



On uniformly GA-convex functions

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Abstract. An ordinary convexity can be interpreted in the form of an inequality between arithmetic means and referred as to AA-convexity. Other classes of convex functions that include means are also known in the literature. Depending on which type of mean is included, arithmetic A or geometric G, there are also GG-convex, AG-convex and GA-convex functions. On the other side, a class with stronger property that ordinary convex class is known as uniform convexity. In this paper, we connect these two concepts, GA-convexity with the uniform convexity, and introduce a new concept named uniform GA-convexity. By analyzing the newly defined class we prove that it inherits some good properties from both classes of convexity. For uniformly GA-convex functions we prove few basic inequalities as Jensen's inequality, the Jensen-Mercer inequality and the Hermite-Hadamard inequality. As applications of the main results we obtain some analytic inequalities and new estimates of some statistical divergences.

1. Introduction and Preliminaries

The theory of convex functions experienced an accelerated development starting from the appearance of the works of J. L. Jensen until today. There is almost no area of mathematical analysis in which convex functions are not used indirectly. Let us recall that a function $f: [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is convex if

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) \quad (1)$$

holds for all $x, y \in [a, b]$ and $t \in [0, 1]$.

The theory of convex functions is closely related to the theory of mathematical inequalities. Together, these theories find various applications in many scientific and applied fields. One of the most important and basic inequality for convex functions is Jensen's inequality

$$q\left(\sum_{i=1}^n p_i x_i\right) \leq \sum_{i=1}^n p_i q(x_i)$$

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which holds for every convex function $q: [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}$, $x_i \in [a, b]$, $i = 1, \dots, n$, and $p_i \geq 0$, $i = 1, \dots, n$, with $\sum_{i=1}^n p_i = 1$ (see [29]). A slightly modified Jensen's inequality is known as the Jensen-Mercer inequality [24],

$$q\left(a + b - \sum_{i=1}^n p_i x_i\right) \leq q(a) + q(b) - \sum_{i=1}^n p_i q(x_i)$$

and holds for every convex function $q: [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}$, $x_i \in [a, b]$, $i = 1, \dots, n$, and $p_i \geq 0$, $i = 1, \dots, n$, with $\sum_{i=1}^n p_i = 1$ (see also [31]).

Another fundamental inequality is the Hermite-Hadamard inequality

$$q\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b q(x) dx \leq \frac{q(a) + q(b)}{2} \quad (2)$$

that holds for every convex function $q: [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}$.

Various inequalities valid for convex functions indicate the possibility of further generalization of convex functions and creation of new classes of convexity. One of such classes is the class of GA-convex functions. To specify, a function $f: [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$ is GA-convex if

$$f(G(x, y)) \leq A(f(x), f(y)),$$

where $G(x, y) = x^t y^{1-t}$ is the geometric mean of x and y and $A(f(x), f(y)) = tf(x) + (1-t)f(y)$ is the arithmetic mean of $f(x)$ and $f(y)$. In other words, a function $f: [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$ is GA-convex if

$$f(x^t y^{1-t}) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in [a, b]$ and $t \in [0, 1]$. These functions are also known as geometric-arithmetic or geometric-arithmetically convex functions.

We can interpret definition of ordinary convexity (1) also in the form of an inequality between means. If we denote the arithmetic mean of x and y by $A(x, y) = tx + (1-t)y$, then (1) is equivalent to

$$f(A(x, y)) \leq A(f(x), f(y)).$$

In this context, an ordinary convexity can be referred as to AA-convexity. Other classes of convex functions that include means are also known in the literature. Depending on which type of mean is included, arithmetic A or geometric G, there are also GG-convex functions and AG-convex functions. More information on this topic the reader can find in [5, 23, 25, 26].

One useful characterization of convex function is by its second derivatives. A twice differentiable function f is convex iff $f'' \geq 0$. We can interpret analogous characterization for the class of GA-convex functions. Using the fact that a function $f: [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$ is GA-convex iff the function $g: [\ln a, \ln b] \rightarrow \mathbb{R}$, defined by $g = f \circ \exp$ is convex, we have that f is GA-convex iff $(f \circ \exp)'' \geq 0$ on $[\ln a, \ln b]$, i.e. iff $x^2 f'(x) + x f''(x) \geq 0$ for all $x \in [a, b]$.

Jensen's inequality for GA-convex function $q: [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$, $x_i \in [a, b]$, $i = 1, \dots, n$, and $p_i \geq 0$, $i = 1, \dots, n$, with $\sum_{i=1}^n p_i = 1$, has the form

$$q\left(\prod_{i=1}^n x_i^{p_i}\right) \leq \sum_{i=1}^n p_i q(x_i).$$

For GA-convex function $q: [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$ also the Hermite-Hadamard inequality

$$q(\sqrt{ab}) \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{q(x)}{x} dx \leq \frac{q(a) + q(b)}{2} \quad (3)$$

holds. Some other variants of the Hermite-Hadamard type inequalities for GA-convex functions can be found in [12, 13, 38].

The Jensen-Mercer inequality for GA-convex functions can be represented as

$$q\left(\frac{ab}{\prod_{i=1}^n x_i^{p_i}}\right) \leq q(a) + q(b) - \sum_{i=1}^n p_i q(x_i)$$

and holds for every GA-convex function $q: [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$, $x_i \in [a, b]$, $i = 1, \dots, n$, and $p_i \geq 0$, $i = 1, \dots, n$, with $\sum_{i=1}^n p_i = 1$ (see [17]).

In recent years, many generalizations of the concept of GA-convexity were introduced and different kinds of Jensen's, the Hermite-Hadamard and the Jensen-Mercer type inequalities have been established (see for example [4, 6, 14, 16, 22, 28, 30, 35, 36, 38]).

The goal of this paper is to introduce a new class of convexity that connects the concept of GA-convexity with uniformly convex functions which we present below.

A function $f: [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is uniformly convex with modulus ϕ , if $\phi: [0, \infty) \rightarrow [0, \infty)$ is an increasing function, vanishes only at 0, and

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - t(1-t)\phi(|x-y|) \quad (4)$$

holds for all $x, y \in [a, b]$ and $t \in [0, 1]$ (see [9, 37]).

It is obviously that uniform convexity implies ordinary convexity but the reverse implication is not true in generally. For example, a linear function is convex but not uniformly convex.

Particularly, when (4) holds with $\phi = k(\cdot)^2$, for some $k > 0$, then f is called strongly convex with modulus k . For example, the function $h(x) = x^2$ is strongly convex with modulus $k = 1$ (see [18, 20]).

Every continuous and differentiable uniformly convex function $f: [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}$ with modulus ϕ satisfies inequality

$$f(y) - f(x) \geq f'(x)(y-x) + \phi(|x-y|), \quad x, y \in [a, b],$$

(see [27], [37]). Moreover, for such function Jensen's inequality

$$\sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right) \geq \sum_{i=1}^n p_i \phi\left(\left|x_i - \sum_{j=1}^n p_j x_j\right|\right) \quad (5)$$

holds with $x_i \in [a, b]$, $i = 1, \dots, n$, $p_i \geq 0$, $i = 1, \dots, n$, such that $\sum_{i=1}^n p_i = 1$.

For more information about uniform convexity see [1, 2, 8, 10, 15, 27, 32, 33].

This paper is divided into four sections. After introduction, in the second section we give the basic notions, preliminary results and study uniformly GA-convex functions with examples. In the third section we present fundamental inequalities as Jensen's, the Jensen-Mercer and the Hermite-Hadamard type inequalities for uniformly GA-convex functions. In the last section, as applications of the main results, we obtain some analytic inequalities and derive new estimates for Shannon's entropy, the Kullback-Leibler divergence and Jeffreys distance.

2. Introduction and Preliminaries

We begin this section with definition of the new class of convex functions.

Definition 2.1. A function $f: [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$ is said to be uniformly GA-convex on $[a, b]$ with modulus ϕ if $\phi: [0, \infty) \rightarrow [0, \infty)$ is an increasing function and

$$f(x^t y^{1-t}) \leq tf(x) + (1-t)f(y) - t(1-t)\phi(|\ln x - \ln y|) \quad (6)$$

holds for all $x, y \in [a, b]$ and $t \in [0, 1]$. If $-f$ is uniformly GA-convex with modulus ϕ , then f is said to be uniformly GA-concave with modulus ϕ .

If

$$f(\sqrt{xy}) \leq \frac{f(x) + f(y)}{2} - \frac{1}{4}\phi(|\ln x - \ln y|) \quad (7)$$

holds for all $x, y \in [a, b]$, then f is said to be uniformly GA-midconvex on $[a, b]$ with modulus ϕ .

Obviously, uniform GA-convexity implies GA-convexity but the reverse implication is not true in general. Namely if $f: [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$ is an uniformly GA-convex function with modulus ϕ , then

$$\begin{aligned} f(x^t y^{1-t}) &\leq tf(x) + (1-t)f(y) - t(1-t)\phi(|\ln x - \ln y|) \\ &\leq tf(x) + (1-t)f(y) \end{aligned}$$

holds for all $x, y \in [a, b]$ and $t \in [0, 1]$. On the other side, for example the logarithm function $\ln(\cdot)$ is GA-convex but not uniformly GA-convex.

Remark 2.2. Note that by definition (6),

$$0 \leq \phi(|\ln x - \ln y|) \leq \frac{tf(x) + (1-t)f(y) - f(x^t y^{1-t})}{t(1-t)}$$

holds for all $x, y \in [a, b]$ and $t \in (0, 1)$. Specially, when $x = y$, then $\phi(0) \leq 0$, i.e. ϕ vanishes at 0.

As a summary of this, in the sequel by modulus we mean every function ϕ nonnegative and increasing on $[0, \infty)$ that vanishes at 0.

Definition 2.3. Let us define exact modulus of GA-convexity of f as

$$\mu_f(s) = \inf \left\{ \frac{tf(x) + (1-t)f(y) - f(x^t y^{1-t})}{t(1-t)} : x, y \in [a, b] \subseteq (0, \infty), s = |\ln x - \ln y|, t \in (0, 1) \right\}.$$

Note that μ_f can take negative values if f is not GA-convex.

Proposition 2.4. A function $f: [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$ is uniformly GA-convex with modulus μ_f iff $\mu_f(s) > 0$ for every $s > 0$.

Proof. It is direct consequence of definition of μ_f and Remark 2.2. $\square$

Remark 2.5. We can replace (6) with any of the following relations:

(D1) For every $\rho > 0$ there exists $\delta > 0$ such that for all $x, y \in [a, b]$, $|\ln y - \ln x| \geq \rho$, and $t \in (0, 1)$,

$$f(x^t y^{1-t}) \leq tf(x) + (1-t)f(y) - t(1-t)\delta; \quad (8)$$

(D2) For every $\rho > 0$ there exists $\delta > 0$ such that for all $x, y \in [a, b]$, $|\ln y - \ln x| \geq \rho$,

$$f(\sqrt{xy}) \leq \frac{f(x) + f(y)}{2} - \delta. \quad (9)$$

Relations (6), (8) and (9) are equivalent. Namely, the first equivalence (6) $\Leftrightarrow$ (8) is a consequence of the previous consideration.

Let's prove (8) $\Leftrightarrow$ (9). First we prove (8) $\Rightarrow$ (9). Let (8) holds. We can take $t = \frac{1}{2}$ in (8) and we get

$$f(\sqrt{xy}) \leq \frac{f(x) + f(y)}{2} - \delta',$$

where $\delta' = \frac{\delta}{4}$ is some nonnegative constant. Now we prove (9) $\Rightarrow$ (8). Let (9) holds and $t \in (0, \frac{1}{2}]$ (similarly for $t \in [\frac{1}{2}, 1)$). We have

$$\begin{aligned} f(x^t y^{1-t}) &= f\left(\left(\sqrt{xy}\right)^{2t} y^{1-2t}\right) \\ &\leq 2t f\left(\sqrt{xy}\right) + (1-2t)f(y) \\ &\leq 2t \left(\frac{f(x) + f(y)}{2} - \delta\right) + (1-2t)f(y) \\ &= tf(x) + tf(y) - 2t\delta + (1-2t)f(y) \\ &= tf(x) + (1-t)f(y) - 2t\delta \\ &\leq tf(x) + (1-t)f(y) - 2t(1-t)\delta, \end{aligned}$$

what we need to prove.

The following proposition connects the classes of uniformly convex functions with uniformly GA-convex functions.

Proposition 2.6. *If a function $f: [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$ is increasing and uniformly convex on $[a, b]$ with modulus ϕ , then f is also uniformly GA-convex on $[e^a, e^b]$ with the same modulus ϕ . If f is decreasing and uniformly concave on $[a, b]$ with modulus ϕ , then f is also uniformly GA-concave on $[e^a, e^b]$ with modulus ϕ .*

Proof. We prove only the case of convexity.

Since f is uniformly convex on $[a, b]$ with modulus ϕ , then

$$\begin{aligned} f(tx + (1-t)y) &\leq tf(x) + (1-t)f(y) - t(1-t)\phi(|x - y|) \\ &= tf(x) + (1-t)f(y) - t(1-t)\phi(|\ln e^x - \ln e^y|) \end{aligned} \quad (10)$$

holds for all $x, y \in [a, b]$ and $t \in [0, 1]$.

By assumption f is also increasing. Then as an easy consequence of AM-GM inequality, i.e. inequality between arithmetic and geometric mean,

$$x^t y^{1-t} \leq tx + (1-t)y,$$

we have

$$f(x^t y^{1-t}) \leq f(tx + (1-t)y) \quad (11)$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$. Combining (10) and (11) we have that f is also uniformly GA-convex on $[e^a, e^b]$ with modulus ϕ . $\square$

The following proposition gives connection in both directions between these two classes of convexity.

Proposition 2.7. *A function $f: [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$ is uniformly GA-convex with modulus ϕ iff the function $g: [\ln a, \ln b] \rightarrow \mathbb{R}$, defined by $g = f \circ \exp$, is uniformly convex on $[\ln a, \ln b]$, with the same modulus ϕ .*

Proof. " $\Leftarrow$ " Let $g: [\ln a, \ln b] \rightarrow \mathbb{R}$, defined by $g = f \circ \exp$, be uniformly convex on $[\ln a, \ln b]$ with modulus ϕ . Then

$$\begin{aligned} &(f \circ \exp)(t \ln x + (1-t) \ln y) \\ &\leq t(f \circ \exp)(\ln x) + (1-t)(f \circ \exp)(\ln y) - t(1-t)\phi(|\ln x - \ln y|) \end{aligned}$$

holds for every $x, y \in [a, b]$ and $t \in [0, 1]$, what is equivalent to

$$f(x^t y^{1-t}) \leq tf(x) + (1-t)f(y) - t(1-t)\phi(|\ln x - \ln y|)$$

Moreover, $f: [a, b] \rightarrow \mathbb{R}$ is uniformly GA-convex with modulus ϕ .

" $\Rightarrow$ " Let f be uniformly GA-convex on $[a, b]$ with modulus ϕ . Then

$$\begin{aligned} & (f \circ \exp)(t \ln x + (1-t) \ln y) \\ &= f(e^{t \ln x + (1-t) \ln y}) \\ &= f(x^t y^{1-t}) \\ &\leq t f(x) + (1-t) f(y) - t(1-t) \phi(|\ln x - \ln y|) \\ &\leq t (f \circ \exp)(\ln x) + (1-t) (f \circ \exp)(\ln y) - t(1-t) \phi(|\ln x - \ln y|). \end{aligned}$$

holds for every $x, y \in [a, b]$ and $t \in [0, 1]$. Moreover, $g = f \circ \exp$ is uniformly convex on $[\ln a, \ln b]$ with modulus ϕ . $\square$

Example 2.8. As emphasized in Introduction, the function $h(x) = x^2$ is uniformly convex with modulus $\phi(r) = r^2$. Let's consider the restriction of h on the interval $[e^a, e^b]$, with $b > a > 0$. Since we have

$$h(x) = (\ln(\exp(x)))^2 = (h \circ \ln \circ \exp)(x) = [h \circ \ln] \circ (\exp)(x) = (f \circ \exp)(x),$$

where $f(x) = \ln^2 x$, then by Proposition 2.7, it follows that $f(x) = \ln^2 x$ is uniformly GA-convex on $[a, b]$ with the same modulus $\phi(r) = r^2$. Furthermore, this function is also increasing on $[c, d] \subseteq [a, b]$, where $d > c > e$, where e is Euler's number, but not convex and neither uniformly convex on $[c, d]$. Moreover, the inverse implications in Proposition 2.6 are not valid in general.

Below we give more examples which are of interest in applications.

Example 2.9. Let $k, p \in \mathbb{R}$, $0 < a < b$ and $f: [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$ be the function defined by $f(x) = x^p$.

1. If $p > 0$ and $0 < k \leq \frac{a^p p^2}{2}$, then f is uniformly GA-convex with modulus $\phi(r) = kr^2$. In particular, the function $f(x) = x^2$ is uniformly GA-convex on $[a, b] \subseteq \left[\frac{\sqrt{2}}{2}, \infty\right)$ with modulus $\phi(r) = r^2$.
2. If $p < 0$ and $0 < k \leq \frac{b^p p^2}{2}$, then f is uniformly GA-convex on $[a, b]$ with modulus $\phi(r) = kr^2$. In particular, the function $f(x) = \frac{1}{x}$ is uniformly GA-convex on $(0, \frac{1}{2}]$ with modulus $\phi(r) = r^2$.

Proof. (1) By assumption $p > 0$ and $a^p p^2 \geq 2k$. It is obvious that $\phi(r) = r^2$ is nonnegative, increasing on $[0, \infty)$ and vanishes at 0. We consider two fixed points $x, y \in [a, b]$ and define

$$g(t) := x^{tp} y^{p-p^t} + kt(1-t)(\ln x - \ln y)^2$$

for every $t \in [0, 1]$.

Since $g(0) = y^p$, $g(1) = x^p$ and

$$\begin{aligned} \frac{d^2 g}{dt^2} &= p^2 x^{tp} y^{p-p^t} (\ln(x) - \ln(y))^2 - 2k (\ln(x) - \ln(y))^2 \\ &\geq (p^2 a^p - 2k) (\ln(x) - \ln(y))^2 \geq 0, \end{aligned}$$

then

$$g(t) \leq t f(x) + (1-t) f(y)$$

for every $x, y \in [a, b]$ and $t \in [0, 1]$. Therefore,

$$x^{tp} y^{p-p^t} + t(1-t)k(|\ln x - \ln y|)^2 \leq tx^p + (1-t)y^p$$

for every $x, y \in [a, b]$ and $t \in [0, 1]$ what we need to prove.

The last statement follows by choosing $k = 1$, $p = 2$ and $a = \frac{\sqrt{2}}{2}$.

(2) The proof of this part is quite similar to that of first part and the last statement follows by choosing $k = 1$, $p = -1$ and $b = \frac{1}{2}$. $\square$

Example 2.10. Let $0 < a < b$ and $0 < k \leq \frac{a \ln a + 2a}{2}$. Let $f: [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$ be defined by $f(x) = x \ln x$. Then f is uniformly GA-convex with modulus $\phi(r) = kr^2$. In particular, the function $f(x) = x \ln x$ is uniformly GA-convex with modulus $\phi(r) = r^2$ on $[a, b]$, with $a \ln a + 2a \geq 2$.

Proof. By assumption $k \leq \frac{a \ln a + 2a}{2}$, $a \ln a + 2a - 2k \geq 0$. It is obvious that ϕ is nonnegative, increasing on $[0, \infty)$ and vanishes at 0. We consider two fixed points $x, y \in [a, b]$ and define

$$g(t) := x^t y^{1-t} (t \ln x + (1-t) \ln y) + kt(1-t)(\ln x - \ln y)^2$$

$$g(t) := x^t y^{1-t} \ln x^t y^{1-t} + kt(1-t)(\ln x - \ln y)^2$$

for every $t \in [0, 1]$.

Since $g(0) = y \ln y$, $g(1) = x \ln x$ and

$$\begin{aligned} \frac{d^2 g}{dt^2} &= [x^t y^{1-t} (t \ln x + (1-t) \ln y + 2) - 2k] (\ln x - \ln y)^2 \\ &\geq [a \ln a + 2a - 2k] (\ln(x) - \ln(y))^2 \geq 0, \end{aligned}$$

then

$$g(t) \leq t f(x) + (1-t) f(y)$$

holds for every $x, y \in [a, b]$ and $t \in [0, 1]$. Therefore,

$$\begin{aligned} &x^t y^{1-t} \ln x^t y^{1-t} + t(1-t)k(\ln x - \ln y)^2 \\ &= x^t y^{1-t} (t \ln x + (1-t) \ln y) + t(1-t)k(\ln x - \ln y)^2 \\ &\leq t x \ln x + (1-t) y \ln y \end{aligned}$$

holds for every $x, y \in [a, b]$ and $t \in [0, 1]$.

The last statement follows by choosing $k = 1$. $\square$

Example 2.11. Let $0 < a < b$ and $0 < k \leq \frac{a \ln a + 2a}{2}$. Let $f: [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$ be defined by $f(x) = (x-1) \ln x$. Then f is uniformly GA-convex with modulus $\phi(r) = kr^2$. In particular, the function $f(x) = (x-1) \ln x$ is uniformly GA-convex with modulus $\phi(r) = r^2$ on $[a, b]$, with $a \ln a + 2a \geq 2$.

Proof. By assumption $0 < k \leq \frac{a \ln a + 2a}{2}$, i.e. $a \ln a + 2a - 2k \geq 0$. It is obvious that ϕ is nonnegative, increasing on $[0, \infty)$ and vanishes at 0. We consider two fixed points $x, y \in [a, b]$ and define

$$g(t) := (x^t y^{1-t} - 1) \ln x^t y^{1-t} + kt(1-t)(\ln x - \ln y)^2$$

for every $t \in [0, 1]$.

Since $g(0) = (y-1) \ln y$, $g(1) = (x-1) \ln x$ and

$$\begin{aligned} \frac{d^2 g}{dt^2} &= (x^t y^{1-t} \ln x^t y^{1-t} + 2x^t y^{1-t} - 2k) (\ln x - \ln y)^2 \\ &\geq (a \ln a + 2a - 2k) (\ln x - \ln y)^2 \geq 0, \end{aligned}$$

then

$$g(t) \leq t f(x) + (1-t) f(y)$$

holds for every $x, y \in [a, b]$ and $t \in [0, 1]$. Therefore,

$$\begin{aligned} &(x^t y^{1-t} - 1) \ln x^t y^{1-t} + kt(1-t)(\ln x - \ln y)^2 \\ &\leq t(x-1) \ln x + (1-t)(y-1) \ln y \end{aligned}$$

holds for every $x, y \in [a, b]$ and $t \in [0, 1]$. This ends the proof.

The last statement follows by choosing $k = 1$. $\square$

In the sequel we prove more characterizations of uniformly GA-convex functions.

Proposition 2.12. Let $f_1, f_2: [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$ be uniformly GA-convex functions with moduli ϕ_1, ϕ_2 , respectively. Then:

- a) λf_1 , where $\lambda > 0$, is uniformly GA-convex with modulus $\lambda\phi_1$;
- b) $f_1 + f_2$, is uniformly GA-convex with modulus $\phi_1 + \phi_2$.

Proof. a) Since f_1 is uniformly GA-convex with modulus ϕ_1 , then

$$f_1(x^t y^{1-t}) \leq t f_1(x) + (1-t) f_1(y) - t(1-t) \phi_1(|\ln x - \ln y|)$$

holds for all $x, y \in [a, b]$ and $t \in [0, 1]$. Multiplying it by $\lambda > 0$, we get

$$\lambda f_1(x^t y^{1-t}) \leq t \lambda f_1(x) + (1-t) \lambda f_1(y) - t(1-t) \lambda \phi_1(|\ln x - \ln y|).$$

b) Since f_1, f_2 are uniformly GA-convex with moduli ϕ_1, ϕ_2 , then inequalities

$$f_1(x^t y^{1-t}) \leq t f_1(x) + (1-t) f_1(y) - t(1-t) \phi_1(|\ln x - \ln y|),$$

$$f_2(x^t y^{1-t}) \leq t f_2(x) + (1-t) f_2(y) - t(1-t) \phi_2(|\ln x - \ln y|)$$

hold for all $x, y \in [a, b]$ and $t \in [0, 1]$. By summarizing, we get

$$\begin{aligned} f_1(x^t y^{1-t}) + f_2(x^t y^{1-t}) &\leq t f_1(x) + (1-t) f_1(y) - t(1-t) \phi_1(|\ln x - \ln y|) \\ &\quad + t f_2(x) + (1-t) f_2(y) - t(1-t) \phi_2(|\ln x - \ln y|), \end{aligned}$$

i.e.

$$\begin{aligned} (f_1 + f_2)(x^t y^{1-t}) &\leq t(f_1 + f_2)(x) + (1-t)(f_1 + f_2)(y) - t(1-t)(\phi_1 + \phi_2)(|\ln x - \ln y|). \end{aligned}$$

This ends the proof. $\square$

Proposition 2.13. Let $f_1, f_2: [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$ be two uniformly GA-convex functions with modulus ϕ . Then the function $f: [a, b] \rightarrow \mathbb{R}$, defined by

$$f(x) = \max_{x \in [a, b]} \{f_1(x), f_2(x)\},$$

is uniformly GA-convex with modulus ϕ .

Proof. Since f_1, f_2 are uniformly GA-convex with modulus ϕ , then

$$\begin{aligned} f_1(x^t y^{1-t}) &\leq t f_1(x) + (1-t) f_1(y) - t(1-t) \phi(|\ln x - \ln y|) \\ &\leq t f(x) + (1-t) f(y) - t(1-t) \phi(|\ln x - \ln y|) \end{aligned} \tag{12}$$

and

$$\begin{aligned} f_2(x^t y^{1-t}) &\leq t f_2(x) + (1-t) f_2(y) - t(1-t) \phi(|\ln x - \ln y|) \\ &\leq t f(x) + (1-t) f(y) - t(1-t) \phi(|\ln x - \ln y|), \end{aligned} \tag{13}$$

hold for all $x, y \in [a, b]$ and $t \in [0, 1]$, where the last inequalities in (12) and (13) are consequences of definition $f(x) = \max_{x \in [a, b]} \{f_1(x), f_2(x)\}$.

Now, by combining (12) and (13), we get

$$\begin{aligned} f(x^t y^{1-t}) &= \max_{x, y \in [a, b]} \{f_1(x^t y^{1-t}), f_2(x^t y^{1-t})\} \\ &\leq t f(x) + (1-t) f(y) - t(1-t) \phi(|\ln x - \ln y|) \end{aligned}$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$, what we needed to prove. $\square$

Proposition 2.14. Let ϕ_1, ϕ_2 be two moduli such that $\phi_2 \leq \phi_1$. If $f: [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$ is uniformly GA-convex functions with modulus ϕ_1 , then f is also uniformly GA-convex with modulus ϕ_2 .

Proof. Since f is uniformly GA-convex function with modulus ϕ_1 , then

$$t(1-t)\phi_1(|\ln x - \ln y|) \leq tf(x) + (1-t)f(y) - f(x^t y^{1-t})$$

holds for every $x, y \in [a, b]$ and $t \in [0, 1]$. Moreover, if $\phi_2 \leq \phi_1$, then we have that

$$t(1-t)\phi_2(|\ln x - \ln y|) \leq t(1-t)\phi_1(|\ln x - \ln y|) \leq tf(x) + (1-t)f(y) - f(x^t y^{1-t})$$

holds for every $x, y \in [a, b]$ and $t \in (0, 1)$. This ends the proof. $\square$

We finish this section with characterization that includes differentiability.

Proposition 2.15. Let a function $f: [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$ be differentiable uniformly GA-convex with modulus ϕ . Then

$$[f'(x)x - f'(y)y](\ln x - \ln y) \geq 2\phi|\ln x - \ln y| \quad (14)$$

holds for all $x, y \in [a, b]$.

Proof. Since f is uniformly GA-convex with modulus ϕ , then

$$f(x^t y^{1-t}) \leq tf(x) + (1-t)f(y) - t(1-t)\phi|\ln x - \ln y|$$

holds for all $x, y \in [a, b]$ and $t \in [0, 1]$. Further we have

$$f\left(\left(\frac{x}{y}\right)^t y\right) - f(y) \leq t(f(x) - f(y)) - t(1-t)\phi|\ln x - \ln y|,$$

i.e. for all $x, y \in [a, b]$ and $t \in (0, 1]$,

$$\frac{f\left(\left(\frac{x}{y}\right)^t y\right) - f(y)}{t} \leq f(x) - f(y) - (1-t)\phi|\ln x - \ln y|.$$

By taking limit

$$\lim_{t \rightarrow 0^+} \frac{f\left(\left(\frac{x}{y}\right)^t y\right) - f(y)}{t} \leq f(x) - f(y) - \phi|\ln x - \ln y|$$

we get

$$f'(y)(\ln x - \ln y)y \leq f(x) - f(y) - \phi|\ln x - \ln y|.$$

If x and y are interchanged in the previous inequality, we get

$$f'(x)(\ln y - \ln x)x \leq f(y) - f(x) - \phi|\ln y - \ln x|.$$

By summing the previous two inequalities, we get

$$f'(y)(\ln x - \ln y)y + f'(x)(\ln y - \ln x)x \leq -2\phi|\ln x - \ln y|,$$

what is equivalent to (14). $\square$

3. Some inequalities for uniformly GA-convex functions

In this section we prove some basic inequalities for uniformly GA-convex functions. We start with Jensen type inequalities.

Theorem 3.1. Let $f: [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$ be an uniformly GA-convex function with modulus ϕ . Let $\{x_k\}_{k=1}^n \subseteq [a, b]$ be a sequence and $p_i \geq 0, i = 1, \dots, n$, with $\sum_{i=1}^n p_i = 1$. Then

$$f\left(\prod_{i=1}^n x_i^{p_i}\right) \leq \sum_{i=1}^n p_i f(x_i) - \sum_{i=1}^n p_i \phi\left(\left|\ln x_i - \ln \prod_{j=1}^n x_j^{p_j}\right|\right). \quad (15)$$

Proof. Since f is uniformly GA-convex on $[a, b]$ with modulus ϕ , by Proposition 2.7 the function $g: [\ln a, \ln b] \rightarrow \mathbb{R}$, defined by $g = f \circ \exp$, is uniformly convex on $[\ln a, \ln b]$ with modulus ϕ . Then by (5) we have

$$\sum_{i=1}^n p_i g(\ln x_i) - g\left(\sum_{i=1}^n p_i \ln x_i\right) \geq \sum_{i=1}^n p_i \phi\left(\left|\ln x_i - \sum_{j=1}^n p_j \ln x_j\right|\right),$$

i.e.

$$\sum_{i=1}^n p_i (f \circ \exp)(\ln x_i) - (f \circ \exp)\left(\sum_{i=1}^n p_i \ln x_i\right) \geq \sum_{i=1}^n p_i \phi\left(\left|\ln x_i - \sum_{j=1}^n p_j \ln x_j\right|\right)$$

what is equivalent to (15). $\square$

Theorem 3.2. Let $f: [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$ be an uniformly GA-convex function with modulus ϕ . Let $\{x_k\}_{k=1}^n \subseteq [a, b]$ be a sequence, τ be a permutation on $\{1, \dots, n\}$ such that

$$x_{\tau(1)} \leq x_{\tau(2)} \leq \dots \leq x_{\tau(n)}$$

and $p_i \geq 0, i = 1, \dots, n$, with $\sum_{i=1}^n p_i = 1$. Then

$$f\left(\prod_{i=1}^n x_i^{p_i}\right) \leq \sum_{i=1}^n p_i f(x_i) - \sum_{i=1}^{n-1} p_{\tau(i)} p_{\tau(i+1)} \phi\left(\ln\left(\frac{x_{\tau(i+1)}}{x_{\tau(i)}}\right)\right). \quad (16)$$

Proof. Without loss of generality, we assume that $x_1 \leq x_2 \leq \dots \leq x_n$. We prove result using the method of induction.

For $n = 2$ the statement follows by definition. Assume that (16) holds for some $n > 2$. Therefore, we have

$$\begin{aligned} f\left(\prod_{i=1}^{n+1} x_i^{p_i}\right) &= f\left(\prod_{i=1}^{n-1} x_i^{p_i} \cdot \left(x_n^{\frac{p_n}{p_n+p_{n+1}}} \cdot x_{n+1}^{\frac{p_{n+1}}{p_n+p_{n+1}}}\right)^{p_n+p_{n+1}}\right) \\ &\leq \sum_{i=1}^{n-1} p_i f(x_i) + (p_n + p_{n+1}) f\left(x_n^{\frac{p_n}{p_n+p_{n+1}}} \cdot x_{n+1}^{\frac{p_{n+1}}{p_n+p_{n+1}}}\right) \\ &\quad - p_1 p_2 \phi\left(\ln\left(\frac{x_2}{x_1}\right)\right) - \dots - p_{n-2} p_{n-1} \phi\left(\ln\left(\frac{x_{n-1}}{x_{n-2}}\right)\right) \\ &\quad - p_{n-1} (p_n + p_{n+1}) \phi\left(\left|\ln(x_{n-1}) - \ln\left(x_n^{\frac{p_n}{p_n+p_{n+1}}} \cdot x_{n+1}^{\frac{p_{n+1}}{p_n+p_{n+1}}}\right)\right|\right). \end{aligned} \quad (17)$$

Since $x_{n-1} \leq x_n \leq x_{n+1}$ and the function $\ln(\cdot)$ is increasing, then

$$\ln\left(x_n^{\frac{p_n}{p_n+p_{n+1}}} \cdot x_{n+1}^{\frac{p_{n+1}}{p_n+p_{n+1}}}\right) - \ln(x_{n-1}) \geq \ln(x_n) - \ln(x_{n-1})$$

and thus

$$\phi\left(\left|\ln(x_{n-1}) - \ln\left(x_n^{\frac{p_n}{p_n+p_{n+1}}} \cdot x_{n+1}^{\frac{p_{n+1}}{p_n+p_{n+1}}}\right)\right|\right) \geq \phi(\ln(x_n) - \ln(x_{n-1})).$$

Hence by (17), we get

$$\begin{aligned} f\left(\prod_{i=1}^{n+1} x_i^{p_i}\right) &\leq \sum_{i=1}^{n-1} p_i f(x_i) + (p_n + p_{n+1}) \left\{ \frac{p_n f(x_n)}{p_n + p_{n+1}} + \frac{p_{n+1} f(x_{n+1})}{p_n + p_{n+1}} \right. \\ &\quad \left. - \frac{p_n p_{n+1}}{(p_n + p_{n+1})^2} \phi\left(\ln\left(\frac{x_{n+1}}{x_n}\right)\right) \right\} - \sum_{i=1}^{n-1} p_i p_{i+1} \phi\left(\ln\left(\frac{x_{i+1}}{x_i}\right)\right) \\ &\leq \sum_{i=1}^{n+1} p_i f(x_i) - \sum_{i=1}^n p_i p_{i+1} \phi\left(\ln\left(\frac{x_{i+1}}{x_i}\right)\right), \end{aligned}$$

achieving inequality (16). $\square$

Corollary 3.3. Let $f: [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$ be an uniformly GA-convex function with modulus ϕ on $[a, b]$. Let $\{x_k\}_{k=1}^n \subseteq [a, b]$ be a sequence and let τ be a permutation on $\{1, \dots, n\}$ such that

$$x_{\tau(1)} \geq x_{\tau(2)} \geq \dots \geq x_{\tau(n)}$$

and $p_i \geq 0, i = 1, \dots, n$, with $\sum_{i=1}^n p_i = 1$. Then

$$f\left(\prod_{i=1}^n x_i^{p_i}\right) \leq \sum_{i=1}^n p_i f(x_i) - \sum_{i=1}^{n-1} p_{\tau(i)} p_{\tau(i+1)} \phi\left(\ln\left(\frac{x_{\tau(i)}}{x_{\tau(i+1)}}\right)\right). \quad (18)$$

Proof. An immediate consequence of Theorem 3.2. $\square$

As an easy consequence of Theorem 3.2 we also get the next corollary by setting $p_i = \frac{1}{n}$, for $i = 1, \dots, n$.

Corollary 3.4. Let $f: [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$ be an uniformly GA-convex function with modulus ϕ . Let $x_1 \leq x_2 \leq \dots \leq x_n$ be a sequence in $[a, b]$. Then

$$f\left(\sqrt[n]{x_1 x_2 \cdots x_n}\right) \leq \frac{1}{n} \sum_{i=1}^n f(x_i) - \frac{1}{n^2} \sum_{i=1}^{n-1} \phi\left(\ln\left(\frac{x_i}{x_{i+1}}\right)\right). \quad (19)$$

Remark 3.5. Inequality (19) presents generalization of (7).

The next we present the Hermite-Hadamard inequality for uniformly GA-convex functions.

Theorem 3.6. Let $f: [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$ be an uniformly GA-convex function with modulus ϕ . Then

$$\begin{aligned} f\left(\sqrt{ab}\right) + \frac{1}{4 \ln\left(\frac{b}{a}\right)} \int_0^{\ln\left(\frac{b}{a}\right)} \phi(u) du &\leq \frac{1}{\ln\left(\frac{b}{a}\right)} \int_a^b \frac{f(u) du}{u} \\ &\leq \frac{f(a) + f(b)}{2} - \frac{1}{6} \phi\left(\ln\left(\frac{b}{a}\right)\right). \end{aligned} \quad (20)$$

Proof. If we set $x: = a$ and $y: = b$ in (6), we get

$$f(a^t b^{1-t}) + t(1-t) \phi\left(\ln\left(\frac{b}{a}\right)\right) \leq t f(a) + (1-t) f(b).$$

By integrating both sides of the above inequality from $t = 0$ to $t = 1$ with respect to t , we obtain the right-hand inequality in (20).

To prove the left-hand inequality, we use (7) what is equivalent to

$$f(\sqrt{xy}) + \frac{1}{4}\phi\left(\left|\ln\left(\frac{y}{x}\right)\right|\right) \leq \frac{f(x) + f(y)}{2} \quad (21)$$

for every $x, y \in [a, b]$.

Setting

$$x = a^t b^{1-t} \quad \text{and} \quad y = a^{1-t} b^t$$

in (21), we obtain

$$f(\sqrt{ab}) + \frac{1}{4}\phi\left(\left|\ln\left(\frac{b}{a}\right)\right|2t - 1\right) \leq \frac{f(a^t b^{1-t}) + f(a^{1-t} b^t)}{2}$$

By integrating both sides of the previous inequality from $t = 0$ to $t = 1$ with respect to t , we get the left-hand inequality in (20), because

$$\int_0^1 \phi\left(\left|\ln\left(\frac{b}{a}\right)\right|2t - 1\right) dt = \frac{1}{\ln\left(\frac{b}{a}\right)} \int_0^{\ln\left(\frac{b}{a}\right)} \phi(u) du$$

and

$$\int_0^1 f(a^t b^{1-t}) dt = \int_0^1 f(a^{1-t} b^t) dt = \frac{1}{\ln\left(\frac{b}{a}\right)} \int_a^b \frac{f(u) du}{u}$$

hold. $\square$

To prove the Jensen-Mercer type inequalities for GA-convex functions we need the following lemma.

Lemma 3.7. Let $f: [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$ be an uniformly GA-convex function with modulus ϕ . Then inequality

$$f\left(\frac{ab}{x}\right) \leq f(a) + f(b) - f(x) - \frac{2\ln\left(\frac{x}{a}\right)\ln\left(\frac{b}{x}\right)}{\left(\ln\left(\frac{b}{a}\right)\right)^2} \phi\left(\ln\left(\frac{b}{a}\right)\right) \quad (22)$$

holds for all $x \in [a, b]$.

Proof. Let $x \in [a, b]$. Then for some $t \in [0, 1]$, we have $x = a^t b^{1-t}$. Thus by (6) we have

$$f(x) + t(1-t)\phi\left(\ln\left(\frac{b}{a}\right)\right) \leq tf(a) + (1-t)f(b). \quad (23)$$

On the other hand,

$$\begin{aligned} & f\left(\frac{ab}{x}\right) + t(1-t)\phi\left(\ln\left(\frac{b}{a}\right)\right) \\ &= f(a^{1-t} b^t) + t(1-t)\phi\left(\ln\left(\frac{b}{a}\right)\right) \leq (1-t)f(a) + tf(b). \end{aligned} \quad (24)$$

Adding (23) and (24), we obtain

$$f(x) + f\left(\frac{ab}{x}\right) + 2t(1-t)\phi\left(\ln\left(\frac{b}{a}\right)\right) \leq f(a) + f(b). \quad (25)$$

Since

$$t = \frac{\ln\left(\frac{b}{x}\right)}{\ln\left(\frac{b}{a}\right)} \quad \text{and} \quad 1 - t = \frac{\ln\left(\frac{x}{a}\right)}{\ln\left(\frac{b}{a}\right)},$$

the result follows from (25). $\square$

Theorem 3.8. Let $f: [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$ be an uniformly GA-convex function with modulus ϕ . Let $\{x_k\}_{k=1}^n \subseteq [a, b]$ be a sequence and $p_i \geq 0, i = 1, \dots, n$, with $\sum_{i=1}^n p_i = 1$. Then

$$\begin{aligned} f\left(\frac{ab}{\prod_{i=1}^n x_i^{p_i}}\right) &\leq f(a) + f(b) - \sum_{i=1}^n p_i f(x_i) - \frac{2\phi\left(\ln\left(\frac{b}{a}\right)\right)}{\left(\ln\left(\frac{b}{a}\right)\right)^2} \sum_{i=1}^n p_i \ln\left(\frac{x_i}{a}\right) \ln\left(\frac{b}{x_i}\right) \\ &\quad - \sum_{i=1}^n p_i \phi\left(\left|\ln \prod_{j=1}^n x_j^{p_j} - \ln x_i\right|\right) \end{aligned}$$

Proof. Applying Theorem 3.1 and Lemma 3.7, we have

$$\begin{aligned} f\left(\frac{ab}{\prod_{i=1}^n x_i^{p_i}}\right) &= f\left(\prod_{i=1}^n \left(\frac{ab}{x_i}\right)^{p_i}\right) \\ &\leq \sum_{i=1}^n p_i f\left(\frac{ab}{x_i}\right) - \sum_{i=1}^n p_i \phi\left(\left|\ln\left(\frac{ab}{x_i}\right) - \ln \prod_{j=1}^n \left(\frac{ab}{x_j}\right)^{p_j}\right|\right) \\ &\leq f(a) + f(b) - \sum_{i=1}^n p_i f(x_i) - \frac{2\phi\left(\ln\left(\frac{b}{a}\right)\right)}{\left(\ln\left(\frac{b}{a}\right)\right)^2} \sum_{i=1}^n p_i \ln\left(\frac{x_i}{a}\right) \ln\left(\frac{b}{x_i}\right) \\ &\quad - \sum_{i=1}^n p_i \phi\left(\left|\ln\left(\frac{ab}{x_i}\right) - \ln \prod_{j=1}^n \left(\frac{ab}{x_j}\right)^{p_j}\right|\right) \\ &= f(a) + f(b) - \sum_{i=1}^n p_i f(x_i) - \frac{2\phi\left(\ln\left(\frac{b}{a}\right)\right)}{\left(\ln\left(\frac{b}{a}\right)\right)^2} \sum_{i=1}^n p_i \ln\left(\frac{x_i}{a}\right) \ln\left(\frac{b}{x_i}\right) \\ &\quad - \sum_{i=1}^n p_i \phi\left(\left|\ln \prod_{j=1}^n x_j^{p_j} - \ln x_i\right|\right), \end{aligned}$$

what we need to prove. $\square$

Theorem 3.9. Let $f: [a, b] \rightarrow \mathbb{R}$ uniformly GA-convex with modulus ϕ . Let $\{x_i\}_{i=1}^n \subseteq [a, b] \subseteq (0, \infty)$ be a monotone sequence and $p_i \geq 0, i = 1, \dots, n$, with $\sum_{i=1}^n p_i = 1$. Then

$$\begin{aligned} f\left(\frac{ab}{\prod_{i=1}^n x_i^{p_i}}\right) &\leq f(a) + f(b) - \sum_{i=1}^n p_i f(x_i) - \frac{2\phi\left(\ln\left(\frac{b}{a}\right)\right)}{\left(\ln\left(\frac{b}{a}\right)\right)^2} \sum_{i=1}^n p_i \ln\left(\frac{x_i}{a}\right) \ln\left(\frac{b}{x_i}\right) \\ &\quad - \sum_{i=1}^{n-1} p_i p_{i+1} \phi\left(\left|\ln\left(\frac{x_{i+1}}{x_i}\right)\right|\right). \end{aligned} \tag{26}$$

Proof. Applying Theorem 3.2, Corollary 3.3 and Lemma 3.7, we have

$$\begin{aligned} f\left(\frac{ab}{\prod_{i=1}^n x_i^{p_i}}\right) &= f\left(\prod_{i=1}^n \left(\frac{ab}{x_i}\right)^{p_i}\right) \\ &\leq \sum_{i=1}^n p_i f\left(\frac{ab}{x_i}\right) - \sum_{i=1}^{n-1} p_i p_{i+1} \phi\left(\left|\ln\left(\frac{x_{i+1}}{x_i}\right)\right|\right) \\ &\leq f(a) + f(b) - \sum_{i=1}^n p_i f(x_i) - \frac{2\phi\left(\ln\left(\frac{b}{a}\right)\right)}{\left(\ln\left(\frac{b}{a}\right)\right)^2} \sum_{i=1}^n p_i \ln\left(\frac{x_i}{a}\right) \ln\left(\frac{b}{x_i}\right) \\ &\quad - \sum_{i=1}^{n-1} p_i p_{i+1} \phi\left(\left|\ln\left(\frac{x_{i+1}}{x_i}\right)\right|\right), \end{aligned}$$

what we need to prove. $\square$

4. Applications

Using results from the previous sections we prove some analytic inequalities.

Proposition 4.1. *The following hold:*

a) For every $x, y \in [a, b] \subseteq (0, \infty)$, where $a(\ln a + 2) \geq 2$, we have

$$\frac{1}{2} \left(\ln \frac{x}{y} \right)^2 + \sqrt{xy} \ln xy \leq x \ln x + y \ln y.$$

b) For every $x, y \in [a, b] \subseteq (0, \infty)$, where p is a real number such that $a^p p^2 \geq 2$, we have

$$4x^{\frac{p}{2}} y^{\frac{p}{2}} + \left(\ln \frac{x}{y} \right)^2 \leq 2(x^p + y^p).$$

Proof. The inequalities follows directly as application of (7) to the function:

a) $f(x) = x \ln x$ uniformly GA-convex with modulus $\phi(r) = r^2$.

b) $f(x) = x^p$ uniformly GA-convex with modulus $\phi(r) = r^2$. $\square$

Proposition 4.2. *Inequality*

$$ab \leq \frac{a^2 + b^2}{2} - \frac{1}{4} \left(\ln \frac{a}{b} \right)^2 \quad (27)$$

holds for a, b greater than $\frac{\sqrt{2}}{2}$.

Proof. Choosing $n = 2$, $p_1 = p_2 = \frac{1}{2}$, $x_1 = a$, $x_2 = b$ in (15) we get

$$f(\sqrt{ab}) \leq \frac{f(a) + f(b)}{2} - \frac{1}{2} \phi\left(\left|\ln \sqrt{\frac{a}{b}}\right|\right) - \frac{1}{2} \phi\left(\left|\ln \sqrt{\frac{b}{a}}\right|\right). \quad (28)$$

Now (27) is a simple consequence of application (28) to $f(x) = x^2$ uniformly GA-convex with modulus $\phi(r) = r^2$. $\square$

Proposition 4.3. Let $0 < a < b$ and $0 < k \leq \frac{a^p p^2}{2}$. Then

$$(ab)^{\frac{p}{2}} + \frac{k \left(\ln \frac{b}{a} \right)^2}{12} \leq \frac{b^p - a^p}{p \ln \left(\frac{b}{a} \right)} \leq \frac{a^p + b^p}{2} - \frac{k}{6} \left(\ln \frac{b}{a} \right)^2.$$

Proof. Let $0 < k \leq \frac{a^p p^2}{2}$, thus $p \neq 0$. As explained in Example 2.9, the function $f(x) = x^p$ is uniformly GA-convex with modulus $\phi(r) = kr^2$. Applying Theorem 3.6 to $f(x) = x^p$, we arrive at

$$(ab)^{\frac{p}{2}} + \frac{k}{4 \ln \frac{b}{a}} \int_0^{\ln \left(\frac{b}{a} \right)} u^2 du \leq \frac{1}{\ln \frac{b}{a}} \int_a^b u^{p-1} du \leq \frac{a^p + b^p}{2} - \frac{k}{6} \left(\ln \frac{b}{a} \right)^2,$$

which completes the proof. $\square$

Proposition 4.4. Let $0 < a \leq x \leq b$ and $0 < k \leq \frac{a^p p^2}{2}$. Then

$$\frac{a^p b^p}{x^p} \leq a^p + b^p - x^p - 2k \ln \frac{x}{a} \ln \frac{b}{x}.$$

Proof. The case $a = b$ is trivial. Let $a \neq b$. As explained in Example 2.9, the function $f(u) = u^p$ is uniformly GA-convex with modulus $\phi(r) = kr^2$. Applying Lemma 3.7 to $f(u) = u^p$, we get the required result. $\square$

Next we give some applications in information theory. For the purpose of those results, let us denote

$$\mathcal{P}_n = \{\mathbf{p} = (p_1, \dots, p_n) : p_1, \dots, p_n > 0, \sum_{i=1}^n p_i = 1\},$$

the set of all complete finite discrete probability distributions. The restriction to positive distributions is only for convenience. If we take $p_i = 0$, for some $i \in \{1, \dots, n\}$, in the following results we need to interpret undefined expressions as $f(0) = \lim_{t \rightarrow 0+} f(t)$, $0f\left(\frac{0}{0}\right) = 0$ and $0f\left(\frac{e}{0}\right) = \lim_{\varepsilon \rightarrow 0+} f\left(\frac{e}{\varepsilon}\right) = e \lim_{t \rightarrow \infty} \frac{f(t)}{t}$, $e > 0$.

Shannon's entropy [34] is defined in terms of probability distribution $\mathbf{p}$ for a random variable X as

$$S(\mathbf{p}) = -\sum_{i=1}^n p_i \ln p_i. \quad (29)$$

It quantifies the unevenness in $\mathbf{p}$ and satisfies inequality

$$0 \leq S(\mathbf{p}) \leq \ln n.$$

We obtain new upper bound for Shannon's entropy $S(\mathbf{p})$ as follows.

Proposition 4.5. Let $\mathbf{p} \in \mathcal{P}_n$ with $p_i \in [a, b] \subseteq (0, \infty)$, where $a(\ln a + 2) \geq 2$. Let τ be a permutation on $\{1, \dots, n\}$ such that

$$p_{\tau(1)} \leq p_{\tau(2)} \leq \dots \leq p_{\tau(n)}.$$

Then

$$S(\mathbf{p}) \leq \sqrt[n]{p_1 \cdot p_2 \cdot \dots \cdot p_n} \ln \left(\frac{1}{\sqrt[n]{p_1 \cdot p_2 \cdot \dots \cdot p_n}} \right) - \frac{1}{n} \sum_{i=1}^{n-1} \ln^2 \left(\frac{p_{\tau(i+1)}}{p_{\tau(i)}} \right). \quad (30)$$

Proof. Applying Theorem 3.2 the function $f(x) = x \ln x$ uniformly GA-convex with modulus $\phi(r) = r^2$ on $[a, b] \subseteq (0, \infty)$, with $a(\ln a + 2) \geq 2$, and substituting x_i by p_i , $i = 1, \dots, n$, we get the required result. $\square$

Nonnegative measures of dissimilarity between pairs of probability measures are known as divergence measures. Important class of divergence measures is defined by means of convex functions f and it is called the class of f -divergences or Csiszár f -divergences (see [11]). Expressed more precisely, for a convex function $f: (0, \infty) \rightarrow \mathbb{R}$ and $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n$, we define Csiszár f -divergence as

$$D_f(\mathbf{q}, \mathbf{p}) = \sum_{i=1}^n p_i f\left(\frac{q_i}{p_i}\right). \quad (31)$$

With the assumption $f(1) = 0$, so called normalized property, the convexity ensures the nonnegativity of Csiszár f -divergence, i.e.

$$D_f(\mathbf{q}, \mathbf{p}) \geq 0$$

with $D_f(\mathbf{q}, \mathbf{p}) = 0$ iff $\mathbf{q} = \mathbf{p}$.

Csiszár f -divergence (31) is useful generalization of some well-known divergences ([3], [18]–[21]) which have deep and fruitful applications in probability, statistics, information theory, physics etc. One of the most commonly used divergence is the Kullback-Leibler divergence which is a particular case of (31) for generating function $f(x) = x \ln x$.

For two $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n$, the Kullback-Leibler divergence is defined by

$$KL(\mathbf{q}, \mathbf{p}) = \sum_{i=1}^n q_i \ln \frac{q_i}{p_i}. \quad (32)$$

The following estimates for the Kullback-Leibler divergence hold.

Proposition 4.6. Let $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n$ such that $\left\{\frac{q_i}{p_i}\right\}_{i=1}^n$ is a sequence in $[a, b] \subseteq (0, \infty)$ with $a(\ln a + 2) \geq 2$. Then

$$\begin{aligned} & \sum_{i=1}^n p_i \left(\ln \prod_{j=1}^n \left(\frac{q_j}{p_j} \right)^{p_j} - \ln \frac{q_i}{p_i} \right)^2 + \prod_{i=1}^n \left(\frac{q_i}{p_i} \right)^{p_i} \ln \left(\prod_{i=1}^n \left(\frac{q_i}{p_i} \right)^{p_i} \right) \\ & \leq KL(\mathbf{q}, \mathbf{p}) \\ & \leq \ln a^a b^b + 2 \sum_{i=1}^n p_i \ln \left(\frac{ap_i}{q_i} \right) \ln \left(\frac{bp_i}{q_i} \right) \\ & \quad - \sum_{i=1}^n p_i \left(\ln \prod_{j=1}^n \left(\frac{q_j}{p_j} \right)^{p_j} - \ln \left(\frac{q_i}{p_i} \right) \right)^2 - \prod_{i=1}^n \left(\frac{q_i}{p_i} \right)^{p_i} \ln \left(\prod_{i=1}^n \left(\frac{q_i}{p_i} \right)^{p_i} \right) \end{aligned} \quad (33)$$

Proof. Applying Theorem 3.1 and Theorem 3.8 to the function $f(x) = x \ln x$ uniformly GA-convex with modulus $\phi(r) = r^2$ on $[a, b] \subseteq (0, \infty)$, with $a(\ln a + 2) \geq 2$, and substituting x_i by $\frac{q_i}{p_i}$, $i = 1, \dots, n$, we obtain (33). $\square$

At the end we derive new estimates for Jeffreys distance defined by

$$J(\mathbf{q}, \mathbf{p}) = \sum_{i=1}^k (q_i - p_i) \ln \frac{q_i}{p_i}, \quad (34)$$

obtained from (31) by generating function $f(x) = (x - 1) \ln x$.

Proposition 4.7. Let $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n$ such that $\left\{\frac{q_i}{p_i}\right\}_{i=1}^n$ is a sequence in $[a, b] \subseteq (0, \infty)$ with $a(\ln a + 2) \geq 2$. Then

$$\begin{aligned} & \sum_{i=1}^n p_i \left[\ln \left(\frac{q_i}{p_i} \right) - \ln \prod_{j=1}^n \left(\frac{q_j}{p_j} \right)^{p_j} \right]^2 + \left(\prod_{i=1}^n \left(\frac{q_i}{p_i} \right)^{p_i} - 1 \right) \ln \left(\prod_{i=1}^n \left(\frac{q_i}{p_i} \right)^{p_i} \right) \\ & \leq J(\mathbf{q}, \mathbf{p}) \\ & \leq \ln a^{a-1} b^{b-1} + 2 \ln \left(\frac{b}{a} \right) \sum_{i=1}^n p_i \ln \left(\frac{ap_i}{q_i} \right) \ln \left(\frac{bp_i}{q_i} \right) \\ & \quad - \sum_{i=1}^n p_i \left[\ln \left(\frac{q_i}{p_i} \right) - \ln \prod_{j=1}^n \left(\frac{q_j}{p_j} \right)^{p_j} \right]^2 - \left(\frac{ab}{\prod_{i=1}^n \left(\frac{q_i}{p_i} \right)^{p_i}} - 1 \right) \ln \left(\frac{ab}{\prod_{i=1}^n \left(\frac{q_i}{p_i} \right)^{p_i}} \right) \end{aligned} \quad (35)$$

Proof. Applying Theorem 3.1 and Theorem 3.8 to the function $f(x) = (x-1) \ln x$ uniformly GA-convex with modulus $\phi(r) = r^2$ on $[a, b] \subseteq (0, \infty)$, with $a(\ln a + 2) \geq 2$, and by substituting x_i by $\frac{q_i}{p_i}$, $i = 1, \dots, n$, we obtain (35). $\square$

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