Filomat 39:20 (2025), 6967–6982
https://doi.org/10.2298/FIL2520967H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Wavelet packets associated with singular partial differential operators

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Abstract. In this paper, we study some types of wavelet packets and the corresponding wavelet transforms associated with the Riemann-Liouville operator on the half plane $[0,+\infty[\times\mathbb{R}$. For these transforms, we establish Plancherel theorems, orthogonality properties, Calderón's reproducing formulas, reconstruction formulas and their scale discrete scaling functions.

1. Introduction

Wavelet analysis is the most widely used mathematical tool which delas various applications in image processing, wave propagation, data compression, computer graphics, and other areas of engineering and sciences. In terms of the classical Fourier transform, many smooth functions fail to become a wavelet. In such situations, the Riemann-Liouville operator associated with singular operators plays an important role. From the papers of [1, 2], we consider the singular partial differential operators defined on $]0,+\infty[\times\mathbb{R}$ by

$$\begin{cases} \Delta = \frac{\partial}{\partial v}, \\ \mathcal{D} = \frac{\partial^2}{\partial u^2} + \frac{2\alpha+1}{u}\frac{\partial}{\partial u} - \frac{\partial^2}{\partial v^2}, \ \alpha \geqslant 0. \end{cases}$$

The following integral transform associated with Δ and $\mathcal{D}$ is called the Riemann-Liouville operator defined on the space of continuous functions on $\mathbb{R}^2$, even with respect to the first variable, by

$$\mathcal{R}(f)(u,v) = \begin{cases} \frac{\alpha}{\pi}\int_{-1}^{1}\int_{-1}^{1} f(us\sqrt{1-t^2}, v+ut)(1-t^2)^{\alpha-\frac{1}{2}}(1-s^2)^{\alpha-1}\,dt\,ds; \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \alpha > 0, \\ \frac{1}{\pi}\int_{-1}^{1} f(u\sqrt{1-t^2}, v+ut)\frac{dt}{\sqrt{1-t^2}}; \quad \text{if } \alpha = 0. \end{cases}$$

2020 *Mathematics Subject Classification*. Primary 44A05; Secondary 42C40.
Keywords. Riemann-Liouville operator, Wavelet packets, Scale discrete functions.
Received: 04 January 2025; Revised: 23 February 2025; Accepted: 02 June 2025
Communicated by Dijana Mosić
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Many harmonic analysis results related to $\mathcal{R}$ have been discussed in [2, 3, 14] and others. Baccar, Hamadi and Rachdi [2], obtained best approximation for the Weierstrass transform connected with $\mathcal{R}$. Baccar and Rachdi [3], introduced DL^p type spaces and studied the convolution product by concerning the Riemann-Liouville operator. In the same year, Rachdi and Rouz [14], taking the Fourier transform associated with $\mathcal{R}$ and we give a new description of the Schwartz spaces and prove a Paley-Wiener and a Paley-Wiener-Schwartz theorems. Recently, many researchers have been examining the behavior of the Fourier transform associated with the Riemann-Liouville operator (4), with respect to various problems that have already been explored for the classical Fourier transform. Hleili [8], proved Calderon's reproducing formulas and extremal functions for the Riemann Liouville -multiplier operators. Hleili [9], introduced variation of uncertainty principles for the continuous wavelet transform by considering the operator $\mathcal{R}$. Hleili and Omri [10], established $L^p - L^q$ version of Miyachi's theorem in terms of the Riemann-Liouville operator. Hleili, Omri and Rachdi [11], proved some uncertainty principle for the Fourier transform connected with $\mathcal{R}$. Rachdi and Herch [13], by considering the Riemann Liouville operator, uncertainty principles for continuous wavelet transform is discussed.

Wavelet analysis appeared in the early 1980s. This theory is a work which brought together engineers, mathematicians and physicists who had developed similar ideas in their respective fields. The mathematical synthesis led to new results, which provided broader perspectives in each original discipline. The wavelet transform reduces calculation time, facilitates analysis, transmission and compression of information. This transform has found many applications in a variety of signal analysis tasks, such as geophysics, medical image processing and acoustics to quantum theory (see [6, 7] and the references therein). Wavelet packets are an advanced extension of wavelet transforms that allow for a more flexible and detailed decomposition of signals. While the classical wavelet transform splits the signal into a low-frequency approximation and high-frequency details, wavelet packets go further by splitting both the approximation and detail components at each level into additional frequency bands. This makes wavelet packets particularly useful in applications like signal compression, noise reduction, and feature extraction. Many authors observed certain types of wavelet packets in different settings. Some of them are listed below: Chui [5], introduced to wavelets and discussed various important properties of wavelets. Sifi [12], examined two types of generalized wavelet packets and the corresponding generalized wavelet transforms in connection with Laguerre functions on $[0, +\infty[\times\mathbb{R}$, and derived several properties. Trimèche [15], constructed generalized harmonic analysis and wavelet packets associated with the Bessel operator. Chabeh and Mourou [4], observed wavelet packets associated with a Dunkl type operator on $\mathbb{R}$ and many obtained interesting results. Inspired from the papers of [4, 5, 12, 15], we are devoted to define and study some types of wavelet packets associated with the Riemann-Liouville operator.

This work is organized as follows. In Section 2, we briefly summarize some harmonic analysis results related to the Riemann-Liouville operator $\mathcal{R}$. Section 3 is devoted to introducing the first type of wavelet packets associated with the Riemann-Liouville operator, along with some harmonic analysis properties, namely a Plancherel theorem and a reconstruction formula. In Section 4, we introduce the scale discrete scaling function and present its properties. Finally, in Section 5, we define and study the S-wavelet packet, its dual, and the corresponding S-wavelet transforms.

2. Preliminaries

In this section, we recall some harmonic analysis results related to the Riemann-Liouville operator. For more details, see [1, 2, 13]. We denote by

• κ the measure defined on $[0, +\infty[\times\mathbb{R}$ by

$$d\kappa(u, v) = \frac{u^{2\alpha+1}}{2^\alpha\Gamma(\alpha + 1)}du \otimes \frac{dv}{(2\pi)^{\frac{1}{2}}},$$

• $L^p(d\kappa)$, $p \in [1,+\infty]$ the Lebesgue space of measurable functions f on $[0,+\infty[\times\mathbb{R}$ such that $\|f\|_{p,\kappa} < +\infty$, with

$$\|f\|_{p,\kappa} = \begin{cases} \left(\int_0^{+\infty}\int_{\mathbb{R}} |f(u,v)|^p d\kappa(u,v)\right)^{\frac{1}{p}}, & \text{if } p \in [1,+\infty[\\ ess\, sup_{(u,v)\in[0,+\infty[\times\mathbb{R}} |f(u,v)|, & \text{if } p = +\infty. \end{cases}$$

• Υ the set given by

$$\Upsilon = \mathbb{R}\times\mathbb{R} \cup \Big\{(iu,v),\ (u,v)\in\mathbb{R}\times\mathbb{R},\ |u| \leqslant |v|\Big\}.$$

• $\mathcal{B}_{\Upsilon_+}$ the σ-algebra defined on Υ_+ by,

$$\mathcal{B}_{\Upsilon_+} = \Big\{\theta^{-1}(B)\ ,\ B \in \mathcal{B}_{Bor}([0,+\infty[\times\mathbb{R})\Big\},$$

where $\mathcal{B}_{Bor}([0,+\infty[\times\mathbb{R})$, is the usual borelian σ-algebra on $[0,+\infty[\times\mathbb{R}$, and θ is the bijective function defined on the set

$$\Upsilon_+ = [0,+\infty[\times\mathbb{R} \cup \Big\{(is,y)\ ;\ (s,y)\in[0,+\infty[\times\mathbb{R};\ s \leqslant |y|\Big\},$$

by

$$\theta(s,y) = (\sqrt{s^2+|y|^2}, y).$$

• $\varkappa$ the measure defined on $\mathcal{B}_{\Upsilon_+}$ by, $\varkappa(B) = \kappa(\theta(B))$.
• $L^p(d\varkappa)$, $p \in [1,+\infty]$ the Lebesgue space of measurable functions f on Υ_+, such that

$$\|f\|_{p,\varkappa} = \begin{cases} \left(\int\int_{\Upsilon_+} |f(\zeta,\xi)|^p d\varkappa(\zeta,\xi)\right)^{\frac{1}{p}} < +\infty, & \text{if } p \in [1,+\infty[\\ ess\, sup_{(\zeta,\xi)\in\Upsilon_+} |f(\zeta,\xi)| < +\infty, & \text{if } p = +\infty. \end{cases}$$

For every $(\zeta,\xi) \in \mathbb{C}\times\mathbb{C}$, the system

$$\begin{cases} \Delta w(u,w) = -i\xi w(u,v), \\ \mathcal{D}w(u,v) = -\zeta^2 w(u,v), \\ w(0,0) = 1, \\ \\ \dfrac{\partial w}{\partial u}(0,v) = 0, \quad v \in \mathbb{R}, \end{cases}$$

admits a unique solution $\vartheta_{(\zeta,\xi)}$ given by

$$\vartheta_{(\zeta,\xi)}(u,v) = j_\alpha(u\sqrt{\zeta^2+|\xi|^2})e^{-i\xi v},\ \forall (u,v)\in\mathbb{R}\times\mathbb{R},$$

where j_α is the modified Bessel function defined by

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(\alpha+1+k)}(\frac{z}{2})^{2k},\ z\in\mathbb{C}.$$

The function $\vartheta_{(\zeta,\xi)}$ is bounded on $\mathbb{R}\times\mathbb{R}$ if and only if (ζ,ξ) belongs to the set Υ and in this case

$$\sup_{(u,v)\in\mathbb{R}\times\mathbb{R}} \left|\vartheta_{(\zeta,\xi)}(u,v)\right| = 1. \tag{1}$$

Definition 2.1. *[1]*
(1) For every $(u,v) \in [0,+\infty[\times\mathbb{R}$, the generalized translation operator $\mathcal{T}_{(u,v)}$ associated with the Riemann-Liouville operator is defined on $L^p(d\kappa)$, $p \in [1,+\infty]$, by

$$\mathcal{T}_{(u,v)}(f)(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^\pi f(\sqrt{u^2+v^2+2uv\cos\theta}, v+y)\sin^{2\alpha}(\theta)d\theta. \tag{2}$$

(2) The generalized convolution product of $f,g \in L^1(d\kappa)$ is defined by

$$f * g(u,v) = \int_0^{+\infty}\int_{\mathbb{R}} \mathcal{T}_{(u,-v)}(\check{f})(s,y)g(s,y)d\kappa(s,y),\ \ \forall (u,v) \in [0,+\infty[\times\mathbb{R}, \tag{3}$$

where $\check{f}(s,y) = f(s,-y)$.

For every $f \in L^p(d\kappa)$, $p \in [1,+\infty]$, and $(u,v) \in [0,+\infty[\times\mathbb{R}$, the function $\mathcal{T}_{(u,v)}(f)$ belongs to $L^p(d\kappa)$ and we have

$$\|\mathcal{T}_{(u,v)}(f)\|_{p,\kappa} \leqslant \|f\|_{p,\kappa}.$$

Definition 2.2. *[1] The Fourier transform $\mathcal{B}$ associated with the Riemann-Liouville operator is defined on $L^1(d\kappa)$ by*

$$\mathcal{B}(f)(\zeta,\xi) = \int_0^{+\infty}\int_{\mathbb{R}} f(u,v)\vartheta_{(\zeta,\xi)}(u,v)d\kappa(u,v),\ \ \forall(\zeta,\xi) \in \Upsilon. \tag{4}$$

Theorem 2.3. *(Inversion formula) [1] Let $f \in L^1(d\kappa)$ such that $\mathcal{B}(f) \in L^1(d\varkappa)$, then for almost every $(u,v) \in [0,+\infty[\times\mathbb{R}$,*

$$f(u,v) = \int\int_{\Upsilon_+} \mathcal{B}(f)(\zeta,\xi)\overline{\vartheta_{(\zeta,\xi)}(u,v)}d\varkappa(\zeta,\xi). \tag{5}$$

Theorem 2.4. *(Plancherel theorem) [1] The Fourier transform $\mathcal{B}$ can be extended to an isometric isomorphism from $L^2(d\kappa)$ onto $L^2(d\varkappa)$. In particular, for every $f \in L^2(d\kappa)$,*

$$\|\mathcal{B}(f)\|_{2,\varkappa} = \|f\|_{2,\kappa}. \tag{6}$$

Proposition 2.5. *[2]*
(1) For every $f \in L^1(d\kappa)$ and $(u,v) \in [0,+\infty[\times\mathbb{R}$, the function $\mathcal{T}_{(u,v)}(f)$ belongs to $L^1(d\kappa)$, and we have

$$\mathcal{B}(\mathcal{T}_{(u,-v)}(f))(\zeta,\xi) = \vartheta_{(\zeta,\xi)}(u,v)\mathcal{B}(f)(\zeta,\xi),\ \ \forall(\zeta,\xi) \in \Upsilon. \tag{7}$$

(2) The Fourier transform $\mathcal{B}$ is a bounded linear operator from $L^1(d\kappa)$ into $L^\infty(d\varkappa)$ and that for every $f \in L^1(d\kappa)$, we have

$$\|\mathcal{B}(f)\|_{\infty,\varkappa} \leqslant \|f\|_{1,\kappa}.$$

(3) For every $f,g \in L^2(d\kappa)$; the function $f * g$ belongs to the space $C_{e,0}(\mathbb{R}\times\mathbb{R})$ consisting of continuous functions h on $\mathbb{R}\times\mathbb{R}$, even with respect to the first variable and such that $\lim\limits_{u^2+|v|^2\longrightarrow+\infty} h(u,v) = 0$.
Moreover,

$$f * g = \mathcal{B}^{-1}(\mathcal{B}(f)\mathcal{B}(g)),$$

where $\mathcal{B}^{-1}$ is the mapping defined on $L^1(d\varkappa)$ by

$$\mathcal{B}^{-1}(g)(u,v) = \int\int_{\Upsilon_+} g(\zeta,\xi)\overline{\vartheta_{(\zeta,\xi)}(u,v)}d\varkappa(\zeta,\xi).$$

Corollary 2.6. *[2] For all functions f and g in $L^2(d\kappa)$, we have*

$$\int_0^{+\infty}\int_{\mathbb{R}} f(u,v)\overline{g(u,v)}d\kappa(u,v) = \int\int_{\Upsilon_+} \mathcal{B}(f)(\zeta,\xi)\overline{\mathcal{B}(g)(\zeta,\xi)}d\varkappa(\zeta,\xi). \tag{8}$$

Remark 2.7. *[13] Let $f,g \in L^2(d\kappa)$, the function $f * g$ belongs to $L^2(d\kappa)$ if and only if $\mathcal{B}(f)\mathcal{B}(g)$ belongs to $L^2(d\varkappa)$, and we have*

$$\|\mathcal{B}(f)\mathcal{B}(g)\|_{2,\varkappa} = \|f * g\|_{2,\kappa}.$$

3. P-wavelet packets associated with the Riemann-Liouville operator

In this section, we present harmonic analysis results related to the P-wavelet packets associated with the Riemann-Liouville operator $\mathcal{R}$. Specifically, we prove a Plancherel theorem and derive a reconstruction formula.
Let $a > 0$. The dilation operator D_a of a measurable function ψ is defined by

$$D_a(\psi) = \frac{1}{a^{\alpha+\frac{3}{2}}}\psi(\frac{r}{a},\frac{x}{a}),\ \forall (r,x) \in [0,+\infty[\times\mathbb{R}.$$

This operators satisfies the following properties:
(1) For every ψ in $L^2(d\kappa)$, the function $D_a(\psi)$ belongs to $L^2(d\kappa)$, and we have

$$\|D_a(\psi)\|_{2,\kappa} = \|\psi\|_{2,\kappa}, \tag{9}$$

and

$$\mathcal{B}(D_a(\psi))(s,y) = a^{\alpha+\frac{3}{2}}\mathcal{B}(\psi)(as,ay). \tag{10}$$

(2) For every $(r,x) \in [0,+\infty[\times\mathbb{R}$, we have

$$D_a\mathcal{T}_{(r,x)} = \mathcal{T}_{(ar,ax)}D_a.$$

Definition 3.1. *Let $a > 0$. A generalized wavelet on $[0,+\infty[\times\mathbb{R}$ is a measurable function ψ on $[0,+\infty[\times\mathbb{R}$ satisfying, for almost all $(\zeta,\xi) \in \Upsilon\backslash\{0_{[0,+\infty[\times\mathbb{R}}\}$, the condition*

$$0 < C_\psi = \int_0^{+\infty} |\mathcal{B}(\psi)(a\zeta,a\xi)|^2\frac{da}{a} < \infty.$$

Example 3.2. *The function g_t, $t > 0$ defined on $[0,+\infty[\times\mathbb{R}$, by*

$$g_t(u,v) = \frac{1}{(2t)^{\alpha+\frac{3}{2}}}e^{-\frac{1}{4t}(u^2+v^2)},$$

satisfies,

$$\mathcal{B}(g_t)(\zeta,\xi) = e^{-t(\zeta^2+2\xi^2)}.$$

The function $\psi(u,v) = -\frac{d}{dt}g_t(u,v)$, is a generalized wavelet on $[0,+\infty[\times\mathbb{R}$, and we have $C_\psi = \frac{1}{8t^2}$.

Proposition 3.3. *Let $\psi \in L^2(d\kappa)$ be a generalized wavelet on $[0,+\infty[\times\mathbb{R}$ and $(\delta_i)_{i\in\mathbb{Z}}$ be a scale sequence in $]0,+\infty[$, which is decreasing, and such that*

$$\lim_{i\longrightarrow-\infty}\delta_i = +\infty,\ \textit{and}\ \lim_{i\longrightarrow+\infty}\delta_i = 0. \tag{11}$$

Then
(1) The function $(\zeta,\xi) \longrightarrow \left(\frac{1}{C_\psi}\int_{\delta_{i+1}}^{\delta_i}|\mathcal{B}(\psi)(a\zeta,a\xi)|^2\frac{da}{a}\right)^{\frac{1}{2}}$ belongs to $L^2(d\varkappa)$.
(2) There exists a function $\psi_i^P \in L^2(d\kappa)$, such that for every $(\zeta,\xi) \in \Upsilon$,

$$\mathcal{B}(\psi_i^P)(\zeta,\xi) = \left(\frac{1}{C_\psi}\int_{\delta_{i+1}}^{\delta_i}|\mathcal{B}(\psi)(a\zeta,a\xi)|^2\frac{da}{a}\right)^{\frac{1}{2}}. \tag{12}$$

Proof. (1) By using Fubini-Tonelli's theorem, relations (6), (9) and (10), we obtain

$$\begin{aligned}\frac{1}{C_\psi}\int\int_{\Upsilon_+}\int_{\delta_{i+1}}^{\delta_i}|\mathcal{B}(\psi)(a\zeta,a\xi)|^2\frac{da}{a}d\varkappa(\zeta,\xi)&\\
&=\frac{1}{C_\psi}\int_{\delta_{i+1}}^{\delta_i}\left(\int\int_{\Upsilon_+}|\mathcal{B}(\psi)(a\zeta,a\xi)|^2d\varkappa(\zeta,\xi)\right)\frac{da}{a}\\
&=\frac{1}{C_\psi}\int_{\delta_{i+1}}^{\delta_i}\left(\int\int_{\Upsilon_+}|\mathcal{B}(D_a(\psi))(\zeta,\xi)|^2d\varkappa(\zeta,\xi)\right)\frac{da}{a^{\alpha+\frac{5}{2}}}\\
&=\frac{\|\psi\|_{2,\kappa}^2}{(\alpha+\frac{3}{2})C_\psi}\Big(\frac{1}{\delta_{i+1}^{\alpha+\frac{3}{2}}}-\frac{1}{\delta_i^{\alpha+\frac{3}{2}}}\Big)<\infty.\end{aligned}$$

This shows that the function $(\zeta,\xi)\longrightarrow\left(\frac{1}{C_\psi}\int_{\delta_{i+1}}^{\delta_i}|\mathcal{B}(\psi)(a\zeta,a\xi)|^2\frac{da}{a}\right)^{\frac{1}{2}}$ belongs to $L^2(d\varkappa)$.
(2) It follows from the Plancherel theorem, Theorem 2.4. □

Definition 3.4. *The function ψ_i^P, $i\in\mathbb{Z}$ is called P-wavelet packet member of step i and the sequence $(\psi_i^P)_{i\in\mathbb{Z}}$ is called P-wavelet packet.*

Corollary 3.5. *For every $i\in\mathbb{Z}$, the function ψ_i^P satisfies the following properties,*

$$0\leqslant\mathcal{B}(\psi_i^P)(\zeta,\xi)\leqslant 1,\ (\zeta,\xi)\in\Upsilon,\tag{13}$$

and

$$\sum_{i=-\infty}^{+\infty}(\mathcal{B}(\psi_i^P)(\zeta,\xi))^2=1,\ (\zeta,\xi)\in\Upsilon.\tag{14}$$

Let $(\psi_i^P)_{i\in\mathbb{Z}}$ be a P-wavelet packet. We consider for every $i\in\mathbb{Z}$ and $(r,x)\in[0,+\infty[\times\mathbb{R}$, the family $\psi_{i,(r,x)}^P$ given by

$$\psi_{i,(r,x)}^P(s,y)=\mathcal{T}_{(r,x)}(\psi_i^P)(s,y),\ \forall(s,y)\in[0,+\infty[\times\mathbb{R},\tag{15}$$

where $\mathcal{T}_{(r,x)}$ are the generalized translation operators given by (2). We note that

$$\forall i\in\mathbb{Z},\ \forall(r,x)\in[0,+\infty[\times\mathbb{R},\ \|\psi_{i,(r,x)}^P\|_{2,\kappa}\leqslant\|\psi_i^P\|_{2,\kappa}.\tag{16}$$

Definition 3.6. *Let $(\psi_i^P)_{i\in\mathbb{Z}}$ be a P-wavelet packet. The P-wavelet packet transform Φ_ψ^P associated with the Riemann-Liouville operator is defined on $L^2(d\kappa)$, for all $i\in\mathbb{Z}$ and $(r,x)\in[0,+\infty[\times\mathbb{R}$, by*

$$\Phi_\psi^P(f)(i,r,x)=\int_0^{+\infty}\int_{\mathbb{R}}f(s,y)\overline{\psi_{i,(r,x)}^P(s,y)}d\kappa(s,y).$$

This transform can also be written in the form

$$\Phi_\psi^P(f)(i,r,x)=f*\overline{\check{\psi}_i^P}(r,-x),\tag{17}$$

where $*$ is the generalized convolution product given by (3).

Theorem 3.7. *Let $(\psi_i^P)_{i\in\mathbb{Z}}$ be a P-wavelet packet.*
(1) (Plancherel formula for Φ_ψ^P): For every $f\in L^2(d\kappa)$, we have

$$\int_0^{+\infty}\int_{\mathbb{R}}|f(r,x)|^2d\kappa(r,x)=\sum_{i=-\infty}^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}}|\Phi_\psi^P(f)(i,s,y)|^2d\kappa(s,y).$$

(2) (Parseval formula for Φ^P_ψ): For every $f, g \in L^2(d\kappa)$, we have

$$\int_0^{+\infty}\int_{\mathbb{R}} f(r,x)\overline{g(r,x)}d\kappa(r,x) = \sum_{i=-\infty}^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}} \Phi^P_\psi(f)(i,s,y)\overline{\Phi^P_\psi(g)(i,s,y)}d\kappa(s,y).$$

Proof. Using Remark 2.7 and (17), we get

$$\begin{aligned}\int_0^{+\infty}\int_{\mathbb{R}} |\Phi^P_\psi(f)(i,s,y)|^2 d\kappa(s,y) &= \int_0^{+\infty}\int_{\mathbb{R}} |f * \overline{\check{\psi}^P_i}(s,-y)|^2 d\kappa(s,y)\\ &= \int_0^{+\infty}\int_{\mathbb{R}} |f * \overline{\check{\psi}^P_i}(s,y)|^2 d\kappa(s,y)\\ &= \int\int_{\Upsilon_+} |\mathcal{B}(f)(\zeta,\xi)|^2|\mathcal{B}(\psi^P_i)(\zeta,\xi)|^2 d\varkappa(\zeta,\xi). \qquad (18)\end{aligned}$$

Now, by Fubini-Tonelli's theorem, the relations (6) and (14), we obtain

$$\begin{aligned}&\sum_{i=-\infty}^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}} |\Phi^P_\psi(f)(i,s,y)|^2 d\kappa(s,y)\\ &= \int\int_{\Upsilon_+} |\mathcal{B}(f)(\zeta,\xi)|^2\left(\sum_{i=-\infty}^{+\infty}|\mathcal{B}(\psi^P_i)(\zeta,\xi)|^2\right) d\varkappa(\zeta,\xi)\\ &= \int_0^{+\infty}\int_{\mathbb{R}} |f(r,x)|^2 d\kappa(r,x).\end{aligned}$$

Which gives the desired result.
(2) follows from the polarization identity and (1) of Theorem 3.7. □

Lemma 3.8. *Let $(\psi^p_i)_{i\in\mathbb{Z}}$ be a P-wavelet packet such that $\mathcal{B}(\psi) \in L^\infty(d\varkappa)$. Then, for every $m, n \in \mathbb{Z}$ with $m < n$, the function*

$$\nabla^p_{\delta_n,\delta_m}(\zeta,\xi) = \frac{1}{C_\psi}\int_{\delta_n}^{\delta_m} |\mathcal{B}(\psi)(a\zeta,a\xi)|^2\frac{da}{a},$$

belongs to $L^2(d\varkappa)$, and we have

$$\|\nabla^p_{\delta_n,\delta_m}\|^2_{2,\varkappa} \leqslant \frac{1}{C^2_\psi(4\alpha+5)}\left(\frac{1}{\delta_n}-\frac{1}{\delta_m}\right)\left(\frac{1}{\delta_n^{4\alpha+5}}-\frac{1}{\delta_m^{4\alpha+5}}\right)\|\psi\|^2_{2,\kappa}\|\mathcal{B}(\psi)\|^2_{\infty,\varkappa}.$$

Proof. Using Hölder's inequality for the measure da, we get for every $(\zeta,\xi) \in \Upsilon$

$$|\nabla^p_{\delta_n,\delta_m}(\zeta,\xi)|^2 \leqslant \frac{1}{C^2_\psi}\left(\frac{1}{\delta_n}-\frac{1}{\delta_m}\right)\int_{\delta_n}^{\delta_m} |\mathcal{B}(\psi)(a\zeta,a\xi)|^4 da.$$

Now, using Fubini-Tonelli's theorem and (10), we obtain

$$\begin{aligned}\|\nabla^p_{\delta_n,\delta_m}\|^2_{2,\varkappa} &\leqslant \frac{1}{C^2_\psi}\left(\frac{1}{\delta_n}-\frac{1}{\delta_m}\right)\int_{\delta_n}^{\delta_m}\left[\int\int_{\Upsilon_+}|\mathcal{B}(\psi)(a\zeta,a\xi)|^4 d\varkappa(\zeta,\xi)\right]da\\ &\leqslant \frac{1}{C^2_\psi(4\alpha+5)}\left(\frac{1}{\delta_n}-\frac{1}{\delta_m}\right)\left(\frac{1}{\delta_n^{4\alpha+5}}-\frac{1}{\delta_m^{4\alpha+5}}\right)\|\mathcal{B}(D_a(\psi))\|^2_{2,\varkappa}\|\mathcal{B}(\psi)\|^2_{\infty,\varkappa}.\end{aligned}$$

Then, the relations (6) and (9) gives the desired result. □

In the following, we establish reproducing inversion formula of Calderón's type for the mapping Φ^P_ψ.

Theorem 3.9. *Let $(\psi^p_i)_{i\in\mathbb{Z}}$ be a P-wavelet packet such that $\mathcal{B}(\psi) \in L^\infty(d\varkappa)$. Then, for every $f \in L^2(d\kappa)$ and $m, n \in \mathbb{Z}$ with $m < n$, the function*

$$f^p_{m,n}(r,x) = \sum_{i=m}^{n-1} \int_0^{+\infty}\int_{\mathbb{R}} \Phi^P_\psi(i,r,x)\psi^p_{i,(r,x)}(s,y)d\kappa(r,x),$$

belongs to $L^2(d\kappa)$, and satisfies

$$\lim_{(m,n)\longrightarrow(-\infty,+\infty)} \|f^p_{m,n} - f\|_{2,\kappa} = 0.$$

Proof. Let f in $L^2(d\kappa)$. By (7), (8), (15) and (17), we have

$$\begin{aligned}
&\int_0^{+\infty}\int_{\mathbb{R}} \Phi^P_\psi(f)(i,s,y)\psi^P_{i,(r,x)}(s,y)d\kappa(s,y)\\
&\quad= \int_0^{+\infty}\int_{\mathbb{R}} f * \overline{\check{\psi}^P_i}(s,-y)\mathcal{T}_{(r,x)}(\psi^P_i)(s,y)d\kappa(s,y)\\
&\quad= \int_0^{+\infty}\int_{\mathbb{R}} f * \overline{\check{\psi}^P_i}(s,y)\mathcal{T}_{(r,-x)}(\check{\psi}^P_i)(s,y)d\kappa(s,y)\\
&\quad= \int\int_{\Upsilon_+} \mathcal{B}(f)(s,y)\mathcal{B}(\overline{\check{\psi}^P_i})(s,y)\overline{\mathcal{B}(\mathcal{T}_{(r,-x)}(\overline{\check{\psi}^P_i}))(s,y)}d\varkappa(s,y)\\
&\quad= \int\int_{\Upsilon_+} \mathcal{B}(f)(s,y)(\mathcal{B}(\psi^P_i)(s,y))^2\overline{\vartheta_{(s,y)}(r,x)}d\varkappa(s,y).
\end{aligned}$$

Then, from Fubini's theorem and (12), we obtain

$$\begin{aligned}
f^p_{m,n}(r,x) &= \int\int_{\Upsilon_+} \mathcal{B}(f)(s,y)\nabla^p_{\delta_n,\delta_m}(s,y)\overline{\vartheta_{(s,y)}(r,x)}d\varkappa(s,y)\\
&=\mathcal{B}^{-1}(\mathcal{B}(f)\nabla^p_{\delta_n,\delta_m})(r,x).
\end{aligned}$$

On the other hand, the function $\nabla^p_{\delta_n,\delta_m}$ belongs to $L^\infty(d\varkappa)$, from this fact and (2.4), the function $f^p_{m,n} \in L^2(d\kappa)$, and we have

$$\mathcal{B}(f^p_{m,n}) = \mathcal{B}(f)\nabla^p_{\delta_n,\delta_m}.$$

Using the previous result and (2.4), we get

$$\|f^p_{m,n} - f\|^2_{2,\kappa} = \int\int_{\Upsilon_+} |\mathcal{B}(f)(\zeta,\xi)|^2(\nabla^p_{\delta_n,\delta_m}(\zeta,\xi) - 1)^2 d\varkappa(\zeta,\xi).$$

The relation (4.6) follows from the fact that

$$\lim_{(m,n)\longrightarrow(-\infty,+\infty)} \nabla^p_{\delta_n,\delta_m}(\zeta,\xi) = 1,$$

and the dominated convergence theorem. □

Theorem 3.10. *Let $(\psi^p_i)_{i\in\mathbb{Z}}$ be a P-wavelet packet. Then, for every $f \in L^1(d\kappa)\cap L^2(d\kappa)$ such that $\mathcal{B}(f) \in L^1(d\varkappa)$, we have the reconstruction formula for ψ^P_i:*

$$f(r,x) = \sum_{i=-\infty}^{+\infty} \mathcal{J}(i,r,x),\ a.e, (r,x) \in [0,+\infty[\times\mathbb{R},$$

where $\mathcal{J}(i,r,x) = \displaystyle\int_0^{+\infty}\int_{\mathbb{R}} \Phi^P_\psi(f)(i,s,y)\psi^P_{i,(r,x)}(s,y)d\kappa(s,y).$

Proof. Let f in $L^1(d\kappa) \cap L^2(d\kappa)$.

$$\int_0^{+\infty} \int_{\mathbb{R}} \Phi_\psi^P(f)(i,s,y)\psi_{i,(r,x)}^P(s,y)d\kappa(s,y)$$
$$= \int\int_{\Upsilon_+} \mathcal{B}(f)(s,y)(\mathcal{B}(\psi_i^P)(s,y))^2\overline{\vartheta_{(s,y)}(r,x)}d\varkappa(s,y).$$

On the other hand, the function $(s,y) \longmapsto \Phi_\psi^P(f)(i,s,y) = f * \overline{\check{\psi_i^P}}(s,-y)$ belongs to $L^2(d\kappa)$ and also the function $(s,y) \longmapsto \psi_{i,(r,x)}^P(s,y) = \mathcal{T}_{(r,x)}(\psi_i^p)(s,y)$ belongs to $L^2(d\kappa)$, then, from the Cauchy-Schwarz's inequality the integral $\mathcal{J}(i,r,x)$ is absolutely convergent.
Now, from Fubini-Tonelli's theorem, (1) and (14), we obtain

$$\sum_{i=-\infty}^{+\infty} |\mathcal{J}(i,r,x)| = \sum_{i=-\infty}^{+\infty} \Big| \int\int_{\Upsilon_+} \mathcal{B}(f)(s,y)|\mathcal{B}(\psi_i^P)(s,y)|^2\overline{\vartheta_{(s,y)}(r,x)}d\varkappa(s,y)\Big|$$
$$\leqslant \int\int_{\Upsilon_+} |\mathcal{B}(f)(s,y)| \sum_{i=-\infty}^{+\infty} |\mathcal{B}(\psi_i^P)(s,y)|^2 |d\varkappa(s,y)$$
$$= \|\mathcal{B}(f)\|_{1,\varkappa} < \infty.$$

This shows that the series $\sum_{i=-\infty}^{+\infty} \mathcal{J}(i,r,x)$ is absolutely convergent, and therefore

$$\sum_{i=-\infty}^{+\infty} \mathcal{J}(i,r,x) = \sum_{i=-\infty}^{+\infty} \int\int_{\Upsilon_+} \mathcal{B}(f)(s,y)|\mathcal{B}(\psi_i^P)(s,y)|^2\overline{\vartheta_{(s,y)}(r,x)}d\varkappa(s,y).$$

Again, applying Fubini's theorem for the previous result, we get

$$\sum_{i=-\infty}^{+\infty} \mathcal{J}(i,r,x) = \int\int_{\Upsilon_+} \mathcal{B}(f)(s,y)\Big(\sum_{i=-\infty}^{+\infty} |\mathcal{B}(\psi_i^P)(s,y)|^2\Big)\overline{\vartheta_{(s,y)}(r,x)}d\varkappa(s,y).$$

Then, the result follows from (5) and (14). □

4. Scale discrete scaling function on $[0,+\infty[\times\mathbb{R}$

In this section, we define and study a scale discrete scaling function on $[0,+\infty[\times\mathbb{R}$, corresponding to the P-wavelet packet $(\psi_i^p)_{i\in\mathbb{Z}}$ studied in the previous section.

Proposition 4.1. *Let $(\psi_i^p)_{i\in\mathbb{Z}}$ be a P-wavelet packet. Then*
(1) For every $j \in \mathbb{Z}$ and $(\zeta,\xi) \in \Upsilon$, we have

$$\sum_{i=-\infty}^{j-1} (\mathcal{B}(\psi_i^p)(\zeta,\xi))^2 = \frac{1}{C_\psi}\int_{\delta_j}^{+\infty} |\mathcal{B}(\psi)(a\zeta,a\xi)|^2 \frac{da}{a}.$$

(2) For every $j \in \mathbb{Z}$, there exists a function $\phi_j^p \in L^2(d\kappa)$, such that

$$\mathcal{B}(\phi_j^p)(\zeta,\xi) = \left(\sum_{i=-\infty}^{j-1} (\mathcal{B}(\psi_i^p)(\zeta,\xi))^2\right)^{\frac{1}{2}}, \ \forall(\zeta,\xi) \in \Upsilon. \tag{19}$$

Proof. (1) From (11) and (13), we obtain

$$\sum_{i=-\infty}^{j-1}(\mathcal{B}(\psi_i^p)(\zeta,\xi))^2 = \frac{1}{C_\psi}\sum_{i=-\infty}^{j-1}\int_{\delta_{i+1}}^{\delta_i}|\mathcal{B}(\psi)(a\zeta,a\xi)|^2\frac{da}{a}$$
$$= \frac{1}{C_\psi}\int_{\delta_j}^{+\infty}|\mathcal{B}(\psi)(a\zeta,a\xi)|^2\frac{da}{a}.$$

(2) It follows from the Plancherel theorem: Theorem 2.4. □

Definition 4.2. *The sequence $(\phi_j^p)_{j\in\mathbb{Z}}$ is called scale discrete scaling function.*

For $j\in\mathbb{Z}$, the function ϕ_j^p satisfy the following property
(1)

$$0\leqslant \mathcal{B}(\phi_j^p)(\zeta,\xi)\leqslant 1, \text{ and } \lim_{j\longrightarrow+\infty}\mathcal{B}(\phi_j^p)(\zeta,\xi)=1,\ \forall(\zeta,\xi)\in\Upsilon. \tag{20}$$

(2)

$$(\mathcal{B}(\psi_j^p)(\zeta,\xi))^2 = (\mathcal{B}(\phi_{j+1}^p)(\zeta,\xi))^2-(\mathcal{B}(\phi_j^p)(\zeta,\xi))^2,\ \forall(\zeta,\xi)\in\Upsilon,$$

and

$$\sum_{j=-\infty}^{+\infty}(\mathcal{B}(\psi_j^p)(\zeta,\xi))^2=\sum_{j=-\infty}^{+\infty}(\mathcal{B}(\phi_{j+1}^p)(\zeta,\xi))^2-(\mathcal{B}(\phi_j^p)(\zeta,\xi))^2=1. \tag{21}$$

For all $j\in\mathbb{Z}$, we define the function $\phi_{j,(r,x)}^p$, by

$$\phi_{j,(r,x)}^p(s,y)=\mathcal{T}_{(r,x)}(\phi_j^p)(s,y),\ \forall(s,y)\in[0,+\infty[\times\mathbb{R}.$$

The function $\phi_{j,(r,x)}^p$ belongs to $L^2(d\kappa)$, and we have

$$\|\phi_{j,(r,x)}^p\|_{2,\kappa}\leqslant\|\phi_j^p\|_{2,\kappa}.$$

Theorem 4.3. *(1) (Plancherel formula associated with $(\phi_j^p)_{j\in\mathbb{Z}}$): For every $f\in L^2(d\kappa)$, we have*

$$\|f\|_{2,\kappa}^2=\lim_{j\longrightarrow+\infty}\int_0^{+\infty}\int_{\mathbb{R}}|\langle f,\phi_{j,(r,x)}^p\rangle_\kappa|^2 d\kappa(r,x),$$

where $\langle .,.\rangle_\kappa$ is the scalar product on $L^2(d\kappa)$.
(2) (Parseval formula associated with $(\phi_j^p)_{j\in\mathbb{Z}}$): For every $f,g\in L^2(d\kappa)$, we have

$$\int_0^{+\infty}\int_{\mathbb{R}}f(r,x)\overline{g(r,x)}d\kappa(r,x)=\lim_{j\longrightarrow+\infty}\int_0^{+\infty}\int_{\mathbb{R}}\langle f,\phi_{j,(s,y)}^p\rangle_\kappa\overline{\langle g,\phi_{j,(s,y)}^p\rangle_\kappa}d\kappa(s,y).$$

Proof. (1) For every $f\in L^2(d\kappa)$ and $j\in\mathbb{Z}$, we have

$$\langle f,\phi_{j,(r,x)}^p\rangle_\kappa = f*\overline{\check{\phi}_j^p}(r,-x). \tag{22}$$

Then, according to Remark 2.7, we obtain

$$\int_0^{+\infty}\int_{\mathbb{R}}|\langle f,\phi_{j,(r,x)}^p\rangle_\kappa|^2 d\kappa(r,x)=\int_0^{+\infty}\int_{\mathbb{R}}|f*\overline{\check{\phi}_j^p}(r,-x)|^2 d\kappa(r,x)$$
$$=\int\int_{\Upsilon_+}|\mathcal{B}(f)(\zeta,\xi)|^2|\mathcal{B}(\phi_j^p)(\zeta,\xi)|^2 d\varkappa(\zeta,\xi). \tag{23}$$

Then, the desired result follows from dominated convergence theorem, (6) and (20).
(2) We obtain the result from (1). □

Theorem 4.4. *Let $(\phi_i^p)_{i\in\mathbb{Z}}$ be a scale discrete scaling function which corresponds to the P-wavelet packet $(\psi_i^p)_{i\in\mathbb{Z}}$.*
(1) (Plancherel formula associated with $(\phi_j^p)_{j\in\mathbb{Z}}$ and Φ_ψ^p): For every $f\in L^2(d\kappa)$, we have

$$\|f\|_{2,\kappa}^2=\int_0^{+\infty}\int_{\mathbb{R}}|\langle f,\phi^p_{j,(r,x)}\rangle_\kappa|^2d\kappa(r,x)+\sum_{j=i}^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}}|\Phi_\psi^p(f)(i,r,x)|^2d\kappa(r,x).$$

(2) (Parseval formula associated with $(\phi_j^p)_{j\in\mathbb{Z}}$ and Φ_ψ^p): For every $f,g\in L^2(d\kappa)$, we have

$$\begin{aligned}\int_0^{+\infty}\int_{\mathbb{R}}f(r,x)\overline{g(r,x)}d\kappa(r,x)=&\int_0^{+\infty}\int_{\mathbb{R}}\langle f,\phi^p_{j,(s,y)}\rangle_\kappa\overline{\langle g,\phi^p_{j,(s,y)}\rangle_\kappa}d\kappa(s,y)\\&+\sum_{j=i}^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}}\Phi_\psi^p(f)(i,r,x)\overline{\Phi_\psi^p(g)(i,r,x)}d\kappa(r,x).\end{aligned}$$

Proof. (1) From Fubini-Tonelli's theorem, relations (18), (19) and (23), we get

$$\begin{aligned}&\int_0^{+\infty}\int_{\mathbb{R}}|\langle f,\phi^p_{j,(r,x)}\rangle_\kappa|^2d\kappa(r,x)+\sum_{j=i}^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}}|\Phi_\psi^p(f)(i,r,x)|^2d\kappa(r,x)\\&=\int\int_{\Upsilon_+}|\mathcal{B}(f)(\zeta,\xi)|^2\sum_{i=-\infty}^{+\infty}|\mathcal{B}(\psi_i^p)(\zeta,\xi)|^2d\varkappa(\zeta,\xi).\end{aligned}$$

Then, (6) and (21), gives the desired result.
(2) It follows from (1). □

Lemma 4.5. *Let $(\phi_i^p)_{i\in\mathbb{Z}}$ be a scale discrete scaling function which corresponds to the P-wavelet packet $(\psi_i^p)_{i\in\mathbb{Z}}$ such that $\mathcal{B}(\psi)\in L^\infty(d\varkappa)$. Then, for every $n\in\mathbb{Z}$ and $(\zeta,\xi)\in\Upsilon$, the function*

$$\nabla^p_{\delta_n,\infty}(\zeta,\xi)=\frac{1}{C_\psi}\int_{\delta_n}^{+\infty}|\mathcal{B}(\psi)(a\zeta,a\xi)|^2\frac{da}{a},$$

belongs to $L^2(d\varkappa)$.

Proof. The proof is similar to the proof of Lemma 3.8. □

Theorem 4.6. *Let $(\phi_i^p)_{i\in\mathbb{Z}}$ be a scale discrete scaling function which corresponds to the P-wavelet packet $(\psi_i^p)_{i\in\mathbb{Z}}$ such that $\mathcal{B}(\psi)\in L^\infty(d\varkappa)$. Then, for every $f\in L^2(d\kappa)$ and $n\in\mathbb{Z}$, the function*

$$f^p_{\infty,n}(r,x)=\int_0^{+\infty}\int_{\mathbb{R}}\langle f,\phi^p_{n,(r,x)}\rangle_\kappa\phi^p_{n,(r,x)}(s,y)d\kappa(s,y),$$

belongs to $L^2(d\kappa)$ and satisfies

$$\lim_{n\longrightarrow+\infty}\|f^p_{\infty,n}-f\|_{2,\kappa}=0.$$

Proof. Let $f\in L^2(d\kappa)$. By (22), we can write

$$f^p_{\infty,n}(r,x)=\int_0^{+\infty}\int_{\mathbb{R}}f*\check{\overline{\phi^p_n}}(s,-y)\overline{\mathcal{T}_{(r,x)}(\overline{\phi^p_n})(s,y)}d\kappa(s,y).$$

On the other hand, the function $(r,x) \longrightarrow f * \overline{\check{\phi}_j^p}(r,-x)$ belongs to $L^2(d\kappa)$, then, by (8) and (19), we get

$$\begin{aligned}
f_{\infty,n}^p(r,x) &= \int_0^{+\infty}\int_{\mathbb{R}} f * \overline{\check{\phi}_n^p}(r,-x)\overline{\mathcal{T}_{(s,y)}(\overline{\phi_n^p})(r,x)}d\kappa(s,y)\\
&= \int\int_{\Upsilon_+} \mathcal{B}(f)(s,y)(\mathcal{B}(\phi_n^P)(s,y))^2\overline{\vartheta_{(s,y)}(r,x)}d\varkappa(s,y)\\
&= \int\int_{\Upsilon_+} \mathcal{B}(f)(s,y)\nabla_{\delta_n,\infty}^p(s,y)\overline{\vartheta_{(s,y)}(r,x)}d\varkappa(s,y)\\
&= \mathcal{B}^{-1}(\mathcal{B}(f)\nabla_{\delta_n,\infty}^p)(r,x).
\end{aligned}$$

On the other hand, the function $\nabla_{\delta_n,\infty}^p$ belongs to $L^\infty(d\varkappa)$, from this fact and (2.4), the function $f_{\infty,n}^p \in L^2(d\kappa)$, and we have

$$\mathcal{B}(f_{\infty,n}^p) = \mathcal{B}(f)\nabla_{\delta_n,\infty}^p.$$

Now, by (2.4), we get

$$\|f_{\infty,n}^p - f\|_{2,\kappa}^2 = \int\int_{\Upsilon_+} |\mathcal{B}(f)(\zeta,\xi)|^2(\nabla_{\delta_n,\infty}^p(\zeta,\xi)-1)^2 d\varkappa(\zeta,\xi).$$

Then, the result follows from the equality

$$\lim_{n\longrightarrow+\infty}\nabla_{\delta_n,\infty}^p(\zeta,\xi) = 1,$$

and the dominated convergence theorem. □

Theorem 4.7. *Let $(\phi_i^p)_{i\in\mathbb{Z}}$ be a scale discrete scaling function which corresponds to the P-wavelet packet $(\psi_i^p)_{i\in\mathbb{Z}}$. For every $f \in L^1(d\kappa)\cap L^2(d\kappa)$ such that $\mathcal{B}(f) \in L^1(d\varkappa)$, we have the following reconstruction formulas,*
(1)

$$f(r,x) = \lim_{j\longrightarrow+\infty}\mathcal{K}(j,r,x),\ a.e,(r,x)\in[0,+\infty[\times\mathbb{R},$$

where

$$\mathcal{K}(j,r,x) = \int_0^{+\infty}\int_{\mathbb{R}}\langle f,\phi_{j,(s,y)}^p\rangle_\kappa\phi_{j,(s,y)}^p(r,x)d\kappa(s,y).$$

(2)

$$f(r,x) = \mathcal{K}(j,r,x) + \sum_{i=j}^{+\infty}\mathcal{L}(i,r,x),\ a.e,(r,x)\in[0,+\infty[\times\mathbb{R},$$

where

$$\mathcal{L}(i,r,x) = \int_0^{+\infty}\int_{\mathbb{R}}\Phi_\psi^p(f)(i,s,y)\psi_{i,(s,y)}^p(r,x)d\kappa(s,y).$$

Proof. Let $f \in L^1(d\kappa)\cap L^2(d\kappa)$ such that $\mathcal{B}(f)\in L^1(d\varkappa)$. By (22), we can write

$$\begin{aligned}
\mathcal{K}(j,r,x) &= \int_0^{+\infty}\int_{\mathbb{R}} f * \overline{\check{\phi}_j^p}(r,-x)\overline{\mathcal{T}_{(s,y)}(\overline{\phi_j^p})(r,x)}d\kappa(s,y)\\
&= \int\int_{\Upsilon_+}\mathcal{B}(f)(s,y)|\mathcal{B}(\phi_j^P)(s,y)|^2\overline{\vartheta_{(s,y)}(r,x)}d\varkappa(s,y).
\end{aligned}$$

From the dominated convergence theorem, relations (5) and (20), we get

$$\begin{aligned}\lim_{j\longrightarrow+\infty}\mathcal{K}(j,r,x) &= \int\int_{\Upsilon_+}\mathcal{B}(f)(s,y)\lim_{j\longrightarrow+\infty}|\mathcal{B}(\phi_j^P)(s,y)|^2\overline{\vartheta_{(s,y)}(r,x)}d\varkappa(s,y)\\ &= \int\int_{\Upsilon_+}\mathcal{B}(f)(s,y)\overline{\vartheta_{(s,y)}(r,x)}d\varkappa(s,y) = f(r,x).\end{aligned}$$

The proof of (2) follows the same way of Theorem 3.10. □

5. S-wavelet packet related to the Riemann-Liouville operator

In this section, we define and study the S-wavelet packet transform and its dual associated with the Riemann-Liouville operator, and we prove for these transforms Plancherel and reconstruction formulas.

Definition 5.1. *A sequence $(\varpi_j^S)_{j\in\mathbb{Z}}$ in $L^2(d\kappa)$ is called an S-wavelet packet associated with the Riemann-Liouville operator if it verifies the following conditions:*
(1) For every $j\in\mathbb{Z}$, $\mathcal{B}(\varpi_j^S)$ is real-valued.
(2) For every $(\zeta,\xi)\in\Upsilon$, we have

$$\alpha\leqslant\mathcal{B}(\varpi_j^S)(\zeta,\xi)\leqslant\beta,\ \forall j\in\mathbb{Z},$$

where α,β are constants with $0<\alpha<\beta<\infty$.

Definition 5.2. *Let $(\varpi_j^S)_{j\in\mathbb{Z}}$ be a S-wavelet packet.*
(1) The S-wavelet packet transform Φ_ψ^S is defined for a function $f\in L^2(d\kappa)$, by

$$\Phi_\psi^S(f)(j,r,x) = \int_0^{+\infty}\int_{\mathbb{R}} f(s,y)\overline{\varpi_{j,(r,x)}^S(s,y)}d\kappa(s,y),\ \forall j\in\mathbb{Z},\forall(r,x)\in[0,+\infty[\times\mathbb{R},$$

where $\varpi_{j,(r,x)}^S$ is the function defined by

$$\varpi_{j,(r,x)}^S(s,y) = \mathcal{T}_{(r,x)}(\varpi_j^S)(s,y).$$

(2) The corresponding dual S-wavelet packet $(\tilde{\varpi}_j^S)_{j\in\mathbb{Z}}$ is given by

$$\mathcal{B}(\tilde{\varpi}_j^S)(\zeta,\xi) = \frac{\mathcal{B}(\varpi_j^S)(\zeta,\xi)}{\sum_{j=-\infty}^{+\infty}\left(\mathcal{B}(\varpi_j^S)(\zeta,\xi)\right)^2},\ \forall(\zeta,\xi)\in\Upsilon.$$

(3) The dual S-wavelet packet transform $\tilde{\Phi}_\psi^S$ is defined for a function $f\in L^2(d\kappa)$, by

$$\tilde{\Phi}_\psi^S(f)(j,r,x) = \int_0^{+\infty}\int_{\mathbb{R}} f(s,y)\overline{\tilde{\varpi}_{j,(r,x)}^S(s,y)}d\kappa(s,y),\ \forall j\in\mathbb{Z},\forall(r,x)\in[0,+\infty[\times\mathbb{R},$$

where $\tilde{\varpi}_{j,(r,x)}^S$ is the function defined by

$$\tilde{\varpi}_{j,(r,x)}^S(s,y) = \mathcal{T}_{(r,x)}(\tilde{\varpi}_j^S)(s,y).$$

The transforms Φ_ψ^S and $\tilde{\Phi}_\psi^S$, can be written as

$$\Phi_\psi^S(f)(j,r,x) = f*\overline{\check{\varpi}_j^S}(r,-x),\ \forall j\in\mathbb{Z},\forall(r,x)\in[0,+\infty[\times\mathbb{R}, \tag{24}$$

and

$$\tilde{\Phi}_\psi^S(f)(j,r,x) = f*\overline{\check{\tilde{\varpi}}_j^S}(r,-x),\ \forall j\in\mathbb{Z},\forall(r,x)\in[0,+\infty[\times\mathbb{R}. \tag{25}$$

Proposition 5.3. *Let $(\varpi^S_j)_{j\in\mathbb{Z}}$ be a S-wavelet packet and let $(\tilde{\varpi}^S_j)_{j\in\mathbb{Z}}$ the corresponding dual S-wavelet packet. We have the following properties:*
(1) For every $(\zeta,\xi)\in\Upsilon$,

$$\sum_{j=-\infty}^{+\infty}\mathcal{B}(\varpi^S_j)(\zeta,\xi)\mathcal{B}(\tilde{\varpi}^S_j)(\zeta,\xi)=1,\ \forall j\in\mathbb{Z}, \tag{26}$$

and

$$\sum_{j=-\infty}^{+\infty}\Big(\mathcal{B}(\tilde{\varpi}^S_j)(\zeta,\xi)\Big)^2=\Big(\sum_{j=-\infty}^{+\infty}\Big(\mathcal{B}(\varpi^S_j)(\zeta,\xi)\Big)^2\Big)^{-1}. \tag{27}$$

(2) For every $(\zeta,\xi)\in\Upsilon$,

$$\sum_{i=-\infty}^{j-1}\mathcal{B}(\varpi^S_i)(\zeta,\xi)\mathcal{B}(\tilde{\varpi}^S_i)(\zeta,\xi)=\frac{\sum_{i=-\infty}^{j-1}\Big(\mathcal{B}(\varpi^S_i)(\zeta,\xi)\Big)^2}{\sum_{i=-\infty}^{+\infty}\Big(\mathcal{B}(\varpi^S_i)(\zeta,\xi)\Big)^2},\ \forall j\in\mathbb{Z}.$$

Theorem 5.4. *(Plancherel formula)*
Let $(\varpi^S_i)_{i\in\mathbb{Z}}$ be an S-wavelet packet and let $(\tilde{\varpi}^S_i)_{i\in\mathbb{Z}}$ the corresponding dual S-wavelet packet. Then, for every $f\in L^2(d\kappa)$, we have

$$\int_0^{+\infty}\int_{\mathbb{R}}|f(r,x)|^2d\kappa(r,x)=\sum_{i=-\infty}^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}}\Phi^S_\psi(f)(i,s,y)\overline{\tilde{\Phi}^S_\psi(f)(i,s,y)}d\kappa(s,y).$$

Proof. From relations (8), (24) and (25), we get

$$\begin{aligned}&\int_0^{+\infty}\int_{\mathbb{R}}\Phi^S_\psi(f)(i,s,y)\overline{\tilde{\Phi}^S_\psi(f)(i,s,y)}d\kappa(s,y)\\&\qquad=\int_0^{+\infty}\int_{\mathbb{R}}\Phi^S_\psi(f)(i,s,-y)\overline{\tilde{\Phi}^S_\psi(f)(i,s,-y)}d\kappa(s,y)\\&\qquad=\int\int_{\Upsilon_+}|\mathcal{B}(f)(\zeta,\xi)|^2\mathcal{B}(\varpi^S_i)(\zeta,\xi)\mathcal{B}(\tilde{\varpi}^S_i)(\zeta,\xi)d\varkappa(\zeta,\xi).\end{aligned}$$

Now, from Fubini-Tonelli's theorem, Cauchy Schwarz's inequality and (27), we obtain

$$\begin{aligned}&\sum_{i=-\infty}^{+\infty}\Big|\int_0^{+\infty}\int_{\mathbb{R}}\Phi^S_\psi(f)(i,s,y)\overline{\tilde{\Phi}^S_\psi(f)(i,s,y)}d\kappa(s,y)\Big|\\&\quad\leqslant\int\int_{\Upsilon_+}|\mathcal{B}(f)(\zeta,\xi)|^2\sum_{i=-\infty}^{+\infty}|\mathcal{B}(\varpi^S_i)(\zeta,\xi)\mathcal{B}(\tilde{\varpi}^S_i)(\zeta,\xi)|d\varkappa(\zeta,\xi)\\&\quad\leqslant\|f\|^2_{2,\kappa}<\infty.\end{aligned}$$

Again, applying Fubini's theorem and (26), we get

$$\begin{aligned}&\sum_{i=-\infty}^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}}\Phi^S_\psi(f)(i,s,y)\overline{\tilde{\Phi}^S_\psi(f)(i,s,y)}d\kappa(s,y)\\&\qquad=\int\int_{\Upsilon_+}|\mathcal{B}(f)(\zeta,\xi)|^2\sum_{i=-\infty}^{+\infty}\mathcal{B}(\varpi^S_i)(\zeta,\xi)\mathcal{B}(\tilde{\varpi}^S_i)(\zeta,\xi)d\varkappa(\zeta,\xi)\\&\qquad=\|f\|^2_{2,\kappa}.\end{aligned}$$

Which achieves the proof. □

Theorem 5.5. *Let $(\varpi_i^S)_{i\in\mathbb{Z}}$ be an S-wavelet packet and let $(\tilde{\varpi}_i^S)_{i\in\mathbb{Z}}$ the corresponding dual S-wavelet packet. Then, for every $f \in L^1(d\kappa) \cap L^2(d\kappa)$, such that $\mathcal{B}(f) \in L^1(d\varkappa)$, we have the following reconstruction formulas,*
(1)

$$f(r,x) = \sum_{i=-\infty}^{+\infty} \int_0^{+\infty} \int_{\mathbb{R}} \Phi_\psi^S(f)(i,s,y)\tilde{\varpi}_{i,(s,y)}^S(r,x)d\kappa(s,y),\ a.e, (r,x) \in [0,+\infty[\times\mathbb{R}.$$

(2)

$$f(r,x) = \sum_{i=-\infty}^{+\infty} \int_0^{+\infty} \int_{\mathbb{R}} \tilde{\Phi}_\psi^S(f)(i,s,y)\varpi_{i,(s,y)}^S(r,x)d\kappa(s,y),\ a.e, (r,x) \in [0,+\infty[\times\mathbb{R}.$$

Proof. The result can be proved in the same way of Theorem 3.10. □

Definition 5.6. *Let $(\varpi_j^S)_{j\in\mathbb{Z}}$ be a S-wavelet packet and let $(\tilde{\varpi}_j^S)_{j\in\mathbb{Z}}$ the corresponding dual S-wavelet packet. The scale discrete scaling function $(\varrho_j^S)_{j\in\mathbb{Z}}$ corresponding to $(\varpi_j^S)_{j\in\mathbb{Z}}$ is defined by*

$$\mathcal{B}(\varrho_j^S)(\zeta,\xi) = \Big(\sum_{i=-\infty}^{j-1} \mathcal{B}(\varpi_i^S)(\zeta,\xi)\mathcal{B}(\tilde{\varpi}_i^S)(\zeta,\xi)\Big)^{\frac{1}{2}},\ \forall(\zeta,\xi) \in \Upsilon.$$

Proposition 5.7. *The scale discrete scaling function $(\varrho_j^S)_{j\in\mathbb{Z}}$ corresponding to $(\varpi_j^S)_{j\in\mathbb{Z}}$ satisfies the following properties:*

$$0 \leqslant \mathcal{B}(\varrho_j^S)(\zeta,\xi) \leqslant 1,\ \forall j \in \mathbb{Z}, \forall(\zeta,\xi) \in \Upsilon,$$

and

$$\lim_{j\longrightarrow+\infty} \mathcal{B}(\varrho_j^S)(\zeta,\xi) = 1.$$

Theorem 5.8. *(Plancherel formula) For every $f \in L^2(d\kappa)$ and $j \in \mathbb{Z}$, we have*

$$\int_0^{+\infty}\int_{\mathbb{R}} |f(r,x)|^2 d\kappa(r,x) = \lim_{j\longrightarrow+\infty} \int_0^{+\infty}\int_{\mathbb{R}} |\langle f, \varrho_{j,(r,x)}^S\rangle_\kappa|^2 d\kappa(r,x),$$

and

$$\begin{aligned}\int_0^{+\infty}\int_{\mathbb{R}} |f(r,x)|^2 d\kappa(r,x) = \lim_{j\longrightarrow+\infty} &\int_0^{+\infty}\int_{\mathbb{R}} |\langle f, \varrho_{j,(r,x)}^S\rangle_\kappa|^2 d\kappa(r,x) \\ &+ \sum_{i=j}^{+\infty} \int_0^{+\infty}\int_{\mathbb{R}} \Phi_\psi^S(f)(i,s,y)\overline{\tilde{\Phi}_\psi^S(f)(i,s,y)} d\kappa(s,y),\end{aligned}$$

where

$$\varrho_{j,(r,x)}^S(s,y) = \mathcal{T}_{(r,x)}(\varrho_j^S)(s,y),\ \forall j \in \mathbb{Z}, \forall(s,y) \in [0,+\infty[\times\mathbb{R}.$$

Proof. The results can be proved in the same way of Theorems 4.3 and 4.4. □

Theorem 5.9. *Let $(\varpi_j^S)_{j\in\mathbb{Z}}$ be an S-wavelet packet and let $(\tilde{\varpi}_j^S)_{j\in\mathbb{Z}}$ the corresponding dual S-wavelet packet. For every $f \in L^1(d\kappa) \cap L^2(d\kappa)$ such that $\mathcal{B}(f) \in L^1(d\kappa)$, we have the following reconstruction formulas,*
(1) For almost all $(r,x) \in [0,+\infty[\times\mathbb{R}$,

$$f(r,x) = \lim_{j\longrightarrow+\infty} \int_0^{+\infty}\int_{\mathbb{R}} \langle f, \varrho_{j,(s,y)}^S\rangle_\kappa \varrho_{j,(s,y)}^S(r,x) d\kappa(s,y).$$

(2) For every $j \in \mathbb{Z}$ and for almost all $(r, x) \in [0, +\infty[\times\mathbb{R}$,

$$f(r,x) = \int_0^{+\infty}\int_{\mathbb{R}} \langle f, \varrho^S_{j,(s,y)}\rangle_\kappa \varrho^S_{j,(s,y)}(r,x)d\kappa(s,y) + \sum_{i=j}^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}} \Phi^S_\psi(f)(i,s,y)\tilde{\varpi}^S_j(s,y)d\kappa(s,y).$$

(3) For every $j \in \mathbb{Z}$ and for almost all $(r, x) \in [0, +\infty[\times\mathbb{R}$,

$$f(r,x) = \int_0^{+\infty}\int_{\mathbb{R}} \langle f, \varrho^S_{j,(s,y)}\rangle_\kappa \varrho^S_{j,(s,y)}(r,x)d\kappa(s,y) + \sum_{i=j}^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}} \tilde{\Phi}^S_\psi(f)(i,s,y)\varpi^S_j(s,y)d\kappa(s,y).$$

Proof. The Proof of this theorem follows the same way of Theorem 4.4. □

Acknowledgements The authors extend their appreciation to the Deanship of Scientific Research at Northern Border University, Arar, KSA for funding this research work through the project number " NBU-FFR-2025-2492-03".

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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