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# Newton-cotes quadrature formula for third differentiable and $h$-convex functions

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**Abstract.** This study presents an innovative Newton-Cotes quadrature formula for functions that are third-differentiable and $h$-convex, utilizing Riemann integrals. Additionally, novel Newton-type inequalities are introduced employing a summation parameter $p \geq 1$ for $s$-convexity, convexity, and the class of $P$-functions.

## 1. Introduction

The second Simpson's quadrature formula, also referred to as the $\frac{3}{8}$ rule of Simpson, is as follows:

$$\int_a^b f(t)dt \approx \frac{b-a}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right],$$

also known in the literature as Newton-cotes quadrature formula.

Many inequalities have been studied for convex functions, but the Simpson-type inequality stands out due to its geometrical significance and applicability. The following inequality is well-known in the literature as the traditional Simpson type inequality for four times continuously differentiable functions.

For four times continuously differentiable functions, the Simpson $\frac{3}{8}$ rule is a classical closed type quadrature rule is as follows:

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{1}{6480}\left\|f^4\right\|_\infty (b-a)^4.$$

For a twice differentiable functions, the Newton-cotes quadrature formula is expressed in [4, Theorem 2.1] as follows:

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(b-a)^2\left[\left|f''(a)\right|+\left|f''(b)\right|\right]}{384}.$$

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For a continuously differentiable functions, the Newton-cotes quadrature formula is defined in [8, Remark 3] as follows:

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{25(b-a)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]}{576}.$$

Convex functions have been used in a variety of mathematical areas as a result of their efforts and study, leading to the discovery of many mathematical inequalities. In paper [3], the authors introduces a well-known class of functions called $s$-convex functions.

**Definition 1.1.** Let us consider $s\in(0,1]$. Then, we say that $\Phi: I\subseteq\mathbb{R}\to\mathbb{R}$ is an $s$-convex function, if $\Phi$ is non-negative and for all $x_1,x_2\in I$, $\tau\in(0,1)$ we have

$$\Phi(\tau x_1+(1-\tau)x_2)\leq\tau^s\Phi(x_1)+(1-\tau)^s\Phi(x_2). \tag{1}$$

If the inequality (1) is reversed, then $\Phi$ is said to be $s$-concave.

By setting

- $s=1$ reduces to convex function [6].
- $s\to 0$ reduces to $P$-functions [7].

Recently, the authors present a new class of function called $B$-function defined as:

**Definition 1.2.** ([1, 2]) Let $g:[0,\infty)\to\mathbb{R}$ be a non-negative function. The function $g$ is called a $B$-function if

$$g(x-a)+g(b-x)\leq 2g\left(\frac{a+b}{2}\right) \tag{2}$$

where $a<x<b$ with $a,b\in[0,\infty)$.

If the inequality (2) is reversed, $g$ is called $A$-function, or that $g$ belongs to the class $A(a,b)$.
If we have equality in (2), $g$ is called $AB$-function, or that $g$ belongs to the class $AB(a,b)$.

**Corollary 1.3.** *Let $h:(0,1)\to\mathbb{R}$ be a non-negative function. The function $h$ is a B-function, if for all $\lambda\in(0,1)$, we have*

$$h(\lambda)+h(1-\lambda)\leq 2h\left(\frac{1}{2}\right). \tag{3}$$

- *The functions $h(\lambda)=\lambda$ and $h(\lambda)=1$ are AB-function, B-function and A-function.*
- *The function $h(\lambda)=\lambda^s$, $s\in(0,1]$ is B-function.*

Depending on previous research, we present a new form of Newton-cotes quadrature formula for third differentiable and $h$-convex functions utilizing a summation parameter $p\geq 1$ and the Riemann integral.

## 2. The basic identity

**Lemma 2.1.** *Let $f:[a,b]\to\mathbb{R}$ be a three times differentiable function on $(a,b)$ such that $f'''\in L_1([a,b])$. Then the following identity holds.*

$$\begin{aligned}&\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\\&=\frac{(b-a)^3}{96}\left\{\int_0^{\frac{2}{3}}t^2\left(t-\frac{3}{4}\right)\left[f'''\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)-f'''\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)\right]dt\right.\\&\left.+\int_{\frac{2}{3}}^1(t-1)^3\left[f'''\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)-f'''\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)\right]dt\right\}.\end{aligned} \tag{4}$$

*Proof.* By using the integration by parts, we obtain

$$\begin{aligned}
L_1 &= \int_0^{\frac{2}{3}} t^2\left(t-\frac{3}{4}\right)\left[f'''\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)-f'''\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)\right]dt \\
&= \left(\frac{2}{b-a}\right)t^2\left(t-\frac{3}{4}\right)\left[f''\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)+f''\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)\right]\Bigg|_0^{\frac{2}{3}} \\
&\quad -\frac{2}{b-a}\int_0^{\frac{2}{3}} t\left(3t-\frac{3}{2}\right)\left[f''\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)+f''\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)\right]dt \\
&= -\frac{2}{27(b-a)}\left[f''\left(\frac{2a+b}{3}\right)+f''\left(\frac{a+2b}{3}\right)\right] \\
&\quad -\frac{4}{(b-a)^2}t\left(3t-\frac{3}{2}\right)\left[f'\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)-f'\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)\right]\Bigg|_0^{\frac{2}{3}} \\
&\quad +\frac{4}{(b-a)^2}\int_0^{\frac{2}{3}}\left(6t-\frac{3}{2}\right)\left[f'\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)-f'\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)\right]dt \\
&= -\frac{2}{27(b-a)}\left[f''\left(\frac{2a+b}{3}\right)+f''\left(\frac{a+2b}{3}\right)\right]-\frac{4}{3(b-a)^2}\left[f'\left(\frac{2a+b}{3}\right)-f'\left(\frac{a+2b}{3}\right)\right] \\
&\quad +\frac{8}{(b-a)^3}\left(6t-\frac{3}{2}\right)\left[f\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)+f\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)\right]\Bigg|_0^{\frac{2}{3}} \\
&\quad -\frac{48}{(b-a)^3}\int_0^{\frac{2}{3}}\left[f\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)+f\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)\right]dt.
\end{aligned}$$

Thus

$$\begin{aligned}
L_1 &= -\frac{2}{27(b-a)}\left[f''\left(\frac{2a+b}{3}\right)+f''\left(\frac{a+2b}{3}\right)\right]-\frac{4}{3(b-a)^2}\left[f'\left(\frac{2a+b}{3}\right)-f'\left(\frac{a+2b}{3}\right)\right] \\
&\quad +\frac{8}{(b-a)^3}\left\{\frac{5}{2}\left[f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right]+\frac{3}{2}\left[f(a)+f(b)\right]\right\} \\
&\quad -\frac{48}{(b-a)^3}\int_0^{\frac{2}{3}}\left[f\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)+f\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)\right]dt.
\end{aligned}$$

According to the preceding details , we deduce

$$\begin{aligned}
L_2 &= \int_{\frac{2}{3}}^1 (t-1)^3\left[f'''\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)-f'''\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)\right]dt \\
&= \frac{2}{27(b-a)}\left[f''\left(\frac{2a+b}{3}\right)+f''\left(\frac{a+2b}{3}\right)\right]+\frac{4}{3(b-a)^2}\left[f'\left(\frac{2a+b}{3}\right)-f'\left(\frac{a+2b}{3}\right)\right] \\
&\quad +\frac{8}{(b-a)^3}\left\{2\left[f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right]\right\}-\frac{48}{(b-a)^3}\int_{\frac{2}{3}}^1\left[f\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)+f\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)\right]dt.
\end{aligned}$$

Consequently

$$\begin{aligned} L_1+L_2 &= \frac{12}{(b-a)^3}\left\{3\left[f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right]+[f(a)+f(b)]\right\} \\ &\quad -\frac{48}{(b-a)^3}\int_0^1\left[f\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)+f\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)\right]dt. \end{aligned}$$

Since

$$\int_0^1 f\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)dt=\frac{2}{b-a}\int_a^{\frac{a+b}{2}}f(t)dt,$$

$$\int_0^1 f\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)dt=\frac{2}{b-a}\int_{\frac{a+b}{2}}^b f(t)dt$$

we deduce

$$L_1+L_2 = \frac{12}{(b-a)^3}\left\{3\left[f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right]+[f(a)+f(b)]\right\}-\frac{96}{(b-a)^4}\int_a^b f(t)dt.$$

To get the desired result, multiply the previous equality by $\frac{(b-a)^3}{96}$. □

Using the variable $\tau = 1-t$, we can rephrase the previously mentioned Lemma 2.1 in the following form.

**Lemma 2.2.** *Let $f:[a,b]\to\mathbb{R}$ be a three times differentiable function on $(a,b)$ such that $f'''\in L_1([a,b])$. Then the following identity holds:*

$$\begin{aligned} &\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt \\ &=\frac{(b-a)^3}{96}\left\{\int_0^{\frac{1}{3}}t^3\left[f'''\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)-f'''\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right]dt\right. \\ &\left.+\int_{\frac{1}{3}}^1(t-1)^2\left(t-\frac{1}{4}\right)\left[f'''\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)-f'''\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right]dt\right\}. \end{aligned} \tag{5}$$

## 3. Fundamental lemmas

To prove the key results, we must use the following two lemmas. These lemmas are based on the power mean integral inequality and the Hölder inequality.

**Lemma 3.1.** *Assuming $p\geq 1$ and $0\leq\lambda_1<\lambda_2\leq 1$. Suppose $\mu:(0,1)\to(0,1)$ is a mapping where $\mu(t)\in\left\{\frac{t}{2},\frac{1+t}{2}\right\}$, $f$ is three-time absolutely continuous function such that $f'''\in L_1(a,b)$ and $w:(0,1)\to\mathbf{R}$ is an absolutely continuous function with $w\in L_p(\lambda_1,\lambda_2)$. If $\left|f'''\right|^p$ is an h-convex mapping on $[a,b]$, then*

$$\begin{aligned} &\int_{\lambda_1}^{\lambda_2}|w(t)|\left[\left|f'''\left((1-\mu(t))a+\mu(t)b\right)\right|+\left|f'''\left(\mu(t)a+(1-\mu(t))b\right)\right|\right]dt \\ &\leq\left(\int_{\lambda_1}^{\lambda_2}|w(t)|\,dt\right)2\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}\left(\left|f'''(a)\right|^p+\left|f'''(b)\right|^p\right)^{\frac{1}{p}}. \end{aligned} \tag{6}$$

*Proof.* We need the following inequality to prove the next results.
Let $A, B \geq 0$ and $\eta > 0$:

$$A^{\eta} + B^{\eta} \leq \max(1, 2^{1-\eta})(A+B)^{\eta}. \tag{7}$$

Let $p \geq 1$, $0 \leq \lambda_1 < \lambda_2 \leq 1$ and $w \in L_p(\lambda_1, \lambda_2)$, using power-mean integral inequality gives:

$$\int_{\lambda_1}^{\lambda_2} |w(t)| \left[ \left| f''' \left( (1-\mu(t))\, a + \mu(t)\, b \right) \right| + \left| f''' \left( \mu(t)\, a + (1-\mu(t))\, b \right) \right| \right] dt$$

$$\leq \left( \int_{\lambda_1}^{\lambda_2} |w(t)|\, dt \right)^{1-\frac{1}{p}}$$

$$\times \left[ \left( \int_{\lambda_1}^{\lambda_2} |w(t)| \left| f''' \left( (1-\mu(t))\, a + \mu(t)\, b \right) \right|^p dt \right)^{\frac{1}{p}} + \left( \int_{\lambda_1}^{\lambda_2} |w(t)| \left| f''' \left( \mu(t)\, a + (1-\mu(t))\, b \right) \right|^p dt \right)^{\frac{1}{p}} \right].$$

The inequality (7) yields $A^{\frac{1}{p}} + B^{\frac{1}{p}} \leq 2^{1-\frac{1}{p}} (A+B)^{\frac{1}{p}}$, thus

$$\int_{\lambda_1}^{\lambda_2} |w(t)| \left[ \left| f''' \left( (1-\mu(t))\, a + \mu(t)\, b \right) \right| + \left| f''' \left( \mu(t)\, a + (1-\mu(t))\, b \right) \right| \right] dt$$

$$\leq \left( \int_{\lambda_1}^{\lambda_2} |w(t)|\, dt \right)^{1-\frac{1}{p}} 2^{1-\frac{1}{p}}$$

$$\times \left[ \int_{\lambda_1}^{\lambda_2} |w(t)| \left( \left| f''' \left( (1-\mu(t))\, a + \mu(t)\, b \right) \right|^p + \left| f''' \left( \mu(t)\, a + (1-\mu(t))\, b \right) \right|^p \right) dt \right]^{\frac{1}{p}}.$$

Given that $\left| f''' \right|^p$ is a $h$-convex function, the inequality (3) provides us

$$\left| f''' \left( (1-\mu(t))\, a + \mu(t)\, b \right) \right|^p \leq h\left(1-\mu(t)\right) \left| f'''(a) \right|^p + h\left(\mu(t)\right) \left| f'''(b) \right|^p.$$

Then, we have

$$\begin{aligned} &\left| f''' \left( (1-\mu(t))\, a + \mu(t)\, b \right) \right|^p + \left| f''' \left( \mu(t)\, a + (1-\mu(t))\, b \right) \right|^p \\ &\leq \left[ h\left(1-\mu(t)\right) + h\left(\mu(t)\right) \right] \left( \left| f'''(a) \right|^p + \left| f'''(b) \right|^p \right) \\ &\leq 2\, h\left(\tfrac{1}{2}\right) \left( \left| f'''(a) \right|^p + \left| f'''(b) \right|^p \right). \end{aligned} \tag{8}$$

Hence

$$\int_{\lambda_1}^{\lambda_2} |w(t)| \left[ \left| f''' \left( (1-\mu(t))\, a + \mu(t)\, b \right) \right| + \left| f''' \left( \mu(t)\, a + (1-\mu(t))\, b \right) \right| \right] dt$$

$$\leq \left( \int_{\lambda_1}^{\lambda_2} |w(t)|\, dt \right)^{1-\frac{1}{p}} 2^{1-\frac{1}{p}} \left[ \int_{\lambda_1}^{\lambda_2} |w(t)|\, 2\, h\left(\frac{1}{2}\right) \left( \left| f'''(a) \right|^p + \left| f'''(b) \right|^p \right) dt \right]^{\frac{1}{p}}.$$

For every value of $0 \leq \lambda_1 < \lambda_2 \leq 1$, we derive the next result:

$$\int_{\lambda_1}^{\lambda_2} |w(t)| \left[ \left| f''' \left( (1-\mu(t))\, a + \mu(t)\, b \right) \right| + \left| f''' \left( ta + (1-t)\, b \right) \right| \right] dt$$

$$\leq \left( \int_{\lambda_1}^{\lambda_2} |w(t)|\, dt \right) 2 \left( h\left(\frac{1}{2}\right) \right)^{\frac{1}{p}} \left( \left| f'''(a) \right|^p + \left| f'''(b) \right|^p \right)^{\frac{1}{p}}.$$

□

**Lemma 3.2.** *Assuming $p >$ and $0 \leq \lambda_1 < \lambda_2 \leq 1$. Suppose $\mu : (0,1) \to (0,1)$ is a mapping where $\mu(t) \in \left\{\frac{t}{2}, \frac{1+t}{2}\right\}$, $f$ is three-time absolutely continuous function such that $f''' \in L_1(a,b)$ and $w : (0,1) \to \mathbf{R}$ is an absolutely continuous function with $w \in L_p(\lambda_1, \lambda_2)$. If $\left|f'''\right|^p$ is an h-convex mapping on $[a,b]$, then*

$$\begin{aligned}
&\int_{\lambda_1}^{\lambda_2} |w(t)| \left[\left|f'''\left((1-\mu(t))\,a + \mu(t)\,b\right)\right| + \left|f'''\left(\mu(t)\,a + (1-\mu(t))\,b\right)\right|\right] dt \\
&\leq \left(\int_{\lambda_1}^{\lambda_2} |w(t)|^q \, dt\right)^{\frac{1}{q}} (\lambda_2-\lambda_1)^{\frac{1}{p}} \; 2\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}} \left(\left|f'''(a)\right|^p + \left|f'''(b)\right|^p\right)^{\frac{1}{p}}.
\end{aligned} \tag{9}$$

*Proof.* Let $p > 1$ , $0 \leq \lambda_1 < \lambda_2 \leq 1$ and $w \in L_p(\lambda_1, \lambda_2)$, using Hölder inequality gives:

$$\begin{aligned}
&\int_{\lambda_1}^{\lambda_2} |w(t)| \left[\left|f'''\left((1-\mu(t))\,a + \mu(t)\,b\right)\right| + \left|f'''\left(\mu(t)\,a + (1-\mu(t))\,b\right)\right|\right] dt \\
&\leq \left(\int_{\lambda_1}^{\lambda_2} |w(t)|^q \, dt\right)^{\frac{1}{q}} \\
&\times \left[\left(\int_{\lambda_1}^{\lambda_2} \left|f'''\left((1-\mu(t))\,a + \mu(t)\,b\right)\right|^p dt\right)^{\frac{1}{p}} + \left(\int_{\lambda_1}^{\lambda_2} \left|f'''\left(\mu(t)\,a + (1-\mu(t))\,b\right)\right|^p dt\right)^{\frac{1}{p}}\right].
\end{aligned}$$

Since $A^{\frac{1}{p}} + B^{\frac{1}{p}} \leq 2^{1-\frac{1}{p}}(A+B)^{\frac{1}{p}}$, we get

$$\begin{aligned}
&\int_{\lambda_1}^{\lambda_2} |w(t)| \left[\left|f'''\left((1-\mu(t))\,a + \mu(t)\,b\right)\right| + \left|f'''\left(\mu(t)\,a + (1-\mu(t))\,b\right)\right|\right] dt \\
&\leq \left(\int_{\lambda_1}^{\lambda_2} |w(t)|^q \, dt\right)^{\frac{1}{q}} 2^{1-\frac{1}{p}} \\
&\times \left[\int_{\lambda_1}^{\lambda_2} \left(\left|f'''\left((1-\mu(t))\,a + \mu(t)\,b\right)\right|^p + \left|f'''\left(\mu(t)\,a + (1-\mu(t))\,b\right)\right|^p\right) dt\right]^{\frac{1}{p}}.
\end{aligned}$$

Given that $\left|f'''\right|^p$ is a $h$-convex function, the inequality (8) provides us

$$\begin{aligned}
&\left|f'''\left((1-\mu(t))\,a + \mu(t)\,b\right)\right|^p + \left|f'''\left(\mu(t)\,a + (1-\mu(t))\,b\right)\right|^p \\
&\leq 2\,h\left(\tfrac{1}{2}\right)\left(\left|f'''(a)\right|^p + \left|f'''(b)\right|^p\right),
\end{aligned}$$

therefore

$$\begin{aligned}
&\int_{\lambda_1}^{\lambda_2} |w(t)| \left[\left|f'''\left((1-\mu(t))\,a + \mu(t)\,b\right)\right| + \left|f'''\left(\mu(t)\,a + (1-\mu(t))\,b\right)\right|\right] dt \\
&\leq \left(\int_{\lambda_1}^{\lambda_2} |w(t)|^q \, dt\right)^{\frac{1}{q}} 2^{1-\frac{1}{p}} \left[\int_{\lambda_1}^{\lambda_2} 2\,h\left(\frac{1}{2}\right)\left(\left|f'''(a)\right|^p + \left|f'''(b)\right|^p\right) dt\right]^{\frac{1}{p}}.
\end{aligned}$$

For all values of $0 \le \lambda_1 < \lambda_2 \le 1$, we obtain the following result:

$$\int_{\lambda_1}^{\lambda_2} |w(t)| \left[ \left| f''' \left( (1-\mu(t))\, a + \mu(t)\, b \right) \right| + \left| f''' \left( ta + (1-t)\, b \right) \right| \right] dt$$

$$\le \left( \int_{\lambda_1}^{\lambda_2} |w(t)|^q \; dt \right)^{\frac{1}{q}} (\lambda_2 - \lambda_1)^{\frac{1}{p}} \; 2 \left( h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}} \left( \left|f'''(a)\right|^p + \left|f'''(b)\right|^p \right)^{\frac{1}{p}}.$$

□

## 4. Newton-cotes quadrature formula via power mean inequality

Now, we present the first theorem.

**Theorem 4.1.** *Let $q \ge 1$, $h$ be a B-function on $(0,1)$, and assume that $f$ is defined as in Lemma 2.1. If $\left|f'''\right|^p$ is an $h$-convex mapping on $[a,b]$, then the following Newton-type inequality holds.*

$$\left| \frac{1}{8} \left[ f(a) + 3 f\left(\frac{2a+b}{3}\right) + 3 f\left(\frac{a+2b}{3}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(t)dt \right| \le \frac{(b-a)^3}{1728} \left( h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}} \left[ \left|f'''(a)\right|^p + \left|f'''(b)\right|^p \right]^{\frac{1}{p}}. \tag{10}$$

*Proof.* Using the modulus of identity (4) and applying the inequality (6), we deduce

$$\left| \frac{1}{8} \left[ f(a) + 3 f\left(\frac{2a+b}{3}\right) + 3 f\left(\frac{a+2b}{3}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(t)dt \right|$$

$$\le \frac{(b-a)^3}{96} \left\{ \int_0^{\frac{2}{3}} \left| t^2 \left( t - \frac{3}{4} \right) \right| \left[ \left| f''' \left( \left(1 - \frac{t}{2}\right) a + \frac{t}{2} b \right) \right| + \left| f''' \left( \frac{t}{2} a + \left(1 - \frac{t}{2}\right) b \right) \right| \right] dt \right.$$

$$\left. + \int_{\frac{2}{3}}^1 \left| (t-1)^3 \right| \left[ \left| f''' \left( \left(1 - \frac{t}{2}\right) a + \frac{t}{2} b \right) \right| + \left| f''' \left( \frac{t}{2} a + \left(1 - \frac{t}{2}\right) b \right) \right| \right] dt \right\}$$

$$\le \frac{(b-a)^3}{48} \left( h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}} \left( \left|f'''(a)\right|^p + \left|f'''(b)\right|^p \right)^{\frac{1}{p}} \left\{ \int_0^{\frac{2}{3}} \left| t^2 \left( t - \frac{3}{4} \right) \right| dt + \int_{\frac{2}{3}}^1 \left| (t-1)^3 \right| dt \right\}.$$

As

$$\int_0^{\frac{2}{3}} t^2 \left( \frac{3}{4} - t \right) dt + \int_{\frac{2}{3}}^1 (1-t)^3 \, dt = \frac{1}{36},$$

the desired inequality is achieved. □

We present a various cases of Newton's quadrature formula based on $p$ and $h$.

Put $p = 1$ in the above Theorem 4.1, we get the following corollary.

**Corollary 4.2.** *Let $s \in [0,1]$ and assume that the assumptions of Lemma 4 hold. If $|f'''|$ is a $h$-convex mapping on $[a,b]$, then the following Newton-type inequality holds*

$$\left| \frac{1}{8} \left[ f(a) + 3 f\left(\frac{2a+b}{3}\right) + 3 f\left(\frac{a+2b}{3}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(t)dt \right| \le \frac{(b-a)^3}{1728}\, h\left(\frac{1}{2}\right) \left[ \left|f'''(a)\right| + \left|f'''(b)\right| \right]. \tag{11}$$

*4.1. Newton-cotes quadrature formula via s-convex function*

Choosing $h(t) = t^s$ in Theorem 4.1 and Corollary 4.2 yields the following results.

**Corollary 4.3.** *Let $p \geq 1$ and assume that $f$ are defined as in Lemma 2.1. If $\left|f'''\right|^p$ is an s-convex mapping on $[a, b]$, then the following Newton-type inequality holds.*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(b-a)^3}{1728}\left(\frac{1}{2}\right)^{\frac{s}{p}}\left[\left|f'''(a)\right|^p+\left|f'''(b)\right|^p\right]^{\frac{1}{p}}, \quad (12)$$

*where $s \in (0, 1]$.*

**Corollary 4.4.** *Let assume that the assumptions of Lemma 4 hold. If $|f'''|$ is a s-convex mapping on $[a, b]$, then the following Newton-type inequality holds*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(b-a)^3}{1728\cdot 2^s}\left[\left|f'''(a)\right|+\left|f'''(b)\right|\right], \quad (13)$$

*where $s \in (0, 1]$.*

**Remark 4.5.** In the precedent Corollary 4.4, if we assume that $\left|f'''\right|$ is bounded i.e. $\sup_{x\in[a,b]}\left|f'''(x)\right| = \left\|f'''\right\|_\infty$, then we obtain

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(b-a)^3}{864}\left(\frac{1}{2}\right)^{s}\left\|f'''\right\|_\infty. \quad (14)$$

The inequality (14) is a new one that uses $s$-convexity. Putting $s = 1$ yields

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(b-a)^3}{1728}\left\|f'''\right\|_\infty.$$

*4.2. Newton-cotes quadrature formula via convex function*

If we choose $h(t) = t$ in the Theorem 4.1 and Corollary 4.2, we obtain the results bellow.

**Corollary 4.6.** *Let $p \geq 1$ and assume that $f$ are defined as in Lemma 2.1. If $\left|f'''\right|^p$ is a convex mapping on $[a, b]$, then the following Newton-type inequality holds*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(b-a)^3}{1728}\left[\frac{\left|f'''(a)\right|^p+\left|f'''(b)\right|^p}{2}\right]^{\frac{1}{p}}. \quad (15)$$

*For $p = 1$, we get*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(b-a)^3}{3456}\left[\left|f'''(a)\right|+\left|f'''(b)\right|\right]. \quad (16)$$

*4.3. Newton-cotes quadrature formula via class P-functions*

If we choose $h(t) = 1$ in the Theorem 4.1 and Corollary 4.2, we obtain the new results involving the class $P$-functions.

**Corollary 4.7.** *Let $p \geq 1$ and assume that $f$ are defined as in Lemma 2.1. If $\left|f'''\right|^p$ is a P-function mapping on $[a, b]$, then the following Newton-type inequality holds*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(b-a)^3}{1728}\left[\left|f'''(a)\right|^p+\left|f'''(b)\right|^p\right]^{\frac{1}{p}}. \quad (17)$$

*Set $p = 1$, then*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(b-a)^3}{1728}\left[\left|f'''(a)\right|+\left|f'''(b)\right|\right]. \quad (18)$$

## 5. Newton-cotes quadrature formula via Hölder inequality

Now, we present the second theorem.

**Theorem 5.1.** *Let $p > 1$, $h$ be a B-function on $(0,1)$, and assume that $f$ are defined as in Lemma 2.1. If $\left|f'''\right|^p$ is an $h$-convex mapping on $[a,b]$, then the following Newton-type inequality holds.*

$$\begin{aligned}
&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \\
&\leq \frac{(b-a)^3}{48}\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}\left(\left|f'''(a)\right|^p+\left|f'''(b)\right|^p\right)^{\frac{1}{p}}\left\{\left(\int_0^{\frac{2}{3}}\left|t^2\left(t-\frac{3}{4}\right)\right|^q dt\right)^{\frac{1}{q}}\left(\frac{2}{3}\right)^{\frac{1}{p}}+\left(\int_{\frac{2}{3}}^1\left|(t-1)^3\right|^q dt\right)^{\frac{1}{q}}\left(\frac{1}{3}\right)^{\frac{1}{p}}\right\},
\end{aligned} \tag{19}$$

*where $\frac{1}{p}+\frac{1}{q}=1$.*

*Proof.* Using the modulus of identity (4) and applying the inequality (9), we deduce

$$\begin{aligned}
&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \\
&\leq \frac{(b-a)^3}{96}\left\{\int_0^{\frac{2}{3}}\left|t^2\left(t-\frac{3}{4}\right)\right|\left[\left|f'''\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)\right|+\left|f'''\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)\right|\right]dt\right. \\
&\left.+\int_{\frac{2}{3}}^1\left|(t-1)^3\right|\left[\left|f'''\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)\right|+\left|f'''\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)\right|\right]dt\right\} \\
&\leq \frac{(b-a)^3}{48}\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}\left(\left|f'''(a)\right|^p+\left|f'''(b)\right|^p\right)^{\frac{1}{p}} \\
&\times\left\{\left(\int_0^{\frac{2}{3}}\left|t^2\left(t-\frac{3}{4}\right)\right|^q dt\right)^{\frac{1}{q}}\left(\frac{2}{3}\right)^{\frac{1}{p}}+\left(\int_{\frac{2}{3}}^1\left|(t-1)^3\right|^q dt\right)^{\frac{1}{q}}\left(\frac{1}{3}\right)^{\frac{1}{p}}\right\},
\end{aligned}$$

which finalizes the proof of Theorem 5.1. □

We present a various cases of Newton's quadrature formula based on $h$.

### 5.1. Newton-cotes quadrature formula via s-convex function

Choosing $h(t)=t^s$ in Theorem 5.1 yields the following results.

**Corollary 5.2.** *Let $p > 1$ and assume that $f$ are defined as in Lemma 2.1. If $\left|f'''\right|^p$ is an $s$-convex mapping on $[a,b]$, the following Newton-type inequality holds.*

$$\begin{aligned}
&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \\
&\leq \frac{(b-a)^3}{48}\left(\frac{1}{2}\right)^{\frac{s}{p}}\left(\left|f'''(a)\right|^p+\left|f'''(b)\right|^p\right)^{\frac{1}{p}}\left\{\left(\int_0^{\frac{2}{3}}\left|t^2\left(t-\frac{3}{4}\right)\right|^q dt\right)^{\frac{1}{q}}\left(\frac{2}{3}\right)^{\frac{1}{p}}+\left(\int_{\frac{2}{3}}^1\left|(t-1)^3\right|^q dt\right)^{\frac{1}{q}}\left(\frac{1}{3}\right)^{\frac{1}{p}}\right\},
\end{aligned} \tag{20}$$

*where $\frac{1}{p}+\frac{1}{q}=1$.*

**Remark 5.3.** In Corollary 5.2, setting $s = 1$ and $s \to 0$ yields the Newton-cotes quadrature formula with convex and class $P$ functions, respectively.

## 6. Applications on special means

For any positive values $\eta_1, \eta_2, a, b > 0$, we consider the following means:

- The weighted arithmetic mean:

$$W(\eta_1, \eta_2, a, b) = \frac{\eta_1 a + \eta_2 b}{\eta_1 + \eta_2}.$$

- The arithmetic mean:

$$A(a, b) = \frac{a + b}{2}.$$

- The harmonic mean:

$$H(a, b) = \frac{2ab}{a + b}.$$

- The $n$-logarithmic mean:

$$L_n(a, b) = \left(\frac{b^{n+1} - a^{n+1}}{(b - a)(n + 1)}\right)^{\frac{1}{n}}, \quad n \in \mathbf{R} - \{-1, 0\}, \ b > a.$$

- The logarithmic mean:

$$L(a, b) = \left(\frac{b - a}{\ln b - \ln a}\right), \quad n \in \mathbf{R} - \{-1, 0\}, \ b > a.$$

In [5], the following example is given: Let $s \in (0, 1)$ and $d, k, c \in \mathbf{R}$. We define a function $\Phi : [0, +\infty) \to R$, as

$$\Phi(t) = \begin{cases} d, & t = 0; \\ k\, t^s + c & t > 0. \end{cases}$$

If $k \geq 0$ and $0 \leq c \leq d$, then $\Phi$ is an $s$-convex function.

**Example 6.1.** Let $t > 0$, $p \geq 1$, $0 < s < 1$ and consider the function $f(t) = t^{\left(\frac{s}{p}+3\right)}$, then

$$f'''(t) = \left(\frac{s}{p} + 3\right)\left(\frac{s}{p} + 2\right)\left(\frac{s}{p} + 1\right) t^{\frac{s}{p}}.$$

In reference to

$$\Phi(t) := \left|f'''\right|^p = \left[\left(\frac{s}{p} + 3\right)\left(\frac{s}{p} + 2\right)\left(\frac{s}{p} + 1\right)\right]^p t^s,$$

thus, for $d = c = 0$, $k = \left[\left(\frac{s}{p} + 3\right)\left(\frac{s}{p} + 2\right)\left(\frac{s}{p} + 1\right)\right]^p$, we have $\left|f'''\right|^p$ is an $s$-convex function.

The following results are obtained by applying the preceding example to inequality (10).

**Proposition 6.2.** *Let $b > a > 0$, $p \geq 1$, $0 < s < 1$ and $n = \frac{s}{p} + 3$. Then the following inequality holds:*

$$\left|\frac{1}{4}A(a^n, b^n) + \frac{3}{8}\left[W^n(1, 2, a, b) + W^n(2, 1, a, b)\right] - L_n^n(a, b)\right| \leq \frac{(b - a)^3\, n(n - 1)(n - 2)}{1728} \left(\frac{1}{2}\right)^{\frac{s-1}{p}} A^{\frac{1}{p}}\left(a^s, b^s\right).$$

## 7. Conclusions

In this paper, we have developed a new Newton-Cotes quadrature formula tailored for third-differentiable and h-convex functions within the framework of Riemann integration. The proposed approach extends classical numerical integration methods by incorporating the properties of $h$-convexity. Furthermore, we established novel Newton-type inequalities involving a summation parameter $p \geq 1$, which apply to $s$-convex functions, standard convex functions, and the broader class of $P$-functions. These results contribute to the theoretical foundation of convex analysis and numerical integration, offering new tools for both theoretical exploration and practical computation.

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