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Nonlinear singular elliptic problem of Schrödinger type involving $p(x)$-Laplacian operator

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Abstract. This paper investigates the existence of weak solution for a class of nonlinear singular elliptic problem of Schrödinger type involving the $p(x)$-Laplacian operator in a bounded domain in $\mathbb{R}^N$. Under certain additional assumptions on the nonlinearities, the corresponding functional satisfies the Palais-Smale condition. Then, by applying the Mountain Pass Theorem, we can demonstrate the existence of weak solution for the considered problem.

1. Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^N (N \geq 2)$, with a Lipschitz boundary denoted by $\partial\Omega$. In this paper, we investigate the existence of weak solution of the following singularity elliptic problem of Schrödinger type

$$\begin{cases} -\Delta_{p(x)} u + \frac{|u|^{s-2}u}{|x|^s} = \lambda V(x)|u|^{q(x)-2}u + \mu f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where

- $\Delta_{p(x)} u = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)$, denotes $p(x)$-Laplacian operator
- $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function
- The potential $V : \Omega \to \mathbb{R}$ is measurable and positive a.e. in Ω

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- p and q are reals functions satisfying $p(x),\ q(x) \in C_+(\overline{\Omega})$
- λ, μ are the reals parameters
- $1 < s < p(x) < \infty$..

Problems with the $p(x)$-Laplacian operator is a significant topic in the theory of partial differential equations and has been studied by many researchers [4–6, 13, 14, 16]. These issues arise from various fields of applied mathematics and physics, such as elastic mechanics, electrorheological fluids, and image restoration.

The solution of elliptic problems with the p(x)-Laplacian without singularity has been studied by several authors (see for example [1, 2, 17, 18]). In [8], the authors studied a type of singular elliptic problems involving the $p(x)$-Laplacian operator, but with subcristal Sobolev-Hardy exponents.

The paper is divided into tree parts. In Section 2, We present basic results for Lebesgue-Sobolev variable exponent spaces and the Mountain Pass Theorem, which we will use to address our problem. In Section 3, we prove the existence of weak solutions for the problem (1), by presenting several lemmas.

2. Definitions and preliminary results

2.1. Generalized Lebesgue-Sobolev spaces

In this section, we present some definitions and properties of Lebesgue-Sobolev spaces with variable exponents, commonly known as generalized Sobolev spaces (see for example [9, 11]). Let

$$C_+(\Omega) = \Big\{p \in C(\Omega) : p(x) > 1, \text{ for every } x \in \Omega\Big\}.$$

For every $p \in C_+(\Omega)$, we define

$$p^+ = \max\Big\{p(x) \in \Omega\Big\} \text{ and } p^- = \min\Big\{p(x) \in \Omega\Big\}.$$

We define the generalized Lebesgue space $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) = \left\{v : \Omega \to \mathbb{R}, \int_\Omega |v(x)|^{p(x)}dx < \infty\right\}$$

equipped with the Luxemburg norm

$$\|v\|_{q(x)} = \inf\Big\{C > 0 : \int_\Omega \Big|\frac{v(x)}{C}\Big|^{q(x)} dx \le 1\Big\}.$$

We define by $L^\infty_+(\Omega)$ the subset of $L^\infty(\Omega)$, of expression:

$$L^\infty_+(\Omega) = \Big\{p \in L^\infty(\Omega) : \text{ inf ess}\, p \ge 1\Big\}.$$

We denote by modular ρ the quantity

$$\rho(v) = \int_\Omega |v|^{p(x)}dx.$$

Proposition 2.1. *[9, 11] If $v \in L^{p(x)}(\Omega)$, then*

1. $\|v\|_{p(x)} < 1 (= 1; > 1)$ *equivalent* $\rho(v) < 1(= 1; > 1)$.
2. *if* $\|v\|_{p(x)} > 1$, *then* $\|v\|^{p^-}_{p(x)} \le \rho(v) \le \|v\|^{p^+}_{p(x)}$.
3. *if* $\|v\|_{p(x)} < 1$, *then* $\|v\|^{p^+}_{p(x)} \le \rho(v) \le \|v\|^{p^-}_{p(x)}$.

Proposition 2.2. *[9] (**Hölder inequality**) Let $u \in L^{p(x)}(\Omega)$, $v \in L^{q(x)}(\Omega)$, $p^- > 1$ et $q \in L^\infty_+(\Omega)$, then*

$$\left|\int_\Omega u(x)v(x)dx\right| \le \left(\frac{1}{p^-}+\frac{1}{q^-}\right)\|u\|_{p(x)}\|v\|_{q(x)} \le 2\|u\|_{p(x)}\|v\|_{q(x)}. \tag{2}$$

Definition 2.3. *[12] Let X be a real Banach space and $1 < s < N$, we recall the classical Hardy's inequality, which says that*

$$\int_\Omega \frac{|v(x)|^s}{|x|^s}dx \le \frac{1}{H}\int_\Omega |\nabla v(x)|^s dx, \quad \text{for all } v \in X, \tag{3}$$

where $H := \left(\frac{n-s}{s}\right)^s$.

Remark 2.4. *Subject to conditions $1 \le p^- \le p^+ < +\infty$, we have*

$$(a+b)^{p(x)} \le 2^{p^+-1}\left(a^{p(x)} + b^{p(x)}\right).$$

The space $W^{1,p(x)}$ is defined by

$$W^{1,p(x)}(\Omega) = \left\{v \in L^{p(x)}(\Omega), \frac{\partial v}{\partial x_j} \in L^{p(x)}(\Omega), \quad j = 1, \dots, N\right\}$$

equipped with the norm

$$\|v\| = \|v\|_{p(x)} + \|\nabla v\|_{p(x)}.$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Proposition 2.5. *[11] $W^{1,p(x)}(\Omega)$ is a reflexive and separable Banach.*

Theorem 2.6. *[3] For $p, q \in L^\infty_+(\Omega), p(x) \le q(x)$, we have*

$$W^{1,p(x)}(\Omega) \hookrightarrow W^{1,q(x)}(\Omega),$$

with continuous injection.

Let p^* be the Sobolev critical exponent associated with p, of expression

$$p^*(x) = \begin{cases} \frac{Np(x)}{(N-p(x))} & \text{si} \quad p(x) < N, \\ +\infty \text{ if} & p(x) \ge N. \end{cases}$$

Theorem 2.7. *[3] For $p,\ q \in C(\overline{\Omega}),\ p^-,\ q^- \ge 1$, we have*

$$q(x) < p^*(x) \text{ for all } x \in \overline{\Omega}.$$

In addition

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega),$$

with continuous and compact injection.

Theorem 2.8. *[3] If $p, q \in C(\overline{\Omega})$ are such that $1 \le p(x) \le q(x) \le p^*(x)$, for all $x \in \overline{\Omega}$, then*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega),$$

with continuous injection.

Proposition 2.9. *[3, 11] (**Poincaré inequality**) If $p^- > 1$, then there exists a positive constant C such that*

$$\|v\|_{p(x)} \leq C\|\nabla v\|_{p(x)}, \ \textit{for all} \ \ v \in W_0^{1,p(x)}(\Omega).$$

In this paper, we will try to find a weak solution to the problem (1) in the following space

$$\mathcal{G} := \left\{\varphi \in W^{1,q(x)}(\Omega) : \ \ \varphi|_{\partial\Omega} = 0\right\}. \tag{4}$$

The space $\mathcal{G}$ is a closed subspace of the separable and reflexive Banach space $W^{1,q(x)}(\Omega)$ (see [9, 11]) , so $\mathcal{G}$ is also separable and reflexive Banach space with the norm $\|v\|_{p(x)} = \|\nabla v\|_{p(x)}$.

2.2. Nemytskii operator and Montain Pass Theorem

We started by defining the Nemytskii operator. Let $v : \Omega \to \mathbb{R}$ and $f \in C(\overline{\Omega} \times \mathbb{R})$.

- The operator N_f defined by
$$\left(N_f v\right)(x) = f(x, v(x))$$
is called the Nemytskii operator relative to f.
- If $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, a Carathéodory function verifying
$$|f(x,s)| \leq a(x) + b|s|^{\frac{p(x)}{q(x)}} \text{ for every } s \in \mathbb{R},$$
with $p,\ q \in C_+(\overline{\Omega}),\ a \in L^{q(x)}(\Omega),\ a(x) \geq 0$ and $b \geq 0$, then the Nemytskii operator N_f of $L^{p(x)}(\Omega)$ in $L^{q(x)}(\Omega)$ is a continuous and bounded operator.

Next, let X be a Banach space.

Definition 2.10. *[10] Let $\varphi : X \to \mathbb{R}$.*

(i) The sequence v_n in X is called a Palais-Smale sequence of level c denoted $(PS)_c$ if

$$\varphi(v_n) \to c \quad \textit{and} \quad \left\|D\varphi(v_n)\right\|_{X^*} \to 0.$$

(ii) We say that the functional φ satisfies the Palais-Smale condition at level c if every sequence of $(PS)_c$ has a convergent sub-sequence.

(iii) If $\varphi \in C^1(X, \mathbb{R})$ satisfies the condition $(\mathrm{PS})_c$, then any accumulation point $\bar{v}$ of a sequence v_n of $(\mathrm{PS})_c$ is a critical point of φ. Implicitly, we have $D\varphi(\bar{v}) = 0$ and $\varphi(\bar{v}) = c$.

Consider the set Γ of all paths connecting the origin to v_0 of X:

$$\Gamma = \left\{\gamma \in C([0,1], X), \quad \gamma(0) = 0, \quad \gamma(1) = v_0\right\},$$

and we put

$$c = \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} \varphi(\gamma(s)).$$

Theorem 2.11. *[10] Assume that $\varphi \in C^1(X, \mathbb{R}),\ \varphi(0) = 0$ and satisfying the three conditions*

1. *There exists $\varrho, b > 0$ such as $\varphi(v) \geq b$ for $\|v\|_X = \varrho$.*
2. *There exists $v_0 \in X$ with $\|v_0\|_X > \varrho$ and such that $\varphi(v_0) \leq 0$.*
3. *φ satisfies the condition of $(PS)_c$.*

Then c is a critical value of φ.

3. Basic assumptions and main result

The present paper will be examined under the following hypotheses:

(M_1) We assume that the functions p, q are continuous and satisfy $p(x) < N$, along with $1 < p^- < p^+ < q^- < q^+ \leq p^*(x)$.In particular, p proves that $|p(x) - p(y)| \leq \frac{c}{|\log|x-y||}$ holds for $|x - y| \leq \frac{1}{2}$ and $x, y \in \mathbb{R}^N$.

(M_2) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function of class $C^1(\Omega \times \mathbb{R}, \mathbb{R})$ such that

$$|f(x, v)| \leq a(x)|v|^{\frac{p(x)}{\alpha(x)}}, \quad \text{for all } (x, v) \in \Omega \times \mathbb{R}.$$

Here, $a \in L^{\alpha(x)}(\Omega)$ is non negative mesurable function with $\frac{1}{\alpha(x)} + \frac{1}{p(x)} = 1$.

(M_3) Suppose that $0 \leq \theta G(x, v) \leq v f(x, v)$, such that $p^+ < \theta < q^-, x \in \Omega$ with $G(x, v) = \int_0^v f(x, t)dt$.

(M_4) The potential $V \in L^\infty(\Omega) \cap L^{\beta(x)}(\Omega)$ is nonnegative and $\frac{1}{\beta(x)} + \frac{1}{q(x)} = 1$.

Now, we introduce the definition of a weak solution for (1).

Definition 3.1. *$v \in \mathcal{G}$ is a weak solution of* (1)*, if for all $u \in \mathcal{G}$*

$$\int_\Omega |\nabla v|^{p(x)-2}\nabla v \nabla u dx + \int_\Omega \frac{|v|^{s-2}vu}{|x|^s}dx - \lambda \int_\Omega V(x)|v|^{q(x)-2}vudx - \mu \int_\Omega f(x, v)udx = 0.$$

The following theorem presents the main result of this paper.

Theorem 3.2. *If the hypotheses (M_1)-(M_4) are satisfied, then the problem* (1) *has a weak solution in $\mathcal{G}$ for all $\lambda, \mu > 0$.*

Befor geving the proof of the main result, we need some results. First we define the energy functional corresponding to problem (1) by

$$\varphi_{\lambda,\mu}(v) = \int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}dx + \int_\Omega \frac{|v|^s u}{s|x|^s}dx - \int_\Omega \lambda \frac{V(x)}{q(x)}|v|^{q(x)}dx - \int_\Omega \mu F(x, v)dx. \tag{5}$$

We put

$$\eta(v) = \int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}dx, \quad \Lambda(v) = \int_\Omega \frac{|v|^s}{s|x|^s}dx,$$

$$\delta(v) = \int_\Omega \frac{V(x)}{q(x)}|v|^{q(x)}dx, \quad \zeta(v) = \int_\Omega \mu F(x, v)dx.$$

Moreover, by Proposition 2.1 we have the following result.

Proposition 3.3. *Let $v \in W_0^{1,p(x)}$ and $\rho_p(v) := \int_\Omega |v(x)|^{p(x)}dx$. Then*

- $\|v\| < 1 (= 1; > 1) \Longleftrightarrow \rho(|\nabla v|) < 1 (= 1; > 1)$
- *If $\|v\| > 1$, then* $\frac{1}{p^+}\|v\|^{p^-} \leq \eta(v) + \Lambda(v) \leq \frac{1}{p^-}\|v\|^{p^+} + \int_\Omega \frac{|v|^s}{s|x|^s}dx$
- *If $\|v\| < 1$, then* $\frac{1}{p^+}\|v\|^{p^+} \leq \eta(v) + \Lambda(v) \leq \frac{1}{p^-}\|v\|^{p^-} + \int_\Omega \frac{|v|^s}{s|x|^s}dx.$

By [**?**] and [7, Theorem 3.1], we have the following lemma.

Lemma 3.4. *The functional $\varphi_{\lambda,\mu}$ is well defined and $C^1(\mathcal{G},\mathbb{R})$. Moreover,*

$$\left\langle \varphi'_{\lambda,\mu}(v),u\right\rangle = \int_\Omega |\nabla v|^{p(x)-2}\nabla v\nabla u dx + \int_\Omega \frac{|v|^{s-2}vu}{|x|^s}dx - \int_\Omega \lambda V(x)|v|^{q(x)-2}vudx - \int_\Omega \mu f(x,v)udx.$$

By (M_2) togheter with (M_4), it is easy to see that $\varphi'_{\lambda,\mu}$ belongs to the topological dual of $\mathcal{G}$.

Lemma 3.5. *There exists positives constants ϱ and b such that $\varphi_{\lambda,\mu}(v)\geq b$ on $\|v\|_{p(x)}=\varrho$.*

Proof. By the Hölder inequality, we get

$$\begin{aligned}\int_\Omega |F(x,v)|dx &\leq \int_\Omega \left|\frac{a(x)}{q(x)}|v|^{q(x)}\right| |\, dx\\ &\leq \frac{2}{q^-}\|a\|_{\alpha(x)}\left||v|^{q(x)}\right|_{p(x)}\\ &\leq \frac{2c_1}{q^-}\|a\|_{\alpha(x)}\|v\|^{q^i}_{p(x)},\end{aligned}$$

and

$$\begin{aligned}\int_\Omega |\frac{V(x)}{q(x)}|v|^{q(x)}|dx &\leq \frac{2}{q^-}\|V\|_{\beta(x)}\left||v|^{q(x)}\right|_{\beta'(x)}\\ &\leq \frac{2}{q^-}\|V\|_{\beta(x)}\|v\|^{q^i}_{q(x)\beta'(x)}\\ &\leq \frac{2c_2}{q^-}\|V\|_{\beta(x)}\|v\|^{q^i}_{p(x)},\end{aligned}$$

where

$$i=\pm \quad \text{if} \quad \|v\|_{p(x)} \gtreqless 1.$$

Using the above and Proposition 3.3, we obtain

$$\begin{aligned}\varphi_{\lambda,\mu}(v) &= \int_\Omega\left(\frac{1}{p(x)}|\nabla v|^{p(x)}+\frac{|v|^s}{s|x|^s}\right)dx - \int_\Omega \lambda\frac{V(x)}{q(x)}|v|^{q(x)}dx - \int_\Omega \mu F(x,v)dx\\ &\geq \frac{1}{p^+}\int_\Omega |\nabla v|^{p(x)}dx - \frac{2\lambda c_2}{q^-}\|V\|_{\beta(x)}\|v\|^{q^i}_{p(x)} - \frac{2\mu c_1}{q^-}\|a\|_{\alpha(x)}\|v\|^{q^i}_{p(x)}\\ &\geq \frac{1}{p^+}\|v\|^{p^i}_{p(x)} - \frac{2\lambda c_2}{q^-}\|V\|_{\beta(x)}\|v\|^{q^i}_{p(x)} - \frac{2\mu c_1}{q^-}\|a\|_{\alpha(x)}\|v\|^{q^i}_{p(x)}\\ &\geq \frac{1}{p^+}\|v\|^{p^i}_{p(x)} - \left(\frac{2\lambda c_2}{q^-}\|V\|_{\beta(x)}+\frac{2\mu c_1}{q^-}\|a\|_{\alpha(x)}\right)\|v\|^{q^i}_{p(x)},\end{aligned}$$

where

$$i=\pm \quad \text{if} \quad \|v\|_{p(x)} \gtreqless 1,$$

and c_1, c_2 are positives constants.
So, for all $\lambda,\mu>0$ and $v\in\mathcal{G}$, with $\|v\|_{p(x)}=\varrho$, sufficiently small, there exists $b>0$ such that

$$\varphi_{\lambda,\mu}(v)\geq b>0.$$

□

Lemma 3.6. *There exists $v_0\in\mathcal{G}$ with $\|v_0\|_{p(x)}>\varrho$ such that $\varphi_{\lambda,\mu}(v_0)<0$.*

Proof. Choose $v_0 \in \mathcal{G}, \|v_0\|_{p(x)} > 1$. For t large enough we have

$$\varphi_{\lambda,\mu}(tv_0) = \int_\Omega \frac{1}{p(x)} |\nabla tv_0|^{p(x)} dx + \int_\Omega \frac{|tv_0|^s}{s|x|^s} dx - \int_\Omega \lambda \frac{V(x)}{q(x)} |tv_0|^{q(x)} dx - \int_\Omega \mu F(x, tv_0) dx,$$

and according to (3) and the Proposition 2.1, We obtain

$$\begin{aligned}
\varphi_{\lambda,\mu}(tv_0) =&\leq \frac{1}{p^-} \int_\Omega |\nabla tv_0|^{p(x)} dx + \int_\Omega \frac{|tv_0|^s}{s|x|^s} dx - \lambda \frac{1}{q^+} \int_\Omega V(x) |tv_0|^{q(x)} dx \\
&\leq \frac{t^{p+}}{p^-} \|v_0\|_{p(x)}^{p^+} + \frac{t^s}{sH} \int_\Omega |\nabla v_0|^s dx - \frac{2\lambda c t^{q^-}}{q^+} \int_\Omega V(x) |v_0|^{q(x)} dx \\
&\leq \frac{t^{p+}}{p^-} \|v_0\|_{p(x)}^{p^+} + \frac{1}{sH} \|v_0\|_{p(x)}^{s} - \frac{2\lambda c t^{q^-}}{q^+} \int_\Omega V(x) |v_0|^{q(x)} dx.
\end{aligned}$$

This yields $\varphi_{\lambda,\mu}(tv_0) \to -\infty$, as $t \to +\infty$ since

$$0 \leq \int_\Omega V(x) |v_0|^{q(x)} dx \leq 2c' \|V\|_{\beta(x)} \|v_0\|_{p(x)}^{q^+}.$$

□

Lemma 3.7. *The functional $\varphi_{\lambda,\mu}$ satisfies the Palais-Smale condition $(PS)_c$, for any $c \in \mathbb{R}$.*

Proof. Let (v_n) be a $(\mathrm{PS})_c$ sequence for the functional $\varphi_{\lambda,\mu}$ in $\mathcal{G}$ i.e. $\varphi_{\lambda,\mu}(v_n)$ is bounded and $\varphi'_{\lambda,\mu}(v_n) \to 0$. Then the sequence v_n is bounded in $\mathcal{G}$.
In fact, since $\varphi_{\lambda,\mu}(v_n)$ is bounded, we have

$$\begin{aligned}
C_1 \geq \varphi_{\lambda,\mu}(v_n) &= \int_\Omega \left(\frac{1}{p(x)} |\nabla v_n|^{p(x)} + \frac{|tv_n|^s}{s|x|^s} \right) dx - \int_\Omega \lambda \frac{V(x)}{q(x)} |v_n|^{q(x)} dx - \int_\Omega \mu F(x, v_n) dx \\
&\geq \int_\Omega \left(\frac{1}{p(x)} |\nabla v_n|^{p(x)} + \frac{|v_n|^s}{s|x|^s} \right) dx - \int_\Omega \lambda \frac{V(x)}{q(x)} |v_n|^{q(x)} dx - \int_\Omega \mu F(x, v_n) dx \\
&\geq \int_\Omega \left(\frac{1}{p(x)} |\nabla v_n|^{p(x)} dx + \frac{|v_n|^s}{s|x|^s} \right) dx - \int_\Omega \lambda \frac{V(x)}{q(x)} |v_n|^{q(x)} dx - \int_\Omega \frac{\mu v_n}{\theta} f(x, v_n) dx.
\end{aligned}$$

In addition

$$\left\langle \varphi'_{\lambda,\mu}(v_n), v_n \right\rangle = \int_\Omega \left(|\nabla v_n|^{p(x)} - \frac{|v_n|^s}{|x|^s} \right) dx - \int_\Omega \lambda V(x) |v_n|^{q(x)} dx - \int_\Omega \mu f(x, v_n) v_n dx,$$

then

$$\begin{aligned}
C_1 \geq& \frac{1}{p^+} \int_\Omega |\nabla v_n|^{p(x)} dx + \int_\Omega \frac{|v_n|^s}{|x|^s} dx - \frac{1}{q^-} \int_\Omega \lambda V(x) |v_n|^{q(x)} dx + \frac{1}{\theta} \left\langle \varphi'_{\lambda,\mu}(v_n), v_n \right\rangle \\
&- \frac{1}{\theta} \int_\Omega |\nabla v_n|^{p(x)} dx - \frac{1}{\theta} \int_\Omega \frac{|v_n|^s}{s|x|^s} dx + \frac{1}{\theta} \int_\Omega \lambda V(x) |v_n|^{q(x)} dx \\
\geq& \left(\frac{1}{p^+} - \frac{1}{\theta} \right) \int_\Omega |\nabla v_n|^{p(x)} dx + \left(1 - \frac{1}{s\theta} \right) \int_\Omega \frac{|v_n|^s}{|x|^s} dx \\
&+ \left(\frac{1}{\theta} - \frac{1}{q^-} \right) \int_\Omega \lambda V(x) |v_n|^{q(x)} dx + \frac{1}{\theta} \left\langle \varphi'_{\lambda,\mu}(v_n), v_n \right\rangle.
\end{aligned}$$

By contradiction, we assume that (v_n) is unbounded in $\mathbb{G}$. In particular, for n to be large enough, we can choose $\|v_n\| \geq 1$. Therefore, there exists $C_3 > 0$ in such a way that

$$-C_3 \|v_n\|_{p(x)} \leq \left\langle \varphi'_{\lambda,\mu}(v_n), v_n \right\rangle \leq C_3 \|v_n\|_{p(x)}$$

because $\varphi'_{\lambda,\mu}(v_n) \to 0$. To that end,

$$C_1 \geq \left(\frac{1}{p^+} - \frac{1}{\theta}\right) \|v_n\|^{p^+}_{p(x)} + \left(\frac{1}{\theta} - \frac{1}{q^-}\right) \int_\Omega \lambda V(x) |v_n|^{q(x)} dx - \frac{1}{\theta} C_3 \|v_n\|_{p(x)}$$
$$\geq \left(\frac{1}{p^+} - \frac{1}{\theta}\right) \|v_n\|^{p^+}_{p(x)} - \frac{1}{\theta} C_3 \|v_n\|_{p(x)}.$$

That implies a contradiction. Hence the sequence (v_n) is bounded in $\mathcal{G}$.

Thus, there exists a subsequence, again denoted by (v_n), which is weakly convergent to v in $\mathcal{G}$. We will prove that (v_n) is strongly convergent to v in $\mathcal{G}$. For this purpose, we have in mind the following equality

$$\left\langle \varphi'_{\lambda,\mu}(v_n) - \varphi'_{\lambda,\mu}(v), v_n - u \right\rangle = \tag{6}$$
$$\langle \eta'(v_n) - \eta'(v), v_n - v\rangle + \langle \Lambda'(v_n) - \Lambda'(v), v_n - v\rangle - \langle \delta'(v_n) - \delta'(v), v_n - v\rangle - \langle \zeta'(v_n) - \zeta'(v), v_n - v\rangle.$$

Obviously, the left-hand term tends to zero for sufficiently large n. The first thing, we show is that $\langle \zeta'(v_n) - \zeta'(v), v_n - v\rangle \to 0$ as $n \to \infty$.

Let B_r be the ball in Ω of radius r centered at the origin and $B'_r = \Omega \setminus B_r$. In unbounded domains, we use the well-known compactness argument. Roughly speaking, we write the following

$$\begin{aligned} |\langle \zeta'(v_n) - \zeta'(v), v_n - v\rangle| &= \left| \int_\Omega (f(x, v_n) - f(x, v))(v_n - v)\, dx \right| \\ &\leq \int_{B_r} \left| f(x, v_n) - f(x, v) \right| |v_n - v|\, dx \\ &\quad + \int_{B'_r} \left| f(x, v_n) - f(x, v) \right| |v_n - v|\, dx. \end{aligned}$$

Considering Theorem 2.8, together with the compact embedding $W^{1,p(x)}(B_r) \hookrightarrow L^{p(x)}(B_r)$, the first term on the right-hand side of the above inequality vanishes as $n \to \infty$. Conversely, the second term disappears as $r \to \infty$. In fact, we have

$$\int_{B_r} \left| f(x, v_n) - f(x, v) \right| |v_n - v|\, dx \leq 2 \left| f(x, v_n) - f(x, v) \right|_{\alpha(x)} |v_n - v|_{p(x), B_r}.$$

Due to (M_2), the Nemytskii operator is bounded. Hence, we obtain

$$\int_{B_r} \left| f(x, v_n) - f(x, v) \right| |v_n - v|\, dx \leq \frac{\varepsilon}{2}.$$

On the other hand, we have

$$\begin{aligned} \int_{B'_r} \left| f(x, v_n) - f(x, v) \right| |v_n - v|\, dx &\leq \int_{B'_r} a(x) |v_n|^{p(x)} + a(x) |v_n|^{p(x)-1} |v| + a(x)|v|^{p(x)} + a(x)|v|^{p(x)-1} |v_n|\, dx \\ &\leq \frac{\varepsilon}{2}, \end{aligned}$$

for r sufficiently large. Indeed,

$$\int_{B'_r} a(x) |v_n|^{p(x)} dx \leq 2\|a\|_{\alpha(x)} \|v_n|^{p(x)}\Big|_{p(x)} \leq \frac{\varepsilon}{8},$$

for r sufficiently large. Using Young's inequality, we get

$$\begin{aligned} \int_{B'_r} a(x) |v_n|^{p(x)-1} |v| dx &\leq \int_{B'_r} a(x) \left(|v_n|^{p(x)} + |v|^{p(x)} \right) dx \\ &\leq 2\|a\|_{\alpha(x)} \left(|v_n|^{p(x)}\Big|_{p(x)} + \|v|^{p(x)}\Big|_{p(x)} \right) \\ &\leq \frac{\varepsilon}{8}, \end{aligned}$$

for r sufficiently large.
Similarly, according to r, we show that the last two terms are less than $\frac{\varepsilon}{8}$.
Using the same reasoning, the following holds

$$\begin{aligned}\langle \delta'(v_n) - \delta'(v), v_n - v\rangle &\leq \lambda \int_{B_r} \left| V(x)\left(|v_n|^{q(x)-2} v_n - |v|^{q(x)-2} v\right)\right| |v_n - v|\, dx \\ &+ \lambda \int_{B_r'} V(x)\left(|v_n|^{q(x)} + |v|^{q(x)-2} v_n v + |v|^{q(x)} + |v_n|^{q(x)-2} v_n v\right) dx \\ &\leq c_1 \left\| V(x)\left(|v_n|^{q(x)-2} v_n - |v|^{q(x)-2} v\right)\right\|_{\beta(x)} \|v_n - v\|_{q(x)} \\ &+ c_2 \|V(x)\|_{r(x)} \left(|v_n|^{q(x)}\Big|_{q(x)} + |v|^{q(x)}\Big|_{q(x)}\right) \\ &\leq \varepsilon,\end{aligned}$$

and

$$\begin{aligned}\langle \Lambda'(v_n) - \Lambda'(v), v_n - v\rangle &\leq \int_{B_r} \left| \frac{|v_n|^{s-2} v_n - |v|^{s-2} v}{|x^s|}\right| |v_n - v| dx \\ &+ \int_{B_r'} \frac{|v_n|^s + |v|^{s-2} v_n v + |v|^s + |v_n|^{s-2} v_n v}{|x^s|} dx \\ &\leq c_1 \left\| \frac{|v_n|^{s-2} v_n - |v|^{s-2} u}{|x^s|}\right\|_{\beta(x)} \|u_n - u\|_{q(x)} \\ &+ c_2 \left\| \frac{1}{|x^s|}\right\|_{\beta(x)} \left(|v_n|^s\Big|_{q(x)} + |v|^s|_{q(x)}\right) \\ &\leq \varepsilon,\end{aligned}$$

for n, r large enough.

The last step consists in using two elementary inequalities in $\mathbb{R}^N$, namely

$$(p-1)|\nu - \Upsilon|^2(|\nu| + |\Upsilon|)^{p-2} \leq \left(|\nu|^{p-2}\nu - |\Upsilon|^{p-2}\Upsilon\right)(\nu - \Upsilon), \quad p(x) \geq 2.$$

$$(p-1)|\nu - \Upsilon|^2(|\nu| + |\Upsilon|)^{p-2} \leq \left(|\nu|^{p-2}\nu - |\Upsilon|^{p-2}\Upsilon\right)(\nu - \Upsilon), \quad 1 < p(x) < 2.$$

It follows from (6) that $\langle \varphi'(v_n) - \varphi'(v), v_n - v\rangle \to 0$ when $n \to \infty$.

Now if $p(x) \geq 2$, we obtain using the first inequality

$$2^{2-p^+} \int_\Omega |\nabla v_n - \nabla v|^{p(x)}\, dx \leq \int_\Omega \left(|\nabla v_n|^{p(x)-2} \nabla v_n - |\nabla v|^{p(x)-2} \nabla v\right)(\nabla v_n - \nabla v)\, dx \to 0 \text{ when } n \to \infty.$$

On the contrary, if $p(x) \geq 1$ we must consider the two sets

$$U_p = \left\{x \in \Omega, p(x) \geq 2\right\}; \quad V_p = \left\{x \in \Omega, 1 < p(x) < 2\right\},$$

and then apply the first inequality to U_p and the second inequality to V_p. $\square$

Proof of Theorem 3.2. Set

$$\Gamma = \left\{\gamma \in C([0,1], \mathcal{G}) : \gamma(0) = 0, \gamma(1) = v_0\right\}$$
$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi_{\lambda,\mu}(\gamma(t)).$$

According to Lemma 3.5, Lemma 3.6 and Lemma 3.7, the energy functional $\varphi_{\lambda,\mu}$ satisfies the geometric conditions of the Mountain Pass theorem. Hence c is a critical value of $\varphi_{\lambda,\mu}$ associated with a critical point $v \in \mathcal{G}$, which is exactly a solution of (1).

References

[1] Abdelmalek, B., Djellit, A., Ghannam, L.: Existence of solution for an elliptic problem involving $p(x)$-Laplacian in RN. Divisions:. **12**, 3873–3883 (2016).

[2] Alves, C. O., Souto, M. A. S.: Existence of solutions for a class of problems in RN involving the $p(x)$-Laplacian. Progress in Nonlinear Differential Equations and Their Applications. **66**, 17–32 (2005).

[3] Edmunds, D. E., Rázkosnflk, J.: Sobolev embedding with variable exponent. Studia Mathematics. **143**, 267–293 (2000).

[4] El Hammar, H., El Ouaarabi, M., Allalou, C., Melliani, S.: $p(x,\cdot)$-Kirchhoff type problem involving the fractional $p(x)$-Laplacian operator with discontinuous nonlinearities. Filomat. **38**(6), 2109–2125 (2024).

[5] El Ouaarabi M., Allalou C., Melliani S.: Existence of weak solutions to a $p(x)$-Kirchhoff type problems involving the $p(x)$-Laplacian-like operators. Nonlinear Studies. **30**(1), 333–345 (2023).

[6] El Ouaarabi M., Allalou C., Melliani S.: Neumann Problem Involving The $p(x)$-Kirchhoff-Laplacian-Like Operator in Variable Exponent Sobolev Space. Asia Pac. J. Math. **9**, 18 (2022).

[7] Fan, X. L., Zhang, Q.: Existence of solutions for $p(x)$-Laplacian Dirichlet problem. Nonlinear Anal. **52** 1843–1852 (2011).

[8] Fan, X.: Solutions for $p(x)$-Laplacian Dirichlet problems with singular coefficients. Journal of Mathematical Analysis and applications. **312**(2), 464–477 (2005).

[9] Fan X. L., Zhao, D.: On the spaces $L^{p(x)}$ and $W^{1,p(x)}$. Journal of Mathematical Analysis and Applications. **263**, 424–446 (2001).

[10] Kavian, O.: Introduction à la théorie des points critiques et applications aux problèmes elliptiques. Mathématiques et applications. **13**, (1993).

[11] Kováčik, O., Rákosník, J.: On spaces $L^{p(x)}$ and $W^{k,p(x)}$, Czechoslovak Math. J. **41**, 592–618 (1991).o, E.: Lower semicontinuity of functional via concentration-compactness principle. J. Math. Anal. Appl. **263**, 264–276 (2001).

[12] Mitidier, E.: A simple approach to Hardy inequalities. Math. Notes. **67** 479–486 (2000).

[13] Moujane, N., El Ouaarabi M., Allalou, C.: Study of some elliptic system of $(p(x),q(x))$-Kirchhoff type with convection. J. Elliptic Parabol. Equ. **9**, 687–704 (2023).

[14] Moujane, N., El Ouaarabi M., Allalou, C.: Elliptic Kirchhoff-type system with two convections terms and under Dirichlet boundary conditions. Filomat. **37**(28), 9693–9707 (2023).

[15] Ragusa, M.A., Tachikawa, A.: On continuity of minimizers for certain quadratic growth functionals. Journal of the Mathematical Society of Japan. **57**(3), 691–700 (2005).

[16] Ragusa, M.A., Tachikawa, A.: Regularity of Minimizers of some Variational Integrals with Discontinuity. Zeitschrift für Analysis und ihre Anwendungen. **27**(4), 469–482 (2008).

[17] Ragusa M.A., Razani A., Safari E.: Existence of radial solutions for a $p(x)$-Laplacian Dirichlet problem. Advances in Difference Equations. **2021**(1), 215 (2021).

[18] Shen, Y. T., Yan. S. S.: The Variational Methods for Quasilinear Elliptic Equations, South China Univ. Sci. and Tech. Press, Juangzhou. (1995).