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Analytic solutions for third-order partial differential equation using a modified double Laplace transform method

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Abstract. This paper presents analytical solutions for third-order partial differential equations with integer-order derivatives derived using a third-order formulation with constant coefficients. The study provides analytical expressions that satisfy these partial differential equations by applying the modified double Laplace decomposition method under specific initial and boundary conditions. Then we found the error values and figured them out using the MATLAB program.

1. Introduction

Partial differential equations (PDEs) generalize classical integer-order PDEs and are extensively used in applied mathematics and the physical sciences to model complex phenomena. Over the past decades, numerous analytical and approximate techniques have been developed to address solutions for these equations. Among these, the Laplace transform (LT) has garnered significant attention due to its wide applications in modern science and engineering. While the LT is conventionally applied to single-variable functions $u(t)$, the double Laplace transform (DLT) has proven more suitable for handling tasks of two variables $u(t,x)$.

Exact solutions for PDEs with constant coefficients have been successfully derived using methods such as the Elzaki decomposition. Techniques including the Laplace decomposition and Adomian decomposition methods have also been applied effectively to the non-homogeneous Benney-Luke equation, together with specified initial conditions.

In [1], the authors examined the pseudo-hyperbolic telegraph PDE using the Caputo-Fabrizio derivative (CFD), introducing a modified DLT method specifically adapted to this model. Another novel approach, a variant of the fixed variational iteration method, was proposed in [10] to develop an advanced technique to solve non-linear fractional PDEs. This method integrates the modified variational iteration approach with the Laplace-Adomian decomposition method, providing a comparative analysis of its performance relative to the LT method.

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In [4], key properties and the convolution theorem related to the DLT are detailed, while in [5], DLT-based techniques were applied to solve partial integro-differential equations, demonstrating the effectiveness of this method for obtaining solutions of real-valued functions. Additionally, in [9], semi-analytical and numerical solutions for the time fractional Schrödinger pseudo-parabolic PDE were derived using the DL decomposition method. In [11], the authors implemented a numerical algorithm to address a third-order PDE with a problem of boundary value of three points.

Further advancements include the work in [6], where Elzaki employed a combination of DLT and the modified variational iteration technique to solve non-linear convolution PDEs.

In the present study, we explore the initial boundary value problem for three-dimensional PDEs with integer-order derivatives, based on these established methodologies to extend solutions in complex systems.

$$\begin{cases} u_{ttt}(t,x) + \lambda u_{tt}(t,x) + \ u_t(t,x) + u(t,x) = \delta u_{txx}(t,x) \\ \quad +u_{xx}(t,x) + f(t,x), 0 < x < l, \ t > 0 \\ u(0,x) = a_0(x), \ u_t(0,x) = a_1(x) \ , u_{tt}(0,x) = a_2(x), 0 \leqslant x \leqslant l \\ \quad u(t,0) = u(t,l) = 0, t \geqslant 0. \end{cases} \tag{1}$$

Here, the positive constant coefficients $\lambda, , \delta$ and a_0, a_1, a_2, f are recognized functions, $u(t,x)$ is a function whose value is not known.

2. A brief introduction to the DLT

Definition 2.1. ([7]) The DLT of a function $u(t,x)$ is defined by the following double integral:

$$\mathcal{L}_t\mathcal{L}_x[u(t,x)] = U(s,p) = \int_0^\infty \int_0^\infty e^{(-st-px)} u(t,x)dxdt, \tag{2}$$

whenever this integral exists. Here $t, x \geq 0$ and s, p are complex numbers.

Definition 2.2. ([4]) Let c and d be sufficiently real constants. The inverse of DLT is $\mathcal{L}_t^{-1}\mathcal{L}_x^{-1}[U(s,p)] = u(t,x)$, defined by:

$$u(t,x) = \mathcal{L}_t^{-1}\mathcal{L}_x^{-1}[U(s,p)] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} ds \ \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} e^{px} U(s,p) dp, \tag{3}$$

here $i = \sqrt{-1}$ is an imaginary number, $U(s,p)$ must be an analytical function for each s and p in the area denoted by the inequalities $Re(s) \geq c$ and $Re(p) \geq d$.

Next, we obtain by transforming the initial value condition using a Laplace transform as follows [4]:

$$\begin{cases} \mathcal{L}_t\mathcal{L}_x[u(t,x)] = U(s,p), \\ \mathcal{L}_t\mathcal{L}_x[\frac{\partial^z u(t,x)}{\partial t^z}] = s^z U(s,p) - \sum_{k=0}^{n-1} s^{z-1-k} \mathcal{L}_x[\frac{\partial^k u(0,x)}{\partial t^k}] \ , \ \mathcal{L}_t\mathcal{L}_x[f(t,x)] = F(s,p), \mathcal{L}_t\mathcal{L}_x[a_0(x)] = G_0(p), \\ \mathcal{L}_t\mathcal{L}_x[a_1(x)] = G_1(p), \ \mathcal{L}_t\mathcal{L}_x[a_2(x)] = G_2(p). \end{cases} \tag{4}$$

3. Exact solution of the integer order derivative (IOD) and its stability estimates

In this section, we will identify the exact solution to the problem (1). The stability estimations for the problem (5) will then be displayed. The ordinary differential equation is used to represent the PDE to accomplish this. Using this transform by [3], the exact solutions and stability of the differential equations can be readily calculated. However, this utilizes the norm in Hilbert space. The problem (1) can be represented in an abstract form as follows:

$$\begin{cases} u^{'''}(t) + \lambda u^{''}(t) + u^{'}(t) + R\, u(t) = f(t), (0 < t < T) \\ \\ u(0) = a_0,\ u^{'}(0) = a_1,\ u^{''}(0) = a_2, 0 < \beta \leqslant 1. \end{cases} \tag{5}$$

In the Hilbert space $H = L_2\,[0, L]$, where $f(t) = f(t, x)$ represents an abstract function defined in $[0, T]$ with values in $H = L_2\,[0, L]$. Here, $a_0 = a_0(x)$, $a_1 = a_1(x)$, and $a_2 = a_2(x)$ are elements of $H = L_2\,[0, L]$. The function $u(t) = u(t, x)$ is the unknown abstract function defined in $[0, T]$ with values in $H = L_2\,[0, L]$. The operator $R : D(R) \to H$ represents the differential operator defined:

$$R\, u(x) = -(\delta u^{'}(t) + 1)\, u^{''}(x) + u(x),$$

on the domain

$$D(\lambda) = \left\{u : u,\ u^{'}\ , u^{''}\ \in L_2[0, L], u(0) = u(L) = 0\right\},$$

and within a Hilbert space $L_2[0, L]$, comprising square-integrable functions defined in the interval $[0, L]$, characterized by a norm

$\|\omega\|_H = \left(\int_{x \in L_2} \left|f(x)\right|^2 dx\right)^{\frac{1}{2}}$.

Here, we have $\langle Ru, v\rangle = \langle u, Rv\rangle$ $(u, v \in D(R))$ and $\langle Ru, u\rangle \geqslant \alpha\langle u, u\rangle$, $\alpha > 0$, $\alpha \geqslant 0$. Next, we will derive the ES from the problem (5). To do this, we define $A(t) = f(t) - u^{'}(t) - \lambda u^{''}(t)$, allowing us to rewrite formula (5) as:

$$\begin{cases} u^{'''}(t) + Ru(t) = A(t), (0 < t < T) \\ u(0) = a_0,\ u^{'}(0) = a_1,\ u^{''}(0) = a_2, 0 < \beta \leqslant 1. \end{cases} \tag{6}$$

Subsequently, we can express (6) as the following system of first-order differential equations to achieve the exact response.

$$\begin{cases} u^{'}(t) - b\ \mu\ u(t) = m(t), \\ m^{'}(t) - q\ \mu\ m(t) = s(t), \\ s^{'}(t) + \mu\ s(t) = A(t), \end{cases} \tag{7}$$

here, $\mu = R^{\frac{1}{3}}$. here, $b = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $q = \frac{1}{2} - i\frac{\sqrt{3}}{2}$, by substituting the values of b and q into the above formula, we obtain:

$$\begin{cases} m(0) = u^{'}(0) - b\ \mu\ u(0), \\ s(0) = u^{''}(0) - \mu\ u^{'}(0) + \mu^2 u(0), \end{cases} \tag{8}$$

integrating the formula (7), using the initial conditions, and in the study of differentiation and integration steps were given one by one for the formula (8) and we obtain

$$u(t) = L_1 u(0) + L_2 u^{'}(0) + L_3 u^{''}(0) + \int_0^t L_4 f(r) dr - \int_0^t L_4\ u^{'}(r) dr - \lambda \int_0^t L_4 u^{''}(r) dr,$$

then $u(t)$ is

$$u(t) = L_1 u(0) + L_2 u^{'}(0) + L_3 u^{''}(0) + \int_0^t L_4 f(r)dr - L_4(u(t) - u(0)) - \lambda L_4(u^{'}(t) - u^{'}(0)). \tag{9}$$

Based on that, we express

$$u^{'}(t) + \frac{(1+L_4)}{\lambda L_4} u(t) = \frac{1}{\lambda L_4}\left((L_1 + L_4)u(0) + (L_2 + \lambda L_4)u^{'}(0) + L_3 u^{''}(0) + \int_0^t L_4 f(r)dr\right), \tag{10}$$

by an interchange of the order integration of the last two formulas of the (10), we obtain

$$\begin{aligned} u(t) = (e^{-\frac{(1+L_4)}{\lambda L_4}t} + (L_1 + L_4)(1 - e^{-\frac{(1+L_4)}{\lambda L_4}t}))\, u(0) + (1 - e^{-\frac{(1+L_4)}{\lambda L_4}t})(L_2 + \lambda L_4)u^{'}(0) + (1 - e^{-\frac{(1+L_4)}{\lambda L_4}t})L_3 \\ + L_4 \int_0^t (1 - e^{-\frac{(1+L_4)}{\lambda L_4}(t-r)}) f(r)dr. \end{aligned} \tag{11}$$

Lemma 3.1. *The following inequalities hold:*

$$\begin{cases} 1.) \left\| e^{-\frac{(1+L_4)}{\lambda L_4}t} + (L_1 + L_4)(1 - e^{-\frac{(1+L_4)}{\lambda L_4}t}) \right\|_{H \to H} \leqslant K, \\ 2.) \left\| (1 - e^{-\frac{(1+L_4)}{\lambda L_4}t})(L_2\, R^{1/3} + \lambda L_4\, R^{1/3}) \right\|_{H \to H} \leqslant K, \\ 3.) \left\| L_3\, R^{2/3}(1 - e^{-\frac{(1+L_4)}{\lambda L_4}t}) \right\|_{H \to H} \leqslant K, \\ 4.) \left\| L_4\, R^{2/3}(1 - e^{-\frac{(1+L_4)}{\lambda L_4}t}) \right\|_{H \to H} \leqslant K. \end{cases} \tag{12}$$

Theorem 3.2. ([3]) *Suppose $a_0 \in D(R)$, $a_1 \in D(R^{2/3})$, $a_2 \in D(R^{1/3})$, and $f(t)$ are continuously differentiable on $[0, T]$. In that case, there is a unique solution to problem (11) along with the subsequent stability inequalities.*

$$\max_{0 \leqslant t \leqslant T} \|u(t)\|_H \leqslant K \left\{ \|a_0\|_H + \left\| a_1 R^{-1/3} \right\|_H + \left\| a_2 R^{-2/3} \right\|_H + \max_{0 \leqslant t \leqslant T} \left\| R^{-2/3} f(t) \right\|_H \right\}, \tag{13}$$

these are valid, where K is independent of $f(t)$, $t \in [0, T]$, a_0, a_1, and a_2.

Proof. From the formulas (11) and (12), we are able to write

$$\begin{aligned} \|u(t)\|_H = & \left\| e^{-\frac{(1+L_4)}{\lambda L_4}t} + (L_1 + L_4)(1 - e^{-\frac{(1+L_4)}{\lambda L_4}t}) \right\|_{H \to H} \|u(0)\|_H \\ & + \left\| (1 - e^{-\frac{(1+L_4)}{\lambda L_4}t})(L_2 + \lambda L_4) \right\|_{H \to H} \left\| u^{'}(0) \right\|_H \\ & + \left\| (1 - e^{-\frac{(1+L_4)}{\lambda L_4}t})L_3 \right\|_{H \to H} \left\| u^{''}(0) \right\|_H \\ & + \int_0^t \left\| (1 - e^{-\frac{(1+L_4)}{\lambda L_4}(t-r)})L_4 \right\|_{H \to H} \left\| f(r) \right\|_H dr. \end{aligned} \tag{14}$$

Using triangle inequality, the formula (14) and given initial conditions, we invest

$$\|u(t)\|_H \leq K \left\{ \|a_0\|_H + \left\| a_1 R^{-1/3} \right\|_H + \left\| a_2 R^{-2/3} \right\|_H + \max_{0 \leqslant t \leqslant T} \left\| R^{-2/3} f(t) \right\|_H \right\}.$$

□

4. Modified double Laplace decomposition method

Utilizing the DLT with relation to t and x for an IOD problem(1), and using (4) , we obtain,

$$\begin{cases} s^3U(s,p) - s^2G_0(p) - sG_1(p) - G_2(p) = -\lambda\mathcal{L}_t\mathcal{L}_x[u_{tt}(t,x)] \\ -\mathcal{L}_t\mathcal{L}_x[u_t(t,x)] - \mathcal{L}_t\mathcal{L}_x[u(t,x)] + \mathcal{L}_t\mathcal{L}_x[\delta u_{txx}(t,x) \\ +u_{xx}(t,x)] + \mathcal{L}_t\mathcal{L}_x[f(t,x)] \\ \mathcal{L}_t\mathcal{L}_x[u(o,x)] = G_0(p),\ \mathcal{L}_t\mathcal{L}_x[u_t(t,x)] = G_1(p)\ , \mathcal{L}_t\mathcal{L}_x[u_{tt}(t,x)] = G_2(p), \end{cases} \tag{15}$$

by simplifying equation (15), we derive

$$\begin{cases} U(s,p) = \frac{G_0(p)}{s} + \frac{G_1(p)}{s^2} + \frac{G_2(p)}{s^3} - \frac{\lambda}{s^3}\mathcal{L}_t\mathcal{L}_x[u_{tt}(t,x)] - \frac{1}{s^3}\mathcal{L}_t\mathcal{L}_x[u_t(t,x)] \\ -\frac{1}{s^3}\mathcal{L}_t\mathcal{L}_x[u(t,x)] + \frac{1}{s^3}\mathcal{L}_t\mathcal{L}_x[\delta u_{txx}(t,x) + u_{xx}(t,x)] + \frac{1}{s^3}F(s,p) \\ \mathcal{L}_t\mathcal{L}_x[u(o,x)] = G_0(p),\ \mathcal{L}_t\mathcal{L}_x[u_t(t,x)] = G_1(p)\ , \mathcal{L}_t\mathcal{L}_x[u_{tt}(t,x)] = G_2(p). \end{cases} \tag{16}$$

The solution to the problem (16) obtained by applying the decomposition series for $u(t,x)$ is,

$$\begin{aligned} u(t,x) = \textstyle\sum_{n=0}^{\infty} u_n(t,x) = a_0(x) + a_1(x)t + a_2(x)t^2 + \mathcal{L}_t^{-1}\mathcal{L}_x^{-1}\left[\frac{F(s,p)}{s^3}\right] \\ -\mathcal{L}_t^{-1}\mathcal{L}_x^{-1}\left\{\frac{\lambda}{s^3}\mathcal{L}_t\mathcal{L}_x\left[\frac{\partial^2}{\partial t^2}\textstyle\sum_{n=0}^{\infty} u_n(t,x)\right]\right\} - \mathcal{L}_t^{-1}\mathcal{L}_x^{-1}\left\{\frac{1}{s^3}\mathcal{L}_t\mathcal{L}_x\left[\frac{\partial}{\partial t}\textstyle\sum_{n=0}^{\infty} u_n(t,x)\right]\right\} \\ -\mathcal{L}_t^{-1}\mathcal{L}_x^{-1}\left\{\frac{1}{s^3}\mathcal{L}_t\mathcal{L}_x\left[\textstyle\sum_{n=0}^{\infty} u_n(t,x)\right]\right\} + \mathcal{L}_t^{-1}\mathcal{L}_x^{-1}\left\{\frac{\delta}{s^3}\mathcal{L}_t\mathcal{L}_x\left[\frac{\partial^2}{\partial x^2}\frac{\partial}{\partial t}\textstyle\sum_{n=0}^{\infty} u_n(t,x)\right]\right\} \\ +\mathcal{L}_t^{-1}\mathcal{L}_x^{-1}\left\{\frac{1}{s^3}\mathcal{L}_t\mathcal{L}_x\left[\frac{\partial^2}{\partial x^2}\textstyle\sum_{n=0}^{\infty} u_n(t,x)\right]\right\}. \end{aligned} \tag{17}$$

From (15), especially, we shall take the following recursive relationship,

$$u_0(t,x) = a_0(x) + a_1(x)t + a_2(x)t^2 + \mathcal{L}_t^{-1}\mathcal{L}_x^{-1}\left[\frac{1}{s^3}F(s,p)\right], \tag{18}$$

in this instance, we can express the remaining terms as follows.

$$\begin{aligned} u_{n+1}(t,x) = -\mathcal{L}_t^{-1}\mathcal{L}_x^{-1}\left\{\frac{\lambda}{s^3}\mathcal{L}_t\mathcal{L}_x\left[\frac{\partial^2}{\partial t^2}\textstyle\sum_{n=0}^{\infty} u_n(t,x)\right]\right\} - \mathcal{L}_t^{-1}\mathcal{L}_x^{-1}\left\{\frac{1}{s^3}\right. \\ \left.\mathcal{L}_t\mathcal{L}_x\left[\frac{\partial}{\partial t}\textstyle\sum_{n=0}^{\infty} u_n(t,x)\right]\right\} - \mathcal{L}_t^{-1}\mathcal{L}_x^{-1}\left\{\frac{1}{s^3}\mathcal{L}_t\mathcal{L}_x\right. \\ \left.\left[\textstyle\sum_{n=0}^{\infty} u_n(t,x)\right]\right\} + \mathcal{L}_t^{-1}\mathcal{L}_x^{-1}\left\{\frac{\delta}{s^3}\mathcal{L}_t\mathcal{L}_x\left[\frac{\partial^2}{\partial x^2}\frac{\partial}{\partial t}\textstyle\sum_{n=0}^{\infty} u_n(t,x)\right]\right\} \\ +\mathcal{L}_t^{-1}\mathcal{L}_x^{-1}\left\{\frac{1}{s^3}\mathcal{L}_t\mathcal{L}_x\left[\frac{\partial^2}{\partial x^2}\textstyle\sum_{n=0}^{\infty} u_n(t,x)\right]\right\}, \end{aligned} \tag{19}$$

here $n \geq 0$. Formulas (18) and (19) represent the modified DL decomposition method for integer order derivatives of the Problem (1). Below are the inverse LT of t and x, each with the order of $\mathcal{L}_t^{-1}\mathcal{L}_x^{-1}$.

5. Applications example

Example 5.1. The 3^{rd}- order PD equation that follows makes sense in terms of IOD:

$$\{u_{ttt}(t,x)+\lambda u_{tt}(t,x)+u_t(t,x)+u(t,x)=\delta u_{txx}(t,x)+u_{xx}(t,x)+f(t,x),\ \ 0<x<\pi, t>0, \beta\in(0,1], \tag{20}$$

where

$$f(t,x)=\left(2t^3+(1+3\delta)t^2+(6\lambda-2\delta)t-2\lambda+\delta+5\right)\sin x, \tag{21}$$

conditioned upon the initial state

$$u(0,x)=-\sin x,\ u_t(0,x)=sinx,\ u_{tt}(0,x)=-2sinx,\ 0\le x\le\pi, \tag{22}$$

and conditions specified at the boundaries

$$u(t,0)=u(t,\pi)=0,\ t\ge 0. \tag{23}$$

Solution:

Applying the DLT with respect to t and x to a $f(t,x)$, we get

$$\mathcal{L}_t\mathcal{L}_x[f(t,x)]=\mathcal{L}_t\mathcal{L}_x\left[\left(2t^3+(1+3\delta)t^2+(6\lambda-2\delta)t-2\lambda+\delta+5\right)\sin x\right], \tag{24}$$

$$F(s,p)=\left[\left(\frac{12}{s^4}+\frac{2(1+3\delta)}{s^3}+\frac{(6\lambda-2\delta)}{s^2}+\frac{(-2\lambda+\delta+5)}{s}\right)\left(\frac{1}{p^2+1}\right)\right]. \tag{25}$$

The result of solving both sides of the problem with the inverse DLT, after multiply $\frac{1}{s^3}$, we obtain

$$f(t,x)=\left(\frac{1}{60}t^6+\frac{(1+3\delta)}{60}t^5+\frac{(6\lambda-2\delta)}{24}t^4+\frac{(-2\lambda+\delta+5)}{6}t^3\right)\sin x. \tag{26}$$

Thus, by the technique (18), we get u_0

$$u_0=u_0(t,x)=\left(\frac{1}{60}t^6+\frac{(1+3\delta)}{60}t^5+\frac{(6\lambda-2\delta)}{24}t^4+\frac{(-2\lambda+\delta+5)}{6}t^3-2t^2+t-1\right)\sin x, \tag{27}$$

using the decomposition series (17) and u_0, we obtain u_1

$$\begin{aligned} u_1=u_1(t,x)&=-\mathcal{L}_t^{-1}\mathcal{L}_x^{-1}\left\{\tfrac{\lambda}{s^3}\mathcal{L}_t\mathcal{L}_x\left[\tfrac{\partial^2}{\partial t^2}u_0(t,x)\right]\right\}-\mathcal{L}_t^{-1}\mathcal{L}_x^{-1}\left\{\tfrac{1}{s^3}\mathcal{L}_t\mathcal{L}_x\right.\\ &\left.\left[\tfrac{\partial}{\partial t}u_0(t,x)\right]\right\}-\mathcal{L}_t^{-1}\mathcal{L}_x^{-1}\left\{\tfrac{1}{s^3}\mathcal{L}_t\mathcal{L}_x\left[u_0(t,x)\right]\right\}+\mathcal{L}_t^{-1}\mathcal{L}_x^{-1}\left\{\tfrac{\delta}{s^3}\mathcal{L}_t\mathcal{L}_x\left[\tfrac{\partial^2}{\partial x^2}\tfrac{\partial}{\partial t}u_0(t,x)\right]\right\}\\ &+\mathcal{L}_t^{-1}\mathcal{L}_x^{-1}\left\{\tfrac{1}{s^3}\mathcal{L}_t\mathcal{L}_x\left[\tfrac{\partial^2}{\partial x^2}u_0(t,x)\right]\right\}, \end{aligned} \tag{28}$$

find all terms of formula (28), we obtain u_1

$$\begin{aligned} u_1=&-\left(\tfrac{1}{15120}t^9+(4+3\delta)\tfrac{1}{5040}t^8+(24\lambda+4\delta+6\delta^2+2)\tfrac{1}{5040}t^7\right.\\ &+(4\lambda+12\lambda\delta-2\delta^2+10)\tfrac{t^6}{720}+(6\lambda^2-4\lambda\delta-2\lambda+6\delta-3+\delta^2)\tfrac{t^5}{120}\\ &\left.+(-2\lambda^2+\lambda\delta+5\lambda-2-4\delta)\tfrac{1}{24}t^4+(-4\lambda+\delta-1)\tfrac{1}{6}t^3\right)\sin x. \end{aligned} \tag{29}$$

Now you can find the value of u_2 in terms u_1 by the same idea (28).

$$\begin{aligned} u_2 = u_2(t,x) = \Big(& \tfrac{1}{9979200}t^{12} + \Big[\tfrac{(1+\delta)}{1663200} + \tfrac{(5+3\delta)}{2494800}\Big]t^{11} + (72\lambda + 120\delta + 60\delta^2 \\ & +68)\tfrac{1}{3628800}t^{10} + (96\lambda + 6\delta + 8\delta^2 + 96\lambda\delta + 6\delta^3 + 22)\tfrac{1}{362880}t^9 \\ & +(24\lambda + 26\delta + 6\delta^2 + 8\lambda\delta + 12\lambda^2 + 12\lambda\delta^2 - 2\delta^3 + 6)\tfrac{1}{40320}t^8 \\ & +(6\lambda^2 - 6\lambda\delta^2 + 18\lambda - 4\lambda\delta - 5\delta + 7\delta^2 + 18\lambda^2\delta + \delta^3 - 7)\tfrac{1}{5040}t^7 \\ & +(6\lambda^3 - 6\lambda^2\delta - 4\lambda^2 + 12\lambda\delta - 6\lambda + 2\lambda\delta^2 - 4\delta + 4\delta^2 - 4)\tfrac{1}{720}t^6 \\ & +(-3\lambda - 4\lambda\delta + \delta^2 - 1)\tfrac{1}{120}t^5 + (-4\lambda^2 + \lambda\delta - 1)\tfrac{1}{24}t^4\Big)\sin x. \end{aligned} \tag{30}$$

And further on. It is feasible to write the final series respond to such as that:

$$U_{DLT} = u_0(t,x) + u_1(t,x) + u_2(t,x) + \ldots \tag{31}$$

Here other terms vanish in the limit and hence the required exact solution to (31) that, we get

$$u(t,x) = \left(t^3 - t^2 + t - 1\right)\sin(x). \tag{32}$$

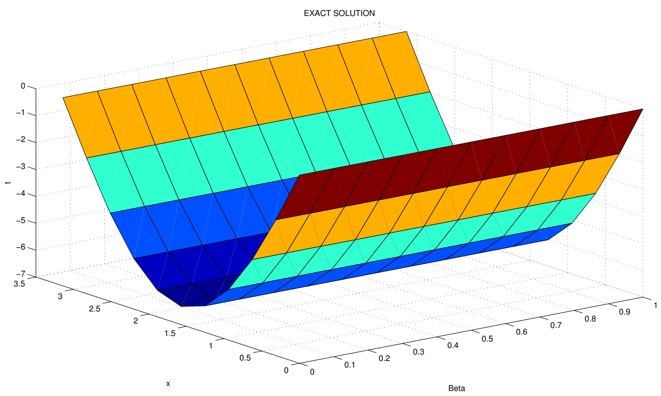

Figure 1: Explain the exact solution of (32), when $N = M = 10$

In our process, we used a modified Gauss elimination technique. We have used the following procedure to calculate the error estimates:

$$error = \max_{\substack{k=0,1,\ldots,N \\ n=0,1,\ldots,M}} |u(t,x) - U_{DLT}|. \tag{33}$$

Here, U_{DLT} represents the analytical solution. The solution presented in the following graph illustrates some physical properties of the analytical solution for equation (31) concerning IOD,

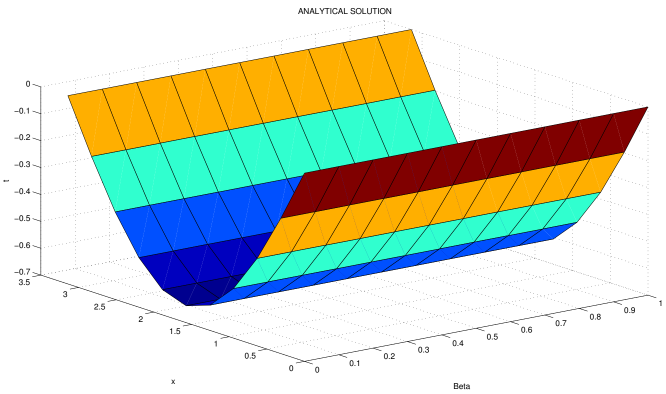

Figure 2: The analytical solution of (31) for $N = M = 10$, $\lambda = 1$ and $\delta = 0.75$, for IOD

Figure 2, which represent the analytical solutions to (31) solved by DLT. Include clearer axis labels and specified the units where applicable. The horizontal axis labeled (x) represents the spatial coordinate, while the axis labeled β corresponds to the fractional parameter that influences the behavior of the solution. The vertical axis (u) denotes the analytical solution of the governing equation (31), which represents the physical quantity of interest (displacement, temperature, or concentration, depending on the context of the model). Furthermore, we have added a more detailed explanation in the caption and the main text to interpret the physical meaning of the solution. Specifically, the surface plot illustrates how the analytical solution varies concerning the spatial variable and the fractional parameter β under the given conditions $N = M = 10$, $\lambda = 1$, and $\delta = 0.75$. This behavior provides insights into the influence of fractional-order derivatives on the solution's profile, demonstrating, for instance, that higher values of β lead to [insert specific observation from the figure, such as smoothing or steepening of the profile].

For finite difference method(FDM): We obtained an approximate solution for IOD using the explicit finite difference method(EFDM). The error is determined similarly (33). To find $f(t_k, x_n)$ as follows:

$$f(t_k, x_n) = \Big(2(k\tau)^3 + (1 + 3\delta)(k\tau)^2 + (6\lambda - 2\delta)k\tau - 2\lambda + \delta + 5\Big)\sin(nh), \tag{34}$$

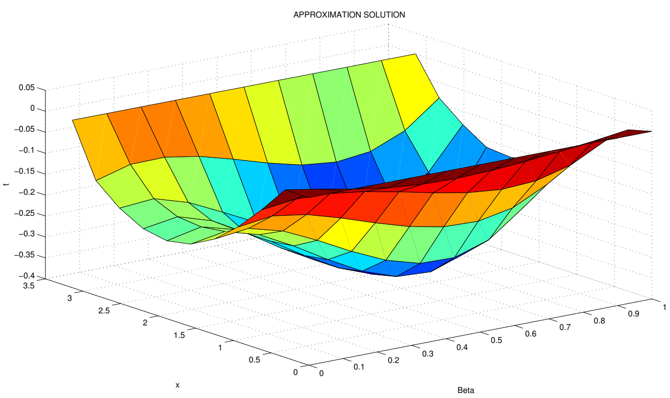

Figure 3: Explain the approximate solution of (34), where $N = M = 10$, $\lambda = 1$ and $\delta = 0.75$, for the integer order derivative using the explicit finite difference method

Figure 3, which represents the approximate solution to (34) solved by the explicit finite difference method (EFDM). For various values, these figures were seen to be relatively close to one another. Physical characteristic of the numerical solution for the interval $N = M = 10$ can be seen when they are compared along the x-axis and t-axis for ($\beta = 1$) respectively, $\lambda = 1$ of the IOD in figure 3. When the x-axis values increase between 0 and 1.99, we see decreasing values on the t-axis. At x = 2, there is no change. When the x-axis values increase, the t-axis values also increase. This means that the figure has a minimum value at x = 2.

When the x-axis values increase between 0 and 1.99, we see decreasing values on the t-axis. At the value x = 2, there are changes of increase and decrease. When the x-axis values increase, the t-axis values also increase.

The numerical and analytical results for Example 1, are displayed in a table (1).

Table 1: Numerical and analytical error results of the integer order derivative, using the finite difference technique and DLT for (31), when $\lambda = 1$, and $\delta = 0.75$.

$N = M$	Finite Difference Method	Double Laplace Transformation
$N = M = 10$	0.1725	0.1969
$N = M = 50$	0.0054	0.0400
$N = M = 100$	0.0011	0.0200
$N = M = 200$	0.0002	0.0100

Table 1, illustrates that the obtained error results for both analytical and approximate solutions are closer to one another and the error value is small if $\delta = 0.75$, $\lambda = 1$ and $N = M = 200$. Furthermore, for $N = M = 10$, the error is 0.1725, as $N = M = 200$ the error are decreases to the value 0.0002 for the approximate solution defined by EFDM. Results from DLT exhibit similar trends: for $\lambda = 1$, $\delta = 0.75$, the error starts at 0.1969 for $N = M = 10$, decreasing to 0.0100 for $N = M = 200$ for the analytical solution, in both error results there are reflexive relations between $N = M$ and values.

6. Conclusion

This study presents an exact analytical solution for a third-order partial differential equation involving integer order derivatives (IOD), derived using the modified double Laplace decomposition method. The validity of the solution is supported by a finite difference scheme and error analysis conducted in MATLAB, which shows a marked decrease in error from 0.1969 to 0.0100 as the discretizations increase from N=M=10 to N=M=200. The work contributes a validated analytical framework for IOD-based models. Limitations include focusing on the form of a specific equation and the numerical method. Future work may extend this approach to more general systems, explore alternative solution techniques, and compare results between hybrid methods to enhance accuracy and applicability.

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