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Torqued and anti-torqued vector fields on hyperbolic spaces

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Abstract. In this paper, we study the existence of torqued and anti-torqued vector fields on the hyperbolic ambient space $\mathbb{H}^n$. Although there are examples of proper torqued vector fields on open subsets of $\mathbb{H}^n$, we prove that there is not a proper torqued vector field globally defined on $\mathbb{H}^n$. Similarly, we have another non-existence result for anti-torqued vector fields as long as their conformal scalar is a non-constant function satisfying certain conditions. When the conformal scalar is constant, some examples of anti-torqued vector fields are provided.

1. Introduction

Let $(M, \langle , \rangle)$ be a Riemannian manifold, ω a 1-form and f a smooth function on M. Let also $\mathcal{V}$ be a nowhere zero smooth vector field on M. In 1944, Yano [26] introduced a wide class of smooth vector fields known as *torse-forming* vector fields. A torse-forming vector field $\mathcal{V}$ is defined by the property that

$$\nabla^0_X \mathcal{V} = fX + \omega(X)\mathcal{V}, \quad \forall X \in \mathfrak{X}(M), \tag{1}$$

where ∇^0 is the Levi-Civita connection on $(M, \langle , \rangle)$. We call the 1-form ω and the function f the *generating form* and *conformal scalar* (or *potential function*) of $\mathcal{V}$, respectively (see [3, 22]). Denote by $\mathcal{W}$ the dual vector field to ω on M, i.e. $\langle \mathcal{W}, X\rangle = \omega(X)$, for every $X \in \mathfrak{X}(M)$. We call $\mathcal{W}$ the *generative* of $\mathcal{V}$.

An immediate class among such vector fields appear when ω is particularly assumed to be a zero 1-form on M, which we call *concircular* vector fields [24, 25]. This class has remarkable applications in Physics [5, 14, 18–21]. Similarly, we have the class of *recurrent* vector fields when f is identically zero [9].

Another class of torse-forming vector fields is the *proper* ones, i.e., the generating form and conformal scalar are nowhere zero. More clearly, this condition means that there is not an open subset of M such that ω and f are identically zero. There are non-trivial examples of such vector fields in various ambient

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spaces. For every rotational hypersurface in the Euclidean n-dimensional space $\mathbb{R}^n$ with the axis passing through the origin, the tangential part of the position vector field Φ is a proper torse-forming vector field [8]. Another non-trivial example is on the hypersphere $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$; if $(x_1, ..., x_n)$ are the canonical coordinates of $\mathbb{R}^n$, then $\mathcal{V} = e^{-x_1}(\partial_1 - x_1\Phi)$, with $\partial_1 := \partial/\partial x_1$, is a proper torse-forming vector field on $\mathbb{S}^{n-1}$ [15].

In this paper, we are interested in two particular types of proper torse-forming vector fields, called torqued and anti-torqued vector fields. In principle, these arise by emposing additional conditions on the generative $\mathcal{W}$. If $\mathcal{V}$ is a proper torse-forming vector field then these two types are defined based on whether $\mathcal{W}$ is perpendicular or parallel to $\mathcal{V}$.

More explicitly, Chen [6] defined a *torqued* vector field as a torse-forming vector field $\mathcal{V}$ with $\langle \mathcal{V}, \mathcal{W} \rangle = 0$ on M. It was proved that a twisted product of an open interval and a Riemannian (or Lorentzian) manifold admits a torqued vector field (see [6]). Such vector fields were used to characterize Ricci solitons [7]. On Einstein manifolds, every torqued vector field is proportional to a concircular vector field $\mathcal{T}$ by a smooth function λ with $\mathcal{T}\lambda = 0$ [7]. In addition, Deshmukh et al. proved that there is not a torqued vector field globally defined on the two standard simply-connected models of Riemannian space forms, the sphere $\mathbb{S}^n$ and the Euclidean space $\mathbb{R}^n$ [12]. In contrast, an example of such a vector field can be established on an open subset of $\mathbb{R}^n$ with nowhere vanishing conformal scalar [12]. Yoldas et al. [27] showed that a characteristic vector field of a Kenmotsu manifold cannot be torqued. Submanifolds of Kenmotsu manifolds were also characterized by using the torqued vector fields [27].

The second type of proper torse-forming vector fields that we focus on is the *anti-torqued* vector fields introduced by Deshmukh et al. [11] (see also [4]). These are proper torse-forming vector fields with $\mathcal{W} = -f\mathcal{V}$ on M, which is contrary to the case of torqued vector fields with $\mathcal{W} \perp \mathcal{V}$ on M. Denote by ν the dual 1-form to $\mathcal{V}$. Then, by (1), an anti-torqued vector field $\mathcal{V}$ fulfills

$$\nabla^0_X \mathcal{V} = f(X - \nu(X)\mathcal{V}), \quad \forall X \in \mathfrak{X}(M). \tag{2}$$

Since an anti-torqued vector field $\mathcal{V}$ is nowhere zero on M, its dual ν is also nowhere vanishing. Therefore, $\mathcal{V}$ is always proper as long as it is not parallel. In contrast, there are non-proper torqued vector fields that are not parallel, e.g. concircular vector fields. In addition, there are anti-torqued vector fields $\mathcal{V}$, where $\mathcal{V} = \nabla^0 g$, for some smooth function g on M, which is not valid for the torqued vector fields. Anti-torqued vector fields are also known as *concurrent-recurrent* vector fields when f is particularly assumed to be a nonzero constant [23]. The necessary and sufficient condition for a Riemannian manifold to carry an anti-torqued vector field is that it is locally a warped product $I \times_\lambda F$, where I is an open interval and F a Riemannian manifold, as one can be seen in [2, Theorem 3.1] and [23, Theorem 3].

Assuming the existence of a vector field nowhere zero on Riemannian manifolds governs their geometry and topology (see [2, 10–13, 15]). Motivated by this, first we study the problem of determining whether a torqued vector field is globally defined on the third standard model $\mathbb{H}^n$ of Riemannian space forms. The same problem is also considered for the anti-torqued case.

This paper is organized as follows. After giving the basic concepts in Section 2 related to Riemannian (sub)manifolds, we provide non-trivial examples of torqued and anti-torqued vector fields in Section 3. Moreover, we have a non-existence result for proper torqued vector fields on Riemannian manifolds as long as they are of constant length (Proposition 3.2). In contrast, there exist other torse-forming vector fields of constant length on Riemannian manifolds (Propositions 3.5 and 3.6). In Section 4, we prove that a proper torqued vector field with a nowhere zero conformal scalar cannot be globally defined on $\mathbb{H}^n$ (Theorem 4.1). Similarly, when the conformal scalar is a non-constant function satisfying certain conditions, there is not an anti-torqued vector field globally defined on $\mathbb{H}^n$ (Theorem 4.2).

2. Preliminaries

Let $(M, \langle , \rangle)$ be a Riemannian manifold of dimension n, and let ∇^0 the Levi-Civita connection of M. The Riemannian curvature tensor is defined by

$$R(X, Y)Z = \nabla^0_X \nabla^0_Y Z - \nabla^0_Y \nabla^0_X Z - \nabla^0_{[X,Y]} Z, \quad X, Y, Z \in \mathfrak{X}(M),$$

where $[,]$ is the bracket operation. Let Γ be a plane section of T_pM with a given basis $\{x, y\}$. The sectional curvature $K(\Gamma)$ of Γ is defined by

$$K(x,y) = \frac{\langle R(x,y)y, x\rangle}{\langle x,x\rangle\langle y,y\rangle - \langle x,y\rangle^2}.$$

A Riemannian manifold $\mathbb{M}^n(c)$ is called a space form if K is a constant c for every plane section at every point. The standard simply-connected models of a Riemannian space form $\mathbb{M}^n(c)$ are the Euclidean space $\mathbb{R}^n = \mathbb{M}^n(0)$, the sphere $\mathbb{S}^n = \mathbb{M}^n(1)$ and the hyperbolic space $\mathbb{H}^n = \mathbb{M}^n(-1)$.

There are several models for the hyperbolic ambient space, two of them are utilized in this paper. Let $(x_1, ..., x_n)$ be the canonical coordinates of $\mathbb{R}^n$. The first one is the upper half space model $\mathbb{H}^n = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_n > 0\}$, which is endowed with the metric

$$\langle , \rangle = \frac{1}{x_n^2}\sum_{i=1}^{n} dx_i^2.$$

Let $\mathbb{R}_1^{n+1}$ be the Lorentz-Minkowski space endowed with the canocial Lorentzian metric

$$\langle , \rangle = -dx_1^2 + \sum_{i=2}^{n+1} dx_i^2.$$

Then, the second model that we concentrate on is the hyperboloid model

$$\mathbb{H}^n = \{P = (x_1, ..., x_{n+1}) \in \mathbb{R}_1^{n+1} : \langle P, P\rangle = -1\}.$$

Let N be a submanifold in a Riemannian manifold M. The Gauss formula is given by

$$\nabla_X^0 Y = \nabla_X Y + h(X,Y), \quad X, Y \in \mathfrak{X}(N),$$

where ∇ is the induced connection on N and h is the second fundamental form.

3. Some Examples and Results on Torqued and Anti-Torqued Vector Fields

Let $\mathcal{V}$ be a torqued vector field on a Riemannian manifold M, i.e.

$$\nabla_X^0 \mathcal{V} = fX + \omega(X)\mathcal{V}, \quad \omega(\mathcal{V}) = 0,$$

for every $X \in \mathfrak{X}(M)$. Assume that M is a twisted product $I \times_\lambda F$, where I is an open interval and F a Riemannian manifold, and μ is a smooth function on F. From [7], it is known that

$$\mathcal{V} = \lambda\mu\frac{\partial}{\partial s}, \quad s \in I,$$

is always a torqued vector field tangent to I, where s is the arc-length parameter of I.

If the ambient space M is a sphere or a Euclidean space, then such vector fields cannot be defined on M globally [11, 12]. However, there are examples of proper torqued vector fields on open subsets of Euclidean space (see [12, Example 1]) as well as of hyperbolic ambient spaces.

Example 3.1. *We consider the hyperboloid model of $\mathbb{H}^n$. If $\Phi : \mathbb{H}^n \to \mathbb{R}_1^{n+1}$ is the standard isometric immersion, then by the Gauss formula we write*

$$\nabla_X^0 Y = \nabla_X Y + \langle X, Y\rangle\Phi, \quad X, Y \in \mathfrak{X}(\mathbb{H}^n),$$

where ∇^0 stands for the Levi-Civita connection of $\mathbb{R}^{n+1}_1$ and ∇ the induced one on $\mathbb{H}^n$. Set $\mathcal{T} = P_0 + \langle P_0, \Phi\rangle\Phi$, where $P_0 \neq 0$ is a fixed vector field in $\mathbb{R}^{n+1}_1$. From [16], it follows that

$$\nabla_X \mathcal{T} = \langle P_0, \Phi\rangle X, \quad \forall X \in \mathfrak{X}(\mathbb{H}^n). \tag{3}$$

Next, we consider the function

$$g : \widetilde{M} \subset \mathbb{H}^n \to \mathbb{R}, \quad (x_1, ..., x_{n+1}) \mapsto g(x_1, ..., x_{n+1}) = \frac{x_1}{x_{n+1}},$$

where

$$\widetilde{M} = \{(x_1, ..., x_{n+1}) \in \mathbb{H}^n : x_1 x_2 x_{n+1} \neq 0, x_1 \neq x_{n+1}\}.$$

We set $P_0 = \partial_2$ and $\mathcal{V} = e^g \mathcal{T}$. From (3)*, it follows that*

$$\nabla_X \mathcal{V} = e^g \langle P_0, \Phi\rangle X + X(g)\mathcal{V}, \quad \forall X \in \mathfrak{X}(\widetilde{M}).$$

Defining $f := e^g\langle P_0, \Phi\rangle$ and $\omega := dg$, the above expression becomes

$$\nabla_X \mathcal{V} = fX + \omega(X)\mathcal{V}, \quad \forall X \in \mathfrak{X}(\widetilde{M}).$$

Because $\mathcal{V}(g) = 0$, we conclude $\omega(\mathcal{V}) = 0$, which implies that $\mathcal{V}$ is a torqued vector field on $\widetilde{M}$. Next, we show that $\mathcal{V}$ is a proper torqued vector field on $\widetilde{M}$. The conformal scalar of $\mathcal{V}$ is given by

$$f(x_1, ..., x_{n+1}) = x_2 e^{x_1/x_{n+1}},$$

which is clearly nowhere zero on $\widetilde{M}$. Moreover, the generating form of $\mathcal{V}$ is

$$\omega = \frac{1}{x_{n+1}} dx_1 - \frac{x_1}{x_{n+1}^2} dx_{n+1} \neq 0 \text{ on } \widetilde{M}.$$

Consequently, $\mathcal{V}$ is proper torqued vector field on $\widetilde{M}$.

On any Riemannian manifold, we have a non-existence result for a proper torqued vector field when it is of constant length. See also [1, Theorem 3].

Proposition 3.2. *A proper torqued vector field on a Riemannian manifold is never of constant length.*

Proof. Let $\mathcal{V}$ be a torqued vector field on a Riemannian manifold M, i.e.

$$\nabla^0_X \mathcal{V} = fX + \omega(X)\mathcal{V}, \quad \omega(\mathcal{V}) = 0,$$

for every $X \in \mathfrak{X}(M)$. By contradiction, assume that

$$0 = X(|\mathcal{V}|) = f|\mathcal{V}|^{-1}\langle X, \mathcal{V}\rangle + |\mathcal{V}|\langle X, \mathcal{W}\rangle, \quad \forall X \in \mathfrak{X}(M),$$

where $\mathcal{W}$ is generative of $\mathcal{V}$. Writing $X = \mathcal{V}$ and $X = \mathcal{W}$, we see that f and ω are identically zero, which is not possible. □

Let $\mathcal{V}$ be an anti-torqued vector field on a Riemannian manifold M. If M is a Euclidean space, then there are examples of $\mathcal{V}$ globally defined on M but not on a sphere (see [11]). Next we give an example of $\mathcal{V}$ when M is the hyperbolic space $\mathbb{H}^n$.

Example 3.3. *In the upper-half space model of $\mathbb{H}^n$, an orthonormal basis $\{e_1, ..., e_n\}$ can be chosen as*

$$e_j = x_n\partial_j, \quad e_n = -x_n\partial_n, \quad j = 1, ..., n-1.$$

The Levi-Civita connection gives, for $i = 1, ..., n$ and $j = 1, ..., n-1$,

$$\nabla^0_{e_j} e_n = e_j, \quad \nabla^0_{e_n} e_n = 0, \quad \nabla^0_{e_i} e_j = 0 \ (i \neq j), \quad \nabla^0_{e_j} e_j = -e_n.$$

It is easy to see that e_n is an anti-torqued vector field, namely

$$\nabla^0_X e_n = X - \langle X, e_n \rangle e_n, \quad \forall X \in \mathfrak{X}(\mathbb{H}^n),$$

where the conformal scalar is identically 1.

Similarly to Example 3.1, on open subsets of $\mathbb{H}^n$, we also may have other examples of anti-torqued vector fields.

Example 3.4. *As in Example 3.1, we consider the hyperboloid model of $\mathbb{H}^n$. We introduce*

$$g : \widetilde{M} \subset \mathbb{H}^n \subset \mathbb{R}^{n+1}_1 \to \mathbb{R}, \quad (x_1, ..., x_{n+1}) \mapsto g(x_1, ..., x_{n+1}) = \frac{1}{x_{n+1}},$$

where

$$\widetilde{M} = \{(x_1, ..., x_{n+1}) \in \mathbb{H}^n : x_{n+1} \neq 0\}.$$

We set $P_0 = \partial_{n+1}$, $\mathcal{T} = P_0 + \langle P_0, \Phi \rangle \Phi = \partial_{n+1} + x_{n+1}\Phi$, and $\mathcal{V} = g\mathcal{T}$, where Φ is the position vector field of $\mathbb{R}^{n+1}_1$. We will show that $\mathcal{V}$ is an anti-torqued vector field on $\widetilde{M}$. Let ∇ be the induced Levi-Civita connection on $\mathbb{H}^n$ from $\mathbb{R}^{n+1}_1$. Since we know that

$$\nabla_X \mathcal{T} = \langle P_0, \Phi \rangle X, \quad \forall X \in \mathfrak{X}(\mathbb{H}^n),$$

the following expression occurs

$$\nabla_X \mathcal{V} = \frac{1}{x_{n+1}} \langle P_0, \Phi \rangle X + X\left(\frac{1}{x_{n+1}}\right)\mathcal{T}, \quad \forall X \in \mathfrak{X}(\widetilde{M}),$$

or equivalently

$$\nabla_X \mathcal{V} = X + X\left(\frac{1}{x_{n+1}}\right)\mathcal{T}, \quad \forall X \in \mathfrak{X}(\widetilde{M}). \tag{4}$$

A direct calculation gives

$$X\left(\frac{1}{x_{n+1}}\right) = -\frac{1}{x^2_{n+1}} \langle P_0, X \rangle.$$

Since $\langle \Phi, X \rangle = 0$ for every $X \in \mathfrak{X}(\mathbb{H}^n)$, we have

$$\langle P_0, X \rangle = \langle P_0 + \langle P_0, \Phi \rangle \Phi, X \rangle = \langle \mathcal{T}, X \rangle, \quad \forall X \in \mathfrak{X}(\mathbb{H}^n).$$

Writing in Equation (4)*, we obtain*

$$\nabla_X \mathcal{V} = X - \left\langle \frac{1}{x_{n+1}}\mathcal{T}, X \right\rangle \left(\frac{1}{x_{n+1}}\mathcal{T}\right), \quad X \in \mathfrak{X}(\widetilde{M})$$

or equivalently

$$\nabla_X \mathcal{V} = X - \langle \mathcal{V}, X \rangle \mathcal{V}, \quad \forall X \in \mathfrak{X}(\widetilde{M}),$$

which implies that $\mathcal{V}$ is an anti-torqued vector field on $\widetilde{M} \subset \mathbb{H}^n$ with constant conformal scalar 1.

In contrast to Proposition 3.2, there are examples of anti-torqued vector fields of constant length on a Riemannian manifold M as in Example 3.3. Moreover, if an anti-torqued vector field $\mathcal{V}$ on M is of constant length, then it must be a unit geodesic vector field. Recall that a smooth vector field $\mathcal{T}$ on a Riemannian manifold is called a *unit geodesic* vector field if $\nabla^0_{\mathcal{T}}\mathcal{T} = 0$. Hence, if $\mathcal{V}$ is a non-parallel anti-torqued vector field whose length is a nonzero constant, then we have

$$0 = X(|\mathcal{V}|) = f|\mathcal{V}|^{-1}(1 - |\mathcal{V}|^2)\langle X, \mathcal{V}\rangle, \quad \forall X \in \mathfrak{X}(M),$$

which implies $|\mathcal{V}| = 1$. It then follows from (2) that $\nabla^0_{\mathcal{V}}\mathcal{V} = 0$.

Therefore, we have proved

Proposition 3.5. *An anti-torqued vector field of constant length on a Riemannian manifold is always a unit geodesic vector field.*

Notice that Proposition 3.5 is not valid for any proper torqued vector field $\mathcal{V}$ due to Proposition 3.2. Moreover, otherwise it follows from (1) that

$$0 = \nabla^0_{\mathcal{V}}\mathcal{V} = f\mathcal{V} + \omega(\mathcal{V})\mathcal{V},$$

or equivalently $f\mathcal{V} = 0$, which is not possible. On the contrary, every proper torqued vector field is always a non-trivial geodesic vector field whose potential function is the conformal scalar f of $\mathcal{V}$ (see [13]).

A final conclusion regarding the torse-forming vector fields of constant length is the following.

Proposition 3.6. *If a proper torse-forming vector field $\mathcal{V}$ on a Riemannian manifold is of constant length, then it is parallel to the generative. Particularly, every unit torse-forming vector field is an anti-torqued vector field.*

Proof. Let $\mathcal{V}$ be a torse-forming vector field on a Riemannian manifold M of constant length $|\mathcal{V}| > 0$, and let μ be the dual of $\mathcal{V}$. Using (1), we have

$$0 = \langle \nabla^0_X\mathcal{V}, \mathcal{V}\rangle = f\langle X, \mathcal{V}\rangle + |\mathcal{V}|^2\omega(X), \quad \forall X \in \mathfrak{X}(M),$$

or equivalently $\omega = -|\mathcal{V}|^{-2}f\mu$. This means that $\mathcal{V}$ is parallel to its generative. Moreover, if $|\mathcal{V}| = 1$, then it is an anti-torqued vector field. □

4. Non-Existence Results on (Anti-)Torqued Vector Fields on $\mathbb{H}^n$

From [12], we know that there are no proper torqued vector fields globally defined on a sphere and on a Euclidean ambient space. In this section, we establish a similar non-existence result for those vector fields on the other standard complete simply-connected model of Riemannian space forms, more precisely on hyperbolic spaces.

Theorem 4.1. *There is not a proper torqued vector field globally defined on $\mathbb{H}^n$ with a conformal scalar f that is nowhere zero.*

Proof. The proof is by contradiction. Suppose that $\mathcal{V}$ is a proper torqued vector field globally defined on $\mathbb{H}^n$. Then, we have

$$\nabla^0_X\mathcal{V} = fX + \omega(X)\mathcal{V}, \quad \omega(\mathcal{V}) = 0, \quad \forall X \in \mathfrak{X}(M). \tag{5}$$

By assumption, the conformal scalar f is nowhere zero on $\mathbb{H}^n$. Denote by $\mathcal{W}$ the generative of $\mathcal{V}$ which is dual to the generating form ω. Since $\omega(\mathcal{V}) = 0$, we conclude that $\mathcal{V} \perp \mathcal{W}$. The curvature tensor of $\mathbb{H}^n$ is

$$R(X, Y)\mathcal{V} = \langle X, \mathcal{V}\rangle Y - \langle Y, \mathcal{V}\rangle X, \quad \forall X, Y \in \mathfrak{X}(\mathbb{H}^n). \tag{6}$$

From (5), it follows that

$$\begin{aligned} R(X,Y)V &= \nabla^0_X(fY+\omega(Y)\mathcal{V}) - \nabla^0_Y(fX+\omega(X)\mathcal{V}) - f[X,Y] - \omega([X,Y])\mathcal{V} \\ &= \{-Y(f)+f\omega(Y)\}X + \{X(f)-f\omega(X)\}Y + d\omega(X,Y)\mathcal{V}. \end{aligned}$$

By (6) and the above equation, we have

$$\langle X,\mathcal{V}\rangle Y - \langle Y,\mathcal{V}\rangle X = \{-Y(f)+f\omega(Y)\}X + \{X(f)-f\omega(X)\}Y + d\omega(X,Y)\mathcal{V}. \tag{7}$$

Since this identity holds for every $X, Y \in \mathfrak{X}(\mathbb{H}^n)$, so does a linearly independent triple $\{X, Y, \mathcal{V}\}$. Then, it must be $d\omega = 0$. By (7), we conclude

$$\langle X,\mathcal{V}\rangle = X(f) - f\omega(X), \quad \forall X \in \mathfrak{X}(\mathbb{H}^n),$$

which implies

$$\langle X, \mathcal{V} - \nabla f + f\mathcal{W}\rangle = 0, \quad \forall X \in \mathfrak{X}(\mathbb{H}^n),$$

or equivalently

$$\nabla f = \mathcal{V} + f\mathcal{W}.$$

Suppose that $\nabla f(p) = 0$, for some point $p \in \mathfrak{X}(\mathbb{H}^n)$. Since f is nowhere zero, it follows that $f(p) \neq 0$, and hence $\mathcal{V}$ and $\mathcal{W}$ would be linearly dependent at p, which contradicts the fact that $\mathcal{V} \perp \mathcal{W}$. Consequently, we are able to suppose that $\nabla f(p) \neq 0$, for every point $p \in \mathbb{H}^n$. Then, $f : \mathbb{H}^n \to \mathbb{R}$ is a submersion and every level set $\Sigma_x = f^{-1}\{f(p)\}$ is a compact submanifold of dimension $n-1$ (see [17]). Introduce a vector field $\mathcal{T} \in \mathfrak{X}(\mathbb{H}^n)$

$$\mathcal{T}(p) = \frac{\nabla f(p)}{\langle \nabla f(p), \nabla f(p)\rangle}, \quad p \in \mathbb{H}^n.$$

It is direct to observe that $\mathcal{T}(f) = 1$ and so the local one-parameter group of local transformations $\{\varphi_t\}$ of $\mathcal{T}$ satisfies

$$\mathcal{T}(\varphi_t(p)) = f(p) + t, \quad t \in \mathbb{R}. \tag{8}$$

Using escape Lemma (see [17]) and expression (8), we understand that $\mathcal{T}$ is a complete vector field and $\{\varphi_t\}$ is one-parameter group of transformations of $\mathbb{H}^n$. We next define a smooth function

$$g : \mathbb{R} \times \Sigma_x \to \mathbb{H}^n, \quad g(t,q) = \varphi_t(q).$$

Notice that

$$\varphi_{t_1} \circ \varphi_{t_2} = \varphi_{t_1+t_2}, \quad t_1, t_2 \in \mathbb{R}.$$

Then, for every $q \in \mathbb{H}^n$, we can find a parameter $t \in \mathbb{R}$ and a point $\varphi_t(q) = q' \in \Sigma_x$ with $q = \varphi_{-t}(q')$, yielding that $g(-t, q') = q$. Also, if $g(t_1, q_1) = g(t_2, q_2)$, then

$$\varphi_{t_1}(q_1) = \varphi_{t_2}(q_2) \tag{9}$$

and expression (8) gives

$$f(q_1) + t_1 = f(q_2) + t_2.$$

Due to $q_1, q_2 \in \Sigma_x$ and $t_1, t_2 \in \mathbb{R}$, we see that $f(q_1) = f(q_2)$ and $t_1 = t_2$, namely from (9) it follows $q_1 = q_2$. Hence g is a one-to-one and onto mapping with its inverse

$$g^{-1}(q) = (-t, q') = (-t, \varphi_t(q)).$$

Consequently, we observe that $\mathbb{R} \times \Sigma_x$ is diffeomorphic to $\mathbb{H}^n$, implying Σ_x is diffeomorphic to $\mathbb{H}^{n-1}$. This is not possible because Σ_x is a compact subset of $\mathbb{H}^n$. $\square$

Although there is no an anti-torqued vector field globally defined on a sphere (see [11]), there are examples in Euclidean and hyperbolic ambient spaces, as mentioned in Section 3. Moreover, in 3-dimensional setting, if the metric of a Riemannian manifold admitting an anti-torqued vector field with constant conformal scalar is Ricci soliton (or gradient Ricci almost soliton), then it is of constant negative curvature [23]. However, as long as the conformal scalar is a non-constant function satisfying certain conditions, we obtain a non-existence result for globally defined anti-torqued vector fields on $\mathbb{H}^n$.

Theorem 4.2. *There is not an anti-torqued vector field globally defined on $\mathbb{H}^n$ whose conformal scalar is non-constant and nowhere takes values $-1, 0,$ or 1.*

Proof. By contradiction suppose that $\mathcal{V}$ is an anti-torqued vector field on $\mathbb{H}^n$. Using (2), we have

$$R(X,Y)\mathcal{V} = -\{Y(f) + f^2\nu(Y)\}X + \{X(f) + f^2\nu(X)\}Y + \{-X(f)\nu(Y) + Y(f)\nu(X) + d\nu(X,Y)\}\mathcal{V}.$$

By (6) and the above equation, we have

$$\begin{aligned}\langle X,\mathcal{V}\rangle Y - \langle Y,\mathcal{V}\rangle X = &-\{Y(f) + f^2\nu(Y)\}X + \{X(f) + f^2\nu(X)\}Y \\ &+\{-X(f)\nu(Y) + Y(f)\nu(X) + d\nu(X,Y)\}\mathcal{V}.\end{aligned}$$

Assume that $\{X, Y, \mathcal{V}\}$ is a linearly independent set. Then, using $\nu(X) = \langle X,\mathcal{V}\rangle$,

$$\begin{aligned} X(f) + \left(f^2 - 1\right)\nu(X) &= 0, \\ Y(f) + \left(f^2 - 1\right)\nu(Y) &= 0, \\ -X(f)\nu(Y) + Y(f)\nu(X) + d\nu(X,Y) &= 0, \end{aligned}$$

for every $X, Y \in \mathfrak{X}(\mathbb{H}^n)$. Obviously, we have $d\nu = 0$ and

$$\nabla f = (1 - f^2)\mathcal{V}.$$

By the hypothesis, f is non-constant and so we may assume $\nabla f(p) \neq 0$ for every $p \in \mathbb{H}^n$ such that $f(p) \neq \pm 1$. The proof can be completed by following the same way as in that of Theorem. 4.1. □

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