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Center and radius of a subset of a metric space

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Abstract. In this paper, we introduce a notion of the center and radius of a subset A of metric space X. In the Euclidean spaces, this notion can be seen as the extension of the center and radius of open/closed balls. The center and radius of a finite product of subsets of metric spaces, and a finite union of subsets of a metric space are also determined.

For any subset A of metric space X, there is a natural question to identify the open balls of X with the largest radius that are entirely contained in A. To answer this question, we introduce a notion of quasi-center and quasi-radius of a subset A of metric space X. We prove that the center of the largest open balls contained in A belongs to the quasi-center of A, and its radius is equal to the quasi-radius of A. In particular, for the Euclidean spaces, we see that centers of the largest open balls contained in A belong to the center of A, and their radius is equal to the radius of A.

1. Introduction

To extend classical geometric ideas beyond the Euclidean spaces, Maurice Fréchet introduced a notion of distance to more abstract setting in 1906. In his doctoral dissertation (Ref. [2]), he defined the distance between any two points within a given set. A set along with this distance notion is called metric space, and this name was given by Hausdorff in 1914. In a metric space, the concept of distance is axiomatized, and allowing us to study convergence, continuity, compactness etc. which are essential in many areas of pure and applied mathematics.

Understanding the geometric properties of subsets within metric spaces is a fundamental aspect of mathematical analysis and topology. The notion of open/closed balls provides essential tools for analyzing properties of metric spaces. Every ball is defined by its center and radius. The center of ball is in the middle of ball, from which the radius is measured. The purpose of the center of a ball is to act as the fixed reference from which all other points on the boundary of the ball are equidistant. It is natural to question whether this concept can be extended to any arbitrary subset of a metric space. Are there any points within a subset of a metric space that are equidistant from its boundary? However, extending this notion of center and radius to a subset of metric spaces has significant challenges and opens new avenues for exploration.

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This paper addresses these questions by introducing a novel framework for defining the center and radius of subsets within arbitrary metric spaces, in a way that parallels to the idea of center and radius of balls in the Euclidean spaces. This definition not only preserves the intuitive geometric interpretation but also extends it to abstract settings. We further explore the properties of the center and radius in the context of finite products of subsets of metric spaces and a finite union of subsets within a metric space.

A central question addressed in this paper is the identification of the largest open balls that are entirely contained within a given subset A of metric space X. To tackle this, we introduce a notion of quasi-center and quasi-radius. These concepts serve as pivotal tools in characterizing the maximal open balls contained in A, and we demonstrate that the center of these balls is located within the quasi-center of A, with their radius equating to the quasi-radius of A. Notably, when applied to the Euclidean spaces, our results show that the center and radius of the largest open balls contained in A correspond directly to the center and radius of A itself.

This paper is structured as follows: Section 2 consists of notations, terminology and basics of metric spaces that are used in this paper. Section 3 provides a detailed definition of the center and radius of a subset within a metric space, along with key properties and examples. Section 4 and Section 5 extend these concepts to finite products and finite unions of subsets of metric space, respectively. Finally, in Section 6, we introduce the notions of quasi-center and quasi-radius, leading to a key result that connects these concepts to largest open balls contained within a subset.

2. Notations, terminology and basics

Let (X, d_X) be a metric space. The distance between two subsets A and B of X is $d_X(A, B) = \inf\{d_X(a, b) \mid a \in A, b \in B\}$. In many results of this paper, a subset is required to be nonclopen, which means that the subset is not both open and closed. The interior, closure and boundary of A are denoted by $A^\circ, \overline{A}$ and $\partial_X(A)$, respectively. For details about metric spaces, we refer to [3].

Now, we recall some basic results and properties of metric spaces.

- Let $f : X \longrightarrow Y$ be an isometry between two metric spaces X and Y. For a subset A of X, $f(\partial_X(A)) = \partial_Y(f(A))$.
- For a subset A of metric space X, $\partial_X(A^\circ) \subseteq \partial_X(A)$ and $\partial_X(\overline{A}) \subseteq \partial_X(A)$.
- If $A \subseteq B \subseteq X$ then $d_X(a, B) \leq d_X(a, A), \forall a \in X$.
- Let A and B be subsets of metric spaces X and Y, respectively. Then $\partial_{X\times Y}(A\times B) = (\overline{A}\times\partial_Y(B)\cup(\partial_X(A)\times\overline{B})$. Similarly, for subsets A_i of metric spaces X_i, we have $\partial_{\prod_1^n X_i}(\prod_1^n A_i) = \bigcup_1^n(\overline{A}_1 \times \overline{A}_2 \times ... \times \partial_{X_i}(A_i) \times ... \times \overline{A}_k)$.
- For separated subsets A and B of a metric space X, we have $\partial_X(A \cup B) = \partial_X(A) \cup \partial_X(B)$.
- Let (X, d_X) be a path metric space (Ref. [1]). For a proper subset A of X, $d_X(a, \partial_X(A)) = d_X(a, A^c), \forall a \in A$.

3. The definition of center and radius

Let (X, d_X) be a metric space and A be a subset of X. We introduce the notion of center and radius of A in X:

Definition 3.1. [Center of a subset] The center of subset A of a metric space X is the set $\{a \in A \mid d_X(a, \partial_X(A)) \geq d_X(b, \partial_X(A)), \forall\, b \in A\}$, where $\partial_X(A)$ is the boundary of A in X. We denote the center of A in X by $Cent_X(A)$.

Thus the center of A is the set of all those elements of A which are at the maximum distance from the boundary of A.

Definition 3.2. [Radius of a subset] The radius of subset A of a metric space X is the distance between the center of A and the boundary of A. We denote the radius of A in X by $rad_X(A)$.

It is clear that for every point in $Cent_X(A)$ has the same distance from $\partial_X(A)$. Thus $rad_X(A) = d_X(Cent_X(A), \partial_X(A)) = d_X(a, \partial_X(A)), \forall a \in Cent_X(A)$.

Example 3.3. Let $\mathbb{R}$ be the set of real numbers with usual metric. For subsets $A = [0,1], B = [0,1] \cup [2,3]$ and $C = [0,1] \cup [5,10]$, we get $Cent_{\mathbb{R}}$(A)= $\{0.5\}$ & $rad_{\mathbb{R}}(A) = 0.5$, $Cent_{\mathbb{R}}$(B)= $\{0.5, 2.5\}$ & $rad_{\mathbb{R}}(B) = 0.5$ and $Cent_{\mathbb{R}}$(C)= $\{7.5\}$ & $rad_{\mathbb{R}}(C) = 2.5$.

Example 3.4. Let $\mathbb{R}^2$ be the real plane with the Euclidean metric. For the unit disc $\mathbb{D}^2 \subseteq \mathbb{R}^2$, $Cent_{\mathbb{R}^2}(\mathbb{D}^2) = \{(0,0)\}$ & $rad_{\mathbb{R}^2}(\mathbb{D}^2) = 1$, and for a punctured unit disc $A = \mathbb{D}^2 \backslash \{(0,0)\} \subseteq \mathbb{R}^2$, $Cent_{\mathbb{R}^2}(A) = \{(x,y) \in A \mid x^2 + y^2 = (\frac{1}{2})^2\}$ & $rad_{\mathbb{R}^2}(A) = \frac{1}{2}$.

Example 3.5. The $n+1$ vertices of the standard n-simplex Δ^n are the points e_i, $1 \le i \le n+1$, in the Euclidean space $\mathbb{R}^{n+1}$ whose i-th coordinate is 1 and all other coordinates are 0. The simplex Δ^n lies in the affine hyperplane $H^n \subseteq \mathbb{R}^{n+1}$ spanned by its vertices e_i. The center of $Cent_{H^n}(\Delta^n)$ is the barycenter $\frac{1}{n+1}(1,1,...,1)$ of Δ^n.

As the center of A consists of all those points of A which are at the maximum distance from its boundary, $rad_X(A)$ is the maximum distance of any point $a \in A$ from its boundary. It is clear that if $Cent_X(A) \neq \emptyset$ then $rad_X(A) = \sup_{a \in A} d_X(a, \partial_X(A))$. And, if $Cent_X(A) = \emptyset$ then $rad_X(A) = \infty$ but $\sup_{a \in A} d_X(a, \partial_X(A))$ could be finite. It leads us to introduce the concept of the Semi-radius of a subset A in the metric space X.

Definition 3.6. [Semi-radius of a subset] The semi-radius of subset A of a metric space X is the supremum of the set that consists of distance of any point $a \in A$ from the boundary of A. We denote the Semi-radius of A in X by $Srad_X(A)$.
That is, $Srad_X(A) = \sup_{a \in A} d_X(a, \partial_X(A))$.

Note that $rad_X(A) \geq Srad_X(A)$. We can notice it from the following examples.

Example 3.7. Let $X = \mathbb{R} \backslash \{1\}$ be the metric subspace of the Euclidean line $\mathbb{R}$. Let $A = [0,2] \backslash \{1\}$ be a subset of X. Then $\partial_X(A) = \{0,2\}$. If $Cent_X(A) \neq \emptyset$, then $\exists\, a \in A$ such that $d_X(a, \partial_X(A)) \geq d_X(x, \partial_X(A)), \forall x \in A$, which is not true. So, $Cent_{\mathbb{R}}(A) = \emptyset$ and $rad_{\mathbb{R}}(A) = \infty$, whereas $Srad_X(A) = 1 < rad_X(A)$.

Example 3.8. Let $\mathbb{R}$ be the set of real numbers with the usual metric. For $A = \bigcup_{n \in \mathbb{N}} [n + \frac{1}{n}, n+1] \subseteq \mathbb{R}$, we have $Cent_{\mathbb{R}}(A) = \emptyset$ and $rad_{\mathbb{R}}(A) = \infty$ but $Srad_{\mathbb{R}}(A) = \frac{1}{2}$.

Example 3.9. Let $X = (-\infty, 0) \cup (\mathbb{Q} \cap [0,\pi]) \cup [\pi, \infty)$ be a metric subspace of the Euclidean line $\mathbb{R}$. For $A = (\mathbb{Q} \cap [0,\pi]) \subseteq X$, we have $\partial_X(A) = \{0, \pi\}$, $Cent_X(A) = \emptyset$ and $rad_X(A) = \infty$ but $Srad_X(A) = \frac{\pi}{2} < rad_X(A)$.

Example 3.10. Let $I_n = [0,1], n \in \mathbb{N}$ be intervals. Take a disjoint union $X = \sqcup_{n \in \mathbb{N}} I_n$ and define $d_X(a_i, b_j) = \frac{1}{2}(1 - \delta_i^j) + (2 - \delta_i^j)|a - b|$, where $a_i \in I_i, b_j \in I_j, \forall i, j \in \mathbb{N}$ and δ_i^j denote the kronecker delta. Then (X, d_X) is a metric space. For $A = \sqcup_{n=3}^{\infty} \left[\frac{1}{n}, 1 - \frac{1}{n}\right] \subseteq X$, we have $\partial_X(A) = \sqcup_{n=3}^{\infty} \{\frac{1}{n}, 1 - \frac{1}{n}\}$, $Cent_X(A) = \emptyset$, $rad_X(A) = \infty$ but $Srad_X(A) = \frac{1}{2} < rad_X(A)$.

Notice that if the boundary of a subset A of a metric space X is empty then every point of A is at infinite distance from $\partial_X(A)$. As the boundary of any metric space X is empty in itself, we have $Cent_X(X) = X$ & $rad_X(X) = \infty$. Similarly, for the empty set $\emptyset$, $Cent_X(\emptyset) = \emptyset$ & $rad_X(\emptyset) = \infty$.

Lemma 3.11. *For any clopen subset A of a metric space X,* $Cent_X(A) = A$ *and* $rad_X(A) = \infty$.

But if a subset A of a metric space X has infinite radius then it does not mean that A is clopen in X. Consider a subset $A = [0, \infty)$ of the set $\mathbb{R}$ of real numbers with the usual metric. Then, $\partial_{\mathbb{R}}(A) = \{0\}$. Here, $Cent_{\mathbb{R}}(A) = \emptyset$ and $rad_{\mathbb{R}}(A) = Srad_{\mathbb{R}}(A) = \infty$.
The following result is for nonclopen subsets of a metric space.

Theorem 3.12. *Let A be a nonclopen subset of a metric space X. Then $Cent_X(A)$ is nonempty if and only if $rad_X(A)$ is finite.*

Proof. As $A \subseteq X$ is nonclopen, $\partial_X(A) \neq \emptyset$. If $Cent_X(A)$ is nonempty then $rad_X(A) = d_X(Cent_X(A), \partial_X(A)) = \inf_{b\in\partial_X(A)} d_X(a,b) \leq d_X(a,b), \forall a \in Cent_X(A), \forall b \in \partial_X(A)$, which is finite.

Conversely, if $rad_X(A)$ is finite then by the definition of radius, the center of A is nonempty. □

Next, we discuss examples of subsets of a metric space with zero radius. Consider $\mathbb{N} \subseteq \mathbb{R}$, the set of real numbers with the usual metric. Then $Cent_{\mathbb{R}}(\mathbb{N}) = \mathbb{N}$ and $rad_{\mathbb{R}}(\mathbb{N}) = 0$. In fact, for any totally disconnected subset A of $\mathbb{R}$, we get $Cent_{\mathbb{R}}(A) = A$ and $rad_{\mathbb{R}}(A) = 0$. For the unit circle $\mathbb{S}^1$ in the Euclidean plane $\mathbb{R}^2$, $Cent_{\mathbb{R}^2}(\mathbb{S}^1) = \mathbb{S}^1$ & $rad_{\mathbb{R}^2}(\mathbb{S}^1) = 0$. We know that topological manifolds are metrizable spaces. Let N be an n-dimensional submanifold of a topological manifold M of dimension m where $n < m$. Note that $N \subseteq \partial_M(N)$, and hence $Cent_M(N) = N$ and $rad_M(N) = 0$. Also notice that in Example 3.5, $Cent_{\mathbb{R}^{n+1}}(\Delta^n) = \Delta^n$ and $rad_{\mathbb{R}^{n+1}}(\Delta^n) = 0$.

In general, for any subset A of a metric space X contained in its boundary, it means for A having empty interior, we have

Lemma 3.13. *Let A be a nonempty subset of a metric space X such that A has empty interior. Then, $Cent_X(A) = A$ and $rad_X(A) = 0$.*

The following result is for subsets of metric spaces having nonempty interior.

Lemma 3.14. *Let X be a metric space and $A \subseteq X$ such that A has nonempty interior. Then, $Cent_X(A) \subseteq A^\circ$.*

Proof. If A is clopen then it is true by Lemma 3.11. And, if A is nonclopen then for any $a \in A$, either $a \in \partial_X(A)$ or $a \in A^\circ$. If $a \in \partial_X(A)$ then $d_X(a, \partial_X(A)) = 0$. If $a \in A^\circ$ then $\exists\, \epsilon > 0$ such that $B_{d_X}(a,\epsilon) \subseteq A^\circ$. As $B_{d_X}(a,\epsilon) \cap \partial_X(A) = \emptyset$, we have $d_X(a, \partial_X(A)) \geq \epsilon > 0, \forall a \in A^\circ$. So, by definition of center of A in X, we get $Cent_X(A) \subseteq A^\circ$. □

Theorem 3.15. *Let X be a metric space and A be a nonempty subset of X. Then, $A^\circ = \emptyset$ if and only if $rad_X(A) = 0$.*

Proof. If $A^\circ = \emptyset$ then by Lemma 3.13, $rad_X(A) = 0$. Conversely, assume that $rad_X(A) = 0$ then by Lemma 3.11, A is nonclopen and by Theorem 3.12, $Cent_X(A)$ is nonempty. So, for $x \in Cent_X(A)$, we get $d_X(x, \partial_X(A)) = 0 \implies x \in \overline{\partial_X(A)} = \partial_X(A)$, and hence we get $Cent_X(A) \subseteq \partial_X(A)$. If $A^\circ \neq \emptyset$ then by Lemma 3.14 $Cent_X(A) \subseteq A^\circ$. As $\partial_X(A) \cap A^\circ = \emptyset$, we get $Cent_X(A) = \emptyset$, a contradiction. Thus, $A^\circ = \emptyset$. □

Theorem 3.16. *The center of a subset A of a metric space X is closed in A.*

Proof. Let $b \in A$ be a limit point of $Cent_X(A)$ in A. Then, $\exists$ a sequence (a_n) in $Cent_X(A)$ such that $(a_n) \longrightarrow b$. Consider a map $p : A \longrightarrow \mathbb{R}$ such that $p(x) = d_X(x, \partial_X(A)), \forall x \in A$. It is easy to observe that p is a continuous map. By the continuity of p, we have $p(a_n) \longrightarrow p(b)$. As $a_n \in Cent_X(A)$, which means $p(a_n) = d_X(a_n, \partial_X(A)) = rad_X(A), \forall n \in \mathbb{N}$. So, $p(a_n)$ is a constant sequence, and hence it converges to $rad_X(A)$. Thus, $p(b) = d_X(b, \partial_X(A)) = rad_X(A)$. This implies that $b \in Cent_X(A)$. Hence, $Cent_X(A)$ is a closed subset of A. □

It is not necessary that $Cent_X(A)$ is a closed subset of X. For example: Consider a subset $A = (0,1) \times \{0\}$ of the Euclidean plane $\mathbb{R}^2$. Then $Cent_{\mathbb{R}^2}(A) = A$ which is not closed in $\mathbb{R}^2$.

Now for any subset A of a metric space X, we establish a relationship between the radii of $\overline{A}$ and A° with the radius of A.

Theorem 3.17. *Let X be a metric space and $A \subseteq X$ such that $Cent_X(A) \neq \emptyset$. Then $rad_X(A) \leq rad_X(A^\circ)$ and $rad_X(A) \leq rad_X(\overline{A})$.*

Proof. If $Cent_X(A^\circ)$ is empty then $rad_X(A^\circ) = \infty$, which implies $rad_X(A) \leq rad_X(A^\circ)$. If $Cent_X(A^\circ) \neq \emptyset$ then for $b \in Cent_X(A^\circ)$, we have $d_X(b, \partial_X(A^\circ)) \geq d_X(b', \partial_X(A^\circ)), \forall b' \in A^\circ$. As $\partial_X(A^\circ) \subseteq \partial_X(A)$, we get $d_X(b', \partial_X(A^\circ)) \geq d_X(b', \partial_X(A))$, $\forall b' \in A^\circ \subseteq A$. Hence, $d_X(b, \partial_X(A^\circ)) \geq d_X(b', \partial_X(A)), \forall b' \in A^\circ$. This implies that $rad_X(A^\circ) \geq \sup_{b'\in A^\circ} d_X(b', \partial_X(A))$ and for any $b' \in \partial_X(A)$, we have $d_X(b', \partial_X(A)) = 0$. So, $rad_X(A^\circ) \geq \sup_{b'\in A} d_X(b', \partial_X(A)) = rad_X(A)$. Thus, $rad_X(A) \leq rad_X(A^\circ)$.

Similarly, as $\partial_X(\overline{A}) \subseteq \partial_X(A)$, we get $rad_X(A) \leq rad_X(\overline{A})$. □

Example 3.18. Consider the set $\mathbb{R}$ of real numbers with the usual metric.
Let $A = \{\frac{1}{n} \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$. Then $\overline{A} = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ and $A^\circ = \emptyset$. We get $Cent_{\mathbb{R}}(A^\circ) = \emptyset$ & $Cent_{\mathbb{R}}(A) = A$, $Cent_{\mathbb{R}}(\overline{A}) = \overline{A}$ and $rad_{\mathbb{R}}(A^\circ) = \infty$ & $rad_{\mathbb{R}}(A) = rad_{\mathbb{R}}(\overline{A}) = 0$.
Let $B = (\mathbb{Q} \cap [0, \infty)) \subseteq \mathbb{R}$. Then $B^\circ = \emptyset$ and $\overline{B} = [0, \infty)$. We get $Cent_{\mathbb{R}}(B^\circ) = \emptyset$ & $Cent_{\mathbb{R}}(B) = B$, $Cent_{\mathbb{R}}(\overline{B}) = \emptyset$ and $rad_{\mathbb{R}}(B) = 0$ & $rad_{\mathbb{R}}(B^\circ) = rad_{\mathbb{R}}(\overline{B}) = \infty$.
Let $C = [0, 1) \subseteq \mathbb{R}$ then $rad_{\mathbb{R}}(C) = rad_{\mathbb{R}}(C^\circ) = rad_{\mathbb{R}}(\overline{C}) = \frac{1}{2}$.

In the Euclidean plane $\mathbb{R}^2$, let $Y = \mathbb{R} \times \{0\}$ and $A = (0, 1) \times \{0\}$. Here $A \subseteq Y \subseteq \mathbb{R}^2$ and $Cent_X(A) \neq \emptyset$. Notice that $rad_{\mathbb{R}^2}(A) = 0 \leq rad_Y(A) = \frac{1}{2}$. In general, we have the following result.

Theorem 3.19. *Let X be a metric space and Y be a subspace of X. If $A \subseteq Y \subseteq X$ such that $Cent_X(A)$ is nonempty, then $rad_X(A) \leq rad_Y(A)$.*

Proof. We know that $\partial_Y(A) \subseteq \partial_X(A) \subseteq X$. So, $d_X(a, \partial_X(A)) \leq d_X(a, \partial_Y(A)), \forall\, a \in A \implies d_X(a, \partial_X(A)) \leq d_Y(a, \partial_Y(A)), \forall\, a \in A$. And, if $Cent_X(A) \neq \emptyset$, we have $rad_X(A) = \sup_{a \in A} d_X(a, \partial_X(A)) \leq \sup_{a \in A} d_Y(a, \partial_Y(A)) \leq rad_Y(A)$. □

Remark 3.20. In the above result, $Cent_X(A) \neq \emptyset$ is necessary. In Example 3.9, let $Y = (\mathbb{Q} \cap [0, \pi]) \cup [\pi, \infty)$. Then $A \subseteq Y \subseteq X$. Here $\partial_Y(A) = \{\pi\}$, $Cent_Y(A) = \{0\}$ and $Cent_X(A) = \emptyset$. But $rad_Y(A) = \pi$ whereas $rad_X(A) = \infty$.

Let $f : X \longrightarrow Y$ be an isometry between two metric spaces X and Y (Ref. [3]). We know that isometry preserves the boundary of a subset. Here, we observe that isometry also preserves the center and radius of a subset.

First, we prove the following lemma.

Lemma 3.21. *Let $f : X \longrightarrow Y$ be an isometry between two metric spaces X and Y. For a subset A of X, $Cent_X(A) \neq \emptyset$ if and only if $Cent_Y(f(A)) \neq \emptyset$.*

Proof. As f is an isometry, $f(\partial_X(A)) = \partial_Y(f(A))$. For $a \in A$, we have $d_X(a, \partial_X(A)) = d_Y(f(a), f(\partial_X(A))) = d_Y(f(a), \partial_Y(f(A)))$. Notice that $Cent_X(A) \neq \emptyset \iff \exists a \in A$ such that $d_X(a, \partial_X(A)) \geq d_X(b, \partial_X(A)), \forall b \in A \iff d_Y(f(a), \partial_Y(f(A))) \geq d_Y(f(b), \partial_Y(f(A))), \forall b \in A. \iff f(a) \in Cent_Y(f(A)) \iff Cent_Y(f(A)) \neq \emptyset$. □

Theorem 3.22. *Let $f : X \longrightarrow Y$ be an isometry between two metric spaces X and Y. For a subset A of X, $rad_X(A) = rad_Y(f(A))$ and $f(Cent_X(A)) = Cent_Y(f(A))$.*

Proof. First, let $Cent_X(A) = \emptyset$. Then it is true by Lemma 3.21. Next, let $Cent_X(A) \neq \emptyset$. Then $rad_Y(f(A)) \geq d_Y(f(a), \partial_Y(f(A)) = d_Y(f(a), f(\partial_X(A))) = d_X(a, \partial_X(A))$, $\forall a \in A$. This implies that $rad_Y(f(A)) \geq rad_X(A)$. By Lemma 3.21, $Cent_Y(f(A)) \neq \emptyset$. Similarly, we get $rad_X(A) \geq rad_Y(f(A))$. Hence, $rad_X(A) = rad_Y(f(A))$.
As $rad_Y(f(A)) = rad_X(A) = d_X(Cent_X(A), \partial_X(A)) = d_Y(f(Cent_X(A)), \partial_Y(f(A)))$, we get every point of $f(Cent_X(A))$ is at the maximum distance from $\partial_Y(f(A)))$. So, $f(Cent_X(A)) \subseteq Cent_Y(f(A))$. Similarly, we get $Cent_Y(f(A)) \subseteq f(Cent_X(A))$. Hence, $f(Cent_X(A)) = Cent_Y(f(A))$. □

Remark 3.23. Let X and Y be two metric spaces such that $A \subseteq X$ and $B \subseteq Y$. If $Cent_X(A)$ is connected and $Cent_Y(B)$ is disconnected then by Theorem 3.22, there does not exist any isometry between X and Y such that $f(A) = B$.

4. Center and radius of a finite product of subsets of metric spaces

Let $(X \times Y, d)$ be the product of metric spaces (X, d_X). and (Y, d_Y), where $d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$, $\forall\, (x_1, y_1), (x_2, y_2) \in X \times Y$. Now, we see how $Cent_{X \times Y}(A \times B)$ is related to $Cent_X(A)$ and $Cent_Y(B)$.

Let $A \subseteq X$ and $B \subseteq Y$ be subsets. If $rad_X(A)$ and $rad_Y(B)$ are infinite then there are three possible cases: (i) both A and B are clopen, (ii) one of A and B is clopen and the other has empty center, and (iii) $Cent_X(A) = Cent_Y(B) = \emptyset$.

Theorem 4.1. *Let (X, d_X) and (Y, d_Y) be two metric spaces. Let $A \subseteq X$ and $B \subseteq Y$ be subsets.*

(1) *If both A and B are clopen, then* $Cent_{X\times Y}(A\times B)=A\times B$ *and* $rad_{X\times Y}(A\times B)=\infty$,

(2) *If A is clopen and* $Cent_Y(B)=\emptyset$, *then* $Cent_{X\times Y}(A\times B)=\emptyset$ *and* $rad_{X\times Y}(A\times B)=\infty$, *and*

(3) $Cent_X(A)=Cent_Y(B)=\emptyset$, *then* $Cent_{X\times Y}(A\times B)=\emptyset$ *and* $rad_{X\times Y}(A\times B)=\infty$.

Proof. (1) If A and B are clopen then $A\times B$ is clopen, and hence $Cent_{X\times Y}(A\times B)=A\times B$ and $rad_{X\times Y}(A\times B)=\infty$.

(2) As A is clopen, $\partial_X(A)=\emptyset$. Recall that $\partial_{X\times Y}(A\times B)=(\overline{A}\times\partial_Y(B))\cup(\partial_X(A)\times\overline{B})$. In this case, $\partial_{X\times Y}(A\times B)=A\times\partial_Y(B)$. First, we observe that $d((a,b),A\times\partial_Y(B))=d_Y(b,\partial_Y(B)),\forall(a,b)\in A\times B$.
We have $d((a,b),A\times\partial_Y(B))\leq d((a,b),(a,y))=\max\{d_X(a,a),d_Y(b,y)\}=d_Y(b,y),\forall y\in\partial_Y(B)\implies d((a,b),A\times\partial_Y(B))\leq d_Y(b,\partial_Y(B))$. And, for $x\in A$ and $y\in\partial_Y(B)$, we have $d_Y(b,\partial_Y(B))\leq d_Y(b,y)\leq\max\{d_X(a,x),d_Y(b,y)\}=d((a,b),(x,y))\implies d_Y(b,\partial_Y(B))\leq d((a,b),A\times\partial_Y(B))$. Hence, $d((a,b),A\times\partial_Y(B))=d_Y(b,\partial_Y(B)),\forall(a,b)\in A\times B$.

Suppose that $Cent_{X\times Y}(A\times B)\neq\emptyset$. For $(a,b)\in Cent_{X\times Y}(A\times B)$, we get $d_Y(b,\partial_Y(B))=d((a,b),A\times\partial_Y(B))\geq d((a,b'),A\times\partial_Y(B))=d_Y(b',\partial_Y(B)),\forall b'\in B\implies b\in Cent_Y(B)$, a contradiction. So, $Cent_{X\times Y}(A\times B)=\emptyset$, and hence $rad_{X\times Y}(A\times B)=\infty$.

(3) Similarly, as in case (*ii*), we get $d((a,b),\overline{A}\times\partial_Y(B))=d_Y(b,\partial_Y(B))$, and $d((a,b),\partial_X(A)\times\overline{B})=d_X(a,\partial_X(A))$, $\forall(a,b)\in A\times B$.

Suppose that $Cent_{X\times Y}(A\times B)\neq\emptyset$. , For $(a,b)\in Cent_{X\times Y}(A\times B)\subseteq A\times B$, we get $d((a,b),\partial_{X\times Y}(A\times B))=\min\{d((a,b),(\partial_X(A)\times\overline{B})),d((a,b),(\overline{A}\times\partial_Y(B)))\}=\min\{d_X(a,\partial_X(A)),d_Y(b,\partial_Y(B))\}$. If $d((a,b),\partial_{X\times Y}(A\times B))=d_X(a,\partial_X(A))$ then $a\in Cent_X(A)$, a contradiction, and if $d((a,b),\partial_{X\times Y}(A\times B))=d_Y(b,\partial_Y(B))$ then $b\in Cent_Y(B)$, again a contradiction. So, $Cent_{X\times Y}(A\times B)=\emptyset$, and hence $rad_{X\times Y}(A\times B)=\infty$. □

Example 4.2. (i) Let $A=\{2,3\}\subseteq\mathbb{Z}$ and $B=[0,\infty)\subseteq\mathbb{R}$, where $\mathbb{Z}$ is discrete space and $\mathbb{R}$ is equipped with the usual metric. As A is clopen in $\mathbb{Z}$ and $Cent_{\mathbb{R}}(B)=\emptyset$, by Theorem 4.1(*ii*), we get $Cent_{\mathbb{Z}\times\mathbb{R}}(A\times B)=\emptyset$ and $rad_{\mathbb{Z}\times\mathbb{R}}(A\times B)=\infty$.

(ii) Let $Q=(0,\infty)\times(0,\infty)$ be the first quadrant in $\mathbb{R}^2$ with the maximum metric. Then by Theorem 4.1(*iii*), we get $Cent_{\mathbb{R}^2}(Q)=\emptyset$ and $rad_{\mathbb{R}^2}(Q)=\infty$.

Theorem 4.3. *Let (X,d_X) and (Y,d_Y) be two metric spaces. For $A\subseteq X$ and $B\subseteq Y$, let* $\hat{B}=\{b\in B\mid d_Y(b,\partial_Y(B))\geq rad_X(A)\}$. *If* $rad_X(A)\leq rad_Y(B)$, *then* $Cent_{X\times Y}(A\times B)=Cent_X(A)\times\hat{B}$.

Proof. First, let both $rad_X(A)$ and $rad_Y(B)$ be infinite. In this case, we have three possibilities. If A and B are clopen then $Cent_X(A)=A$ and $\hat{B}=B$, and the result follows by Theorem 4.1(*i*). And in other two cases either $\hat{B}=\emptyset$ or $Cent_X(A)=\emptyset$, and the result follows by Theorem 4.1(*ii*) or Theorem 4.1(*iii*).

Now, WLOG suppose that $rad_X(A)$ is finite, then $Cent_X(A)$ is nonempty. If $rad_Y(B)$ is finite, then $Cent_Y(B)\neq\emptyset$ and $Cent_Y(B)\subseteq\hat{B}\implies\hat{B}\neq\emptyset$. If $rad_Y(B)=\infty$ such that $rad_X(A)<Srad_Y(B)$, then $\hat{B}\neq\emptyset$.

In both the above cases, we observe that if $(a,b)\in Cent_X(A)\times\hat{B}$, then $d((a,b),\partial_{X\times Y}(A\times B))=rad_X(A)$. As $a\in Cent_X(A)$ and $b\in\hat{B}$, $d_X(a,\partial_X(A))=rad_X(A)$ and $d_Y(b,\partial_Y(B))\geq rad_X(A)\implies d((a,b),\partial_{X\times Y}(A\times B))=\min\{d_X(a,\partial_X(A)),d_Y(b,\partial_Y(B))\}=rad_X(A)$. Next, we observe that if $(a,b)\notin(Cent_X(A)\times\hat{B})$, then $d((a,b),\partial_{X\times Y}(A\times B))<rad_X(A)$. If $(a,b)\notin(Cent_X(A)\times\hat{B})$, then we have either $a\notin Cent_X(A)$ or $b\notin\hat{B}$. If $a\notin Cent_X(A)$, then $d_X(a,\partial_X(A))<rad_X(A)$. And, if $b\notin\hat{B}$, then $d_X(b,\partial_X(A))<rad_X(A)$. This implies $d((a,b),\partial_{X\times Y}(A\times B))<rad_X(A)$. So, $Cent_{X\times Y}(A\times B)=Cent_X(A)\times\hat{B}$.

Now, if $rad_Y(B)=\infty$ such that $rad_X(A)\geq Srad_Y(B)$, then $\hat{B}=\emptyset$. In this case, we prove that $Cent_{X\times Y}(A\times B)=\emptyset$. Let $(a,b)\in Cent_{X\times Y}(A\times B)$. Then $d((a,b),\partial_{X\times Y}(A\times B))=\min\{d_X(a,\partial_X(A)),d_Y(b,\partial_Y(B)\}$. If $d((a,b),\partial_{X\times Y}(A\times B))=d_Y(b,\partial_Y(B)$, then $b\in Cent_Y(B)$, a contradiction. So, $d((a,b),\partial_{X\times Y}(A\times B))=d_X(a,\partial_X(A))\implies a\in Cent_X(A)$. So, we get $rad_X(A)=d_X(a,\partial_X(A))=d((a,b),\partial_{X\times Y}(A\times B))<d_Y(b,\partial_Y(B))\leq\sup_{b\in B}d_Y(b,\partial_Y(B))=Srad_Y(B)$, which is not the case. So, $Cent_{X\times Y}(A\times B)=\emptyset$. Hence, our claim. □

Remark 4.4. In the above theorem,

(*i*) if $rad_X(A)=rad_Y(B)$, then $\hat{B}=Cent_Y(B)$. So, $Cent_{X\times Y}(A\times B)=Cent_X(A)\times Cent_Y(B)$.

(*ii*) if $rad_X(A)=0$, then $\hat{B}=B$ and $Cent_X(A)=A$. So, $Cent_{X\times Y}(A\times B)=A\times B$.

(*iii*) if $rad_Y(B) \leq rad_X(A)$, then $Cent_{X\times Y}(A \times B) = \hat{A} \times Cent_Y(B)$, where $\hat{A} = \{a \in A \mid d_X(a, \partial_X(A)) \geq rad_Y(B)\}$.

From above results, we get

Corollary 4.5. *Let A and B are subsets of metric spaces X and Y, respectively. If the radii of A and B are either both finite or both infinite, then $rad_{X\times Y}(A \times B) = \min\{rad_X(A), rad_Y(B)\}$.*

Moreover, if $rad_X(A) < Srad_Y(B)$ or $rad_Y(B) < Srad_X(A)$, then the above result is also true.

Example 4.6. Let A and B be subsets of the Euclidean space $\mathbb{R}$.

(*i*) If $A = B = [0,1]$ then $rad_{\mathbb{R}}(A) = rad_{\mathbb{R}}(B)$ and $\hat{B} = Cent_Y(B) = \{\frac{1}{2}\}$. By Theorem 4.3, $Cent_{\mathbb{R}^2}(A \times B) = \{(\frac{1}{2}, \frac{1}{2})\}$, and $rad_{\mathbb{R}^2}(A \times B) = \frac{1}{2}$.

(*ii*) If $A = [0,1]$ and $B = [0,5]$ then $rad_{\mathbb{R}}(A) \leq rad_{\mathbb{R}}(B)$ and $\hat{B} = [0.5, 4.5]$. Hence, $Cent_{\mathbb{R}^2}(A\times B) = \{\frac{1}{2}\}\times[0.5, 4.5]$ and $rad_{\mathbb{R}^2}(A \times B) = \frac{1}{2}$.

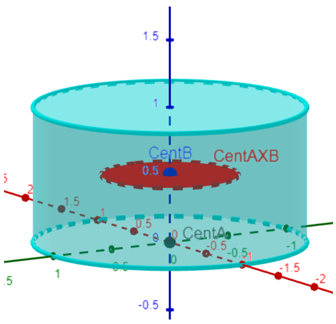
(*iii*) If $A = [0,1]$ and $B = [0,1]\cup[2,4]$ then $rad_{\mathbb{R}}(A) \leq rad_{\mathbb{R}}(B)$ and $\hat{B} = \{\frac{1}{2}\}\cup[2.5, 3.5]$. Hence, $Cent_{\mathbb{R}^2}(A\times B) = \{(\frac{1}{2}, \frac{1}{2})\} \cup (\{\frac{1}{2}\} \times [2.5, 3.5])$ and $rad_{\mathbb{R}^2}(A \times B) = \frac{1}{2}$.

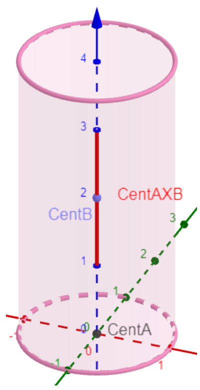
Example 4.7. Let $\mathbb{R}^2$ and $\mathbb{R}$ be the Euclidean spaces. By Theorem 4.3, we have

(*i*) the center and radius of the cylinder $\mathbb{S}^1 \times I$ in $\mathbb{R}^3$ are $\mathbb{S}^1 \times I$ and 0, respectively, where $I = [0,1]$.

(*ii*) for $\mathbb{D}^2 \subseteq \mathbb{R}^2$ and $I \subseteq \mathbb{R}$, $rad_{\mathbb{R}^2}(\mathbb{D}^2) > rad_{\mathbb{R}}(I)$ and $\hat{\mathbb{D}}^2 = \{a \in \mathbb{D}^2 \mid d_{\mathbb{R}^2}(a, \mathbb{S}^1) \geq \frac{1}{2}\} = \{a \in \mathbb{D}^2 \mid |a| \leq \frac{1}{2}\}$. So, $Cent_{\mathbb{R}^3}(\mathbb{D}^2 \times I) = \{(a, \frac{1}{2}) \in (\mathbb{D}^2 \times I) \mid |a| \leq \frac{1}{2}\}$ and $rad_{\mathbb{R}^3}(\mathbb{D}^2 \times I) = \frac{1}{2}$.

(*iii*) for $A = \mathbb{D}^2 \subseteq \mathbb{R}^2, B = [0,4] \subseteq \mathbb{R}$, $rad_{\mathbb{R}^2}(A) < rad_{\mathbb{R}}(B)$ and $\hat{B} = [1,3]$. So, $Cent_{\mathbb{R}^3}(A \times B) = \{(0,0,b) \mid 1 \leq b \leq 3\}$ and $rad_{\mathbb{R}^3}(A \times B) = 1$.





Example 4.7

Next, we generalize Theorem 4.3 for a finite product $\prod_{i=1}^{n} A_i$ of subsets A_i of metric spaces $X_i, 1 \leq i \leq n$.

Theorem 4.8. *Let (X_i, d_i) be metric spaces $, 1 \leq i \leq n$, where $n \in \mathbb{N}$. For $A_i \subseteq X_i, 1 \leq i \leq n$, let $\hat{A}_i = \{a \in A_i \mid d_i(a, \partial_{X_i}(A_i)) \geq \min\{rad_{X_j}(A_j) | 1 \leq j \leq n\}\}$. Then $Cent_{\prod_1^n X_i}(\prod_1^n A_i) = \prod_1^n \hat{A}_i$. Moreover, if the radii of A_i are either all finite or all infinite, then $rad_{\prod_1^n X_i}(\prod_1^n A_i) = \min\{rad_{X_i}(A_i) \mid 1 \leq i \leq n\}$.*

Proof. We prove by induction. If $n = 2$ and $rad_{X_1}(A_1) \leq rad_{X_2}(A_2)$ then $\hat{A}_1 = Cent_{X_1}(A_1)$ and it is true by Theorem 4.3.

Assume that it is true for some $k \in \mathbb{N}$. Let $B = \prod_1^k A_i$. By Induction hypothesis, we have $Cent_{\prod_1^k X_i}(B) = \prod_1^k \hat{A}_i$ and $rad_{\prod_1^k X_i}(B) = \min\{rad_{X_i}(A_i) \mid 1 \le i \le k\}$.

Now, we prove it for $k+1$.

If $rad_{\prod_1^k X_i}(B) \le rad_{X_{k+1}}(A_{k+1})$, then by Theorem 4.3, we have $Cent_{\prod_1^{k+1} X_i}(\prod_1^{k+1} A_i) = Cent_{\prod_1^{k+1} X_i}(B \times A_{k+1}) = (Cent_{\prod_1^k X_i} B) \times \hat{A}_{k+1} = \prod_1^{k+1} \hat{A}_i$, and $rad_{\prod_1^{k+1} X_i}(\prod_1^{k+1} A_i) = rad_{\prod_1^{k+1} X_i}(B \times A_{k+1}) = rad_{\prod_1^k X_i}(B) = \min\{rad_{X_i}(A_i) \mid 1 \le i \le k\} = \min\{rad_{X_i}(A_i) \mid 1 \le i \le k+1\}$.

If $rad_{X_{k+1}}(A_{k+1}) \le rad_{\prod_1^k X_i}(B)$, then by Theorem 4.3, we have $Cent_{\prod_1^{k+1} X_i}(\prod_1^{k+1} A_i) = \hat{B} \times Cent_{X_{k+1}}(A_{k+1}) = \hat{B} \times \hat{A}_{k+1}$, and $rad_{\prod_1^{k+1} X_i}(\prod_1^{k+1} A_i) = rad_{\prod_1^{k+1} X_i}(B \times A_{k+1}) = rad_{X_{k+1}}(A_{k+1}) = \min\{rad_{X_i}(A_i) \mid 1 \le i \le k+1\}$. Next, we observe that $\hat{B} = \prod_1^k \hat{A}_i$.

We have $\hat{B} = \{b \in B | d(b, \partial_{\prod_1^k X_i}(B)) \ge rad_{X_{k+1}}(A_{k+1})\}$. It is easy to see that $d(a, \overline{A}_1 \times \overline{A}_2 \times ... \partial_{X_i}(A_i) \times ... \times \overline{A}_k) = d_i(a_i, \partial_{X_i}(A_i)), \forall a = (a_1, a_2, ..., a_k) \in \prod_1^k A_i$. Note that $d(b, \partial_{\prod_1^k X_i}(B)) = d(b, \bigcup_{i=1}^k (\overline{A}_1 \times \overline{A}_2 \times ... \times \partial_{X_i}(A_i) \times ... \times \overline{A}_k)) = \min\{d(b, (\overline{A}_1 \times \overline{A}_2 \times ... \times \partial_{X_i}(A_i) \times ... \overline{A}_k)) \mid 1 \le i \le k\} = \min\{d_i(b, \partial_{X_i}(A_i)) \mid 1 \le i \le k\}, \forall b \in B$. Now, $b = (b_1, b_2, ..., b_k) \in \hat{B} \iff d(b, \partial_{\prod_1^k X_i}(B)) \ge rad_{X_{k+1}}(A_{k+1}) \iff \min\{d_i(b_i, \partial_{X_i}(A_i)) \mid 1 \le i \le k\} \ge rad_{X_{k+1}}(A_{k+1}) \iff d_i(b_i, \partial_{X_i}(A_i)) \ge rad_{X_{k+1}}(A_{k+1}), 1 \le i \le k \iff b_i \in \hat{A}_i, 1 \le i \le k \iff b = (b_1, b_2, ..., b_k) \in \prod_1^k \hat{A}_i$. Thus, $\hat{B} = \prod_1^k \hat{A}_i$. So, $Cent_{\prod_1^{k+1} X_i}(\prod_1^{k+1} A_i) = \prod_1^k \hat{A}_i \times \hat{A}_{k+1} = \prod_1^{k+1} \hat{A}_i$. Hence, our claim. $\square$

5. Center and radius of a finite union of subsets of a metric space

We know that if A and B are subsets of a metric space (X, d_X) such that $A \cap B = \emptyset$, then we have $diam_X(A \cup B) \le diam_X(A) + diam_X(B) + d_X(A, B)$, where $diam_X(A)$ denotes the diameter of A (Ref. [3]).

In this section, we determine $Cent_X(A \cup B)$ and $rad_X(A \cup B)$ for nonclopen subsets A and B of a metric space X.

For nonclopen subsets A and B of a metric space X, let
$\tilde{A} = \{a \in Cent_X(A) \mid d_X(a, \partial_X(B)) < rad_X(A)\}$, and
$\tilde{B} = \{b \in Cent_X(B) \mid d_X(b, \partial_X(A)) < rad_X(B)\}$.
Using these notations, we have the following results:

Theorem 5.1. *Let A and B be nonclopen separated subsets of a metric space (X, d_X). Then,*

(1) *if $rad_X(A) > rad_X(B)$ and $Cent_X(A) \backslash \tilde{A} \neq \emptyset$, then $Cent_X(A \cup B) = Cent_X(A) \backslash \tilde{A}$ & $rad_X(A \cup B) = rad_X(A)$, and*

(2) *if $rad_X(A) = rad_X(B)$ and $(Cent_X(A) \backslash \tilde{A}) \cup (Cent_X(B) \backslash \tilde{B}) \neq \emptyset$, then $Cent_X(A \cup B) = (Cent_X(A) \backslash \tilde{A}) \cup (Cent_X(B) \backslash \tilde{B})$ and $rad_X(A \cup B) = rad_X(A) = rad_X(B)$.*

Proof. (1) For $a \in Cent_X(A) \backslash \tilde{A}$, we have $d_X(a, \partial_X(B)) \ge rad_X(A)$. Consequently, $d_X(a, \partial_X(A \cup B)) = \min\{d_X(a, \partial_X(A)), d_X(a, \partial_X(B))\} = d_X(a, \partial_X(A)) = rad_X(A)$. If $a \notin Cent_X(A) \backslash \tilde{A}$, then either $a \in \tilde{A}$ or $a \notin Cent_X(A)$. If $a \in \tilde{A}$, then $d_X(a, \partial_X(B)) < rad_X(A)$. So, $d_X(a, \partial_X(A \cup B)) \le d_X(a, \partial_X(B)) < rad_X(A)$. If $a \in A$ such that $a \notin Cent_X(A)$

then $d_X(a, \partial_X(A)) < rad_X(A)$. So, $d_X(a, \partial_X(A \cup B)) \leq d_X(a, \partial_X(A)) < rad_X(A)$. And if $a \in B$ then $d_X(a, \partial_X(A \cup B)) \leq d_X(a, \partial_X(B)) \leq rad_X(B) < rad_X(A)$. So, for $a \notin Cent_X(A)\backslash\tilde{A}$, we have $d_X(a, \partial_X(A \cup B)) < rad_X(A)$. Thus, $Cent_X(A \cup B) = Cent_X(A)\backslash\tilde{A}$ and $rad_X(A \cup B) = rad_X(A)$.

(2) Similarly, for $a \in (Cent_X(A)\backslash\tilde{A}) \cup (Cent_X(B)\backslash\tilde{B})$, we get $d_X(a, \partial_X(A \cup B)) = rad_X(A)$. And, for $a \notin (Cent_X(A)\backslash\tilde{A}) \cup (Cent_X(B)\backslash\tilde{B})$, we get $d_X(a, \partial_X(A \cup B)) < rad_X(A)$. Thus, $Cent_X(A \cup B) = (Cent_X(A)\backslash\tilde{A}) \cup (Cent_X(B)\backslash\tilde{B})$ and $rad_X(A \cup B) = rad_X(A)$. □

Next, we derive relationship of $Srad_X(A \cup B)$ with $rad_X(A)$ & $rad_X(B)$.

Theorem 5.2. *Let A and B be nonclopen separated subsets of a metric space (X, d_X). Then $Srad_X(A \cup B) \leq \max\{rad_X(A), rad_X(B)\}$.*

Proof. As A and B are separated, $\partial_X(A \cup B) = \partial_X(A) \cup \partial_X(B)$. So, for $a \in A$, we have $d_X(a, \partial_X(A \cup B)) \leq d_X(a, \partial_X(A)) \leq rad_X(A)$. Similarly, for $b \in B$, we have $d_X(b, \partial_X(A \cup B)) \leq d_X(b, \partial_X(B)) \leq rad_X(B)$. This implies that $Srad_X(A \cup B) \leq \max\{rad_X(A), rad_X(B)\}$. □

Theorem 5.3. *Let A and B be nonclopen separated subsets of a metric space (X, d_X). Then,*

(1) *if $rad_X(B) < rad_X(A) < \infty$ and $Cent_X(A)\backslash\tilde{A} = \emptyset$, then $Srad_X(A \cup B) < rad_X(A)$, and*

(2) *if $rad_X(A) = rad_X(B) < \infty$ and $(Cent_X(A)\backslash\tilde{A}) \cup (Cent_X(B)\backslash\tilde{B}) = \emptyset$, then $Srad_X(A \cup B) < rad_X(A) = rad_X(B)$.*

Proof. By Theorem 5.2, we get $Srad_X(A \cup B) \leq rad_X(A)$.
First, let $rad_X(A) > rad_X(B)$. As $\tilde{A} \subseteq Cent_X(A)$ and $Cent_X(A)\backslash\tilde{A} = \emptyset$, we get $Cent_X(A) = \tilde{A}$. As $rad_X(A)$ is finite, by Theorem 3.12, we get $Cent_X(A) \neq \emptyset$. So, for $a \in Cent_X(A)$, we get $d_X(a, \partial_X(B)) < rad_X(A)$. Consequently, $d_X(a, \partial_X(A \cup B)) < rad_X(A)$. And for $a \in A$ such that $a \notin Cent_X(A)$, we get $d_X(a, \partial_X(A \cup B)) < rad_X(A)$. For $b \in B$, we have $d_X(b, \partial_X(A \cup B)) \leq rad_X(B)$. Therefore, $Srad_X(A \cup B) < rad_X(A)$.

Now, let $rad_X(A) = rad_X(B)$. We must have both $Cent_X(A)$ and $Cent_X(B)$ are nonempty. As $(Cent_X(A)\backslash\tilde{A}) \cup (Cent_X(B)\backslash\tilde{B})$ is empty, then $Cent_X(A) = \tilde{A}$ & $Cent_X(B) = \tilde{B}$. So, for $a \in Cent_X(A) \cup Cent_X(B)$, we get $d_X(a, \partial_X(A \cup B)) < rad_X(A)$. Also, for $a \in A \cup B$ such that $a \notin Cent_X(A) \cup Cent_X(B)$, we have $d_X(a, \partial_X(A \cup B)) < rad_X(A)$. Therefore, $Srad_X(A \cup B) < rad_X(A) = rad_X(B)$. □

The above result may not hold if $rad_X(A)$ is infinite. For example: Let $A = [2, \infty)$ and $B = [0, 1]$ be subsets of Euclidean line $\mathbb{R}$. Here, $rad_X(B) = 0.5 < rad_X(A) = \infty$ and $Cent_X(A)\backslash\tilde{A} = \emptyset$. But $Srad_X(A \cup B) = \infty \not< rad_X(A)$.

Notice that, in the above theorems, if $Cent_X(A \cup B) \neq \emptyset$, then $Srad_X(A \cup B)$ can be replaced with $rad_X(A \cup B)$. On the other hand, if $Cent_X(A \cup B) = \emptyset$, then above results may not hold by replacing $Srad_X(A)$ with $rad_X(A)$. For example: Consider, a metric subspace $X = \mathbb{R} \times (\{0\} \cup [1, \infty))$ of Euclidean space $\mathbb{R}^2$. Let $A = \bigcup_{n \in \mathbb{N}} [10n, 10n + 5] \times \{0\}$ and $B = \bigcup_{n \in \mathbb{N}} [10n, 10n + 5] \times \{2 - \frac{1}{n}\}$ be subsets of X. Here, $Cent_X(A) = \tilde{A} = \bigcup_{n \in \mathbb{N}} \{10n + 2.5\}$, $rad_X(A) = 2.5$, $rad_X(B) = 0$ and $Srad_X(A \cup B) = 2$. We can also notice that $Cent_X(A \cup B) = \emptyset$ and $rad_X(A \cup B) = \infty \nleq rad_X(A)$.

Remark 5.4. In Theorem 5.3(i), we further establish a relationship between $Srad_X(A \cup B)$ and $rad_X(B)$. Define $\tilde{\tilde{A}} = \{a \in A \mid d_X(a, \partial_X(A \cup B)) > rad_X(B)\}$. Then

(i) if $\tilde{\tilde{A}} \neq \emptyset$, then $rad_X(B) < Srad_X(A \cup B)$, and

(ii) if $\tilde{\tilde{A}} = \emptyset$ then $Srad_X(A \cup B) \leq rad_X(B)$.

Note that for any point $a \in A \cup B$ such that $a \notin \tilde{\tilde{A}}$, we get $d_X(a, \partial_X(A \cup B)) \leq rad_X(B)$. It is easy to observe that if $\tilde{\tilde{A}} \neq \emptyset$, then all those points of $A \cup B$ which are at the maximum distance from $\partial_X(A \cup B)$ are in $\tilde{\tilde{A}}$. So, we get

Remark 5.5. Let A and B are nonclopen separated subsets of a metric space (X, d_X). If $\tilde{\tilde{A}} \neq \emptyset$, then $Cent_X(A \cup B) \subseteq \tilde{\tilde{A}}$.

Theorem 5.6. *Let A and B be nonclopen separated subsets of a metric space (X, d_X), such that $rad_X(B) < rad_X(A) = \infty$ and $rad_X(B) \geq Srad_X(A)$. Then,*

(1) *if $Cent_X(B)\backslash\tilde{B} \neq \emptyset$, then $rad_X(A \cup B) = Srad_X(A \cup B) = rad_X(B)$ & $Cent_X(A \cup B) = Cent_X(B)\backslash\tilde{B}$, and*

(2) *if $Cent_X(B)\backslash\tilde{B} = \emptyset$, then $Srad_X(A \cup B) < rad_X(B)$.*

Proof. As $rad_X(A) = \infty$, by Theorem 3.12, $Cent_X(A) = \emptyset$, which means $d_X(a, \partial_X(A)) < Srad_X(A), \forall a \in A$. Since $rad_X(B)$ is finite, we get $Cent_X(B) \neq \emptyset$.

(1) If $a \in Cent_X(B)\backslash\tilde{B}$, then $a \in Cent_X(B)$ such that $d_X(a, \partial_X(A)) \geq rad_X(B)$. Thus, $d_X(a, \partial_X(A \cup B)) = \min\{d_X(a, \partial_X(A)), d_X(a, \partial_X(B))\} = d_X(a, \partial_X(B)) = rad_X(B)$. And, if $a \notin Cent_X(B)\backslash\tilde{B}$, then $d_X(a, \partial_X(A \cup B)) < rad_X(B)$. So, $Cent_X(A \cup B) = Cent_X(B)\backslash\tilde{B}$ and $rad_X(A \cup B) = Srad_X(A \cup B) = rad_X(B)$.

(2) Similarly, if $Cent_X(B)\backslash\tilde{B} = \emptyset$, then $Srad_X(A \cup B) < rad_X(B)$. □

Example 5.7. Let $X \subseteq \mathbb{R}^2$ be the union of two rectangles with vertices $(0, 0), (1, 0)$, $(1, 2)$ & $(0, 2)$ and $(1, 0), (2, 0), (2, 2)$ & $(1, 2)$. Thus X is a metric subspace of the Euclidean space $\mathbb{R}^2$. We consider nonclopen separated subsets A and B of X.

(i) Let A and B be the line segments joining $(0, 0)$ to $(0, 2)$ and $(2, 1)$ to $(2, 2)$, respectively. Here $rad_X(A) = 1 > 0.5 = rad_X(B), Cent_X(A) = \{(0, 1)\}$ and $\tilde{A} = \emptyset$. By Theorem 5.1(*i*), we get $rad_X(A \cup B) = 1$ & $Cent_X(A \cup B) = \{(0, 1)\}$.

(ii) Let A and B be the line segments joining $(0, 0)$ to $(0, 2)$ and $(2, 0)$ to $(2, 2)$, respectively. Here $rad_X(A) = 1 = rad_X(B), Cent_X(A) = \{(0, 1)\}, Cent_X(B) = \{(2, 1)\}$ and $\tilde{A} = \emptyset = \tilde{B}$. By Theorem 5.1(*ii*), $rad_X(A \cup B) = 1$ and $Cent_X(A \cup B) = \{(0, 1), (2, 1)\}$.

(iii) Let A and B be the line segments joining $(0, 0)$ to $(2, 0)$ and $(1, 0.2)$ to $(1, 1)$, respectively. Here $rad_X(A) = 1 > 0.4 = rad_X(B), Cent_X(A) = \{(1, 0)\}$ and $\tilde{A} = \{(1, 0)\}$. By Theorem 5.3(*i*), $Srad_X(A \cup B) < 1$. Note that for $a = (0.5, 0), d_X(a, \partial_X(A \cup B)) > 0.4 = rad_X(B)$. So, $\tilde{\tilde{A}} \neq \emptyset$, and hence by Remark 5.4(*i*), $Srad_X(A \cup B) > 0.4$.

(iv) Let A and B be the line segments joining $(0, 0)$ to $(2, 0)$ and $(1, 0.2)$ to $(1, 2)$, respectively. Here $rad_X(A) = 1 > 0.9 = rad_X(B), Cent_X(A) = \{(1, 0)\}$ and $\tilde{A} = \{(1, 0)\}$. By Theorem 5.3(*i*), $Srad_X(A \cup B) < 1$. Infact, as $\tilde{\tilde{A}} = \emptyset$, by Remark 5.4(*ii*), $Srad_X(A \cup B) \leq 0.9$.

One can easily verify that the radius and center of $A \cup B$ in above all four cases are the same as we have obtained using Theorems 5.1 and 5.3.

Example 5.8. Let $\mathbb{S}^2 \subseteq \mathbb{R}^3$ be the unit sphere with metric induced from the Euclidean space $\mathbb{R}^3$. Let $A = \{(x, y, z) \in \mathbb{S}^2 \mid y = 0\}\backslash B_{\mathbb{S}^2}((1, 0, 0), 0.1)$ and $B = \{(x, y, z) \in \mathbb{S}^2 \mid z = 0\}\backslash B_{\mathbb{S}^2}((-1, 0, 0), 0.1)$ be nonclopen subsets of $\mathbb{S}^2$ such that $\overline{A} \cap \overline{B} = \emptyset$, where $B_{\mathbb{S}^2}((1, 0, 0), 0.1)$ and $B_{\mathbb{S}^2}((-1, 0, 0), 0.1)$ are open balls centred at $(1, 0, 0)$ and $(-1, 0, 0)$ respectively, with radius 0.1. Here $rad_{\mathbb{S}^2}(A) = rad_{\mathbb{S}^2}(B) \approx 1.97$, $Cent_{\mathbb{S}^2}(A) = \{(-1, 0, 0)\}, Cent_{\mathbb{S}^2}(B) = \{(1, 0, 0)\}$ and $\tilde{A} = \{(-1, 0, 0)\}, \tilde{B} = \{(1, 0, 0)\}$. This implies $Cent_{\mathbb{S}^2}(A)\backslash\tilde{A}$ and $Cent_{\mathbb{S}^2}(B)\backslash\tilde{B}$ are empty. By Theorem 5.3(ii), $Srad_{\mathbb{S}^2}(A \cup B) < 1.97$.

Next, we generalise above results for a finite union $\bigcup_{i=1}^{n} A_i$ of nonclopen subsets $A_i, 1 \leq i \leq n$, of a metric space X.

Theorem 5.9. *Let (X, d_X) be a metric space. For nonclopen subsets $A_i \subseteq X, 1 \leq i \leq n$, such that A_i & A_j are separated, for all $i \neq j$ and $n \in \mathbb{N}$, let $\tilde{A}_j = \{a \in Cent_X(A_j) \mid d_X(a, \partial_X(A_i)) < rad_X(A_j)$, for some $i \neq j\}, 1 \leq j \leq n$. Let M be the collection of all those A_j such that $rad_X(A_j) = \max\{rad_X(A_i) \mid 1 \leq i \leq n\}$ and $Cent_X(A_j)\backslash\tilde{A}_j \neq \emptyset$. Then, $Srad_X(\bigcup_1^n A_i) \leq \max\{rad_X(A_i) \mid 1 \leq i \leq n\}$.*

Moreover, if $\bigcup_{A_j \in M} (Cent_X(A_j)\backslash\tilde{A}_j) \neq \emptyset$, then $Cent_X(\bigcup_1^n A_i) = \bigcup_{A_j \in M} (Cent_X(A_j)\backslash\tilde{A}_j)$ & $rad_X(\bigcup_1^n A_i) = \max\{rad_X(A_i) \mid 1 \leq i \leq n\}$.

Proof. We prove it by induction. If $n = 2$, it is true by Theorems 5.1 & 5.2.

Assume that it is true for some $k \in \mathbb{N}$. Let $B = \bigcup_{1}^{k} A_i$ and K be the collection of all those A_j such that $rad_X(A_j) = \max\{rad_X(A_i) \mid 1 \leq i \leq k\}$ and $Cent_X(A_j)\backslash\tilde{A}_j \neq \emptyset$. By induction hypothesis, we have $Srad_X(B) \leq \max\{rad_X(A_i) \mid 1 \leq i \leq k\}$, and if $\bigcup_{A_j \in K}(Cent_X(A_j)\backslash\tilde{A}_j) \neq \emptyset$, then $Cent_X(B) = \bigcup_{A_j \in K}(Cent_X(A_j)\backslash\tilde{A}_j)$ and $rad_X(B) = \max\{rad_X(A_i) \mid 1 \leq i \leq k\}$.

Now, we prove it for $k+1$. Let K' be the collection of all those A_j such that $rad_X(A_j) = \max\{rad_X(A_i) \mid 1 \leq i \leq k+1\}$ and $Cent_X(A_j)\backslash\tilde{A}_j \neq \emptyset$.

If $rad_X(B) < rad_X(A_{k+1})$, then by Theorems 5.1 & 5.2, we have $Srad_X(\bigcup_{1}^{k+1} A_i) = Srad_X(B \cup A_{k+1}) \leq rad_X(A_{k+1}) \leq \max\{rad_X(A_i) \mid 1 \leq i \leq k+1\}$. And if $Cent_X(A_{k+1})\backslash\tilde{A}_{k+1} \neq \emptyset$, then $Cent_X(\bigcup_{1}^{k+1} A_i) = Cent_X(B \cup A_{k+1}) = Cent_X(A_{k+1})\backslash\tilde{A}_{k+1}$, and $rad_X((\bigcup_{1}^{k+1} A_i)) = rad_X(A_{k+1}) = \max\{rad_X(A_i) \mid 1 \leq i \leq k+1\}$.

If $rad_X(A_{k+1}) < rad_X(B)$, then again by Theorems 5.1 & 5.2, we have $Srad_X(\bigcup_{1}^{k+1} A_i) = Srad_X(B \cup A_{k+1}) \leq rad_X(B) \leq \max\{rad_X(A_i) \mid 1 \leq i \leq k+1\}$. And if $Cent_X(B)\backslash\tilde{B} \neq \emptyset$, then $Cent_X(\bigcup_{1}^{k+1} A_i) = Cent_X(B)\backslash\tilde{B}$ and $rad_X(\bigcup_{1}^{k+1} A_i) = rad_X(B) = \max\{rad_X(A_i) \mid 1 \leq i \leq k+1\}$.

Next, we observe that in this case $Cent_X(B)\backslash\tilde{B} = \bigcup_{A_j \in K'}(Cent_X(A_j)\backslash\tilde{A}_j)$.

We have $\tilde{B} = \{b \in Cent_X(B) \mid d_X(b, \partial_X(A_{k+1})) < rad_X(B)\}$. Let $b \in Cent_X(B)\backslash\tilde{B}$. Then $b \in Cent_X(B) = \bigcup_{A_j \in K}(Cent_X(A_j)\backslash\tilde{A}_j)$ and $b \notin \tilde{B} \implies b \in Cent_X(A_j)\backslash\tilde{A}_j$ for some $A_j \in K$ and $d_X(b, \partial_X(A_{K+1})) \geq rad_X(B) = rad_X(A_j)$. This gives that $Cent_X(A_j)\backslash\tilde{A}_j \neq \emptyset$, and $rad_X(A_j) = \max\{rad_X(A_i \mid 1 \leq i \leq k+1)\} \implies A_j \in K' \implies b \in \bigcup_{A_j \in K'}(Cent_X(A_j)\backslash\tilde{A}_j)$.

Conversely, let $b \in \bigcup_{A_j \in K'}(Cent_X(A_j)\backslash\tilde{A}_j) \implies b \in Cent_X(A_j)\backslash\tilde{A}_j$ for some $A_j \in K' \implies b \notin \tilde{A}_j \implies A_j \in K$ and $d_X(b, \partial_X(A_{k+1})) \geq rad_X(A_j) = rad_X(B) \implies b \in Cent_X(B)$ and $b \notin \tilde{B} \implies b \in Cent_X(B)\backslash\tilde{B}R$.

If $rad_X(A_{k+1}) = rad_X(B)$, then by Theorems 5.1 & 5.2, we have $Srad_X(\bigcup_{1}^{k+1} A_i) = Srad_X(B \cup A_{k+1}) \leq rad_X(B) \leq \max\{rad_X(A_i) \mid 1 \leq i \leq k+1\}$. And if $(Cent_X(B)\backslash\tilde{B}) \cup (Cent_X(A_{k+1})\backslash\tilde{A}_{k+1}) \neq \emptyset$, then $Cent_X(\bigcup_{1}^{k+1} A_i) = Cent_X(B \cup A_{k+1}) = (Cent_X(B)\backslash\tilde{B}) \cup (Cent_X(A_{k+1})\backslash\tilde{A}_{k+1})$ and $rad_X(\bigcup_{1}^{k+1} A_i) = rad_X(A_{k+1}) = rad_X(B) = \max\{rad_X(A_i) \mid 1 \leq i \leq k+1\}$. In this case, it is easy to observe that $(Cent_X(B)\backslash\tilde{B}) \cup (Cent_X(A_{k+1})\backslash\tilde{A}_{k+1}) = \bigcup_{A_j \in K'}(Cent_X(A_j)\backslash\tilde{A}_j)$.

Thus, it is true for $i = k+1$. Hence, our claim. □

6. Largest open balls contained in a subset of metric space

For any subset A of a metric space X, there is a natural question to identify the largest open balls (largest open ball means an open ball of X with the largest radius) that are entirely contained in A. To answer this question, we introduce a notion of quasi-center and quasi-radius of a subset A of metric space X:

Definition 6.1. [Quasi-center of a subset] The quasi-center of A is the set $\{a \in A \mid d_X(a, A^c) \geq d_X(b, A^c), \forall b \in A\}$, where A^c denotes the complement of A in X. We denote the quasi-center of A in X by $QCent_X(A)$.

Thus the quasi-center of A is the set of all those elements of A which are at the maximum distance from A^c.

Definition 6.2. [Quasi-radius of a subset] The quasi-radius of a subset A of metric space X is the distance between its quasi-center and its complement in X. We denote the quasi-radius of A in X by $Qrad_X(A)$.

Notice that $Qrad_X(A) = d_X(QCent_X(A), A^c) = d_X(a, A^c), \forall a \in QCent_X(A)$.

Example 6.3. Let $X \subseteq \mathbb{R}^2$ denote the union of A and B, where A is a semi unit circle $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1$ and $x \geq 0\}$ and B is the union of three line segments joining (*i*) $(0, 1)$ to $(1.5, 1)$, (*ii*) $(1.5, 1)$ to $(1.5, -1)$ and (*iii*) $(0, -1)$ to $(1.5, -1)$. Consider $X = A \cup B$ as a metric subspace of the Euclidean metric space $\mathbb{R}^2$. Here $Cent_X(B) = \{(1.5, 0)\}$ & $rad_X(B) \approx 1.8$, and $QCent_X(B) = \{(1.5, 1), (1.5, -1)\}$ & $Qrad_X(B) \approx 0.8$.

As the complement of a metric space X is empty in itself, we get $QCent_X(X) = X$ & $Qrad_X(X) = \infty$. Also, we have $QCent_X(\emptyset) = \emptyset$ & $Qrad_X(\emptyset) = \infty$.

As the quasi-center of A consists of all those points of A which are at the maximum distance from its complement, $Qrad_X(A)$ is the maximum distance of any point $a \in A$ from its complement. It is clear that if $QCent_X(A) \neq \emptyset$ then $Qrad_X(A) = \sup_{a \in A} d_X(a, A^c)$. And, if $QCent_X(A) = \emptyset$, then $Qrad_X(A) = \infty$, but $\sup_{a \in A} d_X(a, A^c)$ could be finite.

It leads us to introduce the notion of the semi-quasi-radius of a subset A of a metric space X.

Definition 6.4. [Semi-quasi-radius] The semi-quasi-radius of a subset A of metric space X is the supremum of the set that consists of distance of any point $a \in A$ from A^c. We denote the semi-quasi-radius of A in X by $SQrad_X(A)$.

That is, $SQrad_X(A) = \sup_{a \in A} d_X(a, A^c)$.

Note that, $Qrad_X(A) \geq SQrad_X(A)$. In Example 3.7 and 3.8, it is easy to see that $Qrad_X(A) > SQrad_X(A)$.

Lemma 6.5. *Let A be a subset of metric space X. Then,* $SQrad_X(A) \leq rad_X(A)$.

Proof. As $\partial_X(A) \subseteq \overline{A^c} \implies d_X(a, \partial_X(A)) \geq d_X(a, \overline{A^c}) = d_X(a, A^c), \forall a \in A \implies rad_X(A) \geq \sup_{a' \in A} d_X(a', \partial_X(A)) \geq d_X(a, \partial_X(A)) \geq d_X(a, A^c), \forall a \in A$. Thus, $SQrad_X(A) \leq rad_X(A)$. □

If $QCent_X(A) \neq \emptyset$, then by the above lemma $Qrad_X(A) \leq rad_X(A)$. If $QCent_X(A) = \emptyset$, then it may not be true. For example: Consider, $X = (\mathbb{R} \times \{0\}) \cup B$ with subspace metric from Euclidean space $\mathbb{R}^2$, where $B = \bigcup_{n \in \mathbb{N}} [10n, 10n + 5] \times \{2 - \frac{1}{n}\}$. And, let $A = \bigcup_{n \in \mathbb{N}} [10n, 10n + 5] \times \{0\}$. Here, $QCent_X(A) = \emptyset$, $SQrad_X(A) = 2$ but $Qrad_X(A) = \infty \nleq rad_X(A) = 2.5$.

Remark 6.6. A metric space (X, d_X) is a path metric space if the distance between each pair of points equals the infimum of the lengths of the curves joining the points, see [1]. Recall that the Euclidean spaces and connected Riemannian manifolds are path metric spaces. If X is a path metric space, then for a proper subset A of X, we have $d_X(a, \partial_X(A)) = d_X(a, A^c), \forall a \in A$. This gives that for path metric space X, $Cent_X(A) = QCent_X(A)$, and $rad_X(A) = Qrad_X(A)$.

Notice that, in Example 3.3 and 3.4, quasi-center and quasi-radius of subsets are the same as their center and radius, respectively. By Example 6.3, we see that the above remark is not true if X is not a path metric space.

It is easy to observe the following results:

Lemma 6.7. *Let A be a nonempty subset of a metric space X such that* $A \subseteq \partial_X(A)$. *Then,* $QCent_X(A) = A$ *and* $Qrad_X(A) = 0$.

Lemma 6.8. *Let A be a subset of metric space X with nonempty interior. Then* $QCent_X(A) \subseteq A^\circ$.

Theorem 6.9. *Let A be a nonempty subset of metric space X. Then* $A^\circ = \emptyset$ *if and only if* $Qrad_X(A) = 0$.

In the next theorem, using the above notions of quasi-center and quasi-radius, we determine the largest open balls contained in a subset A of metric space X. If A is a proper subset of metric space (X, d_X) such that $QCent_X(A)$ is empty, then there does not exist any open ball with largest radius that is contained in A. As $QCent_X(A) = \emptyset \implies Qrad_X(A) = \infty$. Let open ball $B(a, r)$ be the largest open ball entirely contained in A for some $a \in A$ and $r > 0$. This means $B(a, r) \cap A^c = \emptyset \implies d_X(a, A^c) \geq r$. Now, as $Qrad_X(A) = \infty$, for some $s > r, \exists\, b \in A$ such that $d_X(b, A^c) \geq s \implies B(b, s)$ contained in A, a contradiction. For example, $A = [0, \infty) \subseteq \mathbb{R}$ has no largest open ball contained in A.

For the nonempty quasi-center of a subset A of metric space X, we have the following result.

Theorem 6.10. *Let A be a nonempty proper subset of metric space X. Then the largest open balls of X which are entirely contained in A are the balls whose centers belong to $QCent_X(A)$ and radius is equal to $Qrad_X(A)$.*

Proof. Any point $a \in A$ is either an interior point of A or a boundary of A. If $a \in \partial_X(A)$ then for every $\epsilon > 0$, the open ball $B(a, \epsilon)$ intersects A^c. So, any ball centered at boundary point of A with positive radius can not be entirely contained in A. Thus if $A^\circ = \emptyset$ then $QCent_X(A) = A$ and the largest open balls contained in A are balls with zero radii.

If $A^\circ \neq \emptyset$ then by Lemma 6.8, $QCent_X(A) \subseteq A^\circ$. First, let $a \in A^\circ$ such that $a \notin QCent_X(A)$. As $d_X(a, A^c) < Qrad_X(A)$, and A is proper subset of X, then $\exists\, b \in A^c$ such that $d_X(a, b) < Qrad_X(A)$. Thus $B(a, Qrad_X(A)) \cap A^c \neq \emptyset$. Therefore, any open ball centred at a with radius $\geq Qrad_X(A)$ can not be entirely contained in A. Now, let $a \in QCent_X(A)$. Then $d_X(a, A^c) = Qrad_X(A)$. So, $B(a, Qrad_X(A)) \subseteq A$. Next, we observe that for $\epsilon > 0$, open balls $B(a, Qrad_X(A) + \epsilon)$ has nonempty intersection with A^c. As $\inf_{b \in A^c} d_X(a, b) = Qrad_X(A)$, we get that $\forall \epsilon > 0$, $\exists\, b' \in A^c$ such that $d_X(a, b') < Qrad_X(A) + \epsilon$. So, $B(a, Qrad_X(A) + \epsilon) \cap A^c \neq \emptyset$, and therefore any ball with radius greater than $Qrad_X(A)$ can not be entirely contained in A. Hence, our claim. $\square$

Corollary 6.11. *Let A and B be proper subsets of a metric space X such that $QCent_X(A) \neq \emptyset$ and $A \subseteq B$. Then $Qrad_X(A) \leq Qrad_X(B)$.*

Proof. If $A^\circ = \emptyset$ then the result follows from Theorem 6.9. If $A^\circ \neq \emptyset$ then by Theorem 6.10, $B(a, Qrad_X(A)$ is the largest open ball contained in A, where $a \in QCent_X(A)$. As $B(a, Qrad_X(A)) \subseteq A \subseteq B$ and $B(b, Qrad_X(B))$ is the largest ball contained in B, where $b \in QCent_X(B)$. Hence, our claim. $\square$

But if $A \subseteq B$ are proper subsets of metric space X then it does not imply that $QCent_X(A) \subseteq QCent_X(B)$. For example: Take subsets $A = [0, 1]$ and $B = [0, 2]$ of $\mathbb{R}$ with the usual metric. Then $QCent_{\mathbb{R}}(A) = \{\frac{1}{2}\}$ and $Qent_{\mathbb{R}}(B) = \{1\}$.

Remark 6.12. Note that if $A \subseteq B$ are proper subsets of a path metric space X such that $Cent_X(A) \neq \emptyset$, then by Corollary 6.11, $rad_X(A) \leq rad_X(B)$.

Remark 6.13. We can notice that the radius of a subset may not be equal to half of its diameter. In fact, it is easy to observe that if A is a nonclopen subset of metric space X such that $Cent_X(A) \neq \emptyset$, then $rad_X(A) \leq diam_X(A)$. By Theorem 6.10, we also observe that for a proper subset A of the Euclidean space $\mathbb{R}^n$, having nonempty center, $rad_{\mathbb{R}^n}(A) \leq \frac{1}{2} diam_{\mathbb{R}^n}(A)$.

For subsets A and B of a metric space X, we observe that $diam_X(A) \leq diam_X(B)$ does not imply $rad_X(A) \leq rad_X(B)$ or $Qrad_X(A) \leq Qrad_X(B)$.

Example 6.14. Consider A_1 and $A_2 \subseteq \mathbb{R}^2$, where A_1 is the line segment joining $(-2, 0)$ and $(2, 0)$ in $\mathbb{R}^2$ and A_2 is the closed ball centered at $(4, 0)$ with radius 1. Notice that $diam_{\mathbb{R}^2}(A_1) = 4 > 2 = diam_{\mathbb{R}^2}(A_2)$ but $Qrad_{\mathbb{R}^2}(A_1) = rad_{\mathbb{R}^2}(A_1) = 0 < 1 = rad_{\mathbb{R}^2}(A_2) = Qrad_{\mathbb{R}^2}(A_2)$.

Next, we introduce a notion of concentric subsets.

Definition 6.15. [Concentric Subsets] Two subsets A and B of a metric space X are called concentric subsets if they have the same nonempty centers in X

Example 6.16. Take $A = [-2, 2]$ and $B = [-1, 1]$ in the set $\mathbb{R}$ of real numbers with the usual metric. Here, $Cent_{\mathbb{R}}(A) = Cent_{\mathbb{R}}(B) = \{0\}$. So, A and B are concentric in $\mathbb{R}$. Infact, all intervals of the form $[-n, n]$ or $(-m, m)$ are concentric in $\mathbb{R}$, where m, n are positive real numbers.

Example 6.17. Let $\mathbb{R}^2$ be the real plane with the Euclidean metric. Then the unit circle $\mathbb{S}^1$ and the punctured disc $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\} \backslash \{(0, 0)\}$ are concentric as $Cent_{\mathbb{R}^2}(\mathbb{S}^1) = Cent_{\mathbb{R}^2}(A) = \mathbb{S}^1$.

Remark 6.18. Concentric subsets may not be contained in each other. For example: Let $A = \mathbb{D}^2 \cup \{(2, 0)\}$ and $B = \mathbb{D}^2 \cup \{(0, 2)\}$ be two subsets of the Euclidean plane $\mathbb{R}^2$, where $\mathbb{D}^2$ is the unit disc in $\mathbb{R}^2$. Note that $Cent_{\mathbb{R}^2}(A) = Cent_{\mathbb{R}^2}(B) = \{(0, 0)\}$, but neither $A \subseteq B$ nor $B \subseteq A$. Also, notice that if A and B are concentric subsets with the same radius then A may not be equal to B.

It is easy to see that the relation of being concentric subsets is an equivalence relation on the class of subsets of X with nonempty centers.

Theorem 6.19. *Let A be a subset of a path metric space X such that A has nonempty interior and $Cent_X(A) \neq \emptyset$. Then A and A° are concentric with same radii.*

Proof. By Theorem 3.17 and Remark 6.12, we get $rad_X(A) = rad_X(A^\circ)$. Let $x \in Cent_X(A)$. Then by Lemma 3.14, $x \in A^\circ$. We have $rad_X(A) = rad_X(A^\circ) \geq d_X(x, \partial_X(A^\circ)) \geq d_X(x, \partial_X(A)) = rad_X(A)$. Thus $d_X(x, \partial_X(A^\circ)) = rad_X(A^\circ)$. So, $x \in Cent_X(A^\circ)$. Therefore, $Cent_X(A) \subseteq Cent_X(A^\circ)$. Now, let $x \notin Cent_X(A) = QCent_X(A)$. So, $d_X(x, A^c) < rad_X(A) = Qrad_X(A)$. So, $B(x, rad_X(A)) \cap A^c \neq \emptyset \implies B(x, rad_X(A^\circ)) \cap A^c \neq \emptyset$. This implies that $B(x, rad_X(A^\circ)) \cap (A^\circ)^c \neq \emptyset$. Thus $x \notin QCent_X(A^\circ) = Cent_X(A^\circ)$. Therefore, $Cent_X(A^\circ) \subseteq Cent_X(A)$. Hence, $Cent_X(A^\circ) = Cent_X(A)$. $\square$

Remark 6.20. Theorem 6.19 may not be true if $A^\circ = \emptyset$. For example: Consider the set $\mathbb{R}$ of real numbers with the usual metric. If $A = \{\frac{1}{n} \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$, then $A^\circ = \emptyset$. We have $Cent_X(A^\circ) = \emptyset$ & $Cent_X(A) = A$ and $rad_X(A^\circ) = \infty$ & $rad_X(A) = 0$. Notice that $rad_X(A^\circ) \neq rad_X(A)$, and $Cent_X(A) \neq Cent_X(A^\circ)$.

If X is a path metric space in Theorem 5.9, then we have the following result.

Corollary 6.21. *Let (X, d_X) be a path metric space. For nonempty proper subsets $A_i \subseteq X, 1 \leq i \leq n$, such that A_i and A_j are separated, for all $i \neq j$ and $n \in \mathbb{N}$, and let M be the collection of all those A_j such that $rad_X(A_j) = \max\{rad_X(A_i) \mid 1 \leq i \leq n\}$ and $Cent_X(A_j) \neq \emptyset$. Then, if $M \neq \emptyset$, then $Cent_X(\bigcup_1^n A_i) = \bigcup_{A_j \in M} Cent_X(A_j)$ & $rad_X(\bigcup_1^n A_i) = \max\{rad_X(A_i) \mid 1 \leq i \leq n\}$.*

Proof. We observe that $\tilde{A}_j = \emptyset, \forall\, j$, where $\tilde{A}_j$ is the same as defined in Theorem 5.9. As A_i and A_j are separarted for all $i \neq j$, $\partial_X(A_i) \subseteq (A_j)^c$. So, we get $d_X(a, (A_j)^c) \leq d_X(a, \partial_X(A_i)), \forall\, a \in A_j$. As X is path metric space, by Remark 6.6, we get $rad_X(A_j) = Qrad_X(A_j) \leq d_X(a, \partial_X(A_i)), \forall\, a \in Cent_X(A_j)$. This implies that $\tilde{A}_j = \emptyset, \forall\, j$. Now, the result follows from Theorem 5.9. $\square$

Remark 6.22. If X is a Euclidean space then the center of a disconnected proper subset of X is equal to the union of centers of its connected components with the maximum radius and its radius is equal to the radius of component with the maximum radius.

Example 6.23. Let $A = [0, 1], B = [2, 6]$ and $C = [8, 12]$ be subsets of $\mathbb{R}$ with the usual metric. Here $rad_{\mathbb{R}}(B) = rad_{\mathbb{R}}(C) = 2 > 0.5 = rad_{\mathbb{R}}(A)$. So, by corollary 6.21, $Cent_{\mathbb{R}}(A \cup B \cup C) = Cent_{\mathbb{R}}(B) \cup Cent_{\mathbb{R}}(C) = \{4, 10\}$ and $rad_{\mathbb{R}}(A \cup B \cup C) = rad_{\mathbb{R}}(B) = 2$.

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