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On Cauchy alpha and uniform alpha convergence

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Abstract. In this paper we introduced the notions of Cauchy alpha convergence and uniform alpha convergence for sequences of functions between metric spaces. Using these new concepts, we investigated properties between uniform convergence and alpha convergence and provided figures illustrating the relationships among these different types of convergence based on the structure of the functions and their domains. Finally, we established some Korovkin-type approximation theorems involving these new concepts.

1. Introduction

The study of convergence for function sequences is a fundamental topic in analysis, with significant implications in various branches of mathematics and applied sciences. Continuous convergence, recently referred to as alpha convergence, was first introduced by R. Courant [6] in 1914 . Although H. Hahn [12] defined the concept and explored its properties in 1921, he noted that earlier variations of this idea appeared in the works of Weierstrass and P. Du Bois-Reymond in the 19th century. Additionally, in 1929, C. Carathéodory [5] used the concept of continuous convergence in his work. Later, around the 1950s, H. Schaefer [17], K. Iseki [13], and S. Stoilov [18] obtained further results related with using the alpha convergence. As studies in this area became less common, different types of convergence for function sequences were defined, with their properties examined over the last quarter century.

In addition to alpha convergence, the concepts of exhaustiveness and semi-alpha convergence, as well as the relationships between these concepts, are among the problems studied in recent years. In [11], the concept of exhaustiveness was introduced which is connecting alpha convergence with pointwise convergence.

In 2003, Das and Papanastassiou [7] introduced new type of convergences for sequences of real-valued functions called $\alpha-$uniform equal, $\alpha-$strong uniform equal and $\alpha-$equal. Via using this definition they obtained a characterization of the compact metric space. In 2020, Papanastassiou [16] introduced semi-alpha convergence, semi-exhaustiveness and semi-uniform convergence to gain a new perspective for sequences of functions.

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Borsík [3] studied the properties, transitions and equivalences between various convergence types, providing comparing their relationships and structural behavior in metric spaces. In 1992, Borsík [4] characterized the class of mappings that preserve Cauchy sequences, highlighting their continuity and uniformity properties.

Among the most significant theorems in constructive approximation is the Korovkin's theorem [14]. Although the original theorem was formed based on the concept of uniform convergence, it has recently been adapted to various other convergence concepts and summability methods. A classical reference for Korovkin-type theorems in the context of statistical convergence can be found in [10]. In 2024, Erdem and Tunç studied the Korovkin-type theorems for various convergence types in [9, 20]

In this study, we introduce the notion of uniform alpha convergence and Cauchy alpha convergence for sequences of functions between metric spaces. These new concepts describe the relationship between uniform convergence and alpha convergence.

2. Definitions and auxiliary results

To enhance readability, in this section, we recall some basic definitions and notations. Throughout the paper, we write $X = (X, d)$ and $Y = (Y, \rho)$ to denote metric spaces unless or otherwise mentioned. By $B_d(x_0, \delta)$ denotes the open ball with center x_0 and radius $\delta > 0$ with respect to metric d. The set of all Y-valued functions defined on X is denoted by Y^X:

$$Y^X = \{h \mid h : X \to Y \text{ is a function.}\}$$

The family of continuous, Cauchy continuous and uniformly continuous functions defined on X to Y are denoted by $C(Y^X)$, $CC(Y^X)$ and $UC(Y^X)$, respectively. It is well known that $UC(Y^X) \subset CC(Y^X) \subset C(Y^X)$.

The family of sequences of functions and the family of sequence of continuous functions defined over X to Y is denoted by $\mathfrak{sf}(Y^X)$ and $\mathfrak{scf}(Y^X)$, respectively. For simplicity, if the domain of functions is known, $\mathfrak{sf}(Y^X)$ and $\mathfrak{sf}(Y^X)$ are abbreviated to $\mathfrak{sf}$ and $\mathfrak{scf}$, respectively :

$$\mathfrak{sf} := \mathfrak{sf}(Y^X) = \left\{(h_n) \mid \forall n \in \mathbb{N},\, h_n \in Y^X\right\} \quad \text{and} \quad \mathfrak{scf} := \mathfrak{scf}(Y^X) = \left\{(h_n) \mid \forall n \in \mathbb{N},\, h_n \in C(Y^X)\right\}$$

We will use the following notations to describe various properties of function sequences. As $\mathfrak{c}_{\mathfrak{sf}} := \mathfrak{c}_{\mathfrak{sf}}(Y^X)$, it denotes the set of all point-wise convergent sequences of functions on X to Y , while $\mathfrak{c}^u_{\mathfrak{sf}} := \mathfrak{c}^u_{\mathfrak{sf}}(Y^X)$ represents sequences that are uniformly convergent. The subscript $\mathfrak{scf}$ denotes that the sequences consist of continuous functions, applied consistently across all related notations throughout this paper. We use "$h_n \to h$" for the point-wise convergence of the sequence $(h_n) \in \mathfrak{sf}$ to h; and we use "$h_n \rightrightarrows h$" for the uniformly convergence of the sequence $(h_n) \in \mathfrak{sf}$ to h. For boundedness, let $\mathfrak{b}_{\mathfrak{sf}} := \mathfrak{b}_{\mathfrak{sf}}(Y^X)$ and $\mathfrak{b}^u_{\mathfrak{sf}} := \mathfrak{b}^u_{\mathfrak{sf}}(Y^X)$ be the set of point-wise bounded and uniformly bounded sequences of functions on X to Y, respectively.

Definition 2.1. ([7]) The sequence $(h_n) \in \mathfrak{sf}$ *alpha converges* to $h \in Y^X$, if for every $t \in X$ and for every sequence (t_n) of points of X converging to t, the sequence $(h(t_n))$ converges to $h(t)$.

The notation $h_n \to_\alpha h$ will be used for alpha convergence of the sequence (h_n) to h. The set of alpha convergent sequences of functions on X to Y will be denoted by $\mathfrak{c}^\alpha_{\mathfrak{sf}}(Y^X)$:

$$\mathfrak{c}^\alpha_{\mathfrak{sf}} := \mathfrak{c}^\alpha_{\mathfrak{sf}}(Y^X) = \left\{(h_n) \in \mathfrak{sf} \mid \exists h \in Y^X,\, h_n \to_\alpha h\right\}.$$

It is proved in [2] that the alpha convergence of the sequence (h_n) at $t_0 \in X$ to h is equivalent with the following condition:

$$\forall \varepsilon > 0, \exists \delta = \delta(t_0, \varepsilon) > 0, \exists n_0 = n_0(t_0, \varepsilon) \in \mathbb{N} : t \in B_d(t_0, \delta),\, n \geq n_0 \implies \rho(h_n(t), h(t_0)) < \varepsilon.$$

Here are some facts related to alpha convergence.

Lemma 2.2. *(i) If the set X is compact and $h_n \to_\alpha h$, then $h_n \rightrightarrows h$ ([11]).*
(ii) If the set X is compact and $h_n \to_\alpha h$, then $(h_n) \in \mathfrak{b}^u_{\mathfrak{sf}}$ ([8]).
(iii) If $h_n \rightrightarrows h$ and $h \in C(Y^X)$ on X, then $h_n \to_\alpha h$ ([11]).

If X is a totally bounded set then the Cauchy continuity of a function and the uniform continuity of the function coincide [3].

Lemma 2.3. *Let (x_n) be a Cauchy sequence in X. Then the set $A = \{x_n : n \in \mathbb{N}\} \subset X$ is a totally bounded set.*

Proof. Let $\varepsilon > 0$ is given. Since (x_n) is a Cauchy sequences there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, $d(x_n, x_m) < \varepsilon$. It is clear that $x_n \in B_d(x_{n_0}, \varepsilon)$ for all $n \geq n_0$ then $A \subset \bigcup_{i=1}^{n_0} B_d(x_i, \varepsilon)$. So that A is a totally bounded set. □

3. Cauchy alpha convergence

The alpha convergence a sequence of functions guarantees that the function it converges to is continuous. In this section, we will define a new mode of convergence guaranteed that the function it converges to is Cauchy continuous.

Definition 3.1. The sequence $(h_n) \in \mathfrak{sf}$ is called *Cauchy alpha convergent* to the function $h \in Y^X$ on X if for every Cauchy sequences (t_n) and (w_n) of points in X with the property $d(t_n, w_n) \to 0$, the sequence $\rho(h_n(t_n), h(w_n)) \to 0$.

If (h_n) is Cauchy alpha convergent to h on X then we write $h_n \to_{c\alpha} h$. We will use the notation $\mathfrak{c}^{c\alpha}_{\mathfrak{sf}}(Y^X)$ to denote set of all Cauchy alpha convergent sequences of functions in Y^X:

$$\mathfrak{c}^{c\alpha}_{\mathfrak{sf}} := \mathfrak{c}^{c\alpha}_{\mathfrak{sf}}(Y^X) = \left\{(h_n) \in \mathfrak{sf} \mid \exists h \in Y^X,\ h_n \to_{c\alpha} h\right\}.$$

Proposition 3.2. *If $h_n \to_{c\alpha} h$ then $h \in CC(Y^X)$.*

Proof. Let (t_n) be a Cauchy sequence in X and $\varepsilon > 0$. By the hypothesis, there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $\rho\left(h_n(t_n), h(t_n)\right) < \frac{\varepsilon}{2}$ and $\rho\left(h_n(t_n), h(t_{n+p})\right) < \frac{\varepsilon}{2}$ for all $n \geq n_0$ and $p \in \mathbb{N}_0$. Thus, we have

$$\rho\left(h(t_n), h(t_{n+p})\right) \leq \rho\left(h_n(t_n), h(t_n)\right) + \rho\left(h_n(t_n), h(t_{n+p})\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $n \geq n_0$ and $p \in \mathbb{N}_0$. Therefore $(h(t_n))$ is a a Cauchy sequence in Y, that is h is Cauchy continuous on X. □

We will prove the following lemma using the method of Gregoriades and Papanastassiou ([11], Prop. 1.4). However, since the sequences considered in their proof were convergent, it was possible to produce suitable new sequences using the limit. Since our sequences are not guaranteed to converge, we will implement the method using the notion of elongation of sequences (for definition see [19]).

Lemma 3.3. *If $h_n \to_{c\alpha} h$ then for any subsequence (h_{n_k}), we have $h_{n_k} \to_{c\alpha} h$.*

Proof. Let (t_n) and (w_n) be Cauchy sequences such that $d(t_n, w_n) \to 0$ when $n \to \infty$, and (n_k) be an arbitrary strictly increasing sequence of positive integers. If we define the sequences (x_n) and (y_n) as

$$x_n = t_k \quad \text{and} \quad y_n = w_k \quad \text{if} \quad n_{k-1} < n \leq n_k,\ k \in \mathbb{N},$$

where $n_0 = 0$, then it is clear that (x_n) and (y_n) are Cauchy sequences that $d(x_n, y_n) \to 0$ when $n \to \infty$. By Cauchy alpha convergency of the sequence (h_n) to h, we have $\rho(h_n(x_n), h(y_n)) \to 0$, so that $\rho(h_{n_k}(x_{n_k}), h(y_{n_k})) \to 0$ when $n \to \infty$. Since $x_{n_k} = t_k$ and $y_{n_k} = w_k$ for all $k \in \mathbb{N}$, we get $\rho(h_{n_k}(t_k), h(w_k)) \to 0$ when $k \to \infty$. Hence, the subsequence (h_{n_k}) is Cauchy alpha convergent to h. □

4. Unifom alpha convergence

In this section, we will define a new mode of convergence guaranteed that the function it converges to is uniformly continuous.

Definition 4.1. The sequence $(h_n) \in \mathfrak{sf}$ is called *uniform alpha convergent* to the function $h \in Y^X$ on X if for every sequence (t_n) and (w_n) of points of X with the property $d(t_n, w_n) \to 0$, the sequence $\rho(h_n(t_n), h(w_n)) \to 0$.

If (h_n) is uniform alpha convergent to h on X then we write $h_n \rightrightarrows_\alpha h$. We will use the notation $\mathfrak{c}^{u\alpha}_{\mathfrak{sf}}(Y^X)$ to denote set of all uniform alpha convergent sequences of functions in Y^X:

$$\mathfrak{c}^{u\alpha}_{\mathfrak{sf}} := \mathfrak{c}^{u\alpha}_{\mathfrak{sf}}(Y^X) = \left\{(h_n) \in \mathfrak{sf} \mid \exists h \in Y^X,\, h_n \rightrightarrows_\alpha h\right\}.$$

Lemma 4.2. *If $h_n \rightrightarrows_\alpha h$ then for any subsequence (h_{n_k}), we have $h_{n_k} \rightrightarrows_\alpha h$.*

Proof. Let (t_n) and (w_n) be sequences such that $d(t_n, w_n) \to 0$ when $n \to \infty$, (n_k) be an arbitrary strictly increasing sequence of positive integers and $x_0 \in X$. We define the sequences (x_n) and (y_n) as follows:

$$x_n := \begin{cases} t_k, & n = n_k \\ x_0, & \text{otherwise} \end{cases} \qquad \text{and} \qquad y_n := \begin{cases} w_k, & n = n_k \\ x_0, & \text{otherwise} \end{cases}$$

Using the sequences (x_n) and (y_n), we get

$$d(x_n, y_n) := \begin{cases} d(t_k, w_k), & n = n_k \\ 0, & \text{otherwise.} \end{cases}$$

with the property $d(x_n, y_n) \to 0$. From the uniform alpha convergence of (h_n) to h, which implies $\rho(h_n(x_n), h(y_n)) \to 0$, we have $\rho(h_{n_k}(x_{n_k}), h(y_{n_k})) \to 0$ as $k \to \infty$. Since $x_{n_k} = t_k$ and $y_{n_k} = w_k$ for all $k \in \mathbb{N}$, we get $\rho(h_{n_k}(t_k), h(w_k)) \to 0$ when $k \to \infty$. Hence, the subsequence (h_{n_k}) is uniform alpha convergent to h. □

Theorem 4.3. *Let $(h_n) \in \mathfrak{sf}$ and $h \in Y^X$. Then the following assertions are equivalent:*

i. The sequence (h_n) is uniform alpha convergent to the function h on X.

ii. For every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ and $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that for every $t, w \in X$ with $d(t, w) < \delta$ and for all $n \geq n_0$ we have $\rho\left(h_n(t), h(w)\right) < \varepsilon$.

Proof. ($ii \implies i$) Let (t_n) and (w_n) be sequences such that $d(t_n, w_n) \to 0$ when $n \to \infty$, and $\varepsilon > 0$ be given. From the hypothesis, there exists $\delta = \delta(\varepsilon) > 0$ and $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that for all $t, w \in X$ with $d(t, w) < \delta$ and for all $n \geq n_0$ we have $\rho\left(h_n(t), h(w)\right) < \varepsilon$. From $d(t_n, w_n) \to 0$, there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$ we have $d\left(t_n, w_n\right) < \delta$. If we choose $n^* = \max\{n_0, n_1\}$ then for all $n \geq n^*$ we have $\rho\left(h_n(t_n), h(w_n)\right) < \varepsilon$. That is $\rho(h_n(t_n), h(w_n)) \to 0$.

($i \implies ii$) Let $h_n \rightrightarrows_\alpha h$. Assume that the condition fails i.e. there exists a number $\varepsilon > 0$ such that for all $k \in \mathbb{N}$ there exists a natural number $n_k \geq k$ and $t_k, w_k \in X$ with $d(t_k, w_k) < \frac{1}{k}$ such that $\rho\left(h_{n_k}(t_k), h(w_k)\right) \geq \varepsilon$. Without loss of generality we can assume that the sequence (k_n) is strictly increasing. Hence we have a contradiction because of Lemma 4.2. □

Proposition 4.4. *If $h_n \rightrightarrows_\alpha h$ then $h \in UC(Y^X)$.*

Proof. Let $\varepsilon > 0$ be given. By uniform alpha convergence of (h_n) to h, there exists $\delta = \delta(\varepsilon) > 0$ and $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that for every $t, w \in X$ with $d(t, w) < \delta$ and for all $n \geq n_0$ we have $\rho\left(h_n(t), h(w)\right) < \varepsilon/2$. Hence, we have

$$\rho\left(h(t), h(w)\right) \leq \rho\left(h(t), h_{n_0}(w)\right) + \rho\left(h_{n_0}(w), h(w)\right) < \varepsilon$$

for every $t, w \in X$ with $d(t, w) < \delta$. □

Remark 4.5. (i) It can be easily seen that if the sequence of functions (h_n) converges to a limit function that is not uniformly continuous, then this convergence cannot be uniform alpha.

(ii) Even if not every h_n is continuous, the convergence can still be uniform alpha. For example, let $h_n : (-1, 1) \to \mathbb{R}$, $h_n(t) = 1/n$ if $t > 0$ and $h_n(t) = 2/n$ otherwise ($n \in \mathbb{N}$). Even though every h_n is not continuous, $h_n \rightrightarrows_\alpha 0$.

5. Relationships between modes of convergences

From the definitions, it is clear that the uniform alpha convergence is stronger than the Cauchy alpha convergence:

Proposition 5.1. *If $h_n \rightrightarrows_\alpha h$ then $h_n \to_{c\alpha} h$, that is $\mathfrak{c}^{u\alpha}_{\mathfrak{sf}} \subset \mathfrak{c}^{c\alpha}_{\mathfrak{sf}}$.*

Just as in point-wise and uniform convergence, there is the following relationship between uniform alpha and consequently Cauchy alpha and alpha convergence:

Proposition 5.2. *If $h_n \to_{c\alpha} h$ then $h_n \to_\alpha h$, that is $\mathfrak{c}^{c\alpha}_{\mathfrak{sf}} \subset \mathfrak{c}^{\alpha}_{\mathfrak{sf}}$.*

Proof. Let $t_0 \in X$ be any point and (t_n) be a sequence in X such that $d(t_n, t_0) \to 0$. By Cauchy alpha convergence of (h_n) to h, we have $\rho\,(h_n(t_n), h(t_0)) \to 0$, since every convergent sequence is a Cauchy sequence. Consequently $h_n \to_\alpha h$. □

By using Propositon 5.1, we have the following result.

Corollary 5.3. *If $h_n \rightrightarrows_\alpha h$ then $h_n \to_\alpha h$.*

Remark 5.4. The reverse implications of Proposition 5.1, Proposition 5.2 and Corollary 5.3 may not be true i.e. there exist examples where

(i) A sequence is Cauchy alpha convergent but not uniformly alpha convergent.
(ii) A sequence is alpha convergent but not Cauchy alpha convergent.
(iii) A sequence is alpha convergent but not uniformly alpha convergent.

For (i) and (iii) consider $h_n : (0, 1) \to \mathbb{R}$, $h_n(t) = \frac{n-t}{nt}$ for every $n \in \mathbb{N}$, with standard metric. It is clear that (h_n) is both alpha and Cauchy alpha convergent to $h(t) = \frac{1}{t}$ on $(0, 1)$ but it lacks of uniform alpha convergence. Indeed, let $\varepsilon = 1$ and $\delta > 0$ be given. For each $t \in (0, 1)$ and each $n \in \mathbb{N}$, with $t < \min\{2\delta, 1\}$ and $w = \frac{t}{2}$, we have $|t - w| = |t - \frac{t}{2}| = \frac{t}{2} < \delta$, such that

$$|h_n(t) - h(w)| = \left|\frac{n-t}{nt} - \frac{2}{t}\right| = \frac{n+t}{nt} > 1 = \varepsilon.$$

For (ii) Assume that $h_n : (0, 1) \to \mathbb{R}$, $h_n(t) = 1 - nt$ if $0 < t < 1/n$ and $h_n(t) = 0$ otherwise ($n \in \mathbb{N}$). Although the sequence (h_n) is alpha convergent to $h(t) = 0$ on $(0, 1)$ but it lacks of Cauchy alpha convergence. Indeed for the Cauchy sequences $t_n = w_n = \frac{1}{n^2}$ with $|t_n - w_n| \to 0$ while $|h_n(t_n) - h(w_n)| = 1 - \frac{1}{n} \not\to 0$.

Now, let us give the relationship between uniform alpha convergence and uniform convergence.

Proposition 5.5. *If $h_n \rightrightarrows_\alpha h$ then $h_n \rightrightarrows h$, that is $\mathfrak{c}^{u\alpha}_{\mathfrak{sf}} \subset \mathfrak{c}^{u}_{\mathfrak{sf}}$.*

Proof. Let $\varepsilon > 0$ be given. By uniform alpha convergence of (h_n) to h, there exists $\delta = \delta(\varepsilon) > 0$ and $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that for every $t, w \in X$ with $d(t, w) < \delta$ and for all $n \geq n_0$ we have $\rho\,(h_n(t), h(w)) < \varepsilon$. Here if we choose $w = t$ then for all $n \geq n_0$ we have $\rho\,(h_n(t), h(t)) \leq \varepsilon$. □

Remark 5.6. The reverse implication of Proposition 5.5 may not be true. Consider $h_n : (0,1) \to \mathbb{R}$, $h_n(t) = \frac{1}{t}$ for every $n \in \mathbb{N}$, with standard metric. It is clear that (h_n) is uniformly convergent to $h(t) = \frac{1}{t}$ on $(0,1)$ but since $h(t) = \frac{1}{t}$ is not uniformly continuous on $(0,1)$, it lacks of uniform alpha convergence on $(0,1)$.

Analogous to Lemma 2.2 (iii), we can write the following theorem:

Theorem 5.7. *If $h_n \rightrightarrows h$ and $h \in UC(Y^X)$ then $h_n \rightrightarrows_\alpha h$.*

Proof. Let $\varepsilon > 0$ be given. By uniform convergence of (h_n) to h, there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n_0$ and for all $t \in X$ we have $\rho\left(h_n(t), h(t)\right) < \varepsilon/2$. Also by uniform continuity of h, there exists $\delta = \delta(\varepsilon) > 0$ such that for all $t, w \in X$ with $d(t,w) < \delta$ we have $\rho\left(h(t), h(w)\right) < \varepsilon/2$. From here, we have

$$\rho\left(h_n(t), h(w)\right) \leq \rho\left(h_n(t), h(w)\right) + \rho\left(h(t), h(w)\right) < \varepsilon.$$

for all $n \geq n_0$ and for all $t, w \in X$ with $d(t,w) < \delta$. □

Remark 5.8. Even if (h_n) is Cauchy alpha converges to a uniformly continuous function, the convergence may not be uniform alpha therefore not uniformly. For example, let $h_n : (0,\infty) \to \mathbb{R}$, defined by

$$h_n(t) = \begin{cases} n, & t \geq n \\ t, & t < n. \end{cases}$$

It is clear that $h_n \to_{c\alpha} h$, where $h(t) = t$. Obviously h is uniformly continuous on $(0,\infty)$ while $h_n \not\rightrightarrows_\alpha h$. Hence $h_n \not\rightrightarrows h$.

Since the uniform limit of uniformly continuous functions is uniformly continuous, we have the following result from the Theorem 5.7.

Proposition 5.9. *If $h_n \in UC(Y^X)$ for all $n \in \mathbb{N}$, and $h_n \rightrightarrows h$ then $h_n \rightrightarrows_\alpha h$.*

Remark 5.10. The condition of uniform continuity in Proposition 5.9 can not be alleviated. Let $h_n : \mathbb{R} \to \mathbb{R}$, $h_n(t) = t^2$. It is straightforward that (h_n) is point-wise equicontinuous and uniformly convergent to the function $h(t) = t^2$ but it lacks of uniform alpha convergence from Proposition 4.4 because limit function is not uniformly continuous.

Let us give a lemma that we will use in the proof of the next theorem without proof, since it is very clear.

Lemma 5.11. *Let (t_n) and (w_n) are Cauchy sequences in X with property $d(t_n, w_n) \to 0$. Then the sequence $(z_n) = \{t_1, w_1, t_2, w_2, \cdots, t_n, w_n, \cdots\}$ is a Cauchy sequence.*

Theorem 5.12. *$h_n \rightrightarrows h$ and $h \in CC(Y^X)$ then $h_n \to_{c\alpha} h$.*

Proof. Let (t_n) and (w_n) be arbitrary Cauchy sequences and $\varepsilon > 0$ be given. Since $h_n \rightrightarrows h$, there exists $n_1 = n_1(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n_1$ and for all $t \in X$, we have $\rho\left(h_n(t), h(t)\right) < \frac{\varepsilon}{2}$. By Cauchy continuity of h and Lemma 5.11 there exists $n_2 = n_2(\varepsilon) \in \mathbb{N}$ such that for all $n, m \geq n_2$, we have $\rho\left(h(z_n), h(z_m)\right) < \frac{\varepsilon}{2}$. Hence we have

$$\rho\left(h_n(t_n), h(w_n)\right) \leq \rho\left(h_n(t_n), h(t_n)\right) + \rho\left(h(t_n), h(w_n)\right) < \varepsilon$$

for all $n \geq \max\{n_1, n_2\}$. □

Since the uniform limit of Cauchy continuous functions is Cauchy continuous, we have the following result from the Theorem 5.12.

Corollary 5.13. *If $h_n \in CC(Y^X)$ for all $n \in \mathbb{N}$ and $h_n \rightrightarrows h$ then $h_n \to_{c\alpha} h$.*

By imposing additional conditions such as compactness and totally boundedness on the domains of the function sequences, it can be shown that the converse of the above-obtained propositions is true.

Theorem 5.14. *Let X be a totally bounded set. If $h_n \to_{c\alpha} h$ then $h_n \rightrightarrows_\alpha h$, that is $\mathfrak{c}^{c\alpha}_{\mathfrak{sf}} = \mathfrak{c}^{u\alpha}_{\mathfrak{sf}}$.*

Proof. Let $(h_n) \in \mathfrak{c}^{c\alpha}_{\mathfrak{sf}}$. Assume that $(h_n) \notin \mathfrak{c}^{u\alpha}_{\mathfrak{sf}}$ i.e. there exist sequences (t_n) and (w_n) with $d(t_n, w_n) \to 0$ such that $\lim_{n\to\infty} \rho\,(h_n(t_n), h(w_n)) \neq 0$. Then, there exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}$ there exists $n_k \geq k$ such that $\rho\,(h_{n_k}(t_{n_k}), h(w_{n_k})) \geq \varepsilon$. Without loss of generality, we can assume that (n_k) is strictly increasing. From the totally boundedness of the set X ensures the existence of Cauchy subsequences $(t_{n'_k})$ and $(w_{n'_k})$ of the sequences (t_{n_k}) and (w_{n_k}) [15]. Since $d(t_{n'_k}, w_{n'_k}) \to 0$, from Lemma 3.3 we have $\rho\left(h_{n'_k}(t_{n'_k}), h(w_{n'_k})\right) \to 0$ when $k \to \infty$ which contradicts with the assumption. Consequently, the equality $\mathfrak{c}^{c\alpha}_{\mathfrak{sf}} = \mathfrak{c}^{u\alpha}_{\mathfrak{sf}}$ holds by Proposition 5.1. □

It is not sufficient for $\mathfrak{c}^{\alpha}_{\mathfrak{sf}} = \mathfrak{c}^{u\alpha}_{\mathfrak{sf}}$ equality to have the set X be totally bounded. Let us examine the example below.

Example 5.15. Consider the example in Remark 5.4 (ii), $h_n : (0,1) \to \mathbb{R}$, $h_n(t) = 1 - nt$ if $0 < t < 1/n$ and $h_n(t) = 0$ otherwise ($n \in \mathbb{N}$). Although $(h_n) \in \mathfrak{c}^{\alpha}_{\mathfrak{sf}}$, $(h_n) \notin \mathfrak{c}^{u}_{\mathfrak{sf}}$, hence $(h_n) \notin \mathfrak{c}^{u\alpha}_{\mathfrak{sf}}$.

By using Lemma 2.2 (i), (iii) and Theorem 5.7 with the fact contiunity of a function is uniform on compact sets we have the following propositions.

Proposition 5.16. *(i) Let X be a compact set. If $h_n \to_\alpha h$ then $h_n \rightrightarrows_\alpha h$, that is $\mathfrak{c}^{\alpha}_{\mathfrak{sf}} = \mathfrak{c}^{c\alpha}_{\mathfrak{sf}} = \mathfrak{c}^{u\alpha}_{\mathfrak{sf}}$.*
(ii) Let X be a compact set then $\mathfrak{c}^{\alpha}_{\mathfrak{scf}} = \mathfrak{c}^{c\alpha}_{\mathfrak{scf}} = \mathfrak{c}^{u}_{\mathfrak{scf}} = \mathfrak{c}^{u\alpha}_{\mathfrak{scf}}$.

An example of why the condition that the terms of sequences must be continuous functions should not be removed from the Proposition 5.16 (ii) is given below.

Example 5.17. Let $X = \mathbb{R}$ or $X = [0,1]$. $h_n : X \to \mathbb{R}$ and $h_n(t) = 0$ if $0 < t < 1/2$ and $h_n(t) = 1$ otherwise ($n \in \mathbb{N}$). Then $(h_n) \in \mathfrak{c}^{u}_{\mathfrak{sf}}$ but $(h_n) \notin \mathfrak{c}^{\alpha}_{\mathfrak{sf}}$.

In the following, figures illustrating relationships between different types of convergence based on the structure of the set X and the functions h_n are given.

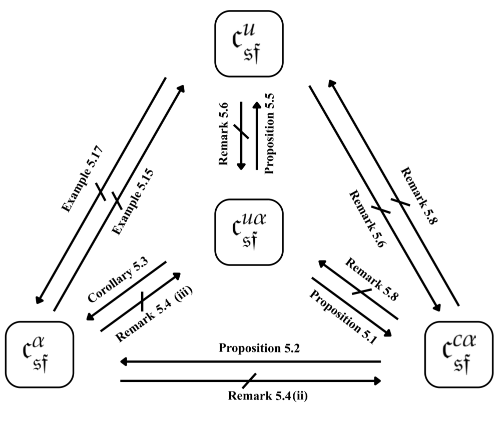

Figure 1: $(h_n) \in \mathfrak{sf}$

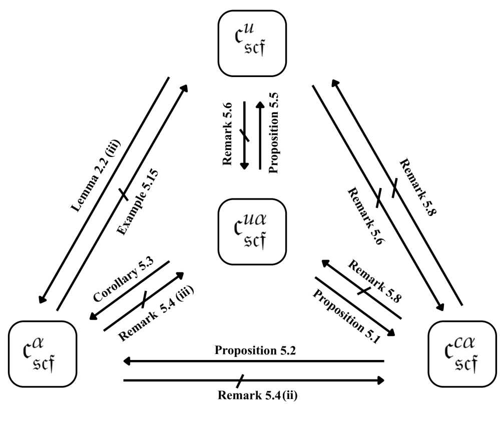

Figure 2: $(h_n) \in \mathfrak{scf}$

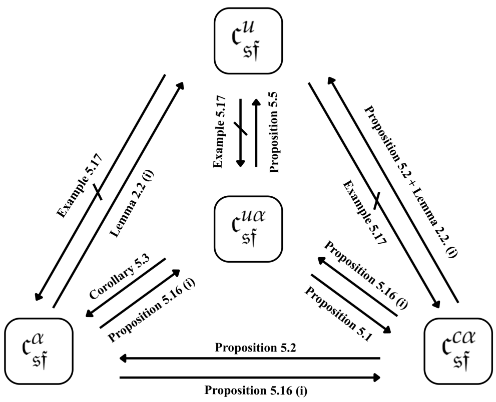

Figure 3: X compact and $(h_n) \in \mathfrak{sf}$

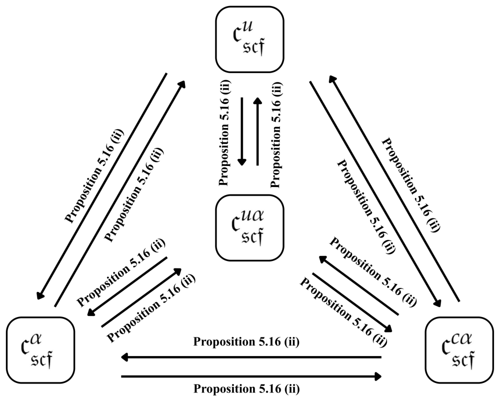

Figure 4: X compact and $(h_n) \in \mathfrak{scf}$

6. Korovkin-type theorems

Let $C(X)$ and $UC(X)$ denote the spaces of continuous and uniformly continuous real-valued functions, respectively, defined on a metric space X. Let $B(X)$ denote the space of bounded functions defined on X. We will be concerned with positive and linear operators defined on these spaces. The positivity of an operator L is understood as the condition that for every positive function f, the function $L(f)$ is also positive. Let $e_k(t) = t^k$, where $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $t \in \mathbb{R}$. For $X = [a, b]$, we present the Korovkin's Theorem to discuss an approximation property of positive and linear operator sequences on $C(X)$:

Theorem 6.1. [14] *Let $X = [a, b]$ and (L_n) be a sequence of positive linear operators on $C(X)$. If $L_n(e_k) \rightrightarrows e_k$ for $k = 0, 1, 2$ then $L_n(h) \rightrightarrows h$ for all $h \in C(X)$.*

For $t, x \in X$ and $r > 0$, let $d_r^t(x) = [d(x, t)]^r$. We now state the Korovkin-type theorem for uniform alpha convergence for metric spaces with using similar method in [1].

Theorem 6.2. *Let X be a totally bounded set, and let (L_n) be a sequence of positive linear operators on $C(X)$. If the conditions*

$$(i) \quad L_n(e_0) \rightrightarrows_\alpha e_0 \qquad (ii) \quad L_n(d_r^t) \rightrightarrows_\alpha 0$$

hold, then $L_n(h) \rightrightarrows_\alpha h$ for every $h \in UC(X)$.

Proof. Let $h \in UC(X)$ and $\epsilon > 0$ be given. Since h is uniformly continuous, there exists a $\delta = \delta(\epsilon) > 0$ such that for every $t, x \in X$ satisfying $d(t, x) < \delta$, we have

$$|h(t) - h(x)| < \frac{\epsilon}{12}.$$

Since h is uniformly continuous on the totally bounded set X, then $h(X) \subset \mathbb{R}$ is totally bounded, so is bounded. Let $M := \sup_{t \in X} |h(t)|$. Moreover, we have the inequality

$$|h(t) - h(x)| \le 2M \le \frac{2M}{\delta^r} d_r^t(x).$$

for $d(t, x) \ge \delta$ and for $r > 0$. Thus, for every $t, x \in X$

$$|h(t) - h(x)| \le \frac{\epsilon}{12} + \frac{2M}{\delta^r} d_r^t. \tag{1}$$

holds. Since $L_n(e_0) \rightrightarrows_\alpha e_0$, we have $L_n(e_0) \rightrightarrows e_0$, and there exists $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that for every $n \geq n_0$ and for all $x \in X$,

$$|L_n(e_0; x) - e_0(x)| < \min\left\{1, \frac{\epsilon}{4M}\right\}.$$

Similarly, since $L_n(d_r^t) \rightrightarrows_\alpha 0$, we have $L_n(d_r^t) \rightrightarrows 0$, and there exists $n_1 = n_1(\epsilon) \in \mathbb{N}$ such that for every $n \geq n_1$ and for all $x \in X$,

$$L_n(d_r^t; x) < \frac{\epsilon}{24\delta^r M}.$$

By using the linearity and positivity properties with the inequality (1), we obtain

$$L_n(|h - h(x)|; x) \leq \frac{\epsilon}{12} L_n(e_0; x) + \frac{2M}{\delta^r} L_n(d_r^t; x).$$

Thus, for every $t, x \in X$ with $d(t, x) < \delta$ and $n \geq \max\{n_0, n_1\}$, we have

$$\begin{aligned}
|L_n(h; x) - h(t)| &\leq |L_n(h; x) - h(x)| + |h(x) - h(t)| \\
&\leq |L_n(h; x) - L_n(h(x); x)| + |L_n(h(x); x) - h(x)| + |h(x) - h(t)| \\
&\leq L_n(|h - h(x)|; x) + |h(x)| \cdot |L_n(e_0; x) - e_0(x)| + |h(x) - h(t)| \\
&\leq \tfrac{\epsilon}{12}|L_n(e_0; x) - e_0(x)| + \tfrac{2M}{\delta^r} L_n(d_r^t; x) + M|L_n(e_0; x) - e_0(x)| + |h(x) - h(t)| + \tfrac{\epsilon}{12} \\
&< \tfrac{\epsilon}{12} \cdot 1 + \tfrac{2M}{\delta^r} \cdot \tfrac{\epsilon\delta^r}{24M} + M \cdot \tfrac{\epsilon}{4M} + \tfrac{\varepsilon}{2} + \tfrac{\epsilon}{12} = \varepsilon.
\end{aligned}$$

□

Remark 6.3. If $X = [a, b]$ is equipped with the standard metric, the proof of Theorem 6.2 is easily seen with the help of Proposition 5.5, Theorem 6.1 and Theorem 5.7.

Corollary 6.4. *Let X be a totally bounded set, and let (L_n) be a sequence of positive linear operators on $C(X)$. If the conditions*

$$(i) \quad L_n(e_0) \to_{c\alpha} e_0 \qquad (ii) \quad L_n(d_r^t) \to_{c\alpha} 0$$

hold, then $L_n(h) \to_{c\alpha} h$ for every $h \in CC(X)$.

Proof. Since the notion of Cauchy continuity of a function on a totally bounded set is equivalent to the notion of uniform continuity, the proof of the corollary is easily obtained from Theorem 6.2 by considering Theorem 5.14 and Proposition 5.1. □

Example 6.5. Let $X = [0, 1]$ and $B_n(h; x) = \sum_{k=0}^{n} \binom{n}{k} h\left(\frac{k}{n}\right) x^k (1 - x)^{n-k}$. Then for every $h \in UC([0, 1])$, the sequence $B_n(h)$ uniformly alpha converges to h on $[0, 1]$.

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