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β-expansions of rational numbers with Pisot Chabauty basis in $\mathbb{Q}_p$

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Abstract. The aim of this paper is to study some arithmetic properties about the periodicity of the β-expansion of p-adic numbers. We prove that for every Pisot Chabauty unit number such that the finiteness property (F) is satisfied, there exists a constant $\gamma'(\beta)$ for which every rational in $[0, \gamma'(\beta)[$ have a purely periodic β-expansion, where

$$\gamma'(\beta) = \sup\{c \in [0,1) : \forall x \in (\mathbb{Q} \cap \mathbb{Z}_p) \cap [0,c), d_\beta(x) \textit{ is purely periodic}\}.$$

1. Introduction

The β-expansions of real numbers were first introduced by A. Rényi [8]. Since then, their arithmetic, diophantine and ergodic properties have been extensively studied by several researchers.
Let $\beta > 1$ be a real number. The β-expansion of a real number $x \in [0,1]$ is defined as the sequence $(x_i)_{i\geq 1}$ with values in $\{0,1,\ldots,[\beta]\}$ produced by the β-transformation $T_\beta : x \longrightarrow \beta x \pmod 1$ as follows :

$$\forall\ i \geq 1,\ x_i = [\beta T_\beta^{i-1}(x)],\ \textit{and thus}\ x = \sum_{i\geq 1} \frac{x_i}{\beta^i}.$$

An expansion is finite if $(x_i)_{i\geq 1}$ is eventually 0. A β-expansion is periodic if $p \geq 1$ and $m \geq 1$ exist in a way $x_k = x_{k+p}$, holds for all $k \geq m$. When $x_k = x_{k+p}$ holds for all $k \geq 1$, then it is purely periodic. The sets of real numbers with periodic β-expansions, purely periodic β-expansions and finite β-expansions are respectively denoted by $Per(\beta)$, $Pur(\beta)$ and $Fin(\beta)$.

Let $\mathbb{Q}(\beta)$ be the smallest field containing $\mathbb{Q}$ and β. An easy argument shows that $Per(\beta) \subseteq \mathbb{Q}(\beta) \cap [0,1)$ for every real number $\beta > 1$.In the statement [9], K. Schmidt showed that if β is a Pisot number (an algebraic integer whose conjugates have modulus <1), then $Per(\beta)$=$\mathbb{Q}(\beta) \cap [0,1)$.

S. Ito and H. Rao discussed the purely periodic β-expansions in the statement [5] and they characterized all reals in $[0,1)$ which have purely periodic β-expansions with Pisot unit base. In the statement [3], V. Berthé and A. Siegel completed the characterization in the Pisot non unit base.

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Set $\gamma(\beta) = \sup\{c \in [0,1) : \forall x \in \mathbb{Q} \cap D(0,c), d_\beta(x) \textit{ is purely periodic}\}$.

S. Akiyama proved in the statement [2] that if β verifies the finiteness property ($Fin(\beta) = \mathbb{Z}[\beta^{-1}] \cap \mathbb{R}_+$), then $\gamma(\beta) > 0$. In the quadratic case, K. Schmidt [9] has proved that if β satisfied $\beta^2 = n\beta + 1$ for some integer $n \geq 1$, then $\gamma(\beta) = 1$. Until now, it is the unique known family of reals for which $\gamma(\beta) = 1$. In [1] the authors has proved that if β is not Pisot unit, then $\gamma(\beta) = 0$, they also showed that if β is a cubic Pisot unit satisfying the finiteness property such that the number field $\mathbb{Q}(\beta)$ is not totally real, then $0 < \gamma(\beta) < 1$.

The $\beta-$expansion in the field of $p-$adic number was introduced by K.Scheicher, V. F. Sirvent and P. Surer [7] . They have proved that if β is a PC number, then $Per(\beta) = \mathbb{Q}(\beta) \cap \mathbb{Z}_p$.

The study of $\gamma(\beta)$ is an important problem that is still largely open. We can define analogous notion in the case the field of the p-adic numbers that means in the similar case with the real number we can define the constant $\gamma'(\beta)$. The main objective of this paper is to determine this problem in $\mathbb{Q}_p$ where β is a PC or SC numbers. Particulary, in this paper, we prove that $\gamma'(\beta) > 0$, if β is a PC unit number in $\mathbb{Q}_p$ satisfying the finiteness property (F).
The purely periodicity of the $\beta-$expansion is a very important problem but still largely open until now. Let's remember also that in the case of the field of formal series, on the one hand, in [6], M.Jelleli, M. Mkaouar and K. Scheicher have studied the characterization of purely periodic $\beta-$expansions in the Pisot unit base. On the other hand, in [8], the authors characterize formal power series that have purely periodic $\beta-$expansions in Pisot or Salem unit basis and they prove that every rational series in the unit disk has a purely periodic $\beta-$expansion when β is a quadratic Pisot unit basis or Salem cubic unit basis.
This paper is organized as follows: In section 2, we introduce some basic definitions of p-adic numbers in the field $\mathbb{Q}_p$. In section 3, we define the β-expansion algorithm for p-adic numbers and we recall some recent results. In section 4, we prove that there exists a constant $\gamma'(\beta)$ for which every rational in the disk $D(0, \gamma'(\beta))$ have a purely periodic β-expansion if β is a PC unit number satisfying the finiteness property (F). Furthermore, we show that the unit condition is necessary to have $\gamma'(\beta) > 0$.

2. *p*-adic numbers

In order to introduce $\mathbb{Q}_p$ in an harmonious way, we begin by presenting the following set: Let p be a prime and $\mathbb{A}_p = \{mp^n, m, n \in \mathbb{Z}\} = \mathbb{Z}[\frac{1}{p}]$. Particulary, we denote by $\mathbb{A}'_p = \mathbb{A}_p \cap [0,1)$.
Recall that $\mathbb{A}_p \subset \mathbb{Q}$ is a principal ring, the unit group of $\mathbb{A}_p$ is $\{\pm p^k, k \in \mathbb{Z}\}$ and the field of fraction is $\mathbb{Q}$.
Now, let us define the p-adic valuation:

$$\begin{array}{rcl} v_p : \mathbb{A}_p & \longrightarrow & \mathbb{Z} \bigcup \{\infty\} \\ x & \longmapsto & \left\{ \begin{array}{ll} max\{n \in \mathbb{Z} : p^n \textit{ divise } x\} & \text{if } x \neq 0, \\ \infty & \text{if } x = 0, \end{array} \right\} \end{array}$$

which verifies the following properties:
- $v_p(0) = \infty$,
- $v_p(xy) = v_p(x) + v_p(y)$,
- $v_p(x + y) \geq min\{v_p(x), v_p(y)\}$ *with* $v_p(x + y) = min\{v_p(x), v_p(y)\}$, *if* $v_p(x) \neq v_p(y)$.

Therefore $v_p(.)$ is an exponential valuation on $\mathbb{A}_p$. The $p - adic$ norm $|.|_p$ is defined by

$$|x|_p = \begin{cases} p^{-v_p(x)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then $|.|_p$ is a non archimedean absolute value on $\mathbb{A}_p$. It fulfills the strict triangular inequality

$$|x + y|_p \leq max\{|x|_p, \ |\{y|_p\} \quad with$$

$$|x+y|_p = max\{|x|_p, |\{y|_p\} \quad if \quad |x|_p \neq |y|_p.$$

Let $|.|$ be the archimedean absolute value. Then $|x|_p$ and $|x|$ satisfy the following product formula

$$\prod_{p\in\mathbb{P}\cup\{\infty\}} |x|_p = 1 \quad for\ all \quad x \in \mathbb{Q} \setminus \{0\}$$

where $\mathbb{P}$ denote the set of primes. The completion of $\mathbb{A}_p$ with respects to $|.|_p$ is the field $\mathbb{Q}_p$ of $p-adic$ numbers. Thus

$$\mathbb{Z} \subset \mathbb{A}_p \subset \mathbb{Q} \subset \mathbb{Q}_p.$$

We mention that each element $x \in \mathbb{Q}_p$ $(x \neq 0)$ admits a unique expansion of the form

$$x = \sum_{n=n_0}^{\infty} x_n p^n, \quad such\ that \quad n_0 \in \mathbb{Z}, \quad x_{n_0} \neq 0 \quad and\ x_n \in \{0, \ldots, p-1\}.$$

From expansions of the form, we will use the notation

$$x = \ldots p_2 p_1 p_0 \bullet p_{-1} \ldots p_{n_0}.$$

Definition 2.1. *Each $x \in \mathbb{Q}_p$ of the form mentioned above has a unique Artin decomposition*

$$x = [x]_p + \{x\}_p$$

such that

$$[x]_p = \sum_{n\geq 0} x_n p^n \quad and \quad \{x\}_p = \sum_{n<0} x_n p^n.$$

The number $[x]_p \in \mathbb{Z}_p$ is called $p-$adic integer part and $\{x\}_p \in \mathbb{A}_p \cap [0,1)$ is called $p-$adic fractional part of x.

Furthermore, we can also define the extension v_p in $\mathbb{Q}_p$:

$$v_p(x) = n_0 \ if \ x \neq 0 \ and \ v_p(x) = \infty \ otherwise.$$

In addition $\mathbb{Q}_p$ is equivalent to the fraction field of the p-adic integers $\mathbb{Z}_p$ where

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \ ; \ |x|_p \leq 1\}.$$

Consequently

$$\mathbb{Z} = \mathbb{A}_p \cap \mathbb{Z}_p = \{x \in \mathbb{A}_p \ ; \ |x|_p \leq 1\}.$$

Our purpose now is to define Pisot-Chabauty numbers. For this, we shall need some definitions:

Definition 2.2. *An element α is called algebraic over $\mathbb{A}_p$, if there is a polynomial*

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{A}_p[x] \ with \ f(\alpha) = 0.$$

If f is irreducible over $\mathbb{A}_p$, then f is called a minimal polynomial of α. If $a_n = p^k$ for some $k \in \mathbb{Z}$, then α is called an algebraic integer. Since p^k is a unit of $\mathbb{A}_p$, we can assume without loss of generality, that $a_n = 1$. If $a_0 = p^{k'}$ for some $k' \in \mathbb{Z}$, then α is called an algebraic unit.

Proposition 2.3. *Let K be complete field with respect to (a non archimedean absolute value $|.|$) and L/K $(K \subset L)$ be an algebraic extension of degree m. Then $|.|$ has a unique extension to L defined by : $|\alpha| = \sqrt[m]{|N_{L/K}(\alpha)|}$ and L is complete with respect to this extension.*

We apply this proposition to algebraic extension of $\mathbb{Q}_p$. Since $\mathbb{Q}_p$ is complete, $|.|_p$ and $v_p(.)$ can be extended uniquely to each algebraic field L of $K = \mathbb{Q}_p$. Thus, every algebraic element over $\mathbb{A}_p$ can be valuated.

Remark 2.4. *In what follows, for algebraic elements β over $\mathbb{A}_p$, we will denote by $\beta^{(1)}, \dots, \beta^{(n)}$ the non-archimedean conjugates of β and by $\beta^{(n+1)}, \dots, \beta^{(2n)}$ the archimedean conjugates of β.*

Finally, we reach to give the definition of Pisot-Chabauty numbers.

Definition 2.5. *A Pisot-Chabauty number (for short PC number) is a p-adic number $\beta \in \mathbb{Q}_p$, such that*

- $\beta^{(1)} = \beta$ *is an algebraic integer over* $\mathbb{A}_p$.
- $|\beta^{(1)}|_p > 1$ *for one non-archimedean conjugate of* β.
- $|\beta^{(i)}|_p \leq 1$ *for all non-archimedean conjugates* $\beta^{(i)}$, $i \in \{2, \dots, n\}$ *of* β.
- $|\beta^{(i)}| < 1$ *for all archimedean conjugates* $\beta^{(i)}$, $i \in \{n+1, \dots, 2n\}$ *of* β.

3. $\beta-$expansion in the field $\mathbb{Q}_p$

Similary to the classical $\beta-$expansion for the real numbers, we introduce the $\beta-$expansion for p-adic numbers. For this, Let $\beta \in \mathbb{Q}_p$ where $|\beta|_p > 1$, $x \in \mathbb{Z}_p$ and denote by $N_p = [0, 1) \cap \{x \in \mathbb{A}_p : |x|_p \leq |\beta|_p\}$.
A representation in base β (or β-representation) of x is a sequence $(d_i)_{i\geq 1}$, $d_i \in \mathbb{A}_p$, such that

$$x = \sum_{i\geq 1} \frac{d_i}{\beta^i}.$$

A particular β-representation of x is called the β-expansion of x and noted
$d_\beta(x) = (d_i)_{i\geq 1}$ with values in N_p produced by the β-transformation $T : \mathbb{Z}_p \to \mathbb{Z}_p$, which is given by the mapping $z \mapsto [\beta z]_p$.
For $k \geq 0$, define

$$T^0(x) = x \text{ and } T^k(x) = T(T^{k-1}(x)).$$

Then $d_k = \{\beta T^{k-1}(x)\}_p$ for all $k \geq 1$.

An equivalent definition of the β-expansion can be obtained by a greedy algorithm. This algorithm works as follows :

$$r_0 = x;\ d_k = \{\beta r_{k-1}\}_p \text{ and } r_k = \lfloor \beta r_{k-1} \rfloor_p \text{ for all } k \geq 1.$$

The β-expansion of x will be noted as $d_\beta(x) = (d_k)_{k\geq 1}$.
Notice that, $d_\beta(x)$ is finite if and only if there is a $k \geq 0$ with $T^k(x) = 0$, $d_\beta(x)$ is ultimately periodic if and only if there is some smallest $n \geq 0$ (the pre-period length) and $s \geq 1$ (the period length) when $T^{n+s}(x) = T^n(x)$, namely the period length. In a special case, where $n = 0$, $d_\beta(x)$ is purely periodic.
Afterwards, we will use the following notations :

$$Fin(\beta) = \{x \in \mathbb{Z}_p : d_\beta(x) \text{ is } finite\} \text{ and } Per(\beta) = \{x \in \mathbb{Z}_p : d_\beta(x) \text{ is eventually periodic}\}.$$

Now, let $x \in \mathbb{Q}_p$ be an element, with $|x|_p > 1$. Then there is a unique $k \in \mathbb{N}$ having $|\beta|_p^k \leq |x|_p < |\beta|_p^{k+1}$. We can represent x by shifting $d_\beta(\beta^{-(k+1)}x)$ by k digits to the left. Therefore, if $d_\beta(x) = 0 \cdot d_1 d_2 d_3 \dots$, then $d_\beta(\beta x) = d_1 \cdot d_2 d_3 \dots$.
If we have $d_\beta(x) = d_l d_{l-1} \dots d_0 \cdot d_{-1} \dots d_{-m}$, then we put $\mathrm{ord}_\beta(x) = -m$.

Definition 3.1. *Let $\beta \in \mathbb{Q}_p$. β verifies the finiteness property* (F) *if $Fin(\beta) = \mathbb{A}_p[\beta^{-1}]$.*

Through the use of the previous set $Per(\beta)$ and the PC numbers, K. Scheicher, V. F. Sirvent and P. Surer [7] established the following theorem in the case of p-adic numbers.

Theorem 3.2. *Let β be a PC number. Then $Per(\beta) = \mathbb{Q}(\beta) \cap \mathbb{Z}_p$.*

Moreover, in the same paper [7], geometric condition of the finiteness property (F) has been given by K. Scheicher, V. F. Sirvent and P. Surer.

4. Purely periodic $\beta-$ *expansion*

We define for each $\beta \in \mathbb{Q}_p$ with $|\beta|_p > 1$ the quantity

$$\gamma'(\beta) = \sup\{c \in [0,1) : \forall x \in (\mathbb{Q} \cap \mathbb{Z}_p) \cap [0,c), d_\beta(x) \textit{ is purely periodic}\},$$

.

In order to prove our main theorem, we need to introduce some basic notions: Let β be a PC unit number with minimal polynomial $P(\beta) = \beta^d + a_{d-1}\beta^{d-1} + \cdots + a_0$ where $a_i \in \mathbb{A}_p$ for $i \in \{0,\ldots,d-1\}$. Let $\beta^{(2)},\ldots,\beta^{(d)}$ be the non-archimedean conjuguates of β.

We denote by $\overline{\beta}$, the vector of non-archimedean conjuguates of β given by $\overline{\beta} = \begin{pmatrix} \beta^{(2)} \\ \vdots \\ \beta^{(d)} \end{pmatrix}$.

Put

$$Q'(\beta) = \{\sum_{i=0}^{n_i} a_i\beta^i, a_i \in \mathbb{Q} \cap \mathbb{Z}_p, |a_i| < 1\}.$$

For $x \in Q'(\beta)$, the j-th non-archimedean conjuguate of x is given by $x^{(j)} = \sum_{i=0}^{n_i} a_i(\beta^{(j)})^i$, $\forall j \in \{2,\ldots,d\}$.

We define $\overline{x}$, the vector of non-archimedean conjuguates of x by $\overline{x} = \begin{pmatrix} x^{(2)} \\ \vdots \\ x^{(d)} \end{pmatrix}$ and $\|\overline{x}\|_p = \sup_{2\le k\le d} |x^{(k)}|_p$.

We begin with these results which are essential for the development of the proof of our main Theorem.

Lemma 4.1. *[7]*
Let $A \subset \mathbb{A}_p$. If A is bounded with respect to $|.|_p$ and $|.|$, i.e.

$$\max_{a\in A} |a|_p < \infty \textit{ and } \max_{a\in A} |a| < \infty,$$

then A is finite.

Lemma 4.2. *Let $\beta \in \mathbb{Q}_p$ be a PC unit number. Put*

$$X(k) = \{x \in Fin(\beta) \cap Q'(\beta); ord_\beta(x) = -k\}.$$

Then

$$\lim_{k\to\infty} \min_{x\in X(k)} \|\overline{x}\|_p = \infty.$$

Proof:
Assume that there exists a constant B and an infinite sequence x_i $(i = 1, 2, \ldots)$ then that both

$$|x_i^{(j)}|_p \le B \text{ for } j = 2,3,\ldots,\ |x_i|_p \le 1 \text{ and } \lim_{i\to\infty} ord_\beta(x_i) = -\infty$$

holds.
We have $x_i = \sum_{i=0}^{n_i} a_i\beta^i$, $n_i \in \mathbb{N}$ and $x_i^{(j)} = \sum_{i=0}^{n_i} a_i(\beta^{(j)})^i$ for all $\beta^{(j)}$ conjuqates of β.

Let now $\beta^{(2)},\ldots,\beta^{(d)}$ be the non-archimedean conjuguates of β. As β is unit, then $x_i \in \mathbb{A}_p[\beta]$. Hence $x_i = A_0^i + A_1^i\beta + \ldots + A_{d-1}^i\beta^{d-1}$ and
$x_i^{(j)} = A_0^i + A_1^i\beta^{(j)} + \ldots + A_{d-1}^i(\beta^{(j)})^{d-1}, \forall j \in \{2,\ldots,d\}$ where $\beta^{(2)},\ldots,\beta^{(d)}$ are the non-archimedean conjuguates of β.

Thus $\begin{pmatrix} x_i \\ x_i^{(2)} \\ \vdots \\ \vdots \\ x_i^{(d)} \end{pmatrix}$=M$\begin{pmatrix} A_0^i \\ A_1^i \\ \vdots \\ \vdots \\ A_{d-1}^i \end{pmatrix}$, where $M = \begin{pmatrix} 1 & \beta & \dots & \dots & \beta^{d-1} \\ 1 & \beta^{(2)} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & (\beta^{(d-1)})^{d-1} \\ 1 & \beta^{(d)} & \dots & \dots & (\beta^{(d)})^{d-1} \end{pmatrix}$.

We have $\det M = \prod_{i<j}(\beta^{(i)} - \beta^{(j)}) \neq 0$ which implies that M is invertible, therefore it transforms a bounded vector in a bounded vector. since $|x_i^{(j)}|_p \leq B$ for $j = 2, 3, \dots$ and $|x_i|_p \leq 1$, we have $\begin{pmatrix} x_i \\ x_i^{(2)} \\ \vdots \\ \vdots \\ x_i^{(d)} \end{pmatrix}$ is bounded, so $\begin{pmatrix} A_0^i \\ A_1^i \\ \vdots \\ \vdots \\ A_{d-1}^i \end{pmatrix}$ is also bounded in $\mathbb{Q}_p$. Furthermore, β has d archimedean conjugates $\beta^{(j)}$, such that $|\beta^{(j)}| < 1$ where $d + 1 \leq j \leq 2d$.
We have, on one hand,

$$x_i^{(j)} = A_0^i + A_1^i\beta^{(j)} + ... + A_{d-1}^i(\beta^{(j)})^{d-1}.$$

Moreover, on the other hand, we have $\forall j \in \{ d + 1, \dots, 2d\}$,

$$|x_i^{(j)}| = |\sum_{i=0}^{n_i} a_i(\beta^{(j)})^i| \leq \sum_{i=0}^{n_i} |a_i||\beta^{(j)}|^i \leq \frac{1}{1 - |\beta^{(j)}|}.$$

So $\begin{pmatrix} x_i^{(d+1)} \\ x_i^{(d+2)} \\ \vdots \\ \vdots \\ x_i^{(2d)} \end{pmatrix}$ is bounded. Recall that M is invertible, then $\begin{pmatrix} A_0^i \\ A_1^i \\ \vdots \\ \vdots \\ A_{d-1}^i \end{pmatrix}$ is also bounded in $\mathbb{R}$. Finally, by Lemma 4.1, we conclude that $\begin{pmatrix} A_0^i \\ A_1^i \\ \vdots \\ \vdots \\ A_{d-1}^i \end{pmatrix}$ takes a finite values. Consequently, these x_i are finite. This is absurd, which proves the lemma.

Proposition 4.3. *Let $\beta \in \mathbb{Q}(\beta)$ be a PC unit number. Then there exists $r > 0$ such that for every $h \in Fin(\beta) \cap Q'(\beta)$ satisfying $ord_\beta(h) \leq -1$, we have $\|\overline{h}\|_p > r$.*

Proof:
According to Lemma 4.2, there exists $s > 0$ such that for every $x \in Fin(\beta) \cap Q'(\beta)$ satisfying $|x|_p \leq 1$ and

$ord_\beta(x) \le -s$, we have $\|\overline{x}\|_p > |\beta|_p$.

Put $r = \inf_{j\in\{2,\dots,d\}} |(\beta^{(j)})^{s-1}|_p|\beta|_p$, where $\beta^{(2)}, \dots, \beta^{(d)}$ are the non-archimedean conjugates of β. Now, let $h \in Fin(\beta) \cap Q'(\beta)$ be with $ord_\beta(h) \le -1$. Then $h = \beta^{s-1}g$ where $ord_\beta(g) \le -s$. Moreover h can be written such that $h = \beta^{s-1}(g_1 + g_2)$ where $ord_\beta(g_1) \ge 0$,
$ord_\beta(g_2) = ord_\beta(g) \le -s$ and $|g_2|_p \le 1$. Since $h = \beta^{s-1}(g_1 + g_2)$, we have

$$\overline{h} = \begin{pmatrix} (\beta^{(2)})^{s-1}(g_1^{(2)} + g_2^{(2)}) \\ \vdots \\ (\beta^{(d)})^{s-1}(g_1^{(d)} + g_2^{(d)}) \end{pmatrix}$$

As β is a PC unit number and $g_1 = c_0 + c_1\beta + \cdots + c_{d-1}\beta^{d-1}$ with $|c_i|_p < |\beta|_p$, we have,

$$|g_1^{(2)}|_p = |c_0 + c_1\beta^{(2)} + \cdots + c_{d-1}(\beta^{(2)})^{d-1}|_p \le |\beta|_p$$

$$|g_1^{(3)}|_p = |c_0 + c_1\beta^{(3)} + \cdots + c_{d-1}(\beta^{(3)})^{d-1}|_p \le |\beta|_p$$

$$\vdots$$

$$|g_1^{(d)}|_p = |c_0 + c_1\beta^{(d)} + \cdots + c_{d-1}(\beta^{(d)})^{d-1}|_p \le |\beta|_p.$$

Since $ord_\beta(g_2) \le -s$ and $|g_2|_p \le 1$, we have $\|\overline{g_2}\|_p > |\beta|_p$. Which involves that there exists $j_0 \in \{2, \dots, d\}$ with $|g_2^{j_0}|_p > |\beta|_p$. So $|g_1^{(j_0)} + g_2^{(j_0)}|_p > |\beta|_p$, which implies that
$|(\beta^{(j_0)})^{s-1}|_p|g_1^{(j_0)} + g_2^{(j_0)}|_p > \inf_{j\in\{2,\dots,d\}} |(\beta^{(j)})^{s-1}|_p|\beta|_p = r$. Finally we infer that $\|\overline{h}\|_p > r$.

Before giving our main theorem, we need moreover the following lemmas. We begin by this lemma in which we characterize the β-expansion of $p-$adic numbers.

Lemma 4.4. *Let $\beta \in \mathbb{Q}_p$ where $|\beta|_p > 1$ and $(a_i)_{i\ge1}$ is a β-representation of x. Then $d_\beta(x) = (a_i)_{i\ge1}$ if and only if $a_i \in \mathrm{N}_p$, for all $i \ge 1$.*

Proof:
The necessary condition is trivial. For the sufficient condition, by assumption we have $(a_i)_{i\ge1}$ is a β-representation of x and $|a_i|_p \le |\beta|_p$ *for all* $i \ge 1$, so

$$x = \sum_{i\ge1} \frac{a_i}{\beta^i}.$$

If we multiply by β, we get

$$\beta x = a_1 + \sum_{i\ge2} \frac{a_i}{\beta^{i-1}}.$$

As $|\sum_{i\ge2} \frac{a_i}{\beta^{i-1}}|_p < 1$ and $a_1 \in \mathbb{A}'_p$, we obtain that $a_1 = \{\beta x\}_p$. Put now $r_0 = x$. We have

$$\beta x - a_1 = \sum_{i\ge2} \frac{a_i}{\beta^{i-1}}$$

and if we multiply again by β, we get

$$\beta(\beta x - a_1) = a_2 + \sum_{i\ge3} \frac{a_i}{\beta^{i-2}}.$$

Since $|\sum_{i\geq 3}\frac{a_i}{\beta^{i-2}}|_p < 1$ and $a_2 \in \mathbb{A}'_p$, we get $a_2 = \{\beta r_1\}_p$ where $r_1 = \beta x - a_1$.
Therefore, it's clear that the sequence $(a_k)_{k\geq 1}$ verifies the recurrent condition

$$r_0 = x; a_k = \{\beta r_{k-1}\}_p \text{ and } r_k = \lfloor \beta r_{k-1}\rfloor_p,$$

which implies that $d_\beta(x) = (a_i)_{i\geq 1}$.

Lemma 4.5. *Let $a_i \neq a_j \in \mathrm{N}_p$. Then $|a_i - a_j|_p > 1$.*

Proof:
Let $a_i = \frac{b_i}{p^{v_p(\beta)}}$ such that $b_i \in \mathbb{N}$ and $b_i < p^{v_p(\beta)}$. Let now $a_j = \frac{b_j}{p^{v_p(\beta)}}$ with $b_j \in \mathbb{N}$ and $b_j < p^{v_p(\beta)}$.
We have

$$|a_i - a_j|_p = \frac{|b_i - b_j|_p}{|p^{v_p(\beta)}|_p}.$$

We have $|b_i - b_j|_\infty < p^{v_p(\beta)}$ and this yield that $|b_i - b_j|_p > p^{-v_p(\beta)}$. Then, we obtain the result.

Lemma 4.6. *Let $\beta \in \mathbb{Q}_p$ where $|\beta|_p > 1$, $c_i \in \mathbb{Q}_p$ and $M = \sum_{i=1}^{l}\frac{c_i}{\beta^i}$ with $1 < |c_i|_p \leq |\beta|_p$. If there exists j such that $c_j \neq 0$, then $M \neq 0$.*

Proof:
Let $c_i \in \mathbb{Q}_p$ and suppose that $M = \sum_{i=1}^{l}\frac{c_i}{\beta^i}$ with $1 < |c_i|_p \leq |\beta|_p$. Let i_0 be the smallest integer $j \in \{1, \dots, l\}$ such that $c_j \neq 0$. We have $M = \sum_{i=1}^{l}\frac{c_i}{\beta^i}$, then $|M|_p = |\frac{c_{i_0}}{\beta^{i_0}}|_p \neq 0$ since $|\frac{c_{i_0}}{\beta^{i_0}}|_p > |\frac{c_k}{\beta^k}|_p$ for all $i_0 < k \leq l$. Therefore $M \neq 0$.

It is natural now to present our main theorem.

Theorem 4.7. *Let $\beta \in \mathbb{Q}_p$ be a PC unit number such that the finiteness property* (F) *is satisfied, then $\gamma'(\beta) > 0$.*

Proof:
We will show that there exists a positive constant c such that every rational x with $|x|_p < c$ has a purely periodic β-expansion. Let $x \in \mathbb{Q}$ such that $|x|_p \leq 1$, $|x| < 1$ and assume that x does not have a purely periodic β-expansion. Since β is a PC unit number, we know that $d_\beta(x)$ is eventually periodic and let s be the length of the period and n the length of the pre-period. So $d_\beta(x) = 0 \cdot a_1 ... a_s \overline{a_{s+1} ... a_{n+s}}$ and $a_s \neq a_{n+s}$. Hence $x(\beta^s - 1) \in Q'\beta)$ and

$$x(\beta^s - 1) = a_1\beta^{s-1} + \cdots + a_s + \frac{a_{s+1} - a_1}{\beta} + \frac{a_{s+2} - a_2}{\beta^2} + \cdots + \frac{a_{n+s} - a_n}{\beta^n}.$$

If we note $H = \frac{a_{s+1}-a_1}{\beta} + \frac{a_{s+2}-a_2}{\beta^2} + \cdots + \frac{a_{n+s}-a_n}{\beta^n}$, we have $H \in \mathbb{A}_p[\beta^{-1}]$. So by the property (F), we have $d_\beta(H)$ is finite i.e $H = \frac{b_1}{\beta} + \frac{b_2}{\beta^2} + \cdots + \frac{b_l}{\beta^l}$. Which implies from Lemma 4.4, $d_\beta(x(\beta^s - 1)) = a_1 .... a_s . b_1 ... b_l$.
Moreover, then it's clear that $|H|_p < 1$. Since $a_{n+s} - a_n \neq 0$ and through Lemma 4.6 and Lemma 4.5, we obtain $H \neq 0$. Thus $ord_\beta(x(\beta^s - 1)) < 0$. Moreover, we have $x(\beta^s - 1) \in Fin(\beta) \cap Q'(\beta)$, thereby by Proposition 4.3, there exists $c > 0$ such that $||\overline{x\beta^s - x}||_p > c$, where

$$\overline{x\beta^s - x} = \begin{pmatrix} x^{(2)}(\beta^{(2)})^s - x^{(2)} \\ \vdots \\ x^{(d)}(\beta^{(d)})^s - x^{(d)}. \end{pmatrix}$$

However $x \in \mathbb{Q}$, then for all $j \in \{2, \dots, d\}$; $|x(\beta^{(j)})^s - x|_p = |x|_p$ and for this, we conclude that $|x|_p > c$ and finally the proof of our theorem is reached.

Remark 4.8. *The "unit" condition is necessary in Theorem 4.7. Indeed: Let $\beta \in \mathbb{Q}_p$ a not unit PC number and let $P(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0 \in \mathbb{A}_p[x]$ be the minimal polynomial of β. Since the modulus of all archimedean conjugates of β is < 1, then $|b_0| < 1$, so $|b_0|_p > 1$. For $n \geq 1$, set $x_n = \frac{1}{b_0^n}$. We will prove that x_n does not have purely periodic $\beta-$expansion. For this, we suppose that $d_\beta(x_n) = 0 \cdot \overline{a_1 \dots a_k}$, we get*

$$\begin{aligned} x_n &= \frac{a_1}{\beta} + \cdots + \frac{a_k}{\beta^k} + \frac{x_n}{\beta^k}, \\ &= (\frac{a_1}{\beta} + \cdots + \frac{a_k}{\beta^k})(1 + \frac{1}{\beta^k} + \frac{1}{\beta^{2k}} + \cdots), \\ &= (\sum_{i=1}^{k} a_i \beta^{-i})(\sum_{i \geq 0} \frac{1}{\beta^{ik}}), \\ &= \frac{\sum_{i=1}^{k} a_i \beta^{-i}}{1 - \beta^{-k}}, \\ &= \frac{\sum_{i=0}^{k-1} a_{k-i} \beta^{i}}{\beta^k - 1}. \end{aligned}$$

This gives $x_n(1-\beta^k) = \sum_{i=0}^{k-1}(-a_{k-i})\beta^i = \frac{1-\beta^k}{b_0^n} \in \mathbb{A}_p[\beta]$,
so $\frac{1-\beta^k}{b_0^n} = c_{n-1}\beta^{n-1} + c_{n-2}\beta^{n-2} + \cdots + c_0$ with $c_{n-1}, \dots, c_0 \in \mathbb{A}_p$. Consequently,

$$\begin{aligned} 1 - \beta^k &= b_0^n (c_{n-1}\beta^{n-1} + c_{n-2}\beta^{n-2} + \cdots + c_0) \\ &= (-b_n\beta^n - b_{n-1}\beta^{n-1} - \cdots - b_1\beta)^n (c_{n-1}\beta^{n-1} + c_{n-2}\beta^{n-2} + \cdots + c_0). \end{aligned}$$

As a result $1 = \beta(z_t\beta^t + \cdots + z_0)$ and this contradicts the hypothesis that β is not unit.

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