{Q_j}|$ and $|y - c_{Q_j}| \leq l_j$. By using the smooth condition of kernel $\mathcal{K}$ we obtain that

$$\begin{aligned} |T(a_j)(x)| &= \left| \int_{Q_j} [\mathcal{K}(x, y) - \mathcal{K}(x, c_{Q_j})] a_j(y) dy \right| \\ &\lesssim \int_{Q_j} \frac{|y - c_{Q_j}|^\epsilon}{|x - c_{Q_j}|^{n+\epsilon}} |a_j(y)| dy \\ &\lesssim \frac{l_j^\epsilon}{|x - c_{Q_j}|^{n+\epsilon}} \|a_j\|_{L^2} |Q_j|^{1/2} \\ &\lesssim \frac{l_j^{n+\epsilon}}{|x - c_{Q_j}|^{n+\epsilon}} \frac{1}{\|\chi_{Q_j}\|_X} \\ &\lesssim \frac{1}{\|\chi_{Q_j}\|_X} [M(\chi_{Q_j})(x)]^{\frac{n+\epsilon}{n}}. \end{aligned}$$

Case 2: $|Q_j| \geq 1$ In this case, we have $|x - y| \sim |x - c_{Q_j}|$ and $|x - y| \geq \frac{1}{2}$, where $x \in \tilde{Q}_j^C$, and $y \in Q_j$. By using the size condition of $\mathcal{K}$, for any $x \in \tilde{Q}_j^C$, we obtain that

$$\begin{aligned} |T(a_j)(x)| &= \left| \int_{Q_j} \mathcal{K}(x, y) a_j(y) dy \right| \lesssim \int_{Q_j} |\mathcal{K}(x, y)| |a_j(y)| dy \\ &\lesssim \int_{Q_j} \frac{C}{|x - c_{Q_j}|^{n+\delta}} |a_j(y)| dy \lesssim \frac{C}{|x - c_{Q_j}|^{n+\delta}} \int_{Q_j} |a_j(y)| dy \\ &\lesssim \frac{C}{|x - c_{Q_j}|^{n+\delta}} \|a_j\|_{L^2} \cdot |Q_j|^{1/2} \lesssim \frac{C}{|x - c_{Q_j}|^{n+\delta}} \frac{|Q_j|}{\|\chi_{Q_j}\|_X} \\ &\lesssim \frac{C l_j^{n+\delta}}{|x - c_{Q_j}|^{n+\delta}} \frac{1}{\|\chi_{Q_j}\|_X} \lesssim \frac{1}{\|\chi_{Q_j}\|_X} \left[M(\chi_{Q_j})(x) \right]^{\frac{n+\delta}{n}}. \end{aligned}$$

Therefore, by this, Definition 2.4 (i), (1), $\gamma = \max\left\{\frac{n}{n+\delta}, \frac{n}{n+\epsilon}\right\}$, $0 < \theta < s \leq 1$, $\theta \in [\gamma, 1)$ and (19), we find that

$$\begin{aligned} I_2 &\lesssim \alpha \left\| \chi_{\left\{x \in \mathbb{R}^n : \sum_{j=1}^{\infty} \frac{\lambda_j}{\|\chi_{Q_j}\|_X} \left[M(\chi_{Q_j})(x) \right]^{\frac{1}{\gamma}} > \frac{\alpha}{2} \right\}} \right\|_X \\ &\lesssim \left(\left(\frac{\alpha}{2} \right)^{\gamma} \left\| \chi_{\left\{x \in \mathbb{R}^n : \left\{ \sum_{j=1}^{\infty} \frac{\lambda_j}{\|\chi_{Q_j}\|_X} \left[M(\chi_{Q_j})(x) \right]^{\frac{1}{\gamma}} \right\}^{\gamma} > \left(\frac{\alpha}{2} \right)^{\gamma} \right\}} \right\|_{X^{\frac{1}{\gamma}}} \right)^{\frac{1}{\gamma}} \\ &\lesssim \left\| \left(\sum_{j=1}^{\infty} \frac{\lambda_j}{\|\chi_{Q_j}\|_X} \chi_{Q_j} \right)^{\gamma} \right\|_{X^{\frac{1}{\gamma}}}^{\frac{1}{\gamma}} \\ &\lesssim \left\| \left(\sum_{j=1}^{\infty} \left(\frac{\lambda_j \chi_{Q_j}}{\|\chi_{Q_j}\|_X} \right)^s \right)^{1/s} \right\|_X \lesssim \|f\|_{h_X}. \end{aligned} \tag{26}$$

Finally, combining (25) and (26), we conclude that for any $\alpha \in (0, \infty)$,

$$\alpha \left\| \chi_{\{x \in \mathbb{R}^n : |T(f)(x)| > \alpha\}} \right\|_X \lesssim \|f\|_{h_X},$$

namely, (23) holds true. This finishes the proof of Theorem 1.1. $\square$

Next, we will prove Theorem 1.2.

Proof of Theorem 1.2 Let θ, s and d be as in Lemma 2.18 and $f \in h_X$. Then, by Lemma 2.18, we find that there exist a sequence $\{a_j\}_{j=1}^{\infty}$ of $(X, 2, d)$ -atoms supported, respectively, in a sequence $\{Q_j\}_{j=1}^{\infty}$ of cubes, and a sequence $\{\lambda_j\}_{j=1}^{\infty}$ of non-negative numbers, independent of f but depending on s , such that (18) and (19)

hold true. From Theorem 1.1, we know

$$T(f) = \sum_{j \in \mathbb{N}} \lambda_j T(a_j) \quad \text{in } L_v^{s_0} \quad \text{and } \mathcal{S}'.$$

Let $\Phi \in \mathcal{S}$ satisfy $\int_{\mathbb{R}^n} \Phi(x) dx \neq 0$. Then, to prove Theorem 1.2, by Assumption 2.14 and Theorem 3.3 (ii), we only need to show that, for any $f \in h_X$,

$$\|m(Tf, \Phi)\|_{WX} \lesssim \|f\|_{h_X}, \quad (27)$$

where $m(Tf, \Phi)$ is as in Definition 3.1(i) with f replaced by Tf . For any $\alpha \in (0, \infty)$, by Lemma 2.7 (iii) and Remark 2.6 (i), we have

$$\begin{aligned} \alpha \left\| \chi_{\{x \in \mathbb{R}^n : m(Tf, \Phi)(x) > \alpha\}} \right\|_X &\leq \alpha \left\| \chi_{\{x \in \mathbb{R}^n : \sum_{j \in \mathbb{N}} \lambda_j m(Ta_j, \Phi)(x) > \alpha\}} \right\|_X \\ &\lesssim \alpha \left\| \chi_{\{x \in 2\sqrt{n}Q_j : \sum_{j \in \mathbb{N}} \lambda_j m(Ta_j, \Phi)(x) > \frac{\alpha}{2}\}} \right\|_X + \alpha \left\| \chi_{\{x \in (2\sqrt{n}Q_j)^c : \sum_{j \in \mathbb{N}} \lambda_j m(Ta_j, \Phi)(x) > \frac{\alpha}{2}\}} \right\|_X \\ &\lesssim \left\| \sum_{j \in \mathbb{N}} \lambda_j m(Ta_j, \Phi) \chi_{2\sqrt{n}Q_j} \right\|_X + \alpha \left\| \chi_{\{x \in (2\sqrt{n}Q_j)^c : \sum_{j \in \mathbb{N}} \lambda_j m(Ta_j, \Phi)(x) > \frac{\alpha}{2}\}} \right\|_X \\ &=: I_1 + I_2. \end{aligned}$$

We first estimate I_1 . Observing that $m(Ta_j, \Phi) \lesssim m(Ta_j)$ and $a_j \in L^q$, by the fact that T is bounded on L^q and the size condition of a_j , we conclude that

$$\|m(Ta_j, \Phi)\|_{L^q} \lesssim \|M(Ta_j)\|_{L^q} \lesssim \|Ta_j\|_{L^q} \lesssim \|a_j\|_{L^q} \lesssim \frac{|Q_j|^{1/q}}{\|\chi_{Q_j}\|_X},$$

which, combined with Lemma 2.18 and (19), implies that

$$I_1 \lesssim \left\| \left\{ \sum_{j=1}^{\infty} \left(\frac{\lambda_j}{\|\chi_{Q_j}\|_X} \right)^s \chi_{Q_j} \right\}^{1/s} \right\|_X \lesssim \|f\|_{h_X}, \quad (28)$$

where the atom a_j in Lemma 2.18 is replaced by $m(Ta_j, \Phi) \chi_{2\sqrt{n}Q_j}$. When $x \in (2\sqrt{n}Q_j)^c$, we will consider two cases: $|Q_j| \geq 1$ and $|Q_j| < 1$.

Consider the case $|Q_j| \geq 1$. In this case, $|Q_j| \geq 1$, then we have

$$\begin{aligned} |\Phi_t * T(a_j)(x)| &= \left| \int_{\mathbb{R}^n} \Phi_t(x-y) T(a_j)(y) dy \right| \\ &\leq t^{-n} \int_{Q(x,t)} |T(a_j)(y)| dy \\ &\leq \sup_{y \in Q(x,t)} |T(a_j)(y)|. \end{aligned}$$

Noting that $0 < t < 1 \leq l(Q_j)$ and $x \in (2\sqrt{n}Q_j)^c$, it implies that $y \in (\tilde{Q})^c$. From the proof of Theorem 1.1, we conclude that

$$\sup_{y \in B(x,t)} |T(a_j)(y)| \leq \frac{C}{\|\chi_{Q_j}\|_X} \left[M(\chi_{Q_j})(x) \right]^{\frac{1}{\max\{\frac{n}{n+\delta}, \frac{n}{n^2\epsilon}\}}}.$$

Consider the case $|Q_j| < 1$. If $0 < t < |x - c_{Q_j}|/2$, then together with $x \in (2\sqrt{n}\tilde{Q}_j)^{\complement}$, we can get that $y \in (\tilde{Q})^{\complement}$. Thus, repeating the same argument as used above, then we have

$$|\Phi_t * T(a_j)(x)| \leq \frac{C}{\|\chi_{Q_j}\|_X} [M(\chi_{Q_j})(x)]^{\frac{1}{\max\{\frac{n}{n+\delta}, \frac{n}{n+\epsilon}\}}}.$$

Finally, we consider the case that $t > |x - c_{Q_j}|/2$. Observing that $\ell(Q_j) < 1$, it follows that a_j has vanishing moment condition and $T_*^{loc}(1) = 0$. For any $x \in (2\sqrt{n}\tilde{Q}_j)^{\complement}$, then by using the mean value theorem together with the Hölder inequality yield that

$$\begin{aligned} |\Phi_t * Ta_j(x)| &= \frac{1}{t^n} \int_{\mathbb{R}^n} \left| \Phi\left(\frac{x-y}{t}\right) Ta_j(y) dy \right| \\ &\leq \frac{1}{t^n} \int_{\mathbb{R}^n} \left| \Phi\left(\frac{x-y}{t}\right) - \Phi\left(\frac{x-c_{Q_j}}{t}\right) \right| |Ta_j(y)| dy \\ &= \frac{1}{t^n} \left(\int_{|y-c_{Q_j}| < nl_j} + \int_{nl_j \leq |y-c_{Q_j}| < \frac{|x-c_{Q_j}|}{2}} + \int_{|y-c_{Q_j}| \geq \frac{|x-c_{Q_j}|}{2}} \right) \\ &\quad \times \left| \Phi\left(\frac{x-y}{t}\right) - \Phi\left(\frac{x-c_{Q_j}}{t}\right) \right| |Ta_j(y)| dy \\ &=: A_1 + A_2 + A_3. \end{aligned} \tag{29}$$

For A_1 , by the the mean value theorem, we find that, for any $j \in \mathbb{N}$ and $y \in \mathbb{R}^n$ with $|y - c_{Q_j}| < nl_j$, such that

$$A_1 \leq \frac{1}{t^n} \int_{|y-c_{Q_j}| < nl_j} |\Phi'((x - c_{Q_j} + \theta(c_{Q_j} - y))/t)| \left(\frac{|y - c_{Q_j}|}{t} \right) |Ta_j(y)| dy,$$

which, combined with the Hölder inequality and the fact that T is bounded on L^2 , implies that, for all $t \in (0, 1)$ and $x \in (2\sqrt{n}\tilde{Q}_j)^{\complement}$,

$$\begin{aligned} A_1 &\lesssim \frac{1}{t^n} \int_{|y-c_{Q_j}| < nl_j} \frac{t^{n+1}}{|x - c_{Q_j}|^{n+1}} \frac{|y - c_{Q_j}|}{t} |Ta_j(y)| dy \\ &\lesssim \frac{l_j}{|x - c_{Q_j}|^{n+1}} \|Ta_j\|_{L^2} |Q_j|^{1/2} \lesssim \frac{l_j^{n+1}}{|x - c_{Q_j}|^{n+1}} \frac{1}{\|\chi_{Q_j}\|_X} \\ &\lesssim \frac{l_j^{n+\epsilon}}{|x - c_{Q_j}|^{n+\epsilon}} \frac{1}{\|\chi_{Q_j}\|_X}. \end{aligned} \tag{30}$$

For A_2 , by using the smooth condition of the kernel $\mathcal{K}$, the vanishing moment condition of a_j and the Hölder inequality, we conclude that, for all $z \in \tilde{Q}_j$, for all $t \in (0, 1)$ and $x \in (2\sqrt{n}\tilde{Q}_j)^{\complement}$,

$$\begin{aligned} A_2 &\lesssim \int_{nl_j \leq |y-c_{Q_j}| < \frac{|x-c_{Q_j}|}{2}} \frac{|y - c_{Q_j}|}{|x - c_{Q_j}|^{n+1}} \left[\int_{Q_j} |a_j(z)| |\mathcal{K}(y, z) - \mathcal{K}(y, c_{Q_j})| dz \right] dy \\ &\lesssim \frac{1}{|x - c_{Q_j}|^{n+1}} \int_{nl_j \leq |y-c_{Q_j}| < \frac{|x-c_{Q_j}|}{2}} |y - c_{Q_j}| \int_{Q_j} |a_j(z)| \frac{|z - c_{Q_j}|^\epsilon}{|y - c_{Q_j}|^{n+\epsilon}} dz dy \end{aligned}$$

$$\begin{aligned}
&\lesssim \frac{l_j^\epsilon}{|x - c_{Q_j}|^{n+1}} \int_{n l_j \leq |y - c_{Q_j}| < \frac{|x - c_{Q_j}|}{2}} \frac{1}{|y - c_{Q_j}|^{n+\epsilon-1}} dy \|a_j\|_{L^2} |Q_j|^{1/2} \\
&\lesssim \frac{l_j^{n+\epsilon}}{|x - c_{Q_j}|^{n+\epsilon}} \frac{1}{\|\chi_{Q_j}\|_X}.
\end{aligned} \tag{31}$$

For A_3 , by the vanishing moment condition of a_j and the Hölder inequality, we find that, for all $z \in \tilde{Q}_j$, for all $t \in (0, 1)$ and $x \in (2\sqrt{n}\tilde{Q}_j)^\complement$,

$$\begin{aligned}
A_3 &\leq \int_{|y - c_{Q_j}| \geq \frac{|x - c_{Q_j}|}{2}} \left| \frac{1}{t^n} \left[\Phi\left(\frac{x - y}{t}\right) - \Phi\left(\frac{x - c_{Q_j}}{t}\right) \right] \right| \\
&\quad \times \int_{Q_j} |a_j(z)| |\mathcal{K}(y, z) - \mathcal{K}(y, c_{Q_j})| dz dy \\
&\lesssim \int_{|y - c_{Q_j}| \geq \frac{|x - c_{Q_j}|}{2}} |\Phi_t(x - y)| \int_{Q_j} |a_j(z)| \frac{|z - c_{Q_j}|^\epsilon}{|y - c_{Q_j}|^{n+\epsilon}} dz dy \\
&\quad + \int_{|y - c_{Q_j}| \geq \frac{|x - c_{Q_j}|}{2}} \left| \frac{1}{t^n} \Phi\left(\frac{x - c_{Q_j}}{t}\right) \right| \int_{Q_j} |a_j(z)| \frac{|z - c_{Q_j}|^\epsilon}{|y - c_{Q_j}|^{n+\epsilon}} dz dy \\
&\lesssim \frac{l_j^\epsilon}{|x - c_{Q_j}|^{n+\epsilon}} \|a_j\|_{L^2} |Q_j|^{1/2} \int_{|y - c_{Q_j}| \geq \frac{|x - c_{Q_j}|}{2}} |\Phi_t(x - y)| dy \\
&\quad + l_j^\epsilon \|a_j\|_{L^2} |Q_j|^{1/2} \\
&\quad \times \int_{|y - c_{Q_j}| \geq \frac{|x - c_{Q_j}|}{2}} \frac{1}{t^n} \frac{t^n}{|x - c_{Q_j}|^n} \frac{1}{|y - c_{Q_j}|^{n+\epsilon}} dy \\
&\lesssim \frac{l_j^{n+\epsilon}}{|x - c_{Q_j}|^{n+\epsilon}} \frac{1}{\|\chi_{Q_j}\|_X}.
\end{aligned} \tag{32}$$

We denote $\theta \in [\max\{\frac{n}{n+\delta}, \frac{n}{n+\epsilon}\}, 1)$ and combined (29), (30), (31) and (32), we conclude that, for all $x \in (2\sqrt{n}\tilde{Q}_j)^\complement$,

$$m(Ta_j, \Phi) = \sup_{t \in (0, 1)} |\Phi_t * Ta_j(x)| \leq \frac{C}{\|\chi_{Q_j}\|_X} \left[M(\chi_{Q_j})(x) \right]^{\frac{1}{\max\{\frac{n}{n+\delta}, \frac{n}{n+\epsilon}\}}}.$$

As with (26), by Definition 2.4 (i), (1), $0 < \theta < s \leq 1$ and (19), we find that

$$I_2 \lesssim \|f\|_{h_X}. \tag{33}$$

Finally, combining (28) and (33), we conclude that for any $\alpha \in (0, \infty)$,

$$\alpha \left\| \chi_{\{x \in \mathbb{R}^n : m(Tf, \Phi)(x) > \alpha\}} \right\|_X \lesssim \|f\|_{h_X},$$

namely, (27) holds true. This finishes the proof of Theorem 1.2. $\square$

5. Applications

In this section, we apply all above results to the following concrete examples of ball quasi-Banach function spaces, namely, Morrey spaces, Orlicz-slice spaces and mixed-norm Lebesgue spaces.

5.1. Morrey spaces.

We first recall the notion of Morrey spaces.

Definition 5.1. Let $0 < q \leq p \leq \infty$. The Morrey space $\mathcal{M}_q^p$ is defined to be the set of all $f \in L_{\text{loc}}^q$ such that

$$\|f\|_{\mathcal{M}_q^p} := \sup_{B \in \mathbb{B}} |B|^{1/p-1/q} \left\{ \int_B |f(y)|^q dy \right\}^{1/q} < \infty,$$

where $\mathbb{B}$ is as in (2).

Let X be a ball quasi-Banach function space. Let $0 < q \leq p \leq \infty$. If $X := \mathcal{M}_q^p$, then $WX := W\mathcal{M}_q^p$ is the weak Morrey space, $h_X := h\mathcal{M}_q^p$ is the local Hardy–Morrey space, and $Wh_X := Wh\mathcal{M}_q^p$ is the weak local Hardy–Morrey space. The definitions of weak Morrey spaces, local Hardy–Morrey spaces and weak local Hardy–Morrey spaces are as in Definitions 2.5, 2.15 and 3.2 with X replaced by $\mathcal{M}_q^p$.

The space $\mathcal{M}_q^p$ was introduced by Morrey [24] in 1938. For more details, for example see [10, 21]. Moreover, the Morrey space $\mathcal{M}_q^p$ satisfies Assumptions 2.12, 2.13 and 2.14, respectively, in Lemmas 5.2, 5.3 and 5.4. The following lemma shows that the Assumption 2.12 for $\mathcal{M}_q^p$ hold true.

Lemma 5.2. [34, Remark 2.4] Let $0 < q \leq p < \infty$. Assume that $s \in (0, 1]$ and $\theta \in (0, \min\{s, q\})$. Then there exists a positive constant C such that for any $\{f_j\}_{j=1}^\infty \subset \mathcal{M}$,

$$\left\| \left\{ \sum_{j=1}^\infty [M^{(\theta)}(f_j)]^s \right\}^{\frac{1}{s}} \right\|_{\mathcal{M}_q^p} \leq C \left\| \left\{ \sum_{j=1}^\infty |f_j|^s \right\}^{\frac{1}{s}} \right\|_{\mathcal{M}_q^p}.$$

The following Fefferman–Stein vector-valued maximal inequalities for $W\mathcal{M}_q^p$ hold true, which shows that the Morrey space $\mathcal{M}_q^p$ satisfies Assumption 2.14.

Lemma 5.3. [14, Theorem 3.2] Let $0 < q \leq p < \infty$. Assume that $r \in (1, \infty)$ and $s \in (0, q)$. Then there exists a positive constant C such that for any $\{f_j\}_{j=1}^\infty \subset \mathcal{M}$,

$$\left\| \left\{ \sum_{j=1}^\infty [M(f_j)]^r \right\}^{1/r} \right\|_{(W\mathcal{M}_q^p)^{1/s}} \leq C \left\| \left\{ \sum_{j=1}^\infty |f_j|^r \right\}^{1/r} \right\|_{(W\mathcal{M}_q^p)^{1/s}}.$$

The following lemma shows that the Morrey space $\mathcal{M}_q^p$ satisfies Assumption 2.13.

Lemma 5.4. [38, Lemma 7.6] Let $0 < q \leq p < \infty$, $r \in (0, q)$ and $s \in (q, \infty]$. Then there exists a positive constant C such that for any $\{f\}_{j=1}^\infty \subset \mathcal{M}$,

$$\|M^{((s/r)')} (f)\|_{[(\mathcal{M}_q^p)^{1/r}]'} \leq C \|f\|_{[(\mathcal{M}_q^p)^{1/r}]'},$$

where $\left[(\mathcal{M}_q^p)^{1/r} \right]'$ is as in (3) with $X := (\mathcal{M}_q^p)^{1/r}$.

To apply Theorems 1.1 and 1.2 to Morrey spaces, we need the following weak-type Fefferman–Stein vector-valued inequality of the Hardy–Littlewood maximal operator M in (5) from $\mathcal{M}_1^p$ to $W\mathcal{M}_1^p$ in [38, Proposition 7.16].

Proposition 5.5. Let $p \in [1, \infty)$ and $r \in (1, \infty)$. Then there exists a positive constant C such that for any $\{f_j\}_{j \in \mathbb{N}} \subset \mathcal{M}_1^p$,

$$\left\| \left\{ \sum_{j=1}^{\infty} [M(f_j)]^r \right\}^{\frac{1}{r}} \right\|_{W\mathcal{M}_1^p} \leq C \left\| \left\{ \sum_{j=1}^{\infty} |f_j|^r \right\}^{\frac{1}{r}} \right\|_{\mathcal{M}_1^p}.$$

Applying Lemmas 5.2, 5.3, 5.4 and proposition 5.5, Theorems 1.1 and 1.2, we immediately obtain the following boundedness of inhomogeneous Calderón–Zygmund operator, as follows.

Theorem 5.6. Let $q \in (0, 1]$, $p \in (0, \infty)$ with $q \leq p$. Let T be an inhomogeneous Calderón–Zygmund operator. If $q \in \left[\max \left\{ \frac{n}{n+\delta}, \frac{n}{n+\epsilon} \right\}, 1 \right]$, then T has a unique extension on $h\mathcal{M}_q^p$. Moreover, there exists a positive constant C such that for any $f \in h\mathcal{M}_q^p$,

$$\|T(f)\|_{W\mathcal{M}_q^p} \leq C\|f\|_{h\mathcal{M}_q^p}.$$

Theorem 5.7. Let $q \in (0, 1]$, $p \in (0, \infty)$ with $q \leq p$. Let T be an inhomogeneous Calderón–Zygmund operator and $T_{loc}^*(1) = 0$. If $q \in \left[\max \left\{ \frac{n}{n+\delta}, \frac{n}{n+\epsilon} \right\}, 1 \right]$, then T has a unique extension on $h\mathcal{M}_q^p$. Moreover, there exists a positive constant C such that for any $f \in h\mathcal{M}_q^p$,

$$\|T(f)\|_{Wh\mathcal{M}_q^p} \leq C\|f\|_{h\mathcal{M}_q^p}.$$

5.2. Orlicz-slice spaces.

We begin with the notions of both Orlicz functions and Orlicz spaces (see, for instance, [4, 16, 25]).

Definition 5.8. A function $\Phi : [0, \infty) \rightarrow [0, \infty)$ is called an Orlicz function if it is nondecreasing and satisfies $\Phi(0) = 0$, $\Phi(t) > 0$ whenever $t \in (0, \infty)$ and $\lim_{t \rightarrow \infty} \Phi(t) = \infty$.

An Orlicz function Φ is said to be of lower (resp., upper) type p with $p \in (-\infty, \infty)$ if there exists a positive constant $C_{(p)}$, depending on p , such that, for any $t \in [0, \infty)$ and $s \in (0, 1)$ [resp., $s \in [1, \infty)$],

$$\Phi(st) \leq C_{(p)} s^p \Phi(t)$$

A function $\Phi : [0, \infty) \rightarrow [0, \infty)$ is said to be of positive lower p (resp., upper) type if it is of lower (resp., upper) type p for some $p \in (0, \infty)$.

Definition 5.9. Let Φ be an Orlicz function with positive lower type p_{Φ}^- and positive upper type p_{Φ}^+ . The Orlicz space $L^{\Phi}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f such that

$$\|f\|_{L^{\Phi}(\mathbb{R}^n)} := \inf \left\{ \lambda \in (0, \infty) : \int_{\mathbb{R}^n} \Phi \left(\frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\} < \infty.$$

Definition 5.10. [37] Let $t, q \in (0, \infty)$ and Φ be an Orlicz function with positive lower type p_{Φ}^- and positive upper type p_{Φ}^+ . The Orlicz-slice space $(E_{\Phi}^q)_t$ is defined to be the set of all measurable functions f such that

$$\|f\|_{(E_{\Phi}^q)_t(\mathbb{R}^n)} := \left\{ \int_{\mathbb{R}^n} \left[\frac{\|f \mathbf{1}_{B(x,t)}\|_{L^{\Phi}(\mathbb{R}^n)}}{\|\mathbf{1}_{B(x,t)}\|_{L^{\Phi}(\mathbb{R}^n)}} \right]^q dx \right\}^{\frac{1}{4}} < \infty.$$

Let X be a ball quasi-Banach function space. If $X := (E_{\Phi}^q)_t$, then $WX := (WE_{\Phi}^q)_t$ is the weak Orlicz-slice space, $h_X := (hE_{\Phi}^q)_t$ is the local Orlicz-slice Hardy space, and $Wh_X := (WhE_{\Phi}^q)_t$ is the weak local Orlicz-slice Hardy space. The definitions of weak Orlicz-slice spaces, local Orlicz-slice Hardy spaces and weak local Orlicz-slice Hardy spaces are as in Definitions 2.5, 2.15 and 3.2 with X replaced by $(E_{\Phi}^q)_t$.

The following lemma shows that the Assumption 2.12 for $(E_{\Phi}^q)_t$ hold true.

Proposition 5.11. [34, Remark 2.4] Let $s \in (0, 1]$ and $\theta \in (0, \min\{s, p_\Phi^-, r\})$. Then there exists a positive constant C such that for any $\{f_j\}_{j=1}^\infty \subset L_{\text{loc}}^1$,

$$\left\| \left\{ \sum_{j=1}^\infty [M^{(\theta)}(f_j)]^s \right\}^{\frac{1}{s}} \right\|_{(E_\Phi^q)_t} \leq C \left\| \left\{ \sum_{j=1}^\infty |f_j|^s \right\}^{\frac{1}{s}} \right\|_{(E_\Phi^q)_t}.$$

The following vector-valued inequality of the Hardy–Littlewood operator $\mathcal{M}$ in (5) on $(WE_\Phi^q)_t$, which shows that $(E_\Phi^q)_t$ satisfies Assumption 2.14.

Proposition 5.12. [38, Proposition 7.47] Let $t \in (0, \infty)$, $q, s \in (1, \infty)$ and Φ be an Orlicz function with positive lower type $p_\Phi^- \in (1, \infty)$ and positive upper type p_Φ^+ . Then there exists a positive constant C , independent of t , such that, for any sequence $\{f_j\}_{j \in \mathbb{N}} \subset \mathcal{M}$,

$$\left\| \left\{ \sum_{j \in \mathbb{N}} [\mathcal{M}(f_j)]^s \right\}^{\frac{1}{s}} \right\|_{(WE_\Phi^q)_t} \leq C \left\| \left\{ \sum_{j \in \mathbb{N}} |f_j|^s \right\}^{\frac{1}{s}} \right\|_{(WE_\Phi^q)_t}.$$

The following lemma shows that the weighted Lebesgue space L_ω^p satisfies Assumption 2.13.

Lemma 5.13. [37, Lemma 4.4] Let $t, q \in (0, \infty)$ and Φ be an Orlicz function with positive lower type p_Φ^- and positive upper type p_Φ^+ . Let $r \in (\max\{q, p_\Phi^+\}, \infty]$ and $s \in (0, \min\{p_\Phi^-, q\})$. Then there exists a positive constant $C_{(s,r)}$, depending on s and r , but independent of t , such that, for any $f \in \mathcal{M}$,

$$\|\mathcal{M}^{((r/s)')} (f)\|_{[(E_\Phi^q)_t]^{1/s}'} \leq C_{(s,r)} \|f\|_{[(E_\Phi^q)_t]^{1/s}}.$$

Here and hereafter, $[(E_\Phi^q)_t]^{1/s}$ denotes the $\frac{1}{s}$ -convexification of $(E_\Phi^q)_t$ as in Definition 2.4(i) with $X := (E_\Phi^q)_t$ and $p := 1/s$, and $[(E_\Phi^q)_t]^{1/s}'$ denotes its dual space.

To apply Theorems 1.1 and 1.2 to Orlicz-slice spaces, we need the following weak-type Fefferman–Stein vector-valued inequality of the Hardy–Littlewood maximal operator M in (5) from $(E_\Phi^q)_t$ to $(WE_\Phi^q)_t$ in [38, Proposition 7.57]. The following lemma shows that the weighted Lebesgue space L_ω^p satisfies Assumption 2.13.

Proposition 5.14. Let $t \in (0, \infty)$, $q \in [1, \infty)$, $r \in (1, \infty)$ and Φ be an Orlicz function with positive lower type $p_\Phi^- \in [1, \infty)$ and positive upper type p_Φ^+ . Then there exists a positive constant C , independent of t , such that, for any $\{f_j\}_{j \in \mathbb{Z}} \subset \mathcal{M}$,

$$\left\| \left\{ \sum_{j \in \mathbb{Z}} [\mathcal{M}(f_j)]^r \right\}^{\frac{1}{r}} \right\|_{(WE_\Phi^q)_t} \leq C \left\| \left\{ \sum_{j \in \mathbb{Z}} |f_j|^r \right\}^{\frac{1}{r}} \right\|_{(E_\Phi^q)_t}.$$

Applying Lemma 5.11, lemma 5.13, Proposition 5.12, Proposition 5.14, Theorems 1.1 and 1.2, we immediately obtain the following boundedness of inhomogeneous Calderón–Zygmund operator, as follows.

Theorem 5.15. Let $t \in (0, \infty)$, $q \in (0, \infty)$ and Φ be an Orlicz function with positive lower type p_Φ^- and positive upper type p_Φ^+ . Let T be an inhomogeneous Calderón–Zygmund operator. If $\min\{p_\Phi^-, q\} \in [\max\{\frac{n}{n+\delta}, \frac{n}{n+\epsilon}\}, 1]$, then T has a unique extension on $(hE_\Phi^q)_t$. Moreover, there exists a positive constant C such that for any $f \in (hE_\Phi^q)_t$,

$$\|Tf\|_{(WE_\Phi^q)_t} \leq C \|f\|_{(hE_\Phi^q)_t}.$$

Theorem 5.16. Let $t \in (0, \infty)$, $q \in (0, \infty)$ and Φ be an Orlicz function with positive lower type p_{Φ}^{-} and positive upper type p_{Φ}^{+} . Let T be an inhomogeneous Calderón–Zygmund operator and $T_{loc}^{*}(1) = 0$. If $\min\{p_{\Phi}^{-}, q\} \in \left[\max\left\{\frac{n}{n+\delta}, \frac{n}{n+\epsilon}\right\}, 1\right]$, then T has a unique extension on $(hE_{\Phi}^q)_t$. Moreover, there exists a positive constant C such that for any $f \in (hE_{\Phi}^q)_t$,

$$\|Tf\|_{(WhE_{\Phi}^q)_t} \leq C\|f\|_{(hE_{\Phi}^q)_t}.$$

5.3. Mixed-norm Lebesgue spaces.

Definition 5.17. Let $\vec{p} := (p_1, \dots, p_n) \in (0, \infty]^n$. The mixed-norm Lebesgue space $L^{\vec{p}}$ is defined to be the set of all measurable functions f such that

$$\|f\|_{L^{\vec{p}}(\mathbb{R}^n)} := \left\{ \int_{\mathbb{R}} \cdots \left[\int_{\mathbb{R}} |f(x_1, \dots, x_n)|^{p_1} dx_1 \right]^{\frac{p_2}{p_1}} \cdots dx_n \right\}^{\frac{1}{p_n}} < \infty$$

with the usual modifications made when $p_i = \infty$ for some $i \in \{1, \dots, n\}$.

Let X be a ball quasi-Banach function space. If $X := L^{\vec{p}}$, then $WX := WL^{\vec{p}}$ is the weak mixed-norm Lebesgue space, $h_X := h^{\vec{p}}$ is the local mixed-norm Hardy space, and $Wh_X := Wh^{\vec{p}}$ is the weak local mixed-norm Hardy space. The definitions of weak mixed-norm Lebesgue spaces, local mixed-norm Hardy spaces and weak local mixed-norm Hardy spaces are as in Definitions 2.5, 2.15 and 3.2 with X replaced by $L^{\vec{p}}$.

The following lemma shows that the Assumption 2.12 for $L^{\vec{p}}$ hold true. For any $\vec{p} := (p_1, \dots, p_n) \in (0, \infty)^n$, we always let $p_- := \min\{p_1, \dots, p_n\}$ and $p_+ := \max\{p_1, \dots, p_n\}$. By [34, Remark 2.4], we can easily obtain the following conclusion and we omit the details here.

Lemma 5.18. Let $\vec{p} \in (0, \infty)^n$. Assume that $s \in (0, 1)$ and $\theta \in (0, \min\{s, p_-\})$. Then there exists a positive constant C such that, for any $\{f_j\}_{j=1}^{\infty} \subset \mathcal{M}$,

$$\left\| \left\{ \sum_{j=1}^{\infty} [M^{(\theta)}(f_j)]^s \right\}^{\frac{1}{s}} \right\|_{L^{\vec{p}}} \leq C \left\| \left\{ \sum_{j=1}^{\infty} |f_j|^s \right\}^{\frac{1}{s}} \right\|_{L^{\vec{p}}},$$

where $(L^{\vec{p}})^{1/s}$ denotes the $\frac{1}{s}$ -convexification of $L^{\vec{p}}$ as in Definition 2.4(i) with X and p replaced, respectively, by $L^{\vec{p}}$ and $1/s$.

The following vector-valued inequality of the Hardy–Littlewood operator $\mathcal{M}$ in (5) on $WL^{\vec{p}}$, which shows that $L^{\vec{p}}$ satisfies Assumption 2.14.

Lemma 5.19. [38, Theorem 7.25] Let $\vec{p} \in (1, \infty)^n$ and $s \in (1, \infty)$. Then there exists a positive constant C such that, for any sequence $\{f_j\}_{j \in \mathbb{N}} \subset \mathcal{M}$,

$$\left\| \left\{ \sum_{j \in \mathbb{N}} [\mathcal{M}(f_j)]^s \right\}^{\frac{1}{s}} \right\|_{WL^{\vec{p}}} \leq C \left\| \left\{ \sum_{j \in \mathbb{N}} |f_j|^s \right\}^{\frac{1}{s}} \right\|_{WL^{\vec{p}}}.$$

We also need the following Fefferman–Stein vector-valued maximal inequality on $WL^{\vec{p}}$. The following lemma shows that the Assumption 2.13 for $L^{\vec{p}}$ hold true. By [15, Lemma 3.5] and [2, Theorem 1.a], we can easily obtain the following conclusion.

Lemma 5.20. [38, Lemma 7.26] Let $\vec{p} \in (0, \infty)^n$, $r \in (0, p_-]$ and $s \in (p_+, \infty]$. Then there exists a positive constant C such that, for any $f \in \mathcal{M}$,

$$\|\mathcal{M}^{((s/r)')} (f)\|_{[(L^{\vec{p}})^{1/r}]'} \leq C \|f\|_{[(L^{\vec{p}})^{1/r}]'},$$

where $[(L^{\vec{p}})^{1/r}]'$ is as in (3) with $X := (L^{\vec{p}})^{1/r}$.

To discuss the boundedness of inhomogeneous Calderón–Zygmund operators, we need the following vector-valued inequality of the Hardy–Littlewood maximal operator $\mathcal{M}$ in (5) from $L^{\vec{p}}$ to $WL^{\vec{p}}$.

Proposition 5.21. [38, Proposition 7.33] Let $\vec{p} \in [1, \infty)^n$ and $r \in (1, \infty)$. Then there exists a positive constant C such that, for any $\{f_j\}_{j \in \mathbb{N}} \subset \mathcal{M}$,

$$\left\| \left\{ \sum_{j \in \mathbb{N}} [\mathcal{M}(f_j)]^r \right\}^{\frac{1}{r}} \right\|_{WL^{\vec{p}}} \leq C \left\| \left\{ \sum_{j \in \mathbb{N}} |f_j|^r \right\}^{\frac{1}{r}} \right\|_{L^{\vec{p}}}.$$

Applying Lemma 5.18, Lemma 5.19 and 5.20, Proposition 5.21, Theorems 1.1 and 1.2, we immediately obtain the following boundedness of inhomogeneous Calderón–Zygmund operator as follows.

Theorem 5.22. Let $\vec{p} \in (0, \infty)^n$. Let T be an inhomogeneous Calderón–Zygmund operator. If $p_- \in \left[\max \left\{ \frac{n}{n+\delta}, \frac{n}{n+\epsilon} \right\}, 1 \right]$, then T has a unique extension on $h^{\vec{p}}$. Moreover, there exists a positive constant C such that for any $f \in h^{\vec{p}}$,

$$\|T(f)\|_{WL^{\vec{p}}} \leq C \|f\|_{h^{\vec{p}}}.$$

Theorem 5.23. Let $\vec{p} \in (0, \infty)^n$. Let T be an inhomogeneous Calderón–Zygmund operator and $T_{loc}^*(1) = 0$. If $p_- \in \left[\max \left\{ \frac{n}{n+\delta}, \frac{n}{n+\epsilon} \right\}, 1 \right]$, then T has a unique extension on $h^{\vec{p}}$. Moreover, there exists a positive constant C such that for any $f \in h^{\vec{p}}$,

$$\|T(f)\|_{Wh^{\vec{p}}} \leq C \|f\|_{h^{\vec{p}}}.$$

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References

- [1] W. Abu-Shammala, A. Torchinsky, *The Hardy–Lorentz spaces $H^{p,q}(\mathbb{R}^n)$* , Studia Math., **182** (2007), 283–294.
- [2] A. Benedek, R. Panzone, *The space L^p , with mixed norm*, Duke Math. J., **28** (1961), 301–324.
- [3] C. Bennett, R. Sharpley, *Interpolation of Operators*, Academic Press, Boston, 1988.
- [4] D. Chang, Z. Fu, D. Yang, S. Yang, *Real-variable characterizations of Musielak–Orlicz–Hardy spaces associated with Schrödinger operators on domains*, Math. Methods Appl. Sci., **39**(2016), no. 3, 533–569.
- [5] X. Chen, J. Tan, *Pseudo-differential operators on local Hardy spaces associated with ball quasi-Banach function spaces*, J. Pseudo-Differ. Oper. Appl. **15** (2024), no. 3, Paper No. 61, 33 pp.
- [6] X. Chen, J. Tan, *The atomic characterization of weighted local Hardy spaces and its applications*, Filomat **38** (2024), no. 17, 5925–5949.
- [7] W. Ding, Y. Han, Y. Zhu, *Boundedness of singular integral operators on local Hardy spaces and dual spaces*, Potential Anal., **55** (2021), 419–441.
- [8] R. Fefferman, F. Soria, *The space Weak H^1* , Studia Math. **85** (1986), no. 1, 1–16.
- [9] Z. Fu, X. Hou, M. Lee, J. Li, *A study of one-sided singular integral and function space via reproducing formula*, J. Geom. Anal. **33** (2023), Paper No. 289, 1–34.
- [10] Z. Fu, R. Gong, E. Pozzi, Q. Wu, *Cauchy–Szegő commutators on weighted Morrey spaces*, Math. Nachr. **296** (2023), no. 5, 1859–1885.
- [11] D. Goldberg, *A local version of real Hardy spaces*, Duke Math. J., **46** (1979), 27–42.
- [12] L. Grafakos, *Classical Fourier Analysis*, (3rd edition), Springer, New York, 2014.
- [13] L. Grafakos, *Modern Fourier Analysis*, (3rd edition), Springer, New York, 2014.
- [14] K.-P. Ho, *Atomic decompositions and Hardy’s inequality on weak Hardy–Morrey spaces*, Sci. China Math., **60** (2017), 449–468.
- [15] L. Huang, J. Liu, D. Yang, W. Yuan, *Atomic and Littlewood–Paley characterizations of anisotropic mixed-norm Hardy spaces and their applications*, J. Geom. Anal., **29** (2019), 1991–2067.

- [16] B. Li, X. Fan, Z. Fu, D. Yang, *Molecular characterization of anisotropic Musielak–Orlicz Hardy spaces and their applications*, Acta Math. Sin. (Engl. Ser.) **32** (2016), no. 11, 1391–1414.
- [17] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces II: Function spaces*, Springer, Berlin, 1979.
- [18] H. Liu, *The weak H^p spaces on homogeneous groups*, Lect. Notes Math. Springer, Berlin, **1494**, (1991), 113–118.
- [19] Y. Li, D. Yang, L. Huang, *Real-variable theory of Hardy spaces associated with generalized Herz spaces of Rafeiro and Samko*, Springer, Singapore, 2023.
- [20] X. Lin, D. Yang, S. Yang, W. Yuan, *Maximal function characterizations of Hardy spaces associated with both non-negative self-adjoint operators satisfying Gaussian estimates and ball quasi-Banach function spaces*, Acta Math. Sci. Ser. B (Engl. Ed.), **44** (2024), no. 2, 484–514.
- [21] F. Liu, Z. Fu, Y. Wu, *Variation operators for commutators of rough singular integrals on weighted Morrey spaces*, J. Appl. Anal. Comput., **14** (2024), 263–282.
- [22] J. Liu, D. Yang, W. Yuan, *Littlewood–Paley characterizations of anisotropic Hardy–Lorentz spaces*, Acta Math. Sci., **38** (2018), no. 1, 1–33.
- [23] E. Lorist, Z. Nieraeth, *Banach function spaces done right*, Indag. Math. (N.S.) **35** (2024), no. 2, 247–268.
- [24] C. B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc., **43** (1938), 126–166.
- [25] M. M. Rao, Z. D. Ren, *Theory of Orlicz spaces*, Marcel Dekker, New York, 1991.
- [26] E. M. Stein, *Harmonic analysis: Real-variable methods, orthogonality, and oscillatory integrals*, Princeton University Press, Princeton, 1993.
- [27] J. Sun, D. Yang, W. Yuan, *Weak Hardy spaces associated with ball quasi-Banach function spaces on spaces of homogeneous type: decompositions, real interpolation, and Calderón–Zygmund operators*, J. Geom. Anal., **32** (2022), no. 2, 1–85.
- [28] Y. Sawano, K.-P. Ho, D. Yang, S. Yang, *Hardy spaces for ball quasi-Banach function spaces*, Dissertationes Math., **525** (2017), 1–102.
- [29] J. Tan, *Real-variable theory of local variable Hardy spaces*, Acta Math. Sin. (Engl. Ser.), **39** (2023), no. 7, 1229–1262.
- [30] J. Tan, *Atomic decompositions of localized Hardy spaces with variable exponents and applications*, J. Geom. Anal., **29** (2019), no. 1, 799–827.
- [31] J. Tan, *Weighted Hardy and Carleson measure spaces estimates for fractional integrations*, Publ. Math. Debrecen., **98** (2021), no. 3–4, 313–330.
- [32] J. Tan, *Product Hardy spaces meet ball quasi-Banach function spaces*, J. Geom. Anal., **34** (2024), no. 3, 1–33.
- [33] J. Tan, L. Zhang, *Bochner–Riesz means on Hardy spaces associated with ball quasi-Banach function spaces*, Mediterr. J. Math., **20** (2023), no. 5, 1–24.
- [34] F. Wang, D. Yang, S. Yang, *Applications of Hardy spaces associated with ball quasi-Banach function spaces*, Results Math., **75** (2020), no. 1, Paper No. 26, 58 pp.
- [35] F. Wang, D. Yang, W. Yuan, *Riesz transform characterization of Hardy spaces associated with ball quasi-Banach function spaces*, J. Fourier Anal. Appl., **29** (2023), no. 5, 1–49.
- [36] D. Yang, Y. Zhou, *A boundedness criterion via atoms for linear operators in Hardy spaces*, Constr. Approx., **29** (2009), no. 2, 207–218.
- [37] Y. Zhang, D. Yang, W. Yuan, S. Wang, *Real-variable characterizations of Orlicz-slice Hardy spaces*, Anal. Appl. (Singap.), **17** (2019), 597–664.
- [38] Y. Zhang, D. Yang, W. Yuan, S. Wang, *Weak Hardy-type spaces associated with ball quasi-Banach function spaces I: Decompositions with applications to boundedness of Calderón–Zygmund operators*, Sci. China Math., **64** (2021), 2007–2064.