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The existence of a $\{P_2, C_3, P_5, \mathcal{T}(3)\}$-factor based on the size or the A_α-spectral radius of graphs

Xianglong Zhang[a], Lihua You[a,*]

[a] *School of Mathematical Sciences, South China Normal University, Guangzhou, 510631, P. R. China*

Abstract. Let G be a connected graph of order n. A $\{P_2, C_3, P_5, \mathcal{T}(3)\}$-factor of G is a spanning subgraph of G such that each component is isomorphic to a member in $\{P_2, C_3, P_5, \mathcal{T}(3)\}$, where $\mathcal{T}(3)$ is a $\{1, 2, 3\}$-tree. The A_α-spectral radius of G is denoted by $\rho_\alpha(G)$. In this paper, we obtain a lower bound on the size or the A_α-spectral radius for $\alpha \in [0, 1)$ of G to guarantee that G has a $\{P_2, C_3, P_5, \mathcal{T}(3)\}$-factor, and construct an extremal graph to show that the bound on A_α-spectral radius is optimal.

1. Introduction

Let G be an undirected simple and connected graph with vertex set $V(G)$ and edge set $E(G)$. The *order* of G is the number of its vertices, and the *size* is the number of its edges.

Let G be a graph of order n with $V(G) = \{v_1, v_2, ..., v_n\}$. The *adjacency matrix* of G is defined as $A(G) = (a_{ij})$, where $a_{ij} = 1$ if $v_iv_j \in E(G)$, and $a_{ij} = 0$ otherwise. The degree diagonal matrix is the diagonal matrix of vertex degrees of G, denoted by $D(G)$. The *signless Laplacian matrix* $Q(G)$ of G is defined by $Q(G) = D(G) + A(G)$. The largest eigenvalue of $Q(G)$ is denoted by $q(G)$. For any $\alpha \in [0, 1)$, Nikiforov[12] introduced the A_α-matrix of G as $A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G)$. It is easy to see that $A_\alpha(G) = A(G)$ if $\alpha = 0$, and $A_\alpha(G) = \frac{1}{2}Q(G)$ if $\alpha = \frac{1}{2}$. The eigenvalues of $A_\alpha(G)$ are called the A_α-eigenvalues of G, and the largest of them, denoted by $\rho_\alpha(G)$, is called the A_α-spectral radius of G. More interesting spectral properties of $A_\alpha(G)$ can be found in [7, 8, 12].

For a given subset $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$, and the subgraph obtained from G by deleting S together with those edges incident to S is denoted by $G - S$. Let G and H be two disjoint graphs. The union $G \cup H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The join $G \vee H$ is derived from $G \cup H$ by joining every vertex of G with every vertex of H by an edge.

A subgraph of a graph G is spanning if the subgraph covers all vertices of G. Let $\mathcal{H}$ be a set of connected graphs. An $\mathcal{H}$-factor of a graph G is a spanning subgraph of G, in which each component is isomorphic to an element of $\mathcal{H}$. An $\mathcal{H}$-factor is also referred as a component factor.

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* Corresponding author: Lihua You
Email addresses: 2023021945@m.scnu.edu.cn (Xianglong Zhang), ylhua@scnu.edu.cn (Lihua You)
ORCID iDs: https://orcid.org/0009-0008-3628-729X (Xianglong Zhang), https://orcid.org/0000-0003-4503-694X (Lihua You)

For a tree T, every vertex of degree 1 is a leaf of T. We denote the set of leaves in T by $Leaf(T)$. An edge of T incident with a leaf is called a pendant edge. A $\{1,3\}$-tree is a tree with every vertex having degree 1 or 3. Let R be a $\{1,3\}$-tree, $\mathbb{T}(3)$ be the set of trees T_R that can be obtained from R as follows (see [4]): T_R is obtained from R by inserting a new vertex of degree 2 into every edge of R, and by adding a new pendant edge to every leaf of R. Then the tree T_R is a $\{1,2,3\}$-tree having $|E(R)| + |Leaf(R)|$ vertices of degree 2 and has the same number of leaves as R. The collection of such $\{1,2,3\}$-trees T_R generated from all $\{1,3\}$-trees R is denoted by $\mathbb{T}(3)$, and any graph in $\mathbb{T}(3)$ is denoted by $\mathcal{T}(3)$.

More and more researchers have been studied the existence of different factors in graphs since 2000. Las Vergnas [6] presented a sufficient and necessary condition for a graph having $\{K_{1,j} : 1 \le j \le k\}$-factor with $k \ge 2$. Kano, Lu and Yu [3] showed that a graph has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor if it satisfies $i(G-S) \le \frac{|S|}{2}$ for every $S \subset V(G)$. Kano, Lu and Yu [4] proved that a graph has a $\{P_2, C_3, P_5, \mathcal{T}(3)\}$-factor if and only if it satisfies $i(G-S) \le \frac{3}{2}|S|$ for all $S \subset V(G)$. Kano and Saito [5] proved that a graph G has a $\{K_{1,l} : m \le l \le 2m\}$-factor if it satisfies $i(G-S) \le \frac{1}{m}|S|$ for every $S \subset V(G)$. Zhang, Yan and Kano [14] gave a sufficient condition for a graph G containing a $\{K_{1,t} : m \le t \le 2m-1\} \cup \{K_{2m+1}\}$-factor. Chen, Lv and Li [2] provided a lower bound on the size (resp. the spectral radius) of G to guarantee that the graph has a $\{P_2, C_n : n \ge 3\}$-factor. Lv, Li and Xu [9] derived a tight A_α-spectral radius and distance signless Laplacian spectral radius for the existence of a $\{K_2, C_{2i+1} : i \ge 1\}$-factor in a graph. Li and Miao [10] determined a sufficient condition about the size or the spectral radius of G to contain $\mathcal{P}_{\ge 2}$-factor and be $\mathcal{P}_{\ge 2}$-factor covered graphs. Miao and Li [11] showed a lower bound on the size or the spectral radius, and an upper bound on the distance spectral radius of G to ensure that G has a $\{K_{1,j} : 1 \le j \le k\}$-factor. Zhou, Zhang and Sun [17] established a relationship between $P_{\ge 2}$-factor and A_α-spectral radius of a graph. Zhang and You [15] showed sufficient conditions via the size or the A_α-spectral radius of graphs to ensure that a graph contains a $\{K_{1,2}, K_{1,3}, K_5\}$-factor. Zhou [16] got a spectral radius condition on the existence of $\{P_2, C_3, P_5, \mathcal{T}(3)\}$-factor in graphs.

Motived by [4, 16, 17] directly, it is natural and interesting to study some sufficient and necessary conditions to ensure that a graph contains a $\{P_2, C_3, P_5, \mathcal{T}(3)\}$-factor. In this paper, we focus on the sufficient conditions via the size or the A_α-spectral radius of graphs and obtain the following two results.

Theorem 1.1. *Let G be a connected graph of order $n \ge 5$, and*

$$F(n) = \begin{cases} \binom{n-2}{2} + 2, & \text{if } n \ge 5 \text{ and } n \notin \{6, 8\}; \\ 9, & \text{if } n = 6; \\ 18, & \text{if } n = 8. \end{cases}$$

If $|E(G)| > F(n)$, then G has a $\{P_2, C_3, P_5, \mathcal{T}(3)\}$-factor.

Theorem 1.2. *Let $\alpha \in [0, 1)$, $\varphi(x) = x^3 - ((\alpha+1)n + \alpha - 4)x^2 + (\alpha n^2 + (\alpha^2 - 2\alpha - 1)n - 2\alpha + 1)x - \alpha^2 n^2 + (5\alpha^2 - 3\alpha + 2)n - 10\alpha^2 + 15\alpha - 8$, G be a connected graph of order n with $n \ge f(\alpha)$, where*

$$f(\alpha) = \begin{cases} 20, & \text{if } \alpha \in [0, \frac{1}{2}]; \\ 25, & \text{if } \alpha \in (\frac{1}{2}, \frac{5}{7}]; \\ \frac{7}{1-\alpha} + 3, & \text{if } \alpha \in (\frac{5}{7}, 1). \end{cases}$$

If $\rho_\alpha(G) > \tau(n)$, then G has a $\{P_2, C_3, P_5, \mathcal{T}(3)\}$-factor, where $\tau(n)$ is the largest root of $\varphi(x) = 0$.

Let $\alpha = 0$ in Theorem 1.2, the main result of [16] via the spectral radius can be obtained, and let $\alpha = \frac{1}{2}$, we have the following corollary via the signless Laplacian spectral radius immediately.

Corollary 1.3. *Let G be a connected graph of order n with $n \ge 20$. If $q(G) > 2\mu(n)$, then G has a $\{P_2, C_3, P_5, \mathcal{T}(3)\}$-factor, where $\mu(n)$ is the largest root of $4x^3 - (6n - 14)x^2 + (2n^2 - 7n)x - n^2 + 7n - 6 = 0$.*

2. Preliminaries

In this section, we introduce some useful definitions and lemmas.

Definition 2.1. [1]) *Let M be a complex matrix of order n described in the following block form*

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1l} \\ \vdots & \ddots & \vdots \\ M_{l1} & \cdots & M_{ll} \end{pmatrix}$$

where the blocks M_{ij} are $n_i \times n_j$ matrices for any $1 \le i, j \le l$ and $n = n_1 + \cdots + n_l$. For $1 \le i, j \le l$, let q_{ij} denote the average row sum of M_{ij}, i.e. q_{ij} is the sum of all entries in M_{ij} divided by the number of rows. Then $Q(M) = (q_{ij})$ (or simply Q) is called the quotient matrix of M. If, in addition, for each pair i, j, M_{ij} has a constant row sum, i.e., $M_{ij}\vec{e}_{n_j} = q_{ij}\vec{e}_{n_i}$, then Q is called the equitable quotient matrix of M, where $\vec{e}_k = (1, 1, \cdots, 1)^T \in C^k$, and C denotes the field of complex numbers.

Let M be a real nonnegative matrix. The largest eigenvalue of M is called the spectral radius of M, denoted by $\rho(M)$.

Lemma 2.2. [13]) *Let B be an equitable quotient matrix of M as defined in Definition 2.1, where M is a nonnegative matrix. Then the eigenvalues of B are also eigenvalues of M, and $\rho(B) = \rho(M)$.*

Lemma 2.3. [12]) *Let K_n be a complete graph of order n. Then $\rho_\alpha(K_n) = n - 1$.*

Lemma 2.4. [12]) *If G is a connected graph, and H is a proper subgraph of G, then $\rho_\alpha(G) > \rho_\alpha(H)$.*

Lemma 2.5. (The Cauchy's interlace theorem[1]) *Let two sequences of real number, $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ and $\eta_1 \ge \eta_2 \ge \ldots \eta_{n-1}$, be the eigenvalues of symmetric matrix A and B, respectively. If B is a principal submatrix of A, then the eigenvalues of B interlace the eigenvalues of A, i.e., $\lambda_1 \ge \eta_1 \ge \lambda_2 \ge \cdots \ge \eta_{n-2} \ge \lambda_{n-1} \ge \eta_{n-1} \ge \lambda_n$.*

Let $i(G)$ denote the number of isolated vertices of G. The following lemma gives a sufficient and necessary condition for a graph containing a $\{P_2, C_3, P_5, \mathcal{T}(3)\}$-factor.

Lemma 2.6. [4]) *A graph G has a $\{P_2, C_3, P_5, \mathcal{T}(3)\}$-factor if and only if $i(G - S) \le \frac{3}{2}|S|$ for all $S \subseteq V(G)$.*

3. The proof of Theorem 1.1

In this section, we prove Theorem 1.1, which gives a sufficient condition via the size of a connected graph to ensure that the graph contains a $\{P_2, C_3, P_5, \mathcal{T}(3)\}$-factor.

Proof. Suppose to the contrary that G contains no $\{P_2, C_3, P_5, \mathcal{T}(3)\}$-factor. By Lemma 2.6, there exists a nonempty subset S of $V(G)$ satisfying $i(G - S) > \frac{3}{2}|S|$.

Choose such a connected graph G of order n so that its size is as large as possible. With the choice of G, the induced subgraph $G[S]$ and every connected component of $G - S$ are complete graphs, and $G = G[S] \vee (G - S)$.

Note that there is at most one non-trivial connected component in $G - S$. Otherwise, we can add edges among all non-trivial connected components to get a bigger non-trival connected component, which contradicts to the choice of G. For convenient, let $|S| = s$ and $i(G - S) = i$. We now consider the following two possible cases.

Case 1. $G - S$ has exactly one non-trivial connected component, say G_1.

In this case, let $|V(G_1)| = n_1 \ge 2$. Obviously, $i \ge \lfloor \frac{3s}{2} \rfloor + 1 = \begin{cases} \frac{3}{2}s + \frac{1}{2}, & \text{if } s \text{ is odd;} \\ \frac{3}{2}s + 1, & \text{if } s \text{ is even.} \end{cases}$ Now we show $i = \lfloor \frac{3s}{2} \rfloor + 1$.

If $i \geq \lfloor \frac{3s}{2} \rfloor + 2$, let H_1 be a new graph obtained from G by joining each vertex of G_1 with one vertex in $V(G-S) \setminus V(G_1)$ by an edge. Then we have $|E(H_1)| = |E(G)| + n_1 > |E(G)|$ and $i(H_1 - S) = i - 1 \geq \lfloor \frac{3s}{2} \rfloor + 1$, a contradiction with the choice of G. Hence $i = \lfloor \frac{3s}{2} \rfloor + 1$ by $i > \frac{3}{2}s$ and $G = K_s \vee (K_{n_1} \cup (\lfloor \frac{3s}{2} \rfloor + 1)K_1)$.

Clearly, we have $n = s + \lfloor \frac{3s}{2} \rfloor + 1 + n_1 \geq \begin{cases} \frac{5}{2}s + \frac{5}{2} \geq 5, & \text{if } s \text{ is odd} \\ \frac{5}{2}s + 3 \geq 8, & \text{if } s \text{ is even} \end{cases}$ and $|E(G)| = s(\lfloor \frac{3s}{2} \rfloor + 1) + \binom{n - \lfloor \frac{3s}{2} \rfloor - 1}{2}$.

Now we show $|E(G)| \leq F(n)$. By $\binom{n-2}{2} + 2 = \begin{cases} 8 < 9, & \text{if } n = 6, \\ 17 < 18, & \text{if } n = 8, \end{cases}$ we only need show $|E(G)| \leq \binom{n-2}{2} + 2$.

Subcase 1.1. s is odd.

$$\binom{n-2}{2} + 2 - |E(G)| = \frac{1}{8}(s-1)(12n - 21s - 37) \geq \frac{1}{8}(s-1)(9s-7) \geq 0.$$

Therefore, $|E(G)| \leq \binom{n-2}{2} + 2$ for odd s, which is a contradiction.

Subcase 1.2. s is even.

$$\binom{n-2}{2} + 2 - |E(G)| = \frac{1}{8}(-21s^2 - 26s + 12ns - 8n + 32) \geq \frac{1}{8}(9(s - \frac{5}{9})^2 + \frac{47}{9}) > 0.$$

Therefore, $|E(G)| < \binom{n-2}{2} + 2$ for even s, which is a contradiction.

Combining the above two subcases, we have $|E(G)| \leq F(n)$, a contradiction.

Case 2. $G - S$ has no non-trivial connected component.

In this case, we prove $i \leq \lfloor \frac{3s}{2} \rfloor + 2$ firstly.

If $i \geq \lfloor \frac{3s}{2} \rfloor + 3$, let H_2 be a new graph obtained from G by adding an edge between two vertices in $V(G-S)$. Clearly, $i(H_2 - S) = i - 2 \geq \lfloor \frac{3s}{2} \rfloor + 1$ and $H_2 - S$ has exactly one non-trivial connected component. Together with $|E(G)| < |E(H_2)|$, we obtain a contradiction with the choice of G, which implies $i = \lfloor \frac{3s}{2} \rfloor + 1$ or $i = \lfloor \frac{3s}{2} \rfloor + 2$ by $i > \frac{3}{2}s$.

Subcase 2.1. $i = \lfloor \frac{3s}{2} \rfloor + 1$.

In this subcase, we have $G = K_s \vee ((\lfloor \frac{3s}{2} \rfloor + 1)K_1)$. Therefore, $n = s + \lfloor \frac{3s}{2} \rfloor + 1$, $|E(G)| = \binom{s}{2} + s(\lfloor \frac{3s}{2} \rfloor + 1)$, $s \geq 2$ by $n \geq 5$, and

$$\binom{n-2}{2} + 2 - |E(G)| = \begin{cases} \frac{1}{8}(s-1)(9s-31), & \text{if } s \text{ is odd;} \\ \frac{1}{8}(9s^2 - 34s + 24), & \text{if } s \text{ is even.} \end{cases}$$

If s is odd, we have $|E(G)| \leq \binom{n-2}{2} + 2$ for $s \geq 5$ (which implies $n \geq 13$), and $|E(G)| = 2s^2 = 18$ for $s = 3$ (which implies $n = 8$).

If s is even, we have $|E(G)| < \binom{n-2}{2} + 2$ for $s \geq 4$ (which implies $n \geq 11$), and $|E(G)| = 9$ for $s = 2$ (which implies $n = 6$).

Combining the above arguments, we have $|E(G)| \leq F(n)$ for all $n \geq 5$, a contradiction.

Subcase 2.2. $i = \lfloor \frac{3s}{2} \rfloor + 2$.

In this subcase, we have $G = K_s \vee ((\lfloor \frac{3s}{2} \rfloor + 2)K_1)$. Therefore, $n = s + \lfloor \frac{3s}{2} \rfloor + 2$, $|E(G)| = \binom{s}{2} + s(\lfloor \frac{3s}{2} \rfloor + 2)$, $s \geq 2$ by $n \geq 5$, and

$$\binom{n-2}{2} + 2 - |E(G)| = \begin{cases} \frac{1}{8}(s-1)(9s-19), & \text{if } s \text{ is odd;} \\ \frac{1}{8}(9s^2 - 22s + 16), & \text{if } s \text{ is even.} \end{cases}$$

If s is odd, we have $|E(G)| \leq \binom{n-2}{2} + 2$ for $s \geq 3$ (which implies $n \geq 9$).

If s is even, we have $|E(G)| \leq \binom{n-2}{2} + 2$ for $s \geq 2$ (which implies $n \geq 7$).

Combining the above arguments, we have $|E(G)| \leq F(n)$ for all $n \geq 5$, a contradiction.

By Case 1 and Case 2, we complete the proof. □

4. The proof of Theorem 1.2

In this section, we prove Theorem 1.2, which presents a sufficient condition in terms of the A_α-spectral radius for a graph to contain a $\{P_2, C_3, P_5, \mathcal{T}(3)\}$-factor.

Proof. Suppose to the contrary that G does not contain a $\{P_2, C_3, P_5, \mathcal{T}(3)\}$-factor. By Lemma 2.6, there exists a nonempty subset S of $V(G)$ satisfying $i(G-S) > \frac{3}{2}|S|$.

Choose such a connected graph G of order n so that its A_α-spectral radius is as large as possible. Together with Lemma 2.4 and the choice of G, the induced subgraph $G[S]$ and every connected component of $G-S$ are complete graphs, and $G = G[S] \vee (G-S)$.

It is easy to see that $G-S$ admits at most one non-trivial connected component. Otherwise, we can construct a new graph G' by adding edges among all non-trivial connected components to obtain a bigger non-trivial connected component. Clearly, G is a proper subgraph of G'. According to Lemma 2.4, $\rho_\alpha(G') > \rho_\alpha(G)$, which contradicts the choice of G. For convenient, let $|S| = s$ and $i(G-S) = i$.

Now, we show Theorem 1.2 by considering the following two cases.

Case 1. $G-S$ has exactly one non-trivial connected component.

In this case, $G = K_s \vee (K_{n_1} \cup iK_1)$, where $n_1 = n - s - i \geq 2$. Now we show $i = \lfloor \frac{3s}{2} \rfloor + 1$.

If $i \geq \lfloor \frac{3s}{2} \rfloor + 2$, then we construct a new graph G'' obtained from G by joining each vertex of K_{n_1} with one vertex in iK_1 by an edge. It is obvious that $i(G''-S) = i - 1 \geq \lfloor \frac{3s}{2} \rfloor + 1$ and G is a proper subgraph of G''. According to Lemma 2.4, $\rho_\alpha(G'') > \rho_\alpha(G)$, which contradicts with the choice of G. Therefore, $i = \lfloor \frac{3s}{2} \rfloor + 1$, $G = K_s \vee (K_{n-s-\lfloor \frac{3s}{2} \rfloor - 1} \cup (\lfloor \frac{3s}{2} \rfloor + 1)K_1)$ by $i > \frac{3s}{2}$, and the quotient matrix of $A_\alpha(G)$ in terms of the partition $\{V((\lfloor \frac{3s}{2} \rfloor + 1)K_1), V(K_{n-s-\lfloor \frac{3s}{2} \rfloor - 1}), V(K_s)\}$ can be written as

$$B_1 = \begin{pmatrix} \alpha s & 0 & (1-\alpha)s \\ 0 & n + (\alpha s - s - \lfloor \frac{3s}{2} \rfloor) - 2 & (1-\alpha)s \\ (1-\alpha)(\lfloor \frac{3s}{2} \rfloor + 1) & (1-\alpha)(n - s - \lfloor \frac{3s}{2} \rfloor - 1) & \alpha n - \alpha s + s - 1 \end{pmatrix}.$$

Then the characteristic polynomial of B_1 is

$$\begin{aligned} f_{B_1}(x) = {} & x^3 - ((\alpha+1)n + \alpha s - \lfloor \frac{3s}{2} \rfloor - 3)x^2 \\ & - ((\alpha n + s - 1)\lfloor \frac{3s}{2} \rfloor - \alpha n^2 - (\alpha^2 + \alpha)sn + (2\alpha + 1)n + (2\alpha + 1)s - 2)x \\ & - ((2\alpha^2 - 3\alpha + 1)\lfloor \frac{3s}{2} \rfloor + 2\alpha^2 - 3\alpha + 1)s^2 \\ & - ((\alpha^2 - 2\alpha + 1)\lfloor \frac{3s}{2} \rfloor^2 - ((2\alpha^2 - 2\alpha + 1)n - 3\alpha^2 + 5\alpha - 3)\lfloor \frac{3s}{2} \rfloor \\ & + \alpha^2 n^2 - (3\alpha^2 - \alpha + 1)n + 2\alpha^2 - 2\alpha + 2)s. \end{aligned} \tag{1}$$

By Lemma 2.2, $\rho_\alpha(G)$ is the largest root of $f_{B_1}(x) = 0$, say, $f_{B_1}(\rho_\alpha(G)) = 0$. Let $\eta_1 = \rho_\alpha(G) \geq \eta_2 \geq \eta_3$ be the three roots of $f_{B_1}(x) = 0$ and $Q = diag(\lfloor \frac{3s}{2} \rfloor + 1, n - s - \lfloor \frac{3s}{2} \rfloor - 1, s)$. It is easy to check that

$$\begin{aligned} & Q^{\frac{1}{2}} B_1 Q^{-\frac{1}{2}} \\ = & \begin{pmatrix} \alpha s & 0 & (1-\alpha)s^{\frac{1}{2}}(\lfloor \frac{3s}{2} \rfloor + 1)^{\frac{1}{2}} \\ 0 & n + (\alpha s - s - \lfloor \frac{3s}{2} \rfloor) - 2 & (1-\alpha)s^{\frac{1}{2}}(n - s - \lfloor \frac{3s}{2} \rfloor - 1)^{\frac{1}{2}} \\ (1-\alpha)s^{\frac{1}{2}}(\lfloor \frac{3s}{2} \rfloor + 1)^{\frac{1}{2}} & (1-\alpha)s^{\frac{1}{2}}(n - s - \lfloor \frac{3s}{2} \rfloor - 1)^{\frac{1}{2}} & \alpha n - \alpha s + s - 1 \end{pmatrix} \end{aligned}$$

is symmetric, and contains

$$\begin{pmatrix} \alpha s & 0 \\ 0 & n + (\alpha s - s - \lfloor \frac{3s}{2} \rfloor) - 2 \end{pmatrix}$$

as a submatrix. Since $Q^{\frac{1}{2}}B_1Q^{-\frac{1}{2}}$ and B_1 admit the same eigenvalues, according to Lemma 2.5, we get

$$\alpha s \le \eta_2 \le n + (\alpha s - s - \lfloor \frac{3s}{2} \rfloor) - 2 < \begin{cases} n-3, & \text{if } s \text{ is odd,} \\ n-5, & \text{if } s \text{ is even.} \end{cases} \tag{2}$$

Subcase 1.1. s is odd.

Let $\varphi(x) = x^3 - ((\alpha+1)n + \alpha - 4)x^2 + (\alpha n^2 + (\alpha^2 - 2\alpha - 1)n - 2\alpha + 1)x - \alpha^2 n^2 + (5\alpha^2 - 3\alpha + 2)n - 10\alpha^2 + 15\alpha - 8$, $\tau(n)$ be the largest root of $\varphi(x) = 0$, $G_2 = K_1 \vee (K_{n-3} \cup 2K_1)$. Clearly, $G \cong G_2$ when $s = 1$, $f_{B_1}(x) = \varphi(x)$ by plugging $s = 1$ into (1), and $\tau(n) = \rho_\alpha(G_2)$ by Lemma 2.2.

Since K_{n-2} is a proper subgraph of G_2, by Lemmas 2.3-2.4 and (2), we have

$$\tau(n) = \rho_\alpha(G_2) > \rho_\alpha(K_{n-2}) = n - 3 > \eta_2. \tag{3}$$

Subcase 1.1.1. $s = 1$.

In this subcase, we have $\rho_\alpha(G) = \rho_\alpha(G_2) = \tau(n)$, which contradicts with $\rho_\alpha(G) > \tau(n)$.

Subcase 1.1.2. $s \ge 3$.

In this subcase, $G \ncong G_2$ and $f_{B_1}(x) \ne \varphi(x)$. By (1) and $\varphi(\tau(n)) = 0$, we have

$$f_{B_1}(\tau(n)) = f_{B_1}(\tau(n)) - \varphi(\tau(n)) = -\frac{1}{4}(s-1)H(\tau(n)), \tag{4}$$

where $H(x) = (4\alpha - 6)x^2 + (-4\alpha^2 n + 2\alpha n + 6s + 8\alpha + 2)x + 4\alpha^2 n^2 - (12\alpha^2 - 12\alpha + 6)sn - (20\alpha^2 - 12\alpha + 8)n + (21\alpha^2 - 36\alpha + 15)s^2 + (37\alpha^2 - 60\alpha + 29)s + 40\alpha^2 - 60\alpha + 32$, its axis of symmetry is $x = -\frac{-4\alpha^2 n + 2\alpha n + 6s + 8\alpha + 2}{2(4\alpha - 6)}$. Now we show $\rho_\alpha(G) < \tau(n)$, and we obtain a contradiction with $\rho_\alpha(G) > \tau(n)$.

Subcase 1.1.2.1. $0 \le \alpha \le \frac{5}{7}$.

Firstly, we show $H(\tau(n)) < H(n-3)$.

By (3), we only need show $-\frac{-4\alpha^2 n + 2\alpha n + 6s + 8\alpha + 2}{2(4\alpha - 6)} < n - 3$, say, show $g_1(s) = 2(n-3)(4\alpha - 6) + (-4\alpha^2 n + 2\alpha n + 6s + 8\alpha + 2) < 0$.

In fact, by $n = n_1 + \frac{5}{2}s + \frac{1}{2} \ge \frac{5}{2}s + \frac{5}{2}$ and $s \ge 3$, we have

$$\begin{aligned} g_1(s) &= (-4\alpha^2 + 10\alpha - 12)n - 16\alpha + 6s + 38 \\ &\le (-4(\alpha - \frac{5}{4})^2 - \frac{23}{4})(\frac{5}{2}s + \frac{5}{2}) - 16\alpha + 6s + 38 \\ &= s(-10\alpha^2 + 25\alpha - 24) - 10\alpha^2 + 9\alpha + 8 \\ &\le 3(-10\alpha^2 + 25\alpha - 24) - 10\alpha^2 + 9\alpha + 8 \\ &< 0. \end{aligned}$$

By the above arguments and (3), we have $H(\tau(n)) < H(n-3)$.

On the other hand, by a direct calculation, we have $H(n-3) = (6\alpha - 6)n^2 + (-12s\alpha^2 + 12s\alpha - 8\alpha^2 - 10\alpha + 30)n + (21\alpha^2 - 36\alpha + 15)s^2 + (37\alpha^2 - 60\alpha + 11)s + 40\alpha^2 - 48\alpha - 28$. Let $P(n) = H(n-3)$. Then the axis of symmetry of $P(x)$ is $x = -\frac{-12s\alpha^2 + 12s\alpha - 8\alpha^2 - 10\alpha + 30}{2(6\alpha - 6)}$.

Now we show $P(n) < 0$. Let $g_2(s) = 2(\frac{5}{2}s + \frac{5}{2})(6\alpha - 6) + (-12s\alpha^2 + 12s\alpha - 8\alpha^2 - 10\alpha + 30)$. Then by $\alpha \le \frac{5}{7}$, we have

$$\begin{aligned} g_2(s) &= (-12\alpha^2 + 42\alpha - 30)s - 8\alpha^2 + 20\alpha \\ &= (-12(\alpha - \frac{7}{4})^2 + \frac{27}{4})s - 8(\alpha - \frac{5}{4})^2 + \frac{25}{2} \\ &< -\frac{300}{49}s + \frac{25}{2} \\ &< 0, \end{aligned}$$

which implies $-\frac{-12s\alpha^2+12s\alpha-8\alpha^2-10\alpha+30}{2(6\alpha-6)} < \frac{5}{2}s+\frac{5}{2}$, and $P(n)$ is monotonically decreasing when $n \geq \frac{5}{2}s+\frac{5}{2}$. Then by $-\frac{\frac{63}{2}s^2+20s-\frac{71}{2}}{2(-9s^2-13s+20)} > \frac{5}{7} \geq \alpha$ and $n \geq \frac{5}{2}s+\frac{5}{2}$, we have

$$\begin{aligned}
P(n) &\leq P(\frac{5}{2}s+\frac{5}{2})\\
&= (6\alpha-6)(\frac{5}{2}s+\frac{5}{2})^2+(-12s\alpha^2+12s\alpha-8\alpha^2-10\alpha+30)(\frac{5}{2}s+\frac{5}{2})\\
&+(21\alpha^2-36\alpha+15)s^2+(37\alpha^2-60\alpha+11)s+40\alpha^2-48\alpha-28\\
&= (-9s^2-13s+20)\alpha^2+(\frac{63}{2}s^2+20s-\frac{71}{2})\alpha-\frac{45}{2}s^2+11s+\frac{19}{2}\\
&\leq \frac{25}{49}(-9s^2-13s+20)+\frac{5}{7}(\frac{63}{2}s^2+20s-\frac{71}{2})-\frac{45}{2}s^2+11s+\frac{19}{2}\\
&= -\frac{1}{49}(225s^2-914s+277).
\end{aligned} \tag{5}$$

If $s \geq 5$, then $P(n) < 0$ by (5).
If $s = 3$, then $-\frac{-12s\alpha^2+12s\alpha-8\alpha^2-10\alpha+30}{2(6\alpha-6)} = -\frac{-44\alpha^2+26\alpha+30}{2(6\alpha-6)} < 8 < 20 \leq f(\alpha) \leq n$ due to $0 \leq \alpha \leq \frac{5}{7}$. Hence,

$$P(n) \leq \begin{cases} P(20) = -540\alpha^2+2368\alpha-1660 < 0, & \text{if } 0 \leq \alpha \leq \frac{1}{2} \text{ and } n \geq f(\alpha) = 20,\\ P(25) = -760\alpha^2+3848\alpha-2860 < 0, & \text{if } \frac{1}{2} < \alpha \leq \frac{5}{7} \text{ and } n \geq f(\alpha) = 25. \end{cases}$$

Thus, we conclude that $P(n) < 0$ for $s \geq 3$. Combining the above arguments, by (5), we have $f_{B_1}(\tau(n)) = -\frac{1}{4}(s-1)H(\tau(n)) > -\frac{1}{4}(s-1)H(n-3) = -\frac{1}{4}(s-1)P(n) > 0$, which implies $\rho_\alpha(G) < \tau(n)$ for $s \geq 3$ by (3), a contradiction with $\rho_\alpha(G) > \tau(n)$.

Subcase 1.1.2.2. $\frac{5}{7} < \alpha < 1$.

By (1), we have

$$\begin{aligned}
f_{B_1}(n-3) &= \frac{3}{4}(7\alpha-5)(1-\alpha)s^3+((3\alpha^2-3\alpha)n-4\alpha^2+6\alpha+1)s^2\\
&+((\frac{3}{2}-\frac{3}{2}\alpha)n^2-(\alpha^2-\frac{11}{2}\alpha+\frac{15}{2})n-\frac{3}{4}\alpha^2-3\alpha+\frac{39}{4})s\\
&+(\frac{3}{2}\alpha-\frac{3}{2})n^2-(\frac{9}{2}\alpha-\frac{15}{2})n-9.
\end{aligned}$$

Let $\Psi(s,n) = f_{B_1}(n-3)$. Thus, we get

$$\begin{aligned}
\frac{\partial\Psi(s,n)}{\partial s} &= \frac{9}{4}(7\alpha-5)(1-\alpha)s^2+2((3\alpha^2-3\alpha)n-4\alpha^2+6\alpha+1)s\\
&+((\frac{3}{2}-\frac{3}{2}\alpha)n^2-(\alpha^2-\frac{11}{2}\alpha+\frac{15}{2})n-\frac{3}{4}\alpha^2-3\alpha+\frac{39}{4}).
\end{aligned}$$

Since $\frac{5}{7} < \alpha < 1$, $n \geq f(\alpha) > \frac{7}{1-\alpha}+3 > \frac{7}{1-\alpha}$. By a simple computation, we have

$$\begin{aligned}
\left.\frac{\partial\Psi(s,n)}{\partial s}\right|_{s=3} &= (\frac{3}{2}-\frac{3}{2}\alpha)n^2+(17\alpha^2-\frac{25}{2}\alpha-\frac{15}{2})n-\frac{333}{2}\alpha^2+276\alpha-\frac{171}{2}\\
&> (\frac{3}{2}-\frac{3}{2}\alpha)(\frac{7}{1-\alpha})^2+(17\alpha^2-\frac{25}{2}\alpha-\frac{15}{2})(\frac{7}{1-\alpha})-\frac{333}{2}\alpha^2+276\alpha-\frac{171}{2}\\
&= \frac{1}{2(1-\alpha)}(333\alpha^3-647\alpha^2+548\alpha-129)\\
&> 0,
\end{aligned}$$

and

$$\begin{aligned}\left.\frac{\partial\Psi(s,n)}{\partial s}\right|_{s=\frac{2}{5}n-1} &= \frac{1}{50}((-6\alpha^2+21\alpha-15)n^2+(120\alpha^2-265\alpha+115)n-425\alpha^2+600\alpha-175)\\ &< \frac{1}{50}((-6\alpha^2+21\alpha-15)(\frac{7}{1-\alpha})^2+(120\alpha^2-265\alpha+115)(\frac{7}{1-\alpha})\\ &\quad -425\alpha^2+600\alpha-175)\\ &= \frac{1}{50(1-\alpha)}(425\alpha^3-185\alpha^2-786\alpha-105)\\ &< 0.\end{aligned}$$

Then $f_{B_1}(n-3) = \Psi(s,n) \geq \min\{\Psi(3,n), \Psi(\frac{2}{5}n-1,n)\}$ since the leading coefficient of $\Psi(s,n)$ (when viewed as a cubic polynomial of s) is positive, and $3 \leq s \leq \frac{2}{5}n-1$.

By $\frac{5}{7} < \alpha < 1$ and $n \geq f(\alpha) > \frac{7}{1-\alpha} + 3 > \frac{7}{1-\alpha}$, we have

$$\begin{aligned}\Psi(3,n) &= (3-3\alpha)n^2+(24\alpha^2-15\alpha-15)n-180\alpha^2+288\alpha-72\\ &> (3-3\alpha)(\frac{7}{1-\alpha})^2+(24\alpha^2-15\alpha-15)(\frac{7}{1-\alpha})-180\alpha^2+288\alpha-72\\ &= \frac{1}{1-\alpha}(180\alpha^3-300\alpha^2+255\alpha-30)\\ &> 0,\end{aligned}$$

and

$$\begin{aligned}\Psi(\frac{2}{5}n-1,n) &= \frac{1}{125}((18\alpha^2-63\alpha+45)n^3-(115\alpha^2-530\alpha+505)n^2\\ &\quad +(75\alpha^2-1025\alpha+1700)n+250\alpha^2-1750)\\ &> \frac{1}{125}((18\alpha^2-63\alpha+45)(\frac{7}{1-\alpha})^3-(115\alpha^2-530\alpha+505)(\frac{7}{1-\alpha})^2\\ &\quad +(75\alpha^2-1025\alpha+1700)(\frac{7}{1-\alpha})+250\alpha^2-1750)\\ &= \frac{1}{125(1-\alpha)^2}(250\alpha^4-1025\alpha^3+565\alpha^2+4221\alpha+840)\\ &> 0.\end{aligned}$$

Therefore, we conclude that $f_{B_1}(n-3) \geq \min\{\Psi(3,n), \Psi(\frac{2}{5}n-1,n)\} > 0$ for $s \geq 3$. By (3), we have $\rho_\alpha(G) < \tau(n)$ for $s \geq 3$, which contradicts with $\rho_\alpha(G) > \tau(n)$.

Subcase 1.2. s is even.

Let $\psi(x) = x^3-((\alpha+1)n+2\alpha-6)x^2+(\alpha n^2+(2\alpha^2-3\alpha-1)n-4\alpha-3)x-2\alpha^2n^2+(18\alpha^2-14\alpha+8)n-72\alpha^2+118\alpha-56$, $\theta(n)$ be the largest root of $\psi(x) = 0$, $G_3 = K_2 \vee (K_{n-6} \cup 4K_1)$. Clearly, $G \cong G_3$ when $s = 2$, $f_{B_1}(x) = \psi(x)$ by plugging $s = 2$ into (1), and $\theta(n) = \rho_\alpha(G_3)$ by Lemma 2.2.

Since K_{n-4} is a proper subgraph of G_3, by Lemmas 2.3-2.4 and (2), we have

$$\theta(n) = \rho_\alpha(G_3) > \rho_\alpha(K_{n-4}) = n-5 > \eta_2. \tag{6}$$

Subcase 1.2.1. $s = 2$.

In this subcase, we have $\rho_\alpha(G) = \rho_\alpha(G_3) = \theta(n)$. Now we prove $\theta(n) < n-3 < \tau(n)$ to get the contradiction.

By a direct computation, we have

$$\psi(n-3) = (2-2\alpha)n^2+(12\alpha^2-6\alpha-10)n-72\alpha^2+112\alpha-20.$$

Let $\Omega(x) = (2-2\alpha)x^2 + (12\alpha^2 - 6\alpha - 10)x - 72\alpha^2 + 112\alpha - 20$. If $\alpha \in [0, \frac{5}{7}]$, then $x = -\frac{12\alpha^2-6\alpha-10}{2(2-2\alpha)} < 8$ and $\Omega(x)$ is increasing when $x \geq 20$, so $\psi(n-3) = \Omega(n) \geq \Omega(20) = 168\alpha^2 - 808\alpha + 580 > 0$ when $n \geq 20$. If $\alpha \in (\frac{5}{7}, 1)$, then $-\frac{12\alpha^2-6\alpha-10}{2(2-2\alpha)} < \frac{7}{1-\alpha}$, and $\Omega(n)$ is increasing when $x \geq \frac{7}{1-\alpha} + 3 > \frac{7}{1-\alpha}$, so $\psi(n-3) = \Omega(n) > \Omega(\frac{7}{1-\alpha}) = \frac{72\alpha^3-100\alpha^2+90\alpha+8}{1-\alpha} > 0$ when $n \geq \frac{7}{1-\alpha} + 3$.

Therefore, we get $\theta(n) < n-3$ by $\psi(n-3) > 0$ and (6), then $\rho_\alpha(G) = \rho_\alpha(G_3) = \theta(n) < \tau(n)$ by (3), which contradicts with $\rho_\alpha(G) > \tau(n)$.

Subcase 1.2.2. $s \geq 4$.

In this subcase, $G \ncong G_3$ and $f_{B_1}(x) \neq \psi(x)$. By (1) and $\psi(\theta(n)) = 0$, we have

$$f_{B_1}(\theta(n)) = f_{B_1}(\theta(n)) - \psi(\theta(n)) = -\frac{1}{4}(s-2)h(\theta(n)), \tag{7}$$

where $h(x) = (4\alpha-6)x^2 + (-4\alpha^2 n + 2\alpha n + 6s + 8\alpha + 10)x + 4\alpha^2 n^2 - (12\alpha^2 - 12\alpha + 6)sn - (36\alpha^2 - 28\alpha + 16)n + (21\alpha^2 - 36\alpha + 15)s^2 + (68\alpha^2 - 114\alpha + 52)s + 144\alpha^2 - 236\alpha + 112$, its axis of symmetry is $x = -\frac{-4\alpha^2 n+2\alpha n+6s+8\alpha+10}{2(4\alpha-6)}$.

Subcase 1.2.2.1. $0 \leq \alpha \leq \frac{5}{7}$.

Firstly, we show $h(\theta(n)) < h(n-5)$.

By (6), we only need show $-\frac{-4\alpha^2 n+2\alpha n+6s+8\alpha+10}{2(4\alpha-6)} < n-5$, say, show $g_3(s) = 2(n-5)(4\alpha-6) + (-4\alpha^2 n + 2\alpha n + 6s + 8\alpha + 10) < 0$.

In fact, by $n = n_1 + \frac{5}{2}s + 1 \geq \frac{5}{2}s + 3$ and $s \geq 4$, we have

$$\begin{aligned} g_3(s) &= (-4\alpha^2 + 10\alpha - 12)n - 32\alpha + 6s + 70 \\ &\leq (-4(\alpha - \frac{5}{4})^2 - \frac{23}{4})(\frac{5}{2} + 3) - 32\alpha + 6s + 70 \\ &= s(-10\alpha^2 + 25\alpha - 30) - 12\alpha^2 - 2\alpha + 34 \\ &\leq 4(-10\alpha^2 + 25\alpha - 30) - 12\alpha^2 - 2\alpha + 34 \\ &< 0. \end{aligned}$$

By the above arguments and (6), we have $h(\theta(n)) < h(n-5)$.

On the other hand, by a direct calculation, we have $h(n-5) = (6\alpha-6)n^2 + (-12s\alpha^2 + 12s\alpha - 16\alpha^2 - 14\alpha + 54)n + (21\alpha^2 - 36\alpha + 15)s^2 + (68\alpha^2 - 114\alpha + 22)s + 144\alpha^2 - 176\alpha - 88$. Let $p(n) = h(n-5)$. Then the axis of symmetry of $p(x)$ is $x = -\frac{-12s\alpha^2+12s\alpha-16\alpha^2-14\alpha+54}{2(6\alpha-6)}$.

Now we show $p(n) < 0$.

If $s \geq 10$, we take $g_4(s) = 2(\frac{5}{2}s + 3)(6\alpha - 6) + (-12s\alpha^2 + 12s\alpha - 16\alpha^2 - 14\alpha + 54)$. By $\alpha \leq \frac{5}{7}$, we have

$$\begin{aligned} g_4(s) &= (-12\alpha^2 + 42\alpha - 30)s - 16\alpha^2 + 22\alpha + 18 \\ &= (-12(\alpha - \frac{7}{4})^2 + \frac{27}{4})s - 16(\alpha - \frac{11}{16})^2 + \frac{409}{16} \\ &< -\frac{300}{49}s + \frac{409}{16} \\ &< 0, \end{aligned}$$

which implies $-\frac{-12s\alpha^2+12s\alpha-16\alpha^2-14\alpha+54}{2(6\alpha-6)} < \frac{5}{2}s + 3$, and $p(n)$ is monotonically decreasing when $n \geq \frac{5}{2}s + 3$. Then by $-\frac{\frac{63}{2}s^2-23s-164}{2(-9s^2-8s+96)} > \frac{5}{7} \geq \alpha$ and $n \geq \frac{5}{2}s+3$, we have

$$\begin{aligned}
p(n) &\leq p(\frac{5}{2}s+3)\\
&= (6\alpha-6)(\frac{5}{2}s+3)^2 + (-12s\alpha^2+12s\alpha-16\alpha^2-14\alpha+54)(\frac{5}{2}s+3)\\
&+ (21\alpha^2-36\alpha+15)s^2 + (68\alpha^2-114\alpha+22)s + 144\alpha^2 - 176\alpha - 88\\
&= (-9s^2-8s+96)\alpha^2 + (\frac{63}{2}s^2-23s-164)\alpha - \frac{45}{2}s^2 + 67s + 20\\
&\leq \frac{25}{49}(-9s^2-8s+96) + \frac{5}{7}(\frac{63}{2}s^2-23s-164) - \frac{45}{2}s^2 + 67s + 20\\
&= -\frac{1}{49}(225s^2-2278s+2360)\\
&< 0.
\end{aligned}$$

If $s \in \{4, 6, 8\}$, then

$$-\frac{-12s\alpha^2+12s\alpha-16\alpha^2-14\alpha+54}{2(6\alpha-6)} = \begin{cases} -\frac{-112\alpha^2+82\alpha+54}{2(6\alpha-6)} < 17, & \text{if } s = 8,\\ -\frac{-88\alpha^2+58\alpha+54}{2(6\alpha-6)} < 15, & \text{if } s = 6,\\ -\frac{-64\alpha^2+34\alpha+54}{2(6\alpha-6)} < 14, & \text{if } s = 4.\end{cases}$$

For $0 \leq \alpha \leq \frac{1}{2}$ and $n \geq f(\alpha) = 20$, we have

$$p(n) \leq p(20) = \begin{cases} -208\alpha^2+648\alpha-272 \leq 0, & \text{if } s = 8,\\ -452\alpha^2+1404\alpha-736 < 0, & \text{if } s = 6,\\ -528\alpha^2+1872\alpha-1080 < 0, & \text{if } s = 4.\end{cases}$$

For $\frac{1}{2} < \alpha \leq \frac{5}{7}$ and $n \geq f(\alpha) = 25$, we have

$$p(n) \leq p(25) = \begin{cases} -768\alpha^2+2408\alpha-1352 < 0, & \text{if } s = 8,\\ -892\alpha^2+3044\alpha-1816 < 0, & \text{if } s = 6,\\ -848\alpha^2+3392\alpha-2160 < 0, & \text{if } s = 4.\end{cases}$$

Thus, we conclude that $p(n) < 0$ for $s \geq 4$. Combining the above arguments, by (7), we have $f_{B_1}(\theta(n)) = -\frac{1}{4}(s-2)h(\theta(n)) > -\frac{1}{4}(s-2)h(n-5) = -\frac{1}{4}(s-2)p(n) \geq 0$, which implies $\rho_\alpha(G) < \theta(n) < \tau(n)$ for $s \geq 4$ by (6), a contradiction with $\rho_\alpha(G) > \tau(n)$.

Subcase 1.2.2.2. $\frac{5}{7} < \alpha < 1$.

By (1), we have

$$\begin{aligned}
f_{B_1}(n-5) = \frac{3}{4}(7\alpha-5)(1-\alpha)s^3 &+ ((3\alpha^2-3\alpha)n - \frac{13}{2}\alpha^2 + \frac{21}{2}\alpha + 2)s^2\\
&+ ((\frac{3}{2}-\frac{3}{2}\alpha)n^2 - (2\alpha^2 - \frac{19}{2}\alpha + \frac{27}{2})n - 2\alpha^2 - 13\alpha + 33)s\\
&+ (3\alpha-3)n^2 - (15\alpha-27)n - 60.
\end{aligned}$$

Let $\Phi(s,n) = f_{B_1}(n-5)$. Thus, we get

$$\begin{aligned}
\frac{\partial\Phi(s,n)}{\partial s} = \frac{9}{4}(7\alpha-5)(1-\alpha)s^2 &+ 2((3\alpha^2-3\alpha)n - \frac{13}{2}\alpha^2 + \frac{21}{2}\alpha + 2)s\\
&+ ((\frac{3}{2}-\frac{3}{2}\alpha)n^2 - (2\alpha^2 - \frac{19}{2}\alpha + \frac{27}{2})n - 2\alpha^2 - 13\alpha + 33).
\end{aligned}$$

Since $\frac{5}{7} < \alpha < 1$, $n \geq f(\alpha) > \frac{7}{1-\alpha} + 3 > \frac{7}{1-\alpha}$. By a simple computation, we have

$$\begin{aligned}\frac{\partial \Phi(s,n)}{\partial s}\bigg|_{s=4} &= (\frac{3}{2} - \frac{3}{2}\alpha)n^2 + (22\alpha^2 - \frac{29}{2}\alpha - \frac{27}{2})n - 306\alpha^2 + 503\alpha - 131 \\ &> (\frac{3}{2} - \frac{3}{2}\alpha)(\frac{7}{1-\alpha})^2 + (22\alpha^2 - \frac{29}{2}\alpha - \frac{27}{2})(\frac{7}{1-\alpha}) - 306\alpha^2 + 503\alpha - 131 \\ &= \frac{1}{2(1-\alpha)}(612\alpha^3 - 1310\alpha^2 + 1065\alpha - 304) \\ &> 0,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \Phi(s,n)}{\partial s}\bigg|_{s=\frac{2}{5}n-\frac{6}{5}} &= \frac{1}{50}((-6\alpha^2 + 21\alpha - 15)n^2 + (36\alpha^2 - 41\alpha - 55)n - 454\alpha^2 + 34\alpha + 600) \\ &< \frac{1}{50}((-6\alpha^2 + 21\alpha - 15)(\frac{7}{1-\alpha})^2 + (36\alpha^2 - 41\alpha - 55)(\frac{7}{1-\alpha}) \\ &\quad - 454\alpha^2 + 34\alpha + 600) \\ &= \frac{1}{50(1-\alpha)}(454\alpha^3 - 236\alpha^2 - 559\alpha - 520) \\ &< 0.\end{aligned}$$

Then $f_{B_1}(n-5) = \Phi(s,n) \geq \min\{\Phi(4,n), \Phi(\frac{2}{5}n - \frac{6}{5}, n)\}$ since the leading coefficient of $\Phi(s,n)$ (when viewed as a cubic polynomial of s) is positive, and $4 \leq s \leq \frac{2}{5}n - \frac{6}{5}$.

By $\frac{5}{7} < \alpha < 1$ and $n \geq f(\alpha) > \frac{7}{1-\alpha} + 3 > \frac{7}{1-\alpha}$, we have

$$\begin{aligned}\Phi(4,n) &= (3 - 3\alpha)n^2 + (40\alpha^2 - 25\alpha - 27)n - 448\alpha^2 + 692\alpha - 136 \\ &> (3 - 3\alpha)(\frac{7}{1-\alpha})^2 + (40\alpha^2 - 25\alpha - 27)(\frac{7}{1-\alpha}) - 448\alpha^2 + 692\alpha - 136 \\ &= \frac{1}{1-\alpha}(448\alpha^3 - 860\alpha^2 + 653\alpha - 178) \\ &> 0,\end{aligned}$$

and

$$\begin{aligned}\Phi(\frac{2}{5}n - \frac{6}{5}, n) &= \frac{1}{125}((18\alpha^2 - 63\alpha + 45)n^3 - (212\alpha^2 - 997\alpha + 965)n^2 \\ &\quad + (386\alpha^2 - 3806\alpha + 6000)n + 264\alpha^2 + 1896\alpha - 11280) \\ &> \frac{1}{125}((18\alpha^2 - 63\alpha + 45)(\frac{7}{1-\alpha} + 3)^3 - (212\alpha^2 - 997\alpha + 965)(\frac{7}{1-\alpha} + 3)^2 \\ &\quad + (386\alpha^2 - 3806\alpha + 6000)(\frac{7}{1-\alpha} + 3) + 264\alpha^2 + 1896\alpha - 11280) \\ &= \frac{1}{125(1-\alpha)^2}(550\alpha^3 - 4825\alpha^2 + 7496\alpha - 2780) \\ &> 0.\end{aligned}$$

Therefore, we conclude that $f_{B_1}(n-5) \geq \min\{\Phi(4,n), \Phi(\frac{2}{5}n - \frac{6}{5}, n)\} > 0$ for $s \geq 4$. By (6), we have $\rho_\alpha(G) < \theta(n) < \tau(n)$ for $s \geq 4$, which contradicts $\rho_\alpha(G) > \tau(n)$.

Case 2. $G - S$ has no non-trivial connected component.

In this case, $G = K_s \vee iK_1$. Now we show $i = \lfloor \frac{3s}{2} \rfloor + 1$ or $i = \lfloor \frac{3s}{2} \rfloor + 2$.

If $i \geq \lfloor \frac{3s}{2} \rfloor + 3$, then we create a new graph G''' by adding an edge between two vertices in iK_1. Thus, $i(G''' - S) = i - 2 \geq \lfloor \frac{3s}{2} \rfloor + 1$ and $G''' - S$ admits exactly one non-trivial connected component. Obviously, G is a proper subgraph of G''', and then $\rho_\alpha(G) < \rho_\alpha(G''') \le \tau(n)$ by applying Case 1, a contradiction. Therefore, $i = \lfloor \frac{3s}{2} \rfloor + 1$ or $i = \lfloor \frac{3s}{2} \rfloor + 2$ by $i > \frac{3s}{2}$.

Subcase 2.1. $i = \lfloor \frac{3s}{2} \rfloor + 1$.

Obviously, $n = s + \lfloor \frac{3s}{2} \rfloor + 1$, and the quotient matrix of $A_\alpha(G)$ in view of the partition $\{V((\lfloor \frac{3s}{2} \rfloor + 1)K_1), V(K_s)\}$ equals to

$$B_2 = \begin{pmatrix} \alpha s & (1-\alpha)s \\ (1-\alpha)(\lfloor \frac{3s}{2} \rfloor + 1) & \alpha n - \alpha s + s - 1 \end{pmatrix}.$$

Then the characteristic polynomial of B_2 is

$$\begin{aligned} f_{B_2}(x) &= x^2 - (\alpha n + s - 1)x + \alpha^2 sn - (\alpha^2 - 2\alpha + 1)s\lfloor \frac{3s}{2} \rfloor - (\alpha^2 - \alpha)s^2 - (\alpha^2 - \alpha)s - s \\ &= x^2 - (\alpha n + s - 1)x + (2\alpha - 1)s\lfloor \frac{3s}{2} \rfloor + \alpha s^2 + \alpha s - s, \end{aligned}$$

and $\rho_\alpha(G)$ is the largest root of $f_{B_2}(x) = 0$ by Lemma 2.2. By a simple computation, we have

$$\rho_\alpha(G) = \frac{\alpha n + s - 1 + \sqrt{(\alpha n + s - 1)^2 - 4((2\alpha - 1)s\lfloor \frac{3s}{2} \rfloor + \alpha s^2 + \alpha s - s)}}{2}. \tag{8}$$

Now we show $\rho_\alpha(G) < n - 3$. It follows from $n = s + \lfloor \frac{3s}{2} \rfloor + 1$ that

$$\begin{aligned} &(2(n-3) - (\alpha n + s - 1))^2 - (\alpha n + s - 1)^2 + 4((2\alpha - 1)s\lfloor \frac{3s}{2} \rfloor + \alpha s^2 + \alpha s - s) \\ &= (4 - 4\alpha)n^2 - (4s + 20 - 12\alpha)n + (8\alpha - 4)s\lfloor \frac{3s}{2} \rfloor + 4\alpha s^2 + 4\alpha s + 8s + 24 \\ &= \begin{cases} 9(1-\alpha)s^2 - 4(-5\alpha + 8)s + 5\alpha + 15, & \text{if } s \text{ is odd}, \\ 9(1-\alpha)s^2 - 2(-7\alpha + 13)s + 8\alpha + 8, & \text{if } s \text{ is even}. \end{cases} \end{aligned} \tag{9}$$

Subcase 2.1.1. s is odd.

Let $t_1(s) = 9(1-\alpha)s^2 - 4(-5\alpha + 8)s + 5\alpha + 15$. Then we obtain $t_1(9) = 456 - 544\alpha$, $t_1(11) = 752 - 864\alpha$ and $t_1(\frac{19-5\alpha}{5(1-\alpha)}) = \frac{1}{25(1-\alpha)}(-400\alpha^2 + 740\alpha + 584)$.

Since

$$n = \frac{5}{2}s + \frac{1}{2} \ge \begin{cases} 20, & \text{if } \alpha \in [0, \frac{1}{2}], \\ 25, & \text{if } \alpha \in (\frac{1}{2}, \frac{5}{7}], \\ \frac{7}{1-\alpha} + 3, & \text{if } \alpha \in (\frac{5}{7}, 1), \end{cases}$$

we have

$$\frac{4(-5\alpha + 8)}{2(9 - 9\alpha)} < \begin{cases} 3 < 9 \le s, & \text{if } \alpha \in [0, \frac{1}{2}], \\ 4 < 11 \le s, & \text{if } \alpha \in (\frac{1}{2}, \frac{5}{7}], \\ \frac{19-5\alpha}{5(1-\alpha)} \le s, & \text{if } \alpha \in (\frac{5}{7}, 1), \end{cases}$$

and

$$t_1(s) \ge \begin{cases} t_1(9) = 456 - 544\alpha > 0, & \text{if } \alpha \in [0, \frac{1}{2}], \\ t_1(11) = 752 - 864\alpha > 0, & \text{if } \alpha \in (\frac{1}{2}, \frac{5}{7}], \\ t_1(\frac{19-5\alpha}{5(1-\alpha)}) = \frac{1}{25(1-\alpha)}(-400\alpha^2 + 740\alpha + 584) > 0, & \text{if } \alpha \in (\frac{5}{7}, 1). \end{cases}$$

Subcase 2.1.2. s is even.

Let $t_2(s) = 9(1-\alpha)s^2 - 2(-7\alpha+13)s + 8\alpha + 8$. Then we obtain $t_2(8) = 376 - 456\alpha$, $t_2(10) = 648 - 752\alpha$ and $t_2(\frac{2(9-2\alpha)}{5(1-\alpha)}) = \frac{1}{25(1-\alpha)}(-336\alpha^2 + 484\alpha + 776)$.

Since

$$n = \frac{5}{2}s + 1 \geq \begin{cases} 20, & \text{if } \alpha \in [0, \frac{1}{2}], \\ 25, & \text{if } \alpha \in (\frac{1}{2}, \frac{5}{7}], \\ \frac{7}{1-\alpha} + 3, & \text{if } \alpha \in (\frac{5}{7}, 1), \end{cases}$$

we have

$$\frac{2(-7\alpha+13)}{2(9-9\alpha)} < \begin{cases} 3 < 8 \leq s, & \text{if } \alpha \in [0, \frac{1}{2}], \\ 4 < 10 \leq s, & \text{if } \alpha \in (\frac{1}{2}, \frac{5}{7}], \\ \frac{2(9-2\alpha)}{5(1-\alpha)} \leq s, & \text{if } \alpha \in (\frac{5}{7}, 1), \end{cases}$$

and

$$t_2(s) \geq \begin{cases} t_2(8) = 376 - 456\alpha > 0, & \text{if } \alpha \in [0, \frac{1}{2}], \\ t_2(10) = 648 - 752\alpha > 0, & \text{if } \alpha \in (\frac{1}{2}, \frac{5}{7}], \\ t_2(\frac{2(9-2\alpha)}{5(1-\alpha)}) = \frac{1}{25(1-\alpha)}(-336\alpha^2 + 484\alpha + 776) > 0, & \text{if } \alpha \in (\frac{5}{7}, 1). \end{cases}$$

Therefore, by (8), (9), $t_1(s) > 0$ and $t_2(s) > 0$, we have $\rho_\alpha(G) < n - 3$.

Subcase 2.2. $i = \lfloor \frac{3s}{2} \rfloor + 2$.

Obviously, $n = s + \lfloor \frac{3s}{2} \rfloor + 2$, and the quotient matrix of $A_\alpha(G)$ in view of the partition $\{V((\lfloor \frac{3s}{2} \rfloor + 2)K_1), V(K_s)\}$ equals to

$$B_3 = \begin{pmatrix} \alpha s & (1-\alpha)s \\ (1-\alpha)(\lfloor \frac{3s}{2} \rfloor + 2) & \alpha n - \alpha s + s - 1 \end{pmatrix}.$$

Then the characteristic polynomial of B_3 is

$$\begin{aligned} f_{B_3}(x) &= x^2 - (\alpha n + s - 1)x + \alpha^2 sn - (\alpha^2 - 2\alpha + 1)s\lfloor \frac{3s}{2} \rfloor - (\alpha^2 - \alpha)s^2 - (2\alpha^2 - 3\alpha + 2)s \\ &= x^2 - (\alpha n + s - 1)x + (2\alpha - 1)s\lfloor \frac{3s}{2} \rfloor + \alpha s^2 + (3\alpha - 2)s, \end{aligned}$$

and $\rho_\alpha(G)$ is the largest root of $f_{B_3}(x) = 0$ by Lemma 2.2. By a simple computation, we have

$$\rho_\alpha(G) = \frac{\alpha n + s - 1 + \sqrt{(\alpha n + s - 1)^2 - 4((2\alpha - 1)s\lfloor \frac{3s}{2} \rfloor + \alpha s^2 + 3\alpha s - 2s)}}{2}. \tag{10}$$

Now we show $\rho_\alpha(G) < n - 3$. It follows from $n = s + \lfloor \frac{3s}{2} \rfloor + 2$ that

$$\begin{aligned} &(2(n-3) - (\alpha n + s - 1))^2 - (\alpha n + s - 1)^2 + 4((2\alpha - 1)s\lfloor \frac{3s}{2} \rfloor + \alpha s^2 + 3\alpha s - 2s) \\ &= (4 - 4\alpha)n^2 - (4s + 20 - 12\alpha)n + (8\alpha - 4)s\lfloor \frac{3s}{2} \rfloor + 4\alpha s^2 + 4\alpha s + 8s + 24 \\ &= \begin{cases} 9(1-\alpha)s^2 - 4(-2\alpha + 5)s + 9\alpha + 3, & \text{if } s \text{ is odd}, \\ 9(1-\alpha)s^2 - 2(-\alpha + 7)s + 8\alpha, & \text{if } s \text{ is even}. \end{cases} \end{aligned} \tag{11}$$

Subcase 2.2.1. s is odd.

Let $t_3(s) = 9(1-\alpha)s^2 - 4(-2\alpha + 5)s + 9\alpha + 3$. Then we obtain $t_3(9) = 552 - 648\alpha$, $t_3(11) = 872 - 992\alpha$ and $t_3(\frac{17-3\alpha}{5(1-\alpha)}) = \frac{1}{25(1-\alpha)}(-264\alpha^2 + 212\alpha + 976)$.

Since

$$n=\frac{5}{2}s+\frac{3}{2}\geq\begin{cases}20, & \text{if } \alpha\in[0,\frac{1}{2}],\\ 25, & \text{if } \alpha\in(\frac{1}{2},\frac{5}{7}],\\ \frac{7}{1-\alpha}+3, & \text{if } \alpha\in(\frac{5}{7},1),\end{cases}$$

we have

$$\frac{4(-2\alpha+5)}{2(9-9\alpha)}<\begin{cases}2<9\leq s, & \text{if } \alpha\in[0,\frac{1}{2}],\\ 3<11\leq s, & \text{if } \alpha\in(\frac{1}{2},\frac{5}{7}],\\ \frac{17-3\alpha}{5(1-\alpha)}\leq s, & \text{if } \alpha\in(\frac{5}{7},1),\end{cases}$$

and

$$t_3(s)\geq\begin{cases}t_3(9)=552-648\alpha>0, & \text{if } \alpha\in[0,\frac{1}{2}],\\ t_3(11)=872-992\alpha>0, & \text{if } \alpha\in(\frac{1}{2},\frac{5}{7}],\\ t_3(\frac{17-3\alpha}{5(1-\alpha)})=\frac{1}{25(1-\alpha)}(-264\alpha^2+212\alpha+976)>0, & \text{if } \alpha\in(\frac{5}{7},1).\end{cases}$$

Subcase 2.2.2. s is even.

Let $t_4(s)=9(1-\alpha)s^2-2(-\alpha+7)s+8\alpha$. Then we obtain $t_4(10)=464-552\alpha$, $t_4(10)=760-872\alpha$ and $t_4(\frac{2(8-\alpha)}{5(1-\alpha)})=\frac{1}{25(1-\alpha)}(-184\alpha^2-76\alpha+1184)$.

Since

$$n=\frac{5}{2}s+2\geq\begin{cases}20, & \text{if } \alpha\in[0,\frac{1}{2}],\\ 25, & \text{if } \alpha\in(\frac{1}{2},\frac{5}{7}],\\ \frac{7}{1-\alpha}+3, & \text{if } \alpha\in(\frac{5}{7},1),\end{cases}$$

we have

$$\frac{4(-2\alpha+5)}{2(9-9\alpha)}<\begin{cases}2<8\leq s, & \text{if } \alpha\in[0,\frac{1}{2}],\\ 3<10\leq s, & \text{if } \alpha\in(\frac{1}{2},\frac{5}{7}],\\ \frac{2(8-\alpha)}{5(1-\alpha)}\leq s, & \text{if } \alpha\in(\frac{5}{7},1),\end{cases}$$

and

$$t_4(s)\geq\begin{cases}t_4(8)=464-552\alpha>0, & \text{if } \alpha\in[0,\frac{1}{2}],\\ t_4(10)=760-872\alpha>0, & \text{if } \alpha\in(\frac{1}{2},\frac{5}{7}],\\ t_4(\frac{2(8-\alpha)}{5(1-\alpha)})=\frac{1}{25(1-\alpha)}(-184\alpha^2-76\alpha+1184)>0, & \text{if } \alpha\in(\frac{5}{7},1).\end{cases}$$

Therefore, by (10), (11), $t_3(s)>0$ and $t_4(s)>0$, we have $\rho_\alpha(G)<n-3$.

Note that $\tau(n)>n-3$. Combining the above arguments, we conclude $\rho_\alpha(G)<n-3<\tau(n)$, which contradicts $\rho_\alpha(G)>\tau(n)$.

By Case 1 and Case 2, we complete the proof of Theorem 1.2. □

5. Extremal graphs

In this section, we claim that the condition in Theorem 1.2 is best possible.

Theorem 5.1. *Let $\alpha\in[0,1)$, n, $\tau(n)$ be as in Theorem 1.2. Then $\rho_\alpha(K_1\vee(K_{n-3}\cup 2K_1))=\tau(n)$, and $K_1\vee(K_{n-3}\cup 2K_1)$ contains no $\{P_2,C_3,P_5,\mathcal{T}(3)\}$-factor.*

Proof. By the proof of Theorem 1.2, we have $\rho_\alpha(K_1\vee(K_{n-3}\cup 2K_1))=\tau(n)$. Let v be the vertex with the maximum degree of $K_1\vee(K_{n-3}\cup 2K_1)$. Set $S=\{v\}$, then we infer $i(K_1\vee(K_{n-3}\cup 2K_1)-S)=2>\frac{3}{2}|S|$. By Lemma 2.6, the graph $K_1\vee(K_{n-3}\cup 2K_1)$ contains no $\{P_2,C_3,P_5,\mathcal{T}(3)\}$-factor. Therefore, the bound on A_α-spectral radius established in Theorem 1.2 is sharp. □

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