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The maximal subsemigroups of the singular part of endomorphism monoids of the star graphs

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Abstract. In this paper, we determine the maximal subsemigroups of the singular part of endomorphism monoids of the star graphs S_n, for a given positive integer n, namely $End(S_n)$, $wEnd(S_n)$, and $swEnd(S_n)$, respectively. The monoid $wEnd(S_n)$ of all weak endomorphisms on S_n, the monoid $End(S_n)$ of all endomorphisms on S_n as well as the monoid $swEnd(S_n)$ of all strong weak endomorphisms on S_n is regular. We also determine the maximal regular submonoids of the singular part of $wEnd(S_n)$, $End(S_n)$, and $swEnd(S_n)$.

1. Introduction and preliminaries

A proper subsemigroup of a semigroup S is maximal if it is contained in no other proper subsemigroup of S. An element $a \in S$ is called *regular* if there is an element $b \in S$ with $aba = a$. The semigroup S is called *regular semigroup* if all elements in S are regular. A proper regular subsemigroup of a (regular) semigroup S is maximal if it is not contained in any other proper regular subsemigroup of S. Maximal subsemigroups of full transformations have been extensively studied. A transformation different from a permutation is called *singular transformation*. In particular, semigroups of singular transformations have been investigated. We would like to mention here only a few authors and some interesting semigroups (of transformations), for which the maximal subsemigroups have been determined.

Graham, Graham, and Rhodes found out that every maximal subsemigroup of a finite semigroup must be a type given in [8]. In [5], Donoven, Mitchell, and Wilson present an algorithm for the calculations of the maximal subsemigroup of an arbitrary finite semigroup, starting on the results by Graham, Graham, and Rhodes. East, Kumar, Mitchell, and Wilson determined the maximal subsemigroups of several finite monoids of transformations in [6]. Yang described the maximal subsemigroups of semigroup of all singular transformations on a finite set in [14]. The maximal subsemigroups of the ideals of the semigroup of all transformations on a finite set were presented by Yang and Yang in [13]. Maximal subsemigroups of infinite symmetric groups was given by Mendes-Goncalves and Sullivan in [11]. The maximal subsemigroups of all transformations on a finite chain, preseving the order are determined in [1]. For the semigroup of all

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orientation-preserving or orientation-reversing transformations on a finite chain, the maximal subsemigroups were determined by Dimitrova, Fernandes, and the first author of the present paper in [3]. Also maximal regular subsemigroups were studied, for example, for the ideals of the monoid of all transformations on a finite set by You in [15], for the ideals of the monoid of all monotone transformations on a finite set by Dimitrova and the first author of the manusrcipt of the present paper in [2], and for particular other semigroups of transformations by Yuan and Zhao in [16].

On the other hand, endomorphism semigroups of graphs were intensively studied, for example, for the cycle graph in [4] or for the path with n vertices in [9]. In particular, regular endomorphism monoids were studied by Wilkeit in [12] and Li in [10].

In the present paper, we study the maximal as well as the maximal regular subsemigroups of semigroups of endomorphisms for the star graph. Let $G = (V, E)$ be a simple graph (i.e., undirected, without loops, and without multiple edges). Let α be a full transformation of V, i.e., a self-mapping on V. We say that α is:

- an *endomorphism* of G if $\{u, v\} \in E$ implies $\{u\alpha, v\alpha\} \in E$, for all $u, v \in V$;
- a *weak endomorphism* of G if $\{u, v\} \in E$ and $u\alpha \neq v\alpha$ imply $\{u\alpha, v\alpha\} \in E$, for all $u, v \in V$;
- a *strong endomorphism* of G if $\{u, v\} \in E$ if and only if $\{u\alpha, v\alpha\} \in E$, for all $u, v \in V$;
- a *strong weak endomorphism* of G if $\{u, v\} \in E$ and $u\alpha \neq v\alpha$ if and only if $\{u\alpha, v\alpha\} \in E$, for all $u, v \in V$.

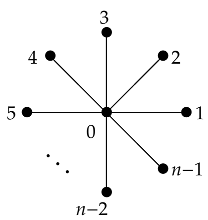
Denote by:

- $End(G)$ the set of all singular endomorphisms of G;
- $wEnd(G)$ the set of all singular weak endomorphisms of G;
- $sEnd(G)$ the set of all singular strong endomorphisms of G;
- $swEnd(G)$ the set of all singular strong weak endomorphisms of G.

Note that we use here the same notation, that is often used in literature for the corresponding monoid containing also permutations. Clearly, $End(G)$, $wEnd(G)$, $sEnd(G)$, and $swEnd(G)$ are semigroups under composition of maps. It is also obvious that $sEnd(G) \subseteq End(G)$ and $sEnd(G) \subseteq swEnd(G) \subseteq wEnd(G)$.

We fix a positive integer n. Let T_{n-1} be the set of all full transformations on the set $X_{n-1} = \{1, 2, \dots, n-1\}$, let $[n\backslash r] = X_{n-1}\backslash\{r\}$ for any $r \in X_{n-1}$, and let $\Omega_n = X_{n-1} \cup \{0\}$.
Denote by S_n the *star graph* (i.e., a tree with diameter two) with n vertices and fix

$$S_n = (\Omega_n, \{\{0, i\} : i \in X_{n-1}\}).$$



Let $\mathscr{T}_{n-1}$ be the set of all full transformations on Ω_n. Further, let $\mathscr{T}^0_{n-1}$ be the set of all $\alpha \in \mathscr{T}_{n-1}$ with $0\alpha = 0$ and let T^0_{n-1} be the set of all $\alpha \in \mathscr{T}^0_{n-1}$ such that α restricted to X_{n-1} belongs to T_{n-1} (in symbols: $\alpha|_{X_{n-1}} \in T_{n-1}$). For $\alpha \in \mathscr{T}_{n-1}$, let $\mathrm{Im}(\alpha) = \{x\alpha : x \in \Omega_n\}$ be the image of α and let $\mathrm{rank}(\alpha) = |\mathrm{Im}(\alpha)|$ be the rank of α. For $r \in \{3, 4, \dots, n-1\}$, let $J_r = \{\alpha \in \mathscr{T}^0_{n-1} : \mathrm{rank}(\alpha) = r\}$, $J_1 = \{\alpha \in \mathscr{T}_{n-1} : \mathrm{rank}(\alpha) = 1\}$, and $J_2 = \{\alpha \in \mathscr{T}_{n-1} : 0\alpha \in X_{n-1}, \mathrm{Im}(\alpha) = \{0, 0\alpha\}\}$. It is easy to verify that
$End(S_n) = T^0_{n-1} \cup \{\alpha \in J_2 : \mathrm{Im}(\alpha|_{X_{n-1}}) = \{0\}\}$,
$swEnd(S_n) = End(S_n) \cup J_1$, and
$wEnd(S_n) = \mathscr{T}^0_{n-1} \cup \{\alpha \in J_1 \cup J_2 : 0\alpha \neq 0\}$.

If $\alpha \in End(S_n)$ and $\{x\alpha, y\alpha\} \in E = \{\{0,i\} : i \in X_{n-1}\}$, then $x = 0$ and $y \in X_{n-1}$ (or conversely), i.e., $\{x, y\} \in E$. So, $\alpha \in sEnd(S_n)$. This shows that $End(S_n)$ and $sEnd(S_n)$ coincide, and we need not to consider the semigroup $sEnd(S_n)$.

This paper is devoted to studying the semigroups $wEnd(S_n), End(S_n)$, and $swEnd(S_n)$. In Section 2, we find the maximal subsemigroups of these structures. By [12], $End(S_n)$ is regular. Since J_1 consists entirely of idempotents, we can conclude that $End(S_n) \cup J_1$ is also regular, i.e., $swEnd(S_n)$ is regular. In [7], it was shown that $wEnd(S_n)$ is also regular. In the third section, we will determine the maximal regular subsemigroups of the regular semigroups $wEnd(S_n)$, $swEnd(S_n)$, and $End(S_n)$.

2. Maximal subsemigroups

In this section, we determine the maximal subsemigroups of the semigroups $wEnd(S_n)$, $swEnd(S_n)$, and $End(S_n)$. First, let us consider the semigroup $wEnd(S_n)$.
If $n = 1$, then S_1 is a point and $wEnd(S_1) = \emptyset$.
If $n = 2$, then S_2 is a two-element chain and $wEnd(S_2)$ consists of two constant mappings π_0 and π_1. So, we have $\{\pi_0\}$ and $\{\pi_1\}$ as the both maximal subsemigroups of $wEnd(S_2)$ as well as of $swEnd(S_2)$. Moreover $End(S_2) = \emptyset$. We consider now the case $n = 3$. Clearly, $wEnd(S_3) = \left\{\begin{pmatrix} 0 & 1 & 2 \\ i_1 & i_2 & i_3 \end{pmatrix} : i_1, i_2, i_3 \in \{0,1\}\right\} \cup \left\{\begin{pmatrix} 0 & 1 & 2 \\ i_1 & i_2 & i_3 \end{pmatrix} : i_1, i_2, i_3 \in \{0,2\}\right\}$.

Lemma 2.1. *Let T be a subsemigroup of $wEnd(S_3)$. Then T is maximal if and only if T is one of the following forms:*

$$T_1 = wEnd(S_3)\setminus\left\{\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix}\right\},$$

$$T_2 = wEnd(S_3)\setminus\left\{\begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 2 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}\right\},$$

$$T_3 = wEnd(S_3)\setminus\left\{\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 0 \end{pmatrix}\right\},$$

$$T_4 = wEnd(S_3)\setminus\left\{\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 2 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 2 \end{pmatrix}\right\},$$

$$T_5 = wEnd(S_3)\setminus\left\{\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}\right\}, \textit{and}$$

$$T_6 = wEnd(S_3)\setminus\left\{\begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 2 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix}\right\}.$$

Proof. It is easy to verify that $T_1, T_2, \ldots, T_6$ are maximal subsemigroups of $wEnd(S_3)$.
Conversely, let T' be a maximal subsemigroup of $wEnd(S_3)$. Assume that there are $\alpha_i \in T'\setminus T_i$ for all $i \in \{1, 2, \ldots, 6\}$. By straightforward calculations, one can show that $wEnd(S_3) =< \alpha_1, \alpha_2, \ldots, \alpha_6 >\subseteq T$, a contradiction to T is a proper subsemigroup of $wEnd(S_3)$. □

So, we suppose that $n \geq 4$.

We put $U_{0,1} = \{\alpha \in J_2 : 0\alpha \neq 0\}$ and $A_{0,1} = wEnd(S_n)\setminus U_{0,1}$. We will obtain five types of maximal subsemigroups of $wEnd(S_n)$.

Lemma 2.2. *$A_{0,1}$ is a maximal subsemigroup of $wEnd(S_n)$.*

Proof. Let $\alpha, \beta \in A_{0,1}$. Then rank$(\alpha) \neq 2$ or $0\alpha = 0$ and rank$(\beta) \neq 2$ or $0\beta = 0$.
If rank$(\alpha) = 1$ or rank$(\beta) = 1$ then rank$(\alpha\beta) = 1$, i.e., $\alpha\beta \in A_{0,1}$. If rank$(\alpha) > 2$ (rank$(\beta) > 2$) then $0\alpha = 0$ ($0\beta = 0$). So, in the remaining case, we have $0\alpha\beta = 0\beta = 0$, i.e., $\alpha\beta \in A_{0,1}$. This shows that $A_{0,1}$ is a subsemigroup of $wEnd(S_n)$ and it remains to show that $A_{0,1}$ is maximal. For this let $\alpha, \beta \in U_{0,1}$ and we

have to show that $\beta \in< A_{0,1}, \alpha >$. There are $j, i_\alpha, i_\beta \in X_{n-1}$ such that $j\alpha = 0, 0\alpha = i_\alpha$, and $0\beta = i_\beta$. Let $U = 0\beta^{-1} = \{x \in X_{n-1} : x\beta = 0\}$. Then we define $\gamma, \delta \in \mathscr{T}_{n-1}$ by

$$x\gamma = \begin{cases} j & \text{if } x \in U, \\ 0 & \text{otherwise} \end{cases} \quad \text{and } x\delta = \begin{cases} i_\beta & \text{if } x = i_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that $\gamma, \delta \in wEnd(S_n)$ and, moreover, $0\gamma = 0\delta = 0$ implies $\gamma, \delta \notin U_{0,1}$, i.e., $\gamma, \delta \in A_{0,1}$. For $x \in X_{n-1} \backslash U$, we have $x\gamma\alpha\delta = 0\alpha\delta = i_\alpha\delta = i_\beta = x\beta$ and, for $x \in U$, we have $x\gamma\alpha\delta = j\alpha\delta = 0\delta = 0 = x\beta$. This shows $\gamma\alpha\delta = \beta$, i.e., $\beta \in< A_{0,1}, \alpha >$. □

For $i \in X_{n-1}$, we put $U_i = \{\alpha \in J_{n-1} : i\alpha = 0\}$ and $A_i = wEnd(S_n) \backslash U_i$.

Lemma 2.3. *A_i is a maximal subsemigroup of $wEnd(S_n)$ for all $i \in X_{n-1}$.*

Proof. Let $i \in X_{n-1}$. Let $\alpha, \beta \in J_{n-1} \backslash U_i$. Then $i\alpha \neq 0$. If $(i\alpha)\beta \neq 0$ then $\alpha\beta \notin U_i$. If $(i\alpha)\beta = 0$, then there are $k < l \in X_{n-1}$ with $k\alpha = l\alpha$ or there is $k \in [n \backslash i]$ with $k\alpha = 0$. In both cases, we obtain that $\text{rank}(\alpha\beta) < n - 1$, i.e., $\alpha\beta \in A_i$. This shows that A_i is a subsemigroup of $wEnd(S_n)$.
We next show that A_i is maximal. Let $\alpha, \beta \in U_i$. Then there is $a \in X_{n-1} \backslash \text{Im}(\alpha) \neq \emptyset$ and let $b \in \text{Im}(\alpha)$. Since $i\alpha = 0$ and $\text{rank}(\alpha) = n - 1$, we obtain that $\text{Im}(\alpha|_{[n \backslash i]}) = [n \backslash a]$. We define $\gamma \in \mathscr{T}_{n-1}$ as follows: $a\gamma = b$, $0\gamma = 0$, and $z\alpha\gamma = z\beta$ for all $z \in [n \backslash i]$, i.e., $\alpha\gamma = \beta$. It is easy to verify that $\gamma \in wEnd(S_n)$ and $\text{Im}(\gamma|_{X_{n-1}}) \subseteq X_{n-1}$, i.e., $\gamma \notin U_i$. Hence, $\beta \in< A_i, \alpha >$. □

For $i < j \in X_{n-1}$, we put $U_{i,j} = \{\alpha \in J_{n-1} : i\alpha = j\alpha\}$ and $A_{i,j} = wEnd(S_n) \backslash U_{i,j}$. We put $U_{j,i} = U_{i,j}$ by technical reasons.

Lemma 2.4. *Let $i < j \in X_{n-1}$. Then $A_{i,j}$ is a maximal subsemigroup of $wEnd(S_n)$.*

Proof. Let $\alpha, \beta \in J_{n-1} \backslash U_{i,j}$. We have $i\alpha \neq j\alpha$. Then there are $r < s \in \Omega_n$ with $r\alpha = s\alpha$. Then $i \neq r$ or $s \neq j$, where $r\alpha\beta = s\alpha\beta$. Then $i\alpha\beta \neq j\alpha\beta$ or $\text{rank}(\alpha\beta) < n - 1$. Hence, $\alpha\beta \notin U_{i,j}$. This shows that $A_{i,j}$ is a subsemigroup of $wEnd(S_n)$ and it remains to show that $A_{i,j}$ is maximal.
For this let $\alpha, \beta \in U_{i,j}$ and we have to show that $\beta \in< A_{i,j}, \alpha >$. We observe that there is $k \in X_{n-1} \backslash \text{Im}(\alpha)$. Since $i\alpha = j\alpha$ and $\text{rank}(\alpha) = n - 1$, we obtain that $\text{Im}(\alpha|_{[n \backslash i]}) = [n \backslash k]$. We define $\gamma \in \mathscr{T}_{n-1}$ by $x\alpha\gamma = x\beta$ for all $x \in [n \backslash i]$ and $0\gamma = k\gamma = 0$. It is easy to verify that γ is well-defined and $\gamma \in wEnd(S_n)$ with $\gamma \notin U_{i,j}$ since $0 \in \text{Im}(\gamma|_{X_{n-1}})$. We have $x\alpha\gamma = x\beta$ for all $x \in [n \backslash i]$ with $0\alpha\gamma = 0 = 0\beta$ and $i\alpha\gamma = j\alpha\gamma = j\beta = i\beta$. This shows that $\alpha\gamma = \beta$, i.e., $\beta \in< A_{i,j}, \alpha >$. □

For $i \in X_{n-1}$, let $J^i_{n-1} = \{\alpha \in J_{n-1} : \Omega_n \backslash \{i\} = \text{Im}(\alpha), i\alpha \neq 0\}$ and let $B_i = wEnd(S_n) \backslash J^i_{n-1}$.

Lemma 2.5. *Let $i \in X_{n-1}$. Then B_i is a maximal subsemigroup of $wEnd(S_n)$.*

Proof. Let $\alpha, \beta \in J_{n-1} \backslash J^i_{n-1}$. Then $\Omega_n \backslash \{i\} \nsubseteq \text{Im}(\beta)$ or $i\beta = 0$. If $\Omega_n \backslash \{i\} \nsubseteq \text{Im}(\beta)$ then $\Omega_n \backslash \{i\} \nsubseteq \text{Im}(\alpha\beta)$, i.e., $\alpha\beta \notin J^i_{n-1}$. Suppose that $i\beta = 0$ and $\Omega_n \backslash \{i\} = \text{Im}(\beta)$. Then $\beta|_{[n \backslash i]}$ is injective. If $i\alpha = 0$ then $i\alpha\beta = 0$, i.e., $\alpha\beta \notin J^i_{n-1}$. If $i\alpha \neq 0$ and $\Omega_n \backslash \{i\} \nsubseteq \text{Im}(\alpha)$ then $i \in \text{Im}(\alpha)$ and there is $a \in i\alpha^{-1}$. Clearly, $a \neq 0$ and $a\alpha\beta = i\beta = 0$. If there is $b \in [n \backslash a]$ with $b\alpha = 0$ then $\text{rank}(\alpha\beta) < n - 1$ and $\alpha\beta \notin J^i_{n-1}$. If $x\alpha \neq 0$ for all $x \in X_{n-1}$ then there is $k \in [n \backslash i]$ with $k \notin \text{Im}(\alpha)$ and $k\beta \notin \text{Im}(\alpha\beta)$, i.e., $i, k\beta \notin \text{Im}(\alpha\beta)$, where $i \neq k\beta$. Thus, $\text{rank}(\alpha\beta) < n - 1$ and $\alpha\beta \notin J^i_{n-1}$. This shows that B_i is a subsemigroup of $wEnd(S_n)$.

Let now $\alpha, \beta \in J^i_{n-1}$ and we will show that $\beta \in< B_i, \alpha >$. For each $x \in X_{n-1}$ with $x\beta \neq 0$, there is $\overline{x} \in X_{n-1}$ with $\overline{x}\alpha = x\beta$. We choose $\overline{x} = i$ if $i\alpha = x\beta$. Then we define $\gamma \in \mathscr{T}_{n-1}$ by

$$x\gamma = \begin{cases} \overline{x} & \text{if } x\beta \neq 0, \\ 0 & \text{if } x\beta = 0, \end{cases}$$

for all $x \in \Omega_n$. It is easy to verify that $\gamma \in wEnd(S_n)$. Since $i\alpha \neq 0, i\alpha \in \text{Im}(\beta)$, we can conclude that $i \in \text{Im}(\gamma)$ and $\gamma \notin J^i_{n-1}$, i.e., $\gamma \in B_i$. For all $x \in X_{n-1}$, it holds $x\gamma\alpha = \overline{x}\alpha = x\beta$, whenever $x\beta \neq 0$ and $x\gamma\alpha = 0\alpha = 0 = 0\beta$, whenever $x\beta = 0$. Therefore, $\gamma\alpha = \beta$, i.e., $\beta \in< B_i, \alpha >$. This shows that B_i is a maximal subsemigroup. □

For $\emptyset \neq Z \subseteq X_{n-1}$ with $|Z| \leq n-3$, we put $R_0^Z = \{\alpha \in J_{n-1} : z\alpha = 0, z \in Z\setminus \mathrm{Im}(\alpha)\} \cup \{\alpha \in J_{n-1}\setminus L_Z : z\alpha = 0 \text{ for some } z \in X_{n-1}\setminus Z\}$,
$R_Z = \bigcup\{U_{i,j} : i < j \in X_{n-1}\setminus Z\}$, and
$L_Z = \bigcup\{J^i_{n-1} : i \in Z\}$.

Lemma 2.6. *Let $\emptyset \neq Z \subseteq X_{n-1}$ with $|Z| \leq n-3$. Then $A_Z = \{\alpha \in wEnd(S_n) : \mathrm{rank}(\alpha) < n-1\} \cup L_Z \cup R_Z \cup R_0^Z$ is a maximal subsemigroup of $wEnd(S_n)$.*

Proof. Let $\alpha, \beta \in J_{n-1} \cap A_Z$ then one of the cases (1)-(4) and (5)-(8), respectively, is satisfied.
(1) $i\alpha = 0$ for some $i \in X_{n-1}\setminus Z$ and $\alpha \notin L_Z$ or
(2) $i\alpha = 0$ for some $i \in Z\setminus \mathrm{Im}(\alpha)$, or
(3) $\alpha \in U_{i,j}$ for some $i < j \in X_{n-1}\setminus Z$, or
(4) $\alpha \in J^i_{n-1}$ for some $i \in Z$ and
(5) $k\beta = 0$ for some $k \in Z\setminus \mathrm{Im}(\beta)$, or
(6) $\beta \in J^k_{n-1}$ for some $k \in Z$, or
(7) $\beta \in U_{k,l}$ for some $k < l \in X_{n-1}\setminus Z$, or
(8) $k\beta = 0$ for some $k \in X_{n-1}\setminus Z$ and $\beta \notin L_Z$.
If (6) holds then $\mathrm{Im}(\alpha\beta) \subseteq X_{n-1}\setminus\{k\}$. Then $\mathrm{rank}(\alpha\beta) < n-1$ or $\alpha\beta \in J^k_{n-1} \subseteq L_Z$ (whenever $k\alpha\beta \neq 0$) or $\alpha\beta \in R_0^Z$ (whenever $k\alpha\beta = 0$). Thus, $\alpha\beta \in A_Z$.
If (1) holds then $i \notin \mathrm{Im}(\alpha)$. If (8) is satisfied and $k = i$ then $i\alpha\beta = 0$ with $\alpha\beta \notin L_Z$. In the remaining cases, we have $\mathrm{rank}(\alpha\beta) < n-1$, i.e., $\alpha\beta \in A_Z$.
If (8) is satisfied then $\mathrm{rank}(\alpha\beta) < n-1$, whenever (2) or (4) with $k \in \mathrm{Im}(\alpha)$ holds. If (4) holds with $k \notin \mathrm{Im}(\alpha)$ then (8) implies $\alpha\beta \in J^i_{n-1}$. Hence $\alpha\beta \in A_Z$.
Suppose now that (3) holds. Then $\alpha\beta \in U_{i,j}$ or $\mathrm{rank}(\alpha\beta) < n-1$. Hence, $\alpha\beta \in A_Z$. Suppose that (2) and (5) are satisfied. Then $i\alpha\beta = 0$. If $k = i$ then $i \notin \mathrm{Im}(\alpha\beta)$, i.e., $\alpha\beta \in R_0^Z$. If $k \neq i$ then $\mathrm{rank}(\alpha\beta) < n-1$, i.e., $\alpha\beta \in A_Z$.
Suppose that (2) and (7) hold. Then $i\alpha = 0$. Since $i \in Z\setminus \mathrm{Im}(\alpha)$ and $k, l \in Z$, there are $p, q \in [n\setminus i]$ with $p\alpha = k$ and $q\alpha = l$. Hence, $\mathrm{rank}(\alpha\beta) < n-1$ and $\alpha\beta \in A_Z$.
Suppose that (4) and (7) hold. Then it is easy to verify that $\mathrm{Im}(\alpha\beta) \subseteq \mathrm{Im}(\beta)\setminus\{i\beta\}$, Since $i \neq k, l$, we get $\mathrm{rank}(\alpha\beta) < n-1$ and $\alpha\beta \in A_Z$. If (4) and (5) are satisfied. Then $\alpha\beta \in J^i_{n-1}$, whenever $k = i$ and $\mathrm{rank}(\alpha\beta) < n-1$, whenever $k \neq i$. This gives $\alpha\beta \in A_Z$. This shows that A_Z is a subsemigroup of $wEnd(S_n)$.

It remains to show that A_Z is maximal. For this let $\alpha, \beta \in wEnd(S_n)\setminus A_Z$. Then $\alpha\beta \in wEnd(S_n)$ and there are $k, l, m \in X_{n-1}$ such that $\alpha \in J^k_{n-1} \bigcap U_{l,m}$ with $k \notin Z$ and $l \in Z$ or $m \in Z$ or there are $k \in X_{n-1}\setminus Z$ and $l \in Z$ with $\alpha \in J^k_{n-1}$ and $l\alpha = 0$. Then it is easy to verify that there are $\gamma, \delta \in A_Z$ such that $\beta = \gamma\alpha\delta$. So, we have shown that $\beta \in < A_Z, \alpha >$. □

Theorem 2.7. *Let T be a subsemigroup of $wEnd(S_n)$. Then T is maximal if and only if T is one of the following types:*
(1) $T = A_{0,1}$ or
(2) $T = A_i$ for some $i \in X_{n-1}$ or
(3) $T = A_{i,j}$ for some $i < j \in X_{n-1}$ or
(4) $T = B_i$ for some $i \in X_{n-1}$ or
(5) $T = \{\alpha \in wEnd(S_n) : \mathrm{rank}(\alpha) < n-1\} \cup R_Z \cup L_Z \cup R_0^Z$ for some $\emptyset \neq Z \subseteq X_{n-1}$ with $|Z| \leq n-3$.

Proof. If T is one of the given forms, then T is a maximal by Lemmas 2.1 - 2.5. Now suppose that T is a maximal subsemigroup of $wEnd(S_n)$. Let $\overline{T} = T \cap T^0_{n-1}$ and $\hat{T} = \{\alpha|_{X_{n-1}} : \alpha \in \overline{T}\} \subseteq T_{n-1}$. Then it is easy to verify that $\hat{T}$ is a maximal subsemigroup of the singular part of T_{n-1} or $\hat{T}$ is the singular part of T_{n-1}. We denote the singular part of T_{n-1} also by T_{n-1}.
Suppose that $\hat{T}$ is a maximal subsemigroup of T_{n-1}. Then by [14], the semigroup $\hat{T}$ has one of the following forms:

(a) $\hat{T} = T_{n-1}\setminus\{\alpha \in T_{n-1} : \mathrm{rank}(\alpha) = n-2 \text{ and } i \notin \mathrm{Im}(\alpha)\}$ for some $i \in X_{n-1}$. So, $J^i_{n-1} \cap T^0_{n-1} \cap T = \emptyset$. Let $\alpha \in J^i_{n-1}\setminus T^0_{n-1}$. Then there is $p \in [n\setminus i]$ with $p\alpha = 0$. A straightforward calculation provides $(J^p_{n-1} \cap T^0_{n-1})\alpha = J^i_{n-1} \cap T^0_{n-1}$. Since $J^i_{n-1} \cap T^0_{n-1} \cap T = \emptyset$, we can conclude that $\alpha \notin T$. So, $(J^i_{n-1}\setminus T^0_{n-1}) \cap T = \emptyset$ and with $J^i_{n-1} \cap T^0_{n-1} \cap T = \emptyset$, we obtain $J^i_{n-1} \cap T = \emptyset$. By Lemma 2.5, we can conclude that $T = wEnd(S_n)\setminus J^i_{n-1}$.

(b) $\hat{T} = T_{n-1}\backslash\{\alpha \in T_{n-1} : \text{rank}(\alpha) = n-2,\ i\alpha = j\alpha\}$ for some $i < j \in X_{n-1}$. So, $U_{i,j} \cap T = \emptyset$. Since $wEnd(S_n)\backslash U_{i,j}$ is a maximal subsemigroup of $wEnd(S_n)$, by Lemma 2.4, we can conclude that $T = wEnd(S_n)\backslash U_{i,j}$.

(c) Let $\hat{T} = \{\alpha \in T_{n-1} : \text{rank}(\alpha) < n-2\} \cup \bigcup\{R^{n-1}_{i,j} : i, j \in X_{n-1}\backslash Z\} \cup \bigcup\{L^{n-1}_i : i \in Z\}$ for some non-empty set $Z \subseteq X_{n-1}$, with $|Z| \leq n-3$, where

$R^{n-1}_{i,j} = \{\alpha \in T_{n-1} : \text{rank}(\alpha) = n-2, i\alpha = j\alpha\}$ for $i < j \in X_{n-1}$ and

$L^{n-1}_j = \{\alpha \in T_{n-1} : \text{rank}(\alpha) = n-2 : j \notin \text{Im}(\alpha)\}$ for some $j \in X_{n-1}$.

Then $(R_Z \cup L_Z) \cap T^0_{n-1} \subseteq T$. Assume that there is $\alpha \in T^0_{n-1} \cap J_{n-1}$ with $\alpha|_{X_{n-1}} \notin \hat{T}$. Then $\hat{T} = T_{n-1}$ since $\hat{T}$ is a maximal subsemigroup of T_{n-1}, a contradiction. Hence, $\{\alpha \in wEnd(S_n) : \text{rank}(\alpha) < n-1\} \cup R_Z \cup (L_Z \cap T^0_{n-1}) \subseteq T$.

Assume that for all $i \in X_{n-1}$, there is $\beta_i \in T \cap J_{n-1}$ with $i\beta_i = 0$ and there is $\alpha \in T\backslash A_Z$. Let $\beta \in (L_Z\backslash T^0_{n-1}) \cup R^Z_0$. Then there is $k \in X_{n-1}$ with $k\beta = 0$. We consider $\beta_k \in T$. If $Z \subseteq \text{Im}(\beta_k)$. Then there are $p, q \in X_{n-1}\backslash Z$ and $p \notin \text{Im}(\beta_k)$. It is easy to verify that there is $\beta_0 \in U_{p,q} \subseteq T$ with $\beta = \beta_k\beta_0$. Suppose that $Z \nsubseteq \text{Im}(\beta_k)$. Then we consider α. There is $m \in Z$ such that $m\alpha = 0$ or $\alpha \in U_{m,l}$ for some $l \in X_{n-1}$, where $Z \subseteq \text{Im}(\alpha)$. Then it is easy to verify that there is $\gamma \in L_Z \cap T^0_{n-1}$ such that $[n\backslash t] \subseteq \text{Im}(\beta_k\gamma\alpha)$ for some $t \in X_{n-1}\backslash Z$ and $k(\beta_k\gamma\alpha) = 0$. As above, there is $\beta_0 \in T$ with $\beta = \beta_k\gamma\alpha\beta_0$. This shows that $L_Z\backslash T^0_{n-1} \cup R^Z_0 \subseteq T$, i.e., $A_Z \subseteq T$. This contradicts to $T\backslash A_Z \neq \emptyset$. Hence $A_Z \subseteq T$ or there is $i \in X_{n-1}$ such that $T \cap U_i = \emptyset$. This shows that $T = A_Z$ or $T = A_i$, respectively.

Now suppose that $\hat{T} = T_{n-1}$. We have already shown that $\hat{T} = T_{n-1}$ implies that there is $i \in X_{n-1}$ with $T \cap U_i = \emptyset$, i.e., $T = A_i$, or $J_{n-1} \subseteq T$. So, we consider the case $J_{n-1} \subseteq T$. Then $\mathscr{T}^0_{n-1} \subseteq T$. Assume that there is $\alpha \in T$ with $0\alpha \neq 0$, rank$(\alpha) = 2$. Let $\gamma \in J_2$ with $0\gamma = i \in X_{n-1}$. Then there is a non-empty set $Y \subseteq X_{n-1}$ with $x\gamma = 0$ for all $x \in Y$ and $x\gamma = i$, otherwise. Let $\delta_i \in \mathscr{T}_{n-1}$ with $0\delta_i = 0$ and $x\delta_i = i$ for all $x \in X_{n-1}$. Further, let $\gamma_Y \in \mathscr{T}_{n-1}$ with $x\gamma_Y = 1$ for all $x \in Y$ and $x\gamma_Y = 0$, otherwise. Clearly $\delta_i, \gamma_Y \in \mathscr{T}^0_{n-1} \subseteq T$ with $\gamma_Y\alpha\gamma_i = \gamma$. This shows that $< \mathscr{T}^0_{n-1}, \alpha >= wEnd(S_n)$, a contradiction. Thus $U_{0,1} \cap T = \emptyset$ and by Lemma 2.2, we get $T = A_{0,1}$. □

We have one maximal subsemigroup of type (1), $(n-1)$ maximal subsemigroups of type (2) as well as of type (4), $\dfrac{(n-1)(n-2)}{2}$ maximal subsemigroups of type (3) and $2^{n-1} - n - 1$ maximal subsemigroups of type (5). Altogether, we have $\dfrac{(n-2)(n+1)}{2} + 2^{n-1}$ maximal subsemigroups of $wEnd(S_n)$.

Next, we determine the maximal subsemigroups of $End(S_n)$. If $n = 3$ then $End(S_3)$ consists of four elements:

$$End(S_3) = \left\{\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 0 \end{pmatrix}\right\}.$$

Lemma 2.8. *Let T be a subsemigroup of $End(S_3)$. Then T is maximal if and only if T is one of the following forms:*

(1) $T_1 = \left\{\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix}\right\}$ *or*

(2) $T_2 = \left\{\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}\right\}$ *or*

(3) $T_3 = \left\{\begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 0 \end{pmatrix}\right\}$.

Proof. It is easy to see that if T is of the form (1) or (2) or (3), then T is maximal.
Conversely, let T' be a maximal subsemigroup of $End(S_3)$ with $T' \nsubseteq T_1, T' \nsubseteq T_2$ and $T' \nsubseteq T_3$. Then there exist $\alpha_1 \in T'\backslash T_1, \alpha_2 \in T'\backslash T_2$, and $\alpha_3 \in T'\backslash T_3$. It is straightforward to check (by all possibilities for α_1, α_2, and α_3) that $< \alpha_1, \alpha_2, \alpha_3 >= End(S_3)$. □

Let $n \geq 4$ and $J^{End}_{n-1} = J_{n-1} \cap T^0_{n-1}$.

Theorem 2.9. *Let T be a subsemigroup of $End(S_n)$. Then T is maximal if and only if T is one of the following four forms:*

(1) $T = End(S_n)\backslash J_2^w$, with $J_2^w = \{\alpha \in J_2 : 0\alpha \neq 0$ and $\mathrm{Im}(\alpha|_{X_{n-1}}) = \{0\}\}$;
(2) $T = End(S_n)\backslash R_{i,j}^{End}$, with $R_{i,j}^{End} = \{\alpha \in J_{n-1}^{End} : i\alpha = j\alpha\}$ for some $i < j \in X_{n-1}$;
(3) $T = End(S_n)\backslash L_i^{End}$, with $L_i^{End} = \{\alpha \in J_{n-1}^{End} : i \notin \mathrm{Im}(\alpha)\}$ for some $i \in X_{n-1}$;
(4) $T = \{\alpha \in End(S_n) : \mathrm{rank}(\alpha) < n-1\} \cup R_Z^{End} \cup L_Z^{End} = A_Z^{reg}$, where $R_Z^{End} = \bigcup\{R_{i,j}^{End} : i, j \in X_{n-1}\backslash Z\}$, $L_Z^{End} = \bigcup\{L_i^{End} : i \in Z\}$ for some $\emptyset \neq Z \subseteq X_{n-1}$ with $|Z| \leq n-3$.

Proof. Note that $End(S_n) = T_{n-1}^0 \cup J_2^w$, where $T_{n-1}^0 \cong T_{n-1}$. Let T be a maximal subsemigroup of $End(S_n)$. Then $\hat{T} = \{\alpha|_{X_{n-1}} : \alpha \in T\backslash J_2^w\}$ is a maximal subsemigroup of T_{n-1} or $T_{n-1}^0 \subseteq T$ since $\mathrm{rank}(\alpha) \leq 2$ for all $\alpha \in End(S_n)\backslash T_{n-1}^0$. Suppose now that $\{\alpha \in End(S_n) : \mathrm{rank}(\alpha) > 2\} \nsubseteq T$. Then by [14], we have

$\hat{T} = T_{n-1}\backslash\{\alpha \in J_{n-2}^{n-1} : x\alpha \neq i$ for all $x \in X_{n-1}\}$ for some $i \in X_{n-1}$ or

$\hat{T} = T_{n-1}\backslash\{\alpha \in J_{n-2}^{n-1} : i\alpha = j\alpha\}$ for some $i < j \in X_{n-1}$ or

$\hat{T} = \{\alpha \in T_{n-1} : \mathrm{rank}(\alpha) \leq n-3\} \bigcup\{\alpha \in J_{n-2}^{n-1} : x\alpha \neq z$ for all $x \in X_{n-1}$ and some $z \in Z\} \cup \{\alpha \in J_{n-2}^{n-1} : i\alpha = j\alpha$ for some $i < j \in X_{n-1}\backslash Z\}$ for some non-empty set $Z \subseteq X_{n-1}$ with $|Z| \leq 3$, where $J_{n-2}^{n-1} = \{\alpha \in T_{n-1} : \mathrm{rank}(\alpha) = n-2\}$.

Let $\tilde{T} = \{\alpha \in T_{n-1}^0 : \alpha|_{X_{n-1}} \in \hat{T}\}$. Since $T_{n-1}^0 \cong T_{n-1}$ and $\hat{T}$ is a maximal subsemigroup of T_{n-1}, we get that $\tilde{T}$ is a maximal subsemigroup of T_{n-1}^0. Note that $\tilde{T} = End(S_n)\backslash J$ for some $J \subseteq J_{n-1}^{End}$. So, $T \subseteq \tilde{T} \cup J_2^w$ and $\tilde{T} \cup J_2^w$ is a maximal subsemigroup of $T_{n-1}^0 \cup J_2^w = End(S_n)$, i.e., $T = \tilde{T} \cup J_2^w = End(S_n)\backslash\{\alpha \in T_{n-1}^0 : \alpha|_{X_{n-1}} \notin \hat{T}\}$. By the three possibilities for $\hat{T}$, we obtain the cases (2), (3), or (4).

Suppose now that $T_{n-1}^0 \subseteq T$. Let $\alpha, \beta \in J_2^w$. We define $\gamma \in \mathscr{T}_{n-1}$ by $0\gamma = 0$ and $x\gamma = 0\beta \neq 0$ for all $x \in X_{n-1}$. It is clear that $\gamma \in T_{n-1}^0$ and $0\alpha\gamma = 0\beta$ as well as $x\alpha\gamma = 0\gamma = 0 = x\beta$ for all $x \in X_{n-1}$. This shows that $\beta = \gamma\alpha$, i.e., $\beta \in < T_{n-1}^0, \alpha >$. Hence, T_{n-1}^0 is a maximal subsemigroup of $T_{n-1}^0 \cup J_2^w = End(S_n)$ with $T_{n-1}^0 \subseteq T$. Hence $T = T_{n-1}^0 = End(S_n)\backslash J_2^w$. □

Similar as for the semigroup $wEnd(S_n)$, we obtain that there is one maximal subsemigroup of type (1), $\frac{(n-1)(n-2)}{2}$ maximal subsemigroups of type (2), $(n-1)$ maximal subsemigroups of type (3) and $2^{n-1}-n-1$ maximal subsemigroups of type (4). So, we have $\frac{(n-3)n}{2} + 2^{n-1}$ maximal subsemigroups of $End(S_n)$.

Lemma 2.10. *$End(S_n)$ is a maximal subsemigroup of $swEnd(S_n)$.*

Proof. Since $End(S_n)$ is a semigroup, it is enough to show that $End(S_n)$ is a maximal subsemigroup of $swEnd(S_n)$. Let $\alpha, \beta \in J_1$. Then there are $i, j \in \Omega_n$ such that $\mathrm{Im}(\alpha) = \{i\}$ and $\mathrm{Im}(\beta) = \{j\}$. Let $\gamma \in \mathscr{T}_{n-1}$ with

$$y\gamma = \begin{cases} 0 & \text{if } x = 0, \\ i & \text{if } x = j, \\ j & \text{otherwise.} \end{cases}$$

Then it is easy to verify that $\gamma \in End(S_n)$ and $\beta = \alpha\gamma$, i.e., $\beta \in < End(S_n), \alpha >$. Consequently, $End(S_n)$ is a maximal subsemigroup of $swEnd(S_n)$. □

Since $J_1 \subseteq swEnd(S_n)$, we can conclude that J_1 is an ideal of $swEnd(S_n)$.

Lemma 2.11. *Let T be a maximal subsemigroup of $End(S_n)$. Then $T \cup J_1$ is a maximal subsemigroup of $swEnd(S_n)$.*

Proof. $T \cup J_1$ is the disjoint union of a subsemigroup and an ideal of $swEnd(S_n)$. This provides that $T \cup J_1$ is a semigroup itself. We will show that $T \cup J_1$ is maximal. For this, let $\alpha \in swEnd(S_n)\backslash(T \cup J_1)$. Then $< T \cup J_1, \alpha > = < T, \alpha > \cup J_1 = End(S_n) \cup J_1 = swEnd(S_n)$ since T is a maximal subsemigroup of $End(S_n)$. This shows that $T \cup J_1$ is a maximal subsemigroup of $swEnd(S_n)$. □

Theorem 2.12. *Let T be a subsemigroup of $swEnd(S_n)$. Then T is maximal if and only if $T = End(S_n)$ or $T = \hat{T} \cup J_1$ for some maximal subsemigroup $\hat{T}$ of $End(S_n)$.*

Proof. $End(S_n)$ is a maximal subsemigroup of $swEnd(S_n)$ by Lemma 2.10. On the other hand, $\hat{T} \cup J_1$ is maximal subsemigroup of $swEnd(S_n)$ for all maximal subsemigroups $\hat{T}$ of $End(S_n)$ by Lemma 2.11.
Let now T be a maximal subsemigroup of $swEnd(S_n)$. Since $swEnd(S_n)$ is the disjoint union of $End(S_n)$ and J_1, there are sets $\tilde{T} \subseteq End(S_n)$ and $G \subseteq J_1$ such that $T = \tilde{T} \cup G$. If $G = \emptyset$ then $T = \tilde{T}$ and by Lemma 2.10, we have $T = End(S_n)$. Suppose now that $G \neq \emptyset$. Assume that $\tilde{T}$ does not be a maximal subsemigroup of $End(S_n)$. We can conclude that $\tilde{T} \subset \hat{T}$ for some maximal subsemigroup $\hat{T}$ of $End(S_n)$. Let $\alpha \in \hat{T} \backslash \tilde{T} \neq \emptyset$. Then $swEnd(S_n) =< T, \alpha >=< \tilde{T}, G, \alpha > \subseteq < \tilde{T}, \alpha > \cup J_1 = \hat{T} \cup J_1$ since J_1 is an ideal of $swEnd(S_n)$ and $\hat{T}$ is a subsemigroup of $swEnd(S_n)$. Thus, $End(S_n) \cup J_1 = swEnd(S_n) \subseteq \hat{T} \cup J_1$, a contradiction to $\hat{T} \subset End(S_n)$. Hence, $T = \hat{T} \cup G \subseteq \hat{T} \cup J_1$. As in the proof of Lemma 2.11, we can conclude that $G = J_1$, i.e., $T = \hat{T} \cup J_1$. □

The number of maximal subsemigroups of $swEnd(S_n)$ is the number of maximal subsemigroups of $End(S_n)$ plus one. So, we have $\dfrac{(n-3)n}{2} + 2^{n-1} + 1$ maximal subsemigroups of $swEnd(S_n)$.

3. Maximal regular subsemigroups

This section is devoted to the maximal regular subsemigroups. In Section 2, we have determined the maximal subsemigroups of $wEnd(S_n)$ and $End(S_n)$, respectively. The maximal regular subsemigroups of $wEnd(S_n)$ and $End(S_n)$, respectively, are the maximal regular subsemigroups of their maximal subsemigroups.

Lemma 3.1. *For $i < j \in X_{n-1}, A_{i,j}$ is regular.*

Proof. Let $i < j \in X_{n-1}$ and let $\alpha \in A_{i,j}$. Then it is sufficient to consider the case $\alpha \in J_{n-1}$. We can write $\alpha = \begin{pmatrix} \mathcal{A}_0 & \mathcal{A}_1 & \dots & \mathcal{A}_{n-2} \\ 0 & i_1 & \dots & i_{n-2} \end{pmatrix}$ and put $a_k = \min \mathcal{A}_k$ for $k \in \{1, 2, \dots, n-2\}$. Let $\beta \in \mathcal{T}_{n-1}$ be defined by

$$x\beta = \begin{cases} a_k & \text{if } x = i_k \text{ for some } k \in \{1, 2, \dots, n-2\}, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that $\beta \in wEnd(S_n) \backslash U_{i,j}$ and $\alpha = \alpha\beta\alpha$. □

Lemma 3.2. *For $i \in X_{n-1}, A_i$ is regular.*

Proof. Let $i \in X_{n-1}$ and let $\alpha \in A_i$. Then we have only to consider the case $\alpha \in J_{n-1}$. We can write $\alpha = \begin{pmatrix} \mathcal{A}_0 & \mathcal{A}_1 & \dots & \mathcal{A}_{n-2} \\ 0 & i_1 & \dots & i_{n-2} \end{pmatrix}$ and put $a_k = \min \mathcal{A}_k$ if $i \notin \mathcal{A}_k$ and $a_k = i$ if $i \in \mathcal{A}_k$, for all $k \in \{1, 2, \dots, n-2\}$. We define now $\beta \in \mathcal{T}_{n-1}$ by

$$x\beta = \begin{cases} a_k & \text{if } x = i_k \text{ for some } k \in \{1, 2, \dots, n-2\}, \\ 0 & \text{if } x = 0, \\ i & \text{otherwise.} \end{cases}$$

It is easy to verify that $\beta \in wEnd(S_n)$, $\text{rank}(\alpha) = \text{rank}(\beta) = n-1$, and $\alpha = \alpha\beta\alpha$. Since $0\alpha = 0$ implies $0 \in \mathcal{A}_0$, we can conclude that $i\beta \neq 0$. Hence, $\beta \notin U_i$. □

Lemma 3.3. *$A_{0,1}$ is regular.*

Proof. Let $\alpha \in A_{0,1}$. Then $\alpha \notin U_{0,1}$. If $0\alpha \neq 0$ then $\text{rank}(\alpha) \leq 2$. Since $\alpha \notin U_{0,1}$, we have $\text{rank}(\alpha) = 1$. Then α is idempotent and this implies that α is regular. If $0\alpha = 0$, then there is $\beta \in wEnd(S_n)$ with $\alpha = \alpha\beta\alpha$ since $wEnd(S_n)$ is regular. Then it is easy to verify that $0\beta = 0$, i.e., $\beta \notin U_{0,1}$. So, $\beta \in A_{0,1}$. □

For $i \in X_{n-1}$, let $C_i = \{\alpha \in B_i \cap J_{n-1} : i\alpha = 0, i \in \mathrm{Im}(\alpha)\}$. It is easy to verify that $C_i \neq \emptyset$.

Lemma 3.4. *Let $i \in X_{n-1}$ and let $\alpha \in C_i$. Then α is not regular in B_i.*

Proof. Assume that α is regular in B_i. Then there is $\beta \in B_i$ with $\alpha = \alpha\beta\alpha$. Let $A = [n\backslash i]$. Since $\mathrm{rank}(\alpha) = n-1$ and $i\alpha = 0$, we have $\alpha|_A$ is injective. Then $\beta|_{A\alpha}$ is injective with $\mathrm{Im}(\beta|_{A\alpha}) = A$, i.e., $\mathrm{Im}(\beta) = \Omega_n\backslash\{i\}$. Since $i\alpha = 0$ and $i \in \mathrm{Im}(\alpha)$, we can conclude that $i \in A\alpha$. So, $\mathrm{Im}(\beta|_{A\alpha}) = A$ implies $i\beta \in A$, i.e., $i\beta \neq 0$. Thus $\beta \in J^i_{n-1}$, i.e., $\beta \notin B_i$, a contradiction. □

Corollary 3.5. *For $i \in X_{n-1}$, B_i is not regular.*

For $i \in X_{n-1}$, let $V_i = \{\alpha \in B_i \cap J_{n-1} : |i\alpha\alpha^{-1}| = 2 \text{ and } i\alpha \neq 0\}$ and let $B^{reg}_{i,1} = B_i\backslash(C_i \cup V_i)$.

Lemma 3.6. *Let $i \in X_{n-1}$. Then $B^{reg}_{i,1}$ is a regular subsemigroup of B_i.*

Proof. Let $\alpha, \beta \in B^{reg}_{i,1} \cap J_{n-1}$. Then $|i\gamma\gamma^{-1}| = 1$ or $i\gamma = 0$ (and $i \notin \mathrm{Im}(\gamma)$) for $\gamma \in \{\alpha, \beta\}$. It is routine to verify that $\mathrm{rank}(\alpha\beta) < n-1$ or $|i\alpha\beta(\alpha\beta)^{-1}| = 1$ or $i\alpha\beta = 0$ (and $i \notin \mathrm{Im}(\alpha\beta)$). This shows that $B^{reg}_{i,1}$ is a semigroup. Next, we show that $B^{reg}_{i,1}$ is regular. Let $\alpha \in B^{reg}_{i,1}$. It is enough to consider the case $\alpha \in J_{n-1}$. So, we can write $\alpha = \begin{pmatrix} \mathcal{A}_0 & \mathcal{A}_1 & \dots & \mathcal{A}_{n-2} \\ 0 & i_1 & \dots & i_{n-2} \end{pmatrix}$. For each $k \in \{1, 2, \dots, n-2\}$, there is $a_k \in \mathcal{A}_k$ with $a_k\alpha = i_k$. Suppose that $|i\alpha\alpha^{-1}| = 1$. Then $i \in \mathrm{Im}(\alpha), i\alpha \neq 0$, and $i \in \{a_1, a_2, \dots, a_{n-2}\}$. Let $\beta \in \mathscr{T}_{n-1}$ be defined by

$$x\beta = \begin{cases} a_k & \text{if } x = i_k \text{ for some } k \in \{1, 2, \dots, n-2\}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $i \in \mathrm{Im}(\beta)$, we verify that $\beta \in B_i$. Since $|i\beta\beta^{-1}| = 1$, we get that $\beta \in B^{reg}_{i,1}$. Moreover, we have $\alpha\beta\alpha = \alpha$. Suppose that $i\alpha = 0$. Then $i \notin \mathrm{Im}(\alpha)$ and α restricted to $[n\backslash i]$ is bijective. We define $\beta \in \mathscr{T}_{n-1}$ by

$$x\beta = \begin{cases} x\alpha^{-1} & \text{if } x \in [n\backslash i], \\ 0 & \text{otherwise.} \end{cases}$$

Then it is easy to verify that $\beta \in B^{reg}_{i,1}$ and $\alpha\beta\alpha = \alpha$. This shows that α is regular. Consequently, $B^{reg}_{i,1}$ is regular. □

For $i \in X_{n-1}$, let $W_i = \{\alpha \in J_{n-1} \cap B_i : i\alpha = 0, i \notin \mathrm{Im}(\alpha)\}$ and let $B^{reg}_{i,2} = B_i\backslash(C_i \cup W_i)$.

Lemma 3.7. *Let $i \in X_{n-1}$. Then $B^{reg}_{i,2}$ is a regular subsemigroup of B_i.*

Proof. Let $\alpha, \beta \in B^{reg}_{i,2} \cap J_{n-1}$. Then $i\alpha \neq 0$ and $i\beta \neq 0$. It is a routine matter to verify that $i\alpha\beta = 0$ implies $\mathrm{rank}(\alpha\beta) < n-1$. Hence $\alpha\beta \in B^{reg}_{i,2}$. This shows that $B^{reg}_{i,2}$ is a semigroup. Next, we show that $B^{reg}_{i,2}$ is regular. Let $\alpha \in B^{reg}_{i,2} \cap J_{n-1}$ with $\alpha = \begin{pmatrix} \mathcal{A}_0 & \mathcal{A}_1 & \dots & \mathcal{A}_{n-2} \\ 0 & i_1 & \dots & i_{n-2} \end{pmatrix}$. For each $k \in \{1, 2, \dots, n-2\}$, there is $a_k \in \mathcal{A}_k$ with $a_k\alpha = i_k$, where $i \in \{a_1, a_2, \dots, a_{n-2}\}$. We define $\beta \in \mathscr{T}_{n-1}$ by

$$x\beta = \begin{cases} a_k & \text{if } x = i_k \text{ for some } k \in \{1, 2, \dots, n-2\}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $i \in \{a_1, a_2, \dots, a_{n-2}\}$, we have $i \in \mathrm{Im}(\beta)$. Since $\alpha \in B_i\backslash W_i$, we have $i \in \{i_1, i_2, \dots, i_{n-2}\}$ and $i\beta \neq 0$. This provides $\beta \in B^{reg}_{i,2}$, where $\alpha\beta\alpha = \alpha$. □

Lemma 3.8. *Let $i \in X_{n-1}$, let $\alpha \in V_i$, and let $\beta \in W_i$. Then $\beta\alpha$ is not regular in B_i.*

Proof. Since $\mathrm{Im}(\beta|_{[n\backslash i]}) = [n\backslash i]$ and $\mathrm{Im}(\alpha|_{X_{n-1}}) = \mathrm{Im}(\alpha|_{[n\backslash i]})$, we can conclude that $\mathrm{Im}(\beta\alpha|_{[n\backslash i]}) = \mathrm{Im}(\alpha|_{[n\backslash i]}) = \mathrm{Im}(\beta\alpha|_{X_{n-1}})$. This shows that $\beta\alpha \in J_{n-1}$ and $i \in \mathrm{Im}(\beta\alpha)$ since $i \in \mathrm{Im}(\alpha)$. Hence $i \in \mathrm{Im}(\beta\alpha)$. Moreover, $0\beta\alpha = i\beta\alpha = 0$. This shows that $\beta\alpha \in C_i$ and $\beta\alpha$ is not regular in B_i by Lemma 2.4. $\square$

Proposition 3.9. *Let $i \in X_{n-1}$ and let T be a maximal regular subsemigroup of B_i. Then there is $k \in \{1, 2\}$ such that $T = B_{i,k}^{reg}$.*

Proof. Let $k \in \{1, 2\}$. Then $B_{i,k}^{reg}$ is regular by Lemmas 3.6 and 3.7, respectively. Since $V_i \cap W_i = \emptyset$, we can conclude that $B_{i,k}$ is a maximal regular subsemigroup of B_i by Lemma 3.8. Assume that $T \neq B_{i,1}^{reg}$ and $T \neq B_{i,2}^{reg}$. Since $T \subseteq B_i\backslash C_i$ by Lemma 3.4, there is $\alpha \in V_i \cap T$ and $\beta \in W_i \cap T$. Then $\beta\alpha \in T$ is not regular by Lemma 3.8, a contradiction. $\square$

We put $\overline{R}_0^Z = \{\alpha \in L_Z : z\alpha = 0 \text{ for some } z \in X_{n-1}\backslash Z\}$.

Lemma 3.10. *Let Z be a non-empty subset of X_{n-1} with $|Z| \leq n-3$. If $\alpha \in (L_Z \cap R_Z) \cup \overline{R}_0^Z$ then α is not regular in A_Z.*

Proof. Let $\alpha \in (L_Z \cap R_Z) \cup \overline{R}_0^Z$. Then there is $a \in Z$ with $a \notin \mathrm{Im}(\alpha)$. Assume that α is regular in A_Z. Then there is $\beta \in A_Z$ with $\alpha\beta\alpha = \alpha$. In particular, we can conclude that $\beta|_{[n\backslash a]}$ is injective, i.e., $a\beta = 0$ or there is $k \in [n\backslash a]$ with $\beta \in U_{k,a}$. It is easy to verify that $Z \not\subseteq \mathrm{Im}(\beta)$. But we have $Z \subseteq \mathrm{Im}(\beta)$ since there is $z \in X_{n-1}\backslash Z$ with $|\mathrm{Im}(\alpha|_{[n\backslash z]})| = n-2$. This is a contradiction. $\square$

Corollary 3.11. *Let Z be a non-empty subset of X_{n-1} with $|Z| \leq n-3$. Then A_Z is not regular.*

Lemma 3.12. *All elements in $A_Z\backslash[(L_Z \cap R_Z) \cup \overline{R}_0^Z]$ are regular in $A_Z\backslash[(L_Z \cap R_Z) \cup \overline{R}_0^Z]$.*

Proof. Let $\alpha \in A_Z\backslash[(L_Z \cap R_Z) \cup \overline{R}_0^Z]$. Then there are the following four cases are possible:

a) $\alpha \in R_Z$. Then $\mathrm{Im}(\alpha) = X_{n-1}\backslash\{a\}$ for some $a \in X_{n-1}\backslash Z$. Then there is $b \in X_{n-1}\backslash(Z \cup \{a\})$. Further, there is $p \in X_{n-1}\backslash Z$ with $p\alpha = q\alpha$ for some $q \in [n\backslash p]$. We define $\beta \in \mathscr{T}_{n-1}^0$ by $a\beta = b\beta$ and $x\beta = y$ with $y \in [n\backslash p]$ and $y\alpha = x$ for all $x \in [n\backslash a]$. It is easy to verify that $\beta \in A_Z\backslash[(L_Z \cap R_Z) \cup \overline{R}_0^Z]$ and $\alpha\beta\alpha = \alpha$.

b) There is $a \in X_{n-1}\backslash Z$ such that $\mathrm{Im}(\alpha) = \Omega_n\backslash\{a\}$. We define $\beta \in \mathscr{T}_{n-1}^0$ by $a\beta = 0$ and $x\beta = y$ with $y\alpha = x$ for all $x \in [n\backslash a]$. It is easy to verify that $\beta \in A_Z\backslash[(L_Z \cap R_Z) \cup \overline{R}_0^Z]$ and $\alpha\beta\alpha = \alpha$.

c) $\alpha \in L_Z$ and $\alpha \in R_{p,q}$ for some $p \in X_{n-1}$ and $q \in Z$. Similar as in the proof of a), we can show that there is $\beta \in A_Z\backslash[(L_Z \cap R_Z) \cup \overline{R}_0^Z]$ with $\alpha\beta\alpha = \alpha$.

d) There is $a \in Z$ such that $\mathrm{Im}(\alpha) = \Omega_n\backslash\{a\}$. Similar as in the proof of b), we can show that there is $\beta \in A_Z\backslash[(L_Z \cap R_Z) \cup \overline{R}_0^Z]$ with $\alpha\beta\alpha = \alpha$. $\square$

We have shown that $A_Z\backslash[(L_Z \cap R_Z) \cup \overline{R}_0^Z]$ is the set of regular elements in A_Z, but it is easy to see that this set does not form a subsemigroup of A_Z. In fact, let $\beta_1 = \begin{pmatrix} 0 & 1 & 2 & \dots & n-2 \\ 0 & 0 & 2 & \dots & n-2 \end{pmatrix}$ and let $\beta_2 = \begin{pmatrix} 0 & 1 & \dots & n-3 & n-2 \\ 0 & 1 & \dots & n-3 & 1 \end{pmatrix}$. If $Z = \{3, 4, \dots, n-1\}$, then it is easy to verify that $\beta_1, \beta_2 \in A_Z\backslash[(L_Z \cap R_Z) \cup \overline{R}_0^Z]$ and $\beta_1, \beta_2 \in L_Z \cap R_Z$. So, the maximal regular subsemigroups of A_Z are within the set $A_Z\backslash[(L_Z \cap R_Z) \cup \overline{R}_0^Z]$. A "nice" description of the maximal regular subsemigroups of A_Z seems almost impossible. Therefore, we skip the description of the maximal regular subsemigroups of A_Z in this paper.

Theorem 3.13. *Let T be a regular subsemigroup of $wEnd(S_n)$. Then T is maximal if and only if T is one of the following types:*

(1) $T = A_{0,1}$ or

(2) $T = A_i$ for some $i \in X_{n-1}$ or

(3) $T = A_{i,j}$ for some $i < j \in X_{n-1}$ or
(4) $T = B_{i,k}^{reg}$ for some $i \in X_{n-1}$ and some $k \in \{1, 2\}$ or
(5) T is a maximal regular subsemigroup of A_Z for some $\emptyset \neq Z \subseteq X_{n-1}$ with $|Z| \leq n - 3$.

Proof. Let T be a maximal regular subsemigroup of $wEnd(S_n)$. Then T needs to be a maximal regular subsemigroup of one of the maximal subsemigroups of $wEnd(S_n)$. So by Lemmas 3.1 - 3.3, $T = A_{0,1}$ or $T = A_i$ for some $i \in X_{n-1}$ or $T = A_{i,j}$ for some $i < j \in X_{n-1}$ or T is a maximal regular subsemigroup of A_Z for some non-empty subset Z of X_{n-1} with $|Z| \leq n - 3$ or T is a maximal regular subsemigroup of B_i for some $i \in X_{n-1}$. In the latter case, we obtain $T = B_{i,1}^{reg}$ or $T = B_{i,2}^{reg}$ by Proposition 3.9. □

We consider now the regular subsemigroups of $End(S_n)$. Note that $End(S_n)\backslash J_2^w = T_{n-1}^0$.

Lemma 3.14. *$End(S_n)\backslash J_2^w$ is regular.*

Proof. Since $End(S_n)$ is regular and since non of the elements in J_2^w can be the inverse of any $\alpha \in End(S_n)\backslash J_2^w$, we can conclude that $End(S_n)\backslash J_2^w$ is regular. □

In [15], all maximal regular subsemigroups of T_{n-1} are characterized. We will use it for a description of the maximal regular subsemigroups of $End(S_n)$.

Theorem 3.15. *Let T be a regular subsemigroup of $End(S_n)$. Then T is a maximal regular subsemigroup of $End(S_n)$ if and only if $T = T_{n-1}^0$ or there is a maximal regular subsemigroup $\hat{T}$ of T_{n-1} such that $T = J_2^w \cup \{\alpha \in T_{n-1}^0 : \alpha|_{X_{n-1}} \in \hat{T}\}$.*

Proof. Suppose that T is a maximal regular subsemigroup of $End(S_n)$. Then there are sets $G \subseteq J_2^w$ and $\overline{T} \subseteq T_{n-1}^0$ such that $T = G \cup \overline{T}$.

Suppose that $\overline{T} \neq T_{n-1}^0$. Then it is easy to verify that $\hat{T} = \{\alpha|_{X_{n-1}} : \alpha \in \overline{T}\}$ is a maximal regular subsemigroup of T_{n-1}, i.e., $\overline{T} = \{\alpha \in T_{n-1}^0 : \alpha|_{X_{n-1}} \in \hat{T}\}$. Note that T contains all $\alpha \in End(S_n)\backslash J_2^w$ with rank$(\alpha) \leq 2$, where rank$(\beta) = 2$ for each $\beta \in J_2^w$, we can conclude that $G = J_2^w$. Hence, $T = J_2^w \cup \{\alpha \in T_{n-1}^0 : \alpha|_{X_{n-1}} \in \hat{T}\}$.
Suppose that $T = T_{n-1}^0$. Then by Lemma 3.14, we get that $T = End(S_n)\backslash J_2^w = T_{n-1}^0$.
Conversely, since T_{n-1} is regular with $T_{n-1} \cong T_{n-1}^0$, we can conclude that T_{n-1}^0 is regular. Let T be a maximal regular subsemigroup of T_{n-1}. Then $\overline{T} = \{\alpha \in T_{n-1}^0 : \alpha|_{x_{n-1}} \in \hat{T}\}$ is a maximal regular subsemigroup of T_{n-1}^0 since $T \cong T_{n-1}^0$. Note that $\alpha^3 = \alpha$ for all $\alpha \in J_2^w$. Hence, J_2^w consists entirely of regular elements. Since $\overline{T}$ contains all $\alpha \in End(S_n)$ with rank$(\alpha) \leq 2$ and $End(S_n) = T_{n-1}^0 \cup J_2^w$, we can conclude that $\overline{T} \cup J_2^w$ is a maximal regular subsemigroup of $End(S_n)$. □

Theorem 3.16. *Let T be a regular subsemigroup of $swEnd(S_n)$. Then T is a maximal regular subsemigroup of $swEnd(S_n)$ if and only if $T = End(S_n)$ or $T = \overline{T} \cup J_1$ for some maximal regular subsemigroup $\overline{T}$ of $End(S_n)$.*

Proof. Note that J_1 consists entirely of idempotents and $swEnd(S_n) = End(S_n) \cup J_1$, where J_1 is an ideal of $swEnd(S_n)$. These observations prove that if $\overline{T}$ is a maximal regular subsemigroup of $End(S_n)$, then $\overline{T} \cup J_1$ is a maximal regular subsemigroup of $End(S_n) \cup J_1 = swEnd(S_n)$. Moreover, $End(S_n)$ is regular as well as a maximal subsemigroup of $swEnd(S_n)$ by Theorem 2.12. Therefore, $End(S_n)$ is a maximal regular subsemigroup of $swEnd(S_n)$.

Conversely, let T be a maximal regular subsemigroup of $swEnd(S_n)$. If $J_1 \cap T = \emptyset$ then $T \subseteq End(S_n)$, i.e., $T = End(S_n)$. Suppose now that $J_1 \cap T \neq \emptyset$. Since $End(S_n)$ is a maximal regular subsemigroup of $swEnd(S_n)$, we get $T\backslash J_1 \neq End(S_n)$. Because J_1 is an ideal, we can conclude that $T\backslash J_1 \subset End(S_n)$ is a maximal regular subsemigroup of $End(S_n)$ and $T \subseteq (T\backslash J_1) \cup J_1$. This implies $T = (T\backslash J_1) \cup J_1$, where $(T\backslash J_1)$ is a maximal regular subsemigroup of $End(S_n)$. □

References

[1] I. Dimitrova, J. Koppitz, On the maximal subsemigroups of some transformation semigroups, Asian-Eur. J. Math. 1(2) (2008), 189-202. https://doi.org/10.1142/S1793557108000187

[2] I. Dimitrova, J. Koppitz, On the maximal regular subsemigroups of ideals of order-preserving or order-reversing transformations, Semigroup Forum. 82 (2011), 172–180. https://doi.org/10.1007/s00233-010-9272-8

[3] I. Dimitrova, V. H. Fernandes, J. Koppitz, The maximal subsemigroups of semigroups of transformations preserving or reversing the orientation on a finite chain, Publ. Math. Debrecen. 81 (2012), 11-29. https://doi.org/10.5486/PMD.2012.4897

[4] I. Dimitrova, V. H. Fernandes, J. Koppitz, T. M. Quinteiro, On monoids of endomorphisms of a cycle graph, Mathematica Slovaca. 74(5) (2024), 1071-1088. https://doi.org/10.1515/ms-2024-0078

[5] C. R. Donoven, J. D. Mitchell, W. A. Wilson, Computing maximal subsemigroups of a finite semigroup, J. Algebra. 505 (2018), 559-596. https://doi.org/10.1016/j.jalgebra.2018.01.044

[6] J. East, J. Kumar, J. D. Mitchell, W. A. Wilson, Maximal subsemigroups of finite transformation and diagram monoids, J. Algebra. 504 (2018), 176-216. https://doi.org/10.1016/j.jalgebra.2018.01.048

[7] V. H. Fernandes, T. Paulista, On the monoid of partial isometries of a finite star graph, Commun. Algebra. 51(3) (2023), 1028–1048. https://doi.org/10.1080/00927872.2022.2121404

[8] N. Graham, R. Graham, J. Rhodes, Maximal subsemigroups of finite semigroups, J. Combin. Theory. 4 (1968), 203-209. https://doi.org/10.1016/S0021-9800(68)80001-8

[9] H. Hou, Y. Luo, Z. Cheng, The endomorphism monoid of $\overline{P_n}$, European J. Combin. 29(5) (2008), 1173-1185.

[10] W. Li, Graphs with regular monoids, Discrete Math. 265 (2003), 105-118.

[11] S. Mendes-Goncalves, R. P. Sullivan, Maximal Subsemigroups of infinite symmetric groups, Bull. Aust. Math. Soc. 110(2) (2024), 324-337. https://doi.org/.10.1017/S0004972723001375

[12] E. Wilkeit, Graphs with a regular endomorphism monoid, Arch. Math. 66 (1996), 344-352.

[13] H. B. Yang, X. L. Yang, Maximal Subsemigroups of Finite Transformation Semigroups $K(n,r)$, Acta Math Sinica. 20 (2004), 475–482. https://doi.org/10.1007/s10114-004-0367-6

[14] X. Yang, Maximal Subsemigroups of the finite singular tranformation semigroup, Commun. Algebra. 29(3) (2001),1175-1182. https://doi.org/10.1081/AGB-100001675

[15] T. You, Maximal regular subsemigroups of certain semigroups of transformations, Semigroup Forum. 64 (2002), 391–396. https://doi.org/10.1007/s002330010117

[16] Y. Yuan, P. Zhao, Maximal subsemigroups and the maximal regular subsemigroups of the semigroup $H^*_{(n,m)}(r)$, J. Shandong university (Natural Science). 55(12) (2020): 19-24. https://doi.org/10.6040/j.issn.1671-9352.0.2020.319