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Ulam-Hyers stability of Riemann-Liouville fractional integro-differential equations with fractional non-local integral boundary conditions

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Abstract. This research investigates the Ulam-Hyers and Ulam-Hyers-Rassias stability of the Riemann-Liouville fractional integro-differential equation with fractional non-local integral boundary conditions, employing the successive approximation method. Additionally, the study explores stability analysis through the Ulam-Hyers stability concept. The validity of the main results is demonstrated with several concrete examples.

1. Introduction and preliminaries

Ulam-Hyers stability refers to a concept in the field of functional equations and mathematical analysis. It is concerned with the stability of functional equations under small perturbations. Specifically, it describes the behavior of a functional equation when its solution is approximately satisfied rather than exactly. The Ulam-Hyers stability property ensures that if a function nearly satisfies a functional equation, there exists a true solution close to it.

In simple terms, Ulam-Hyers stability investigates the robustness of the solutions of a functional equation. If a small perturbation is made to the functional equation, the original solution should still be close to the perturbed solution. This property is useful in many areas of mathematics, including the study of differential equations and integral equations, and in various applications of approximation theory.

Research into stability problems for a broad spectrum of functional equations began with Ulam's renowned lecture in 1940 at the University of Wisconsin. During his talk, Ulam explored numerous significant open mathematical problems. These issues were later compiled in [50]. Among these problems,

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one concerning the stability of group homomorphisms led to the development of the concept known as Ulam stability. Hyers addressed this question partially and affirmatively for Banach spaces in [13].

Many years later, Rassias [35] expanded upon Hyers' findings by allowing for an unbounded Cauchy difference. Since then, extensive research has been conducted on the stability of various functional equations within different abstract spaces [12, 19, 36]. However, it seems that Obloza was the first mathematician to specifically study the Hyers-Ulam stability of functional equations, [32, 33]. For more details on Ulam-Hyers stability and its applications in different areas, we can refer to [1–3, 5–11, 14, 21–25, 28, 30, 31, 39–49, 52] and the references therein.

Recently, researchers have increasingly examined boundary value problems for nonlinear fractional differential equations. Fractional derivatives provide a valuable tool for characterizing the memory and hereditary properties of various materials and processes [34], making fractional-order models often more practical and accurate than their traditional integer-order counterparts. Fractional differential equations impact numerous scientific and technical fields, including physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, and the fitting of experimental data [34, 37].

In particular, significant progress has been made in understanding fractional boundary value problems, from theoretical studies to numerical simulations. The nonlocal nature of fractional differential operators has indeed played a key role in advancing the field, offering insight into the memory and hereditary characteristics of various systems. See, for example, [4, 26, 34, 37].

In the study of fractional differential equations, the concept of a fractional-order derivative with $\gamma = 0$ is approached using the Riemann-Liouville method. The fractional Riemann-Liouville derivative is a natural generalization of the Cauchy formula for the anti-derivative function $u(t)$ and serves as the left inverse of the corresponding fractional integral. Initial conditions in the Riemann-Liouville form are used for the initial value problem of ordinary differential equations with fractional order $\gamma = 0$ and fractional derivatives.

To address physical constraints, Caputo developed a modified definition of the fractional derivative. Caputo and Mainard further refined this concept, which offers a more intuitive approach to handling initial conditions in integro-differential equations of non-integer orders. This modified derivative is often referred to as the Caputo derivative or the regularized fractional derivative for $0 < \gamma < 1$.

Considerable research has been conducted on integral-differential equations, both from theoretical and practical perspectives. Notable references include the Volterra monograph [51], and the papers by [15, 17, 18], Lakshmi Kantham [27], and Medlock [29]. In recent years, there has been a surge of interest in studying Ulam-Hyers and Ulam-Hyers-Rassias stability for differential and integro-differential equations (see [16, 20, 38, 48, 53]). Using the fixed point theorem approach, the authors of [16, 38] explored several forms of Hyers-Ulam-Rassias stability for Volterra integro-differential equations.

Our goal in this paper is to study the Ulam-Hyers and Ulam-Hyers-Rassias stability for fractional non-local integral boundary conditions of the type for the Riemann liouville fractional integro-differential equations :

$$D^{\alpha}u(t) = f\left(t, u(t), (\phi u)(t), (\psi u)(t)\right), \ t \in [0, T], \alpha \in (1, 2], \tag{1.1}$$

subject to the boundary conditions of fractional order given by

$$D^{\alpha-2}u(0+) = 0$$

$$D^{\alpha-1}u(0+) = vI^{\alpha-1}u(\eta), 0 < \eta < T, \ v \text{ is a constant.}$$

and

$$(\phi x)(t) = \int_0^t \gamma(t, s)x(s)ds, (\psi x)(t) = \int_0^t \delta(t, s)x(s)ds.$$

with γ and δ being continuous functions on [0,T] $\times[0, T]$. The unique solution of (1.1), subject to the boundary conditions, is given by D^{α} , where D^{α} denotes the Riemann-Liouville fractional derivative of order α and $f : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function,

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\tau(\alpha)}\sigma(s)ds + \frac{vt^{\alpha-1}}{\tau(\alpha)-A} I^{2\alpha-1}\sigma(\eta) \tag{1.2}$$

where $t \in J = [a,b]$, $\sigma \in C([0,T])$, that is, σ is a continuous function from $[0,T]$ into $\mathbb{R}$, and $T > 0$.

Section 2 discusses the Ulam-Hyers stability of equation (1.2), while Section 4 addresses the Ulam-Hyers-Rassias stability of the same equation. The definitions of the Ulam-Hyers stability categories utilized in this article are provided next.

Consider $\varepsilon > 0$, $\psi \in C(J, \mathbb{R}_+)$, and $\sigma \in C(J, \mathbb{R}_+)$. We examine the following disparities:

$$|v'(t) - \rho(t)| \leq \varepsilon, \quad t \in J, \tag{1.3}$$

and

$$|v'(t) - \rho(t)| \leq \varepsilon\psi(t), \quad t \in J, \tag{1.4}$$

also

$$|v'(t) - \rho(t)| \leq \varepsilon\sigma(t), \quad t \in J, \tag{1.5}$$

where

$$\rho(t) := \int_0^t \frac{(t-s)^{\alpha-1}}{\tau(\alpha)}\sigma(s)ds + \frac{vt^{\alpha-1}}{[\tau(\alpha)-A]} I^{2\alpha-1}\sigma(\eta).$$

Definition 1.1. *The problem (1.2) is Ulam-Hyers stable if there is a constant $K_f > 0$ such that for each $\varepsilon > 0$ and for each solution $v \in C^1(J, \mathbb{R})$ of (1.3) there is a solution u of (1.2) satisfying*

$$|v(t) - u(t)| \leq K_f\varepsilon.$$

Definition 1.2. *The problem (1.2) is Ulam-Hyers-Rassias stable concerning $\phi \in C(J, \mathbb{R}+)$ if there is a constant $C_f > 0$ such that for each $\varepsilon > 0$ and for each solution $v \in C^1(J, \mathbb{R})$ of (1.4) there is a solution u of (1.2) satisfying*

$$|v(t) - u(t)| \leq C_f\varepsilon\phi.$$

Definition 1.3. *The problem (1.2) is σ-semi-Ulam-Hyers stable if there is a constant $K_f > 0$ and σ be a non-decreasing function and for each solution $v \in C^1(J, \mathbb{R})$ of (1.5) there is a solution u of (1.2) satisfying*

$$|v(t) - u(t)| \leq K_f\varepsilon\sigma(t).$$

2. Ulam-Hyers stability for Riemann-Liouville fractional I.D.E with fractional non-local integral boundary conditions

The Ulam-Hyers stability for Riemann-Liouville fractional integro-differential equation with fractional non-local boundary conditions (1.2) will be presented in this section using the successive approximation approach.

Remark 2.1. *We note that there is a continuous function $\delta(t)$ on J such that $|\delta(t)| \leq \varepsilon$ and that if the function v is a solution of 1.3.*

$$v'(t) = \rho(t) + \delta(t).$$

Let $f : J \times R \to R$, $(\phi x)(t) : J \times J \times R \to R$ and $(\delta x)(t) : J \times J \times R \to R$ are continuous functions. We consider the following hypotheses:

(H1) There exist positive constants L_1, L_2 such that for each $(t,s) \in J \times J$ and $w_1, w_2 \in R$ one has

$$\left| \frac{vt^{\alpha-1}}{\tau(\alpha)-A} I^{2\alpha-1} w_1 - \frac{vt^{\alpha-1}}{\tau(\alpha)-A} I^{2\alpha-1} w_2 \right| \leq L_1 |w_1 - w_2|,$$

$$\left| \int_0^t \frac{(t-s)^{\alpha-1}}{\tau(\alpha)} w_1 - \int_0^t \frac{(t-s)^{\alpha-1}}{\tau(\alpha)} w_2 \right| \leq L_1 |w_1 - w_2|.$$

(H2) Let us consider the inequality (1.4) where $\psi \in C(J, \mathbb{R}_+)$. Assume that $C > 0$ is a constant such that $kC^k = (b-0)C^{k-1}$, for all $k \geq 1$, and $0 < CL < 1$, and that, for $t \in J$, the following hypothesis is met.

$$\int_0^t \psi(s)ds \leq C\psi(t).$$

Theorem 2.2. *Assume that $\dfrac{vt^{\alpha-1}}{\tau(\alpha)-A} I^{2\alpha-1}\sigma(\eta)$ and $\displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\tau(\alpha)}\sigma(s)ds$ satisfy the (H1).*

Then , for each $\varepsilon > 0$ if the function v satisfies (1.3), there exists a unique solution u of (1.2) provided $u_0 = v_0$ and u satisfies the following estimate

$$|u(t) - v(t)| \leq \varepsilon b \exp((b-0)(1+L)). \tag{2.1}$$

Proof.

For each $\varepsilon > 0$ and let the function v satisfy 1.3, then basing on Remark 2.1,one has that then there is a continuous function $\delta(t)$ on J such that $|\delta(t)| \leq \varepsilon$ and v' (t)= $\rho(t)$+$\delta(t)$. This yields that the function v satisfies the integral equation

$$v(t) = v_0 + \int_0^t \rho(s)ds + \int_O^t \delta(s)ds, \tag{2.2}$$

where

$$\begin{aligned}\int_0^t \rho(s)ds &= \int_0^t \Big[\int_0^s \frac{(s-s)^{\alpha-1}}{\tau(\alpha)}\sigma(s)ds + \frac{vs^{\alpha-1}}{[\tau(\alpha)-A]} I^{2\alpha-1}\sigma(\eta)\Big] ds \\ &= \int_0^t \frac{vs^{\alpha-1}}{[\tau(\alpha)-A]} I^{2\alpha-1}\sigma(\eta) ds.\end{aligned}$$

We consider the sequence $(u_n)_{n\geq 0}$ defined as follows: $u_0(t) = v(t)$ and for $n = 1, 2, \cdots,$

$$u_n(t) = v_0 + \int_o^t \rho_{n-1}(s)ds, \tag{2.3}$$

where

$$\int_o^t \rho_{n-1}(s)ds = \int_o^t \frac{vs^{\alpha-1}}{[\tau(\alpha)-A]} I^{2\alpha-1}\sigma(\eta)_{n-1} ds.$$

by (2.2) and (2.3) , for n=1 one has

$$\begin{aligned}|u_1(t) - u_0(t)| &= \left| v_0 + \int_0^t \rho_0(s)ds - v(t) \right| \\ &= \left| v_0 + \int_0^t \rho_0(s)ds - v_0 - \int_0^t \rho_0(s)ds - \int_o^t \delta(s)ds \right|\end{aligned}$$

$$= \Big| \int_o^t \delta(s)ds \Big| \leq \varepsilon(t-0), \forall t \in J. \tag{2.4}$$

For $n = 1, 2, \cdots$, from the hypothesis (H1) one has

$$\begin{aligned} |u_{n+1}(t) - u_n(t)| &= \Big| \int_0^t \rho_0(s)ds - \int_0^t \rho_{n-1}(s)ds \Big| \\ &\leq L \int_0^t |u_n(s) - u_{n-1}(s)|ds + L \int_0^t \int_o^s |u_n(r) - u_{n-1}(r)|drds, \end{aligned}$$

where $L = \max L_1, L_2$. In particular, for n =1 and by (2.4) one gets

$$\begin{aligned} |u_2(t) - u_1(t)| &\leq \varepsilon L \int_0^t (s-0)ds + \varepsilon L \int_0^t \int_o^s (r-0)drds \\ &= \varepsilon L \Big(\frac{(t-0)^2}{2!} + \frac{(t-0)^3}{3!} \Big) \end{aligned}$$

and so, for $n = 2$, one also obtains

$$\begin{aligned} |u_3(t) - u_2(t)| &\leq \varepsilon L^2 \int_0^t \Big(\frac{(s-0)^2}{2!} + \frac{(s-0)^3}{3!} \Big) ds + \varepsilon L^2 \int_0^t \int_o^s \Big(\frac{(r-0)^2}{2!} + \frac{(r-0)^3}{3!} \Big) drds \\ &= \varepsilon L^2 \Big(\frac{(t-0)^3}{3!} + \frac{(t-0)^4}{4!} + \frac{(t-0)^5}{5!} \Big) \\ &\leq 3\varepsilon L^2 \Big(\frac{(t-0)^3}{3!} + \frac{(t-0)^4}{4!} + \frac{(t-0)^5}{5!} \Big) \end{aligned}$$

and for $n \geq 4$ we have

$$|u_n(t) - u_{n-1}(t)| \leq \varepsilon n L^{n-1} \Big(\frac{(t-0)^n}{n!} + \frac{(t-0)^{n+1}}{(n+1)!} + \cdots + \frac{(t-0)^{2n}}{(2n)!} + \frac{(t-0)^{2n+1}}{(2n+1)!} \Big). \tag{2.5}$$

Then, the estimation (2.5) can be rewritten by:

$$\begin{aligned} |u_n(t) - u_{n-1}(t)| &\leq \frac{\varepsilon(t-0)(L(t-0))^{n-1}}{(n-1)!} \Big(1 + \frac{(t-0)}{n+1} + \frac{(t-0)^2}{(n+1)(n+2)} \\ &\quad + \cdots + \frac{(t-0)^n}{(n+1)(n+2)\cdots} + \frac{(t-0)^{n+1}}{(n+1)(n+2)\cdots 2n(2n+1)} \Big) \\ &\leq \frac{\varepsilon b(L(t-0))^{n-1}}{(n-1)!} \Big(1 + \frac{(t-0)}{1!} + \frac{(t-0)^2}{2!} + \cdots + \frac{(t-0)^n}{n!} + \frac{(t-0)^{n+1}}{(n+1)!} \Big) \\ &\leq \frac{\varepsilon b(L(t-0))^{n-1}}{(n-1)!} \exp(t-0). \end{aligned}$$

Furthermore, if we assume that

$$|u_n(t) - u_{n-1}(t)| \leq \frac{\varepsilon b(L(t-0))^{n-1}}{(n-1)!} \exp(t-0), \tag{2.6}$$

then, one also gets

$$|u_{n+1}(t) - u_n(t)| \leq \frac{\varepsilon b(L(t-0))^n}{n!} \exp(t-0), \forall t \in J.$$

This yields :

$$\sum_{n=0}^{\infty} |u_{n+1}(t) - u_n(t)| \leq \varepsilon b \exp(t-0) \sum_{n=0}^{\infty} \frac{(L(t-0))^n}{n!}. \tag{2.7}$$

Since the right-hand series is convergent to the function $\exp(L(t-0))$, for each $\varepsilon > 0$ we deduce the series $u_0(t) + \sum_{n=1}^{\infty}[u_{n+1}(t) - u_n(t)]$ is uniformly convergent with respect to the norm $|\cdot|$ and

$$\sum_{n=0}^{\infty} |u_{n+1}(t) - u_n(t)| \leq \varepsilon b \exp((b-0)(1+L)). \tag{2.8}$$

Assume that

$$\overline{u}(t) == u_0(t) + \sum_{n=0}^{\infty}[u_{n+1}(t) - u_n(t)]. \tag{2.9}$$

Then,

$$u_j(t) = u_0(t) + \sum_{n=0}^{j}[u_{n+1}(t) - u_n(t)] \tag{2.10}$$

is the jth partial of the series (2.9). From (2.9) and (2.10), we obtain

$$\lim_{j\to\infty} |\overline{u}(t) - u_j(t)| = 0, \forall t \in J.$$

Define $u(t) = \overline{u}(t)$, for $t \in J$. We remark that the limit of the above sequence is a solution to the following equation:

$$u(t) = v_0 + \int_0^t \rho(s)ds, \text{ for all } t \in J, \tag{2.11}$$

where we denote by :

$$\rho(t) := \int_0^t \frac{(t-s)^{\alpha-1}}{\tau\alpha}\sigma(s)ds + \frac{vt^{\alpha-1}}{[\tau(\alpha) - A]} I^{2\alpha-1}\sigma(\eta).$$

By (2.3), (2.11) and the hypothesis (H1), one has that

$$\begin{aligned}
\left|u(t) - v_0 - \int_0^t \rho(s)ds\right| &= \left|\overline{u}(t) - (u_j(t) - \int_0^t \rho_{j-1}(s)ds - \int_o^t \rho(s)ds\right| \\
&\leq |\overline{u}(t) - u_j(t)| + \int_0^t \left|\rho_{j-1}(s)ds - \rho(s)\right| ds \\
&\leq |\overline{u}(t) - u_j(t)| + L\int_0^t |u_{j-1}(s) - u(s)|ds \\
&\quad + L\int_0^t \int_0^s |u_{j-1}(r) - u(r)|drds
\end{aligned} \tag{2.12}$$

Combining (2.9) and (2.10), we get

$$|\overline{u}(t) - u_j(t)| \leq \sum_{n=j+1}^{\infty} |u_{n+1}(t) - u_n(t)|$$

and by the estimation (2.7), one has for all $t \in J$

$$|u(t) - u_j(t)| \leq \varepsilon b \exp(b-0) \sum_{n=j+1}^{\infty} \frac{(L(b-0))^n}{n!}. \tag{2.13}$$

Hence, it follows from the inequalities (2.12) and (2.13) that

$$
\begin{aligned}
|u(t) - v_0 - \int_0^t \rho(s)ds| &\leq \varepsilon b e^{(b-o)} \sum_{n=j+1}^{\infty} \frac{(L(t-0))^n}{n!} + \varepsilon L b e^{(b-o)} \Big(\int_0^t \sum_{n=j+1}^{\infty} \frac{(L(s-0))^n}{n!} ds \\
&\quad + \int_0^t \int_0^s \sum_{n=j+1}^{\infty} \frac{(L(r-0))^n}{n!} dr ds \Big) \\
&\leq \varepsilon b e^{(b-o)} \Big[\sum_{n=j+1}^{\infty} \frac{t(L(t-0))^n}{n!} ds + \sum_{n=j+1}^{\infty} L^{n+1} \left(\frac{(t-0)^{n+1}}{(n+1)!} + \frac{(t-0)^{n+2}}{(n+2)!} \right) \Big]. \qquad (2.14)
\end{aligned}
$$

Taking limit as $n \to \infty$, we see that the right-hand series of (2.14) is convergent.
Therefore, one deduces that for all $t \in J$

$$|u(t) - v_0 - \int_0^t \rho(s)ds| \leq 0.$$

This means that for all $t \in J$

$$u(t) = v_0 + \int_0^t \rho(s)ds, \qquad (2.15)$$

which is a solution of (1.2). In addition, from the estimation (2.8), we have the estimate as follows:

$$|u(t) - v(t)| \leq \varepsilon b \exp\left((b-0)(1+L)\right).$$

To show the uniqueness of solution to the problem (1.2), we assume that $\overline{u}(t)$ is another solution of (1.1), which has the following form for all $t \in J$

$$\overline{u}(t) = v_0 + \int_0^t \overline{\rho}(s)ds, \qquad (2.16)$$

where

$$\overline{\rho}(t) := \int_0^t \frac{(t-s)^{\alpha-1}}{\tau\alpha} \overline{\sigma}(s)ds + \frac{v t^{\alpha-1}}{[\tau(\alpha) - A]} I^{2\alpha-1} \overline{\sigma}(\eta).$$

By using the hypothesis (H1), one obtains that for all $t \in J$

$$w(t) \leq L \int_0^t w(s)ds + L \int_0^t \int_0^s w(r)dsds.$$

where $w(t) = |u(t) - \overline{u}(t)|$.

Then, applying Gronwall's lemma, we infer that w(t)=0 on J and so that, $u(t) = \overline{u}(t)$. This completes the proof. ■

3. σ-Semi-Ulam-Hyers stability for Riemann-Liouville fractional I.D.E. with fractional non-local integral boundary conditions

We will introduce the σ-semi-Ulam-Hyers stability for Riemann-Liouville fractional I.D.E. with fractional non-local boundary conditions (1.2) in this section using the successive approximation approach.

Theorem 3.1. *Assume that* $\frac{vt^{\alpha-1}}{\tau(\alpha)-A}I^{2\alpha-1}\sigma(\eta)$ *and* $\int_0^t \frac{(t-s)^{\alpha-1}}{\tau(\alpha)}\sigma(s)ds$ *satisfy (H1). Next, for any* $\varepsilon > 0$ *and* $\sigma : [a,b] \rightarrow (o,\infty)$*, if the function v satisfies (1.5), then there is a unique solution. Using u from (1.2), we derive* $u_0 = v_0$*, and u satisfies the constraint provided.*

$$|u(t) - v(t)| \leq \varepsilon b \exp((b-0)(1+L))\frac{\sigma(t)}{\sigma(o)}. \tag{3.1}$$

Proof.

According to Remark (2.1), for any $\varepsilon > 0$ and given the function v satisfying (1.5), there exists a continuous function $\delta(t)$ on J such that $|\delta(t)| \leq \varepsilon$ and v' (t)= $\rho(t)$+$\delta(t)$. This indicates that the integral equation is satisfied by the function v.

$$v(t) = v_0 + \int_0^t \rho(s)ds + \int_O^t \delta(s)ds, \tag{3.2}$$

where

$$\begin{aligned}\int_0^t \rho(s)ds &= \int_0^t \Big[\int_0^s \frac{(s-s)^{\alpha-1}}{\tau(\alpha)}\sigma(s)ds + \frac{vs^{\alpha-1}}{[\tau(\alpha)-A]}I^{2\alpha-1}\sigma(\eta)\Big]ds \\ &= \int_0^t \frac{vs^{\alpha-1}}{[\tau(\alpha)-A]}I^{2\alpha-1}\sigma(\eta)]ds.\end{aligned}$$

We consider the sequence $(u_n)_{n\geq 0}$ defined as follows: $u_0(t) = v(t)$ and for $n = 1, 2, \cdots$,

$$u_n(t) = v_0 + \int_o^t \rho_{n-1}(s)ds, \tag{3.3}$$

where

$$\int_o^t \rho_{n-1}(s)ds = \int_o^t \frac{vs^{\alpha-1}}{[\tau(\alpha)-A]}I^{2\alpha-1}\sigma(\eta)_{n-1}ds.$$

by (3.2) and (3.3) , for n=1 one has

$$\begin{aligned}|u_1(t) - u_0(t)| &= \Big|v_0 + \int_0^t \rho_0(s)ds - v(t)\Big| \\ &= \Big|v_0 + \int_0^t \rho_0(s)ds - v_0 - \int_0^t \rho_0(s)ds - \int_o^t \delta(s)ds\Big|\end{aligned}$$

$$= \Big|\int_o^t \delta(s)ds\Big| \leq \varepsilon(t-0)\frac{\sigma(t)}{\sigma(0)}, \forall t \in J. \tag{3.4}$$

From the hypothesis $(H1)$, one has for all $n = 1, 2, \cdots$

$$\begin{aligned}|u_{n+1}(t) - u_n(t)| &= \Big|\int_0^t \rho_0(s)ds - \int_0^t \rho_{n-1}(s)ds\Big| \\ &\leq L\int_0^t |u_n(s) - u_{n-1}(s)|ds + L\int_0^t\int_o^s |u_n(r) - u_{n-1}(r)|drds,\end{aligned}$$

where $L = \max L_1, L_2$. In particular, for n =1 and by (3.4) one gets

$$\begin{aligned}|u_2(t) - u_1(t)| &\leq \varepsilon L\int_0^t (s-0)ds\frac{\sigma(t)}{\sigma(0)} + \varepsilon L\int_0^t\int_o^s (r-0)drds\frac{\sigma(t)}{\sigma(0)} \\ &= \varepsilon L\Big(\frac{(t-0)^2}{2!} + \frac{(t-0)^3}{3!}\Big)\frac{\sigma(t)}{\sigma(0)}\end{aligned}$$

and so, for $n = 2$, one also obtains that

$$
\begin{aligned}
|u_3(t) - u_2(t)| &\leq \varepsilon L^2 \int_0^t \Big(\frac{(s-0)^2}{2!} + \frac{(s-0)^3}{3!}\Big) ds \frac{\sigma(t)}{\sigma(0)} \\
&\quad + \varepsilon L^2 \int_0^t \int_o^s \Big(\frac{(r-0)^2}{2!} + \frac{(r-0)^3}{3!}\Big) drds \frac{\sigma(t)}{\sigma(0)} \\
&= \varepsilon L^2 \Big(\frac{(t-0)^3}{3!} + \frac{(t-0)^4}{4!} + \frac{(t-0)^5}{5!}\Big) \frac{\sigma(t)}{\sigma(0)} \\
&\leq 3\varepsilon L^2 \Big(\frac{(t-0)^3}{3!} + \frac{(t-0)^4}{4!} + \frac{(t-0)^5}{5!}\Big) \frac{\sigma(t)}{\sigma(0)}
\end{aligned}
$$

and for $n \geq 4$ we have

$$
|u_n(t) - u_{n-1}(t)| \leq \varepsilon n L^{n-1} \Big(\frac{(t-0)^n}{n!} + \cdots + \frac{(t-0)^{2n}}{(2n)!} + \frac{(t-0)^{2n+1}}{(2n+1)!}\Big) \frac{\sigma(t)}{\sigma(0)} \tag{3.5}
$$

Then, the estimation (3.5) can be rewritten as:

$$
\begin{aligned}
&|u_n(t) - u_{n-1}(t)| \\
&\leq \frac{\varepsilon(t-0)(L(t-0))^{n-1}}{(n-1)!} \Big(1 + \frac{(t-0)}{n+1} + \frac{(t-0)^2}{(n+1)(n+2)} \\
&\quad + \cdots + \frac{(t-0)^n}{(n+1)(n+2)\cdots} + \frac{(t-0)^{n+1}}{(n+1)(n+2)\cdots 2n(2n+1)}\Big) \frac{\sigma(t)}{\sigma(0)} \\
&\leq \frac{\varepsilon b(L(t-0))^{n-1}}{(n-1)!} \Big(1 + \frac{(t-0)}{1!} + \cdots + \frac{(t-0)^n}{n!} + \frac{(t-0)^{n+1}}{(n+1)!}\Big) \frac{\sigma(t)}{\sigma(0)} \\
&\leq \frac{\varepsilon b(L(t-0))^{n-1}}{(n-1)!} \exp(t-0) \frac{\sigma(t)}{\sigma(0)}.
\end{aligned}
$$

Furthermore, if we assume that

$$
|u_n(t) - u_{n-1}(t)| \leq \frac{\varepsilon b(L(t-0))^{n-1}}{(n-1)!} \exp(t-0) \frac{\sigma(t)}{\sigma(0)}, \tag{3.6}
$$

then, we obtain for all $t \in J$ that :

$$
|u_{n+1}(t) - u_n(t)| \leq \frac{\varepsilon b(L(t-0))^n}{n!} \exp(t-0) \frac{\sigma(t)}{\sigma(0)}.
$$

This yields :

$$
\sum_{n=0}^{\infty} |u_{n+1}(t) - u_n(t)| \leq \varepsilon b \frac{\sigma(t)}{\sigma(0)} \exp(t-0) \sum_{n=0}^{\infty} \frac{(L(t-0))^n}{n!}. \tag{3.7}
$$

Since the right-hand series is convergent to the function $\exp(L(t-0))$, for each $\varepsilon > 0$ we deduce the series $u_0(t) + \sum_{n=1}^{\infty} [u_{n+1}(t) - u_n(t)]$ is uniformly convergent concerning the norm $|\cdot|$ and

$$
\sum_{n=0}^{\infty} |u_{n+1}(t) - u_n(t)| \leq \varepsilon b \exp((b-0)(1+L)) \frac{\sigma(t)}{\sigma(0)}. \tag{3.8}
$$

Assuming that

$$\overline{u}(t) == u_0(t) + \sum_{n=0}^{\infty}[u_{n+1}(t) - u_n(t)], \tag{3.9}$$

we have :

$$u_j(t) = u_0(t) + \sum_{n=0}^{j}[u_{n+1}(t) - u_n(t)], \tag{3.10}$$

which is the j-th partial of the series (3.9).

From (3.9) and (3.10), we obtain for all $t \in J$:

$$\lim_{j\to\infty} |\overline{u}(t) - u_j(t)| = 0.$$

Define $u(t) = \tilde{u}(t)$, for $t \in J$. We remark that the limit of the above sequence is a solution to the following equation:

$$u(t) = v_0 + \int_0^t \rho(s)ds, \ t \in J, \tag{3.11}$$

where we set :

$$\rho(t) := \int_0^t \frac{(t-s)^{\alpha-1}}{\tau\alpha}\sigma(s)ds + \frac{vt^{\alpha-1}}{[\tau(\alpha) - A]}I^{2\alpha-1}\sigma(\eta).$$

By (3.4), (3.11) and using the hypothesis $(H1)$, we get :

$$\begin{aligned}
\Big|u(t) - v_0 - \int_0^t \rho(s)ds\Big| &= \Big|\overline{u}(t) - \Big(u_j(t) - \int_0^t \rho_{j-1}(s)ds\Big) - \int_o^t \rho(s)ds\Big| \\
&\leq |\overline{u}(t) - u_j(t)| + \int_0^t |\rho_{j-1}(s)ds - \rho(s)|ds \\
&\leq |\overline{u}(t) - u_j(t)| + L\int_0^t |u_{j-1}(s) - u(s)|ds \\
&\quad + L\int_0^t\int_0^s |u_{j-1}(r) - u(r)|drds.
\end{aligned} \tag{3.12}$$

Combining (3.9) and (3.10), we deduce that :

$$\Big|\overline{u}(t) - u_j(t)\Big| \leq \sum_{n=j+1}^{\infty}\Big|u_{n+1}(t) - u_n(t)\Big|,$$

and by the estimation (3.7), we obtain for all $t \in J$ that :

$$\Big|u(t) - u_j(t)\Big| \leq \varepsilon b\exp(b-0)\sum_{n=j+1}^{\infty}\frac{(L(b-0))^n}{n!}\frac{\sigma(t)}{\sigma(0)}. \tag{3.13}$$

Hence, it follows from the inequalities (3.12) and (3.13) that

$$\begin{aligned}
\Big|u(t) - v_0 - \int_0^t \rho(s)ds\Big| &\leq \varepsilon be^{(b-o)}\sum_{n=j+1}^{\infty}\frac{(L(t-0))^n}{n!}\frac{\sigma(t)}{\sigma(0)} + \varepsilon Lbe^{(b-o)}\Big(\int_0^t\sum_{n=j+1}^{\infty}\frac{(L(s-0))^n}{n!}ds \\
&\quad + \int_0^t\int_0^s\sum_{n=j+1}^{\infty}\frac{(L(r-0))^n}{n!}drds\Big)\frac{\sigma(t)}{\sigma(0)}.
\end{aligned}$$

Therefore, we deduce that :

$$\Big|u(t) - v_0 - \int_0^t \rho(s)ds\Big| \leq \varepsilon b e^{(b-o)}\Big[\sum_{n=j+1}^{\infty} \frac{(L(t-0))^n}{n!}ds + \sum_{n=j+1}^{\infty} L^{n+1}\Big(\frac{(t-0)^{n+1}}{(n+1)!} + \frac{(t-0)^{n+2}}{(n+2)!}\Big)\Big]\frac{\sigma(t)}{\sigma(0)}. \tag{3.14}$$

Taking limit as $n \to \infty$, we see that the right-hand series of (3.14) is convergent. Therefore, we obtain for all $t \in J$:

$$\Big|u(t) - v_0 - \int_0^t \rho(s)ds\Big| \leq 0.$$

This means that for all $t \in J$:

$$u(t) = v_0 + \int_0^t \rho(s)ds, \tag{3.15}$$

which is a solution of (1.2).

In addition, from the estimation (3.8), we have the estimate as follows:

$$|u(t) - v(t)| \leq \varepsilon b \exp((b-0)(1+L))\frac{\sigma(t)}{\sigma(0)}.$$

To show the uniqueness of solution to the problem (1.2), we assume that $\overline{u}(t)$ is another solution of 1.2, which has the following form for any $t \in J$:

$$\overline{u}(t) = v_0 + \int_0^t \overline{\rho}(s)ds, \tag{3.16}$$

where we put :

$$\overline{\rho}(t) := \int_0^t \frac{(t-s)^{\alpha-1}}{\tau\alpha}\sigma\,(s)ds + \frac{vt^{\alpha-1}}{[\tau(\alpha) - A]}I^{2\alpha-1}\overline{\sigma}(\eta).$$

By using the hypothesis $(H1)$, we have for any $t \in J$:

$$w(t) \leq L\int_0^t w(s)ds + L\int_0^t\int_0^s w(r)dsds,$$

where $w(t) = |u(t) - \overline{u}(t)|$.

Then, according to Gronwall's Lemma, we infer that $w(t) = 0$ on J and so that $u(t) = \overline{u}(t)$. This achieves the proof. ■

4. Ulam-Hyers-Rassias stability for Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions

The Ulam-Hyers-Rassias stability for Riemann-Liouville fractional integro-differential equations with fractional nonlocal boundary conditions 1.2 shall be presented in this section, using the successive approximation approach.

Remark 4.1. *We note that there exists a continuous function $\xi(t)$ on J such that $|\xi(t)| \leq \varepsilon\psi(t)$ and that if the function v is a solution of (1.4).*

$$v'(t) = \rho(t) + \xi(t).$$

Theorem 4.2. *Assume* $(H1)$ *and* $(H2)$ *hold true. There exists a unique solution* u *of (1.2) with* $u_0 = v_0$ *and* u *verifies the following estimate, for* $t \in J$*, for each* $\varepsilon > 0$ *if the function* v *satisfies (1.4).*

$$|v(t) - u(t)| \leq \varepsilon \frac{b-0}{(1-C)(1-CL)} \psi(t). \tag{4.1}$$

Proof.

Based on Remark (4.1), we have that for any $\varepsilon > 0$, the function v verifies (1.4).

This means that there exists a continuous function $\xi(t)$ on J such that $|\xi(t)| \leq \varepsilon\psi(t)$ and $v'(t) = \rho(t) + \xi(t)$. Consequently, the integral equation is satisfied by the function v defined as :

$$v(t) = v_0 + \int_0^t \rho(s)d(s) + \int_0^t \xi(s)d(s), \tag{4.2}$$

where

$$\begin{aligned}\int_0^t \rho(s)d(s) &= \int_0^t \Big[\int_0^s \frac{(s-s)^{\alpha-1}}{\tau(\alpha)}\sigma(s)d(s) + \frac{vs^{\alpha-1}}{[\tau(\alpha)-A]} I^{2\alpha-1}\sigma(\eta)\Big]ds \\ &= \int_0^t \frac{vs^{\alpha-1}}{[\tau(\alpha)-A]} I^{2\alpha-1}\sigma(\eta)ds.\end{aligned}$$

Similarly to Theorem (2.2), we reconsider the sequence $(u_n)_n \geq 0$ defined as in (2.3) with $u_0(t) = v(t)$, $\forall t \in J$. Now, by (2.3), the hypothesis $(H3)$ and (4.2), for $n = 1$, one has for all $t \in J$:

$$|u_1(t) - u_0(t)| = \Big|v_0 + \int_0^t \rho_0(s)ds - v(t)\Big| \leq \varepsilon \int_0^t \psi(s)d(s) \leq \varepsilon C\psi(t).$$

For $n = 1, 2, \ldots$, and from the hypothesis $(H1)$, we have :

$$|u_{n+1}(t) - u_n(t)| \leq L\int_0^t \Big(|u_n(s) - u_{n-1}(s)|ds + \int_0^s |u_n(r) - u_{n-1}(r)|dr\Big)ds,$$

where $L = \max\{L_1, L_2\}$.

In particular for $n = 1$, for all $t \in J$

$$\begin{aligned}|u_2(t) - u_1(t)| &\leq \varepsilon LC\int_0^t \psi(s)ds + \varepsilon LC\int_0^t\int_0^s \psi(r)drds \\ &= \varepsilon L(C^2 + C^3)\psi(t),\end{aligned}$$

and so that, for $n = 2$, we obtain

$$\begin{aligned}|u_3(t) - u_2(t)| &\leq L\int_0^s |u_2(s) - u_1(s)|ds + L\int_0^t\int_0^s |u_2(r) - u_1(r)|drds \\ &\leq 3\varepsilon L^2(C^3 + C^4 + C^5)\psi(t).\end{aligned}$$

For $n \geq 4$, we have

$$|u_n(t) - u_{n-1}(t)| \leq n\varepsilon(C^n + C^{n+1} + \cdots + C^{2n} + C^{2n+1})L^{n-1}\psi(t). \tag{4.3}$$

Now, by the hypothesis $(H3)$, the estimation (2.5) may be differently rewritten, for all $t \in J$, as :

$$\begin{aligned}|u_n(t) - u_{n-1}(t)| &\leq \varepsilon(b-0)(CL)^{n-1}(1 + C^1 + \cdots + C^{n+1})\psi(t) \\ &\leq \varepsilon(b-0)\Big(\frac{1-C^{n+1}}{1-C}\Big)(CL)^{n-1}\psi(t).\end{aligned}$$

Besides, if the assumption

$$|u_n(t) - u_{n-1}(t)| \leq \varepsilon(b-0)\Big(\frac{1-C^{n+1}}{1-C}\Big)(CL)^{n-1}\psi(t), \tag{4.4}$$

is satisfied for all $t \in J$, then by induction, we get

$$|u_{n+1}(t) - u_n(t) \leq \varepsilon(b-0)\Big(\frac{1-C^{n+2}}{1-C}\Big)(CL)^{n}\psi(t), \forall t \in J.$$

This yields :

$$\sum_{n=0}^{\infty}|u_{n+1}(t) - u_n(t)| \leq \varepsilon(b-0)\Big(\frac{1}{1-C}\Big)\sum_{n=0}^{\infty}(CL)^{n}\psi(t). \tag{4.5}$$

Due to the hypothesis $(H3)$, we observe that $\sum_{n=0}^{\infty}(CL)^n \to \frac{1}{1-CL}$ as $n \to \infty$.

Hence, for every $\varepsilon > 0$ we infer that the series $u_0(t) + \sum_{n=0}^{\infty}[u_{n+1}(t) - u_n(t)]$ is uniformly convergent on J and

$$\sum_{n=0}^{\infty}|u_{n+1}(t) - u_n(t)| \leq \varepsilon\frac{b-0}{(1-C)(1-CL}\psi(t), \forall t \in J. \tag{4.6}$$

Similarly to the proof of Theorem (2.2), we can show that $u(\cdot)$ is a solution of (1.2) which has the following form for each $t \in J$:

$$u(t) = v_0 + \int_0^t \rho(s)d(s),$$

where

$$\rho(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\tau(\alpha)}\sigma(s)d(s) + \frac{vt^{\alpha-1}}{[\tau(\alpha)-A]}I^{2\alpha-1}\sigma(\eta).$$

Additionally, the following estimate :

$$|u(t) - v(t)| \leq \varepsilon\frac{b-0}{(1-C)(1-CL)}\psi(t),$$

is satisfied for any $t \in J$. ■

5. σ-Semi-Ulam-Hyers-Rassias stability for Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions

Using the successive approximation method, this section will provide the σ-Semi-Ulam-Hyers-Rassias stability for Riemann-Liouville fractional integro-differential equations with fractional nonlocal boundary conditions (1.2).

Remark 5.1. *We note that there exists a continuous function $\xi(t)$ on J such that $|\xi(t)| \leq \varepsilon\sigma(t)|$ and that if the function v is a solution of (1.4).*

$$v'(t) = \rho(t) + \xi(t).$$

Theorem 5.2. *Assume that both hypothesis* $(H1)$ *and hypothesis* $(H2)$ *are true.*

There exists a unique solution u *of (1.2) with* $u_0 = v_0$ *and* u *fulfils the following estimate, for* $t \in J$, *for each* $\varepsilon > 0$ *and the* $\sigma : [a, b] \to (0, \infty)$ *if the function* v *satisfies (1.5).*

$$|v(t) - u(t)| \leq \varepsilon \frac{b - 0}{(1 - C)(1 - CL)} \psi(t) \frac{\sigma(t)}{\sigma(0)}. \tag{5.1}$$

Proof.

Let v verifying (1.5) for each $\varepsilon > 0$.

Thanks to Remark (5.1), there exists a continuous function $\xi(t)$ on J such that $|\xi(t)| \leq \varepsilon\sigma(t)$ and $v'(t) = \rho(t) + \xi(t)$.

We deduce that the integral equation is satisfied by the function v :

$$v(t) = v_0 + \int_0^t \rho(s)d(s) + \int_0^t \xi(s)d(s), \tag{5.2}$$

where

$$\begin{aligned}\int_0^t \rho(s)d(s) &= \int_0^t \Big[\int_0^s \frac{(s-s)^{\alpha-1}}{\tau(\alpha)}\sigma(s)d(s) + \frac{vs^{\alpha-1}}{[\tau(\alpha)-A]} I^{2\alpha-1}\sigma(\eta)\Big]ds \\ &= \int_0^t \frac{vs^{\alpha-1}}{[\tau(\alpha)-A]} I^{2\alpha-1}\sigma(\eta)ds.\end{aligned}$$

Analogously to Theorem (3.1), we shall consider the sequence $(u_n)_n \geq 0$ defined as as in (3.3) with $u_0(t) = v(t)$, $\forall t \in J$. Now, by (3.3), the hypothesis $(H3)$ and (5.2), for $n = 1$, we have

$$|u_1(t) - u_0(t)| = \Big|v_0 + \int_0^t \rho_0(s)ds - v(t)\Big| \leq \varepsilon \int_0^t \psi(s)d(s)\frac{\sigma(t)}{\sigma(0)} \leq \varepsilon C\psi(t)\frac{\sigma(t)}{\sigma(0)}, \forall t \in J.$$

From the hypothesis $(H1)$, we have for $n = 1, 2, \cdots$

$$|u_{n+1}(t) - u_n(t)| \leq L\int_0^t \Big(|u_n(s) - u_{n-1}(s)|ds + \int_0^s |u_n(r) - u_{n-1}(r)|dr\Big)ds,$$

where $L = \max\{L_1, L_2\}$. In particular for $n = 1$, one has

$$\begin{aligned}|u_2(t) - u_1(t)| &\leq \varepsilon LC\int_0^t \psi(s)ds\frac{\sigma(t)}{\sigma(0)} + \varepsilon LC\int_0^t\int_0^s \psi(r)drds\frac{\sigma(t)}{\sigma(0)} \\ &= \varepsilon L(C^2 + C^3)\psi(t)\frac{\sigma(t)}{\sigma(0)}, \forall t \in J\end{aligned}$$

and so that, for $n = 2$, we also obtain

$$\begin{aligned}|u_3(t) - u_2(t)| &\leq L\int_0^s |u_2(s) - u_1(s)|ds + L\int_0^t\int_0^s |u_2(r) - u_1(r)|drds \\ &\leq 3\varepsilon L^2(C^3 + C^4 + C^5)\psi(t)\frac{\sigma(t)}{\sigma(0)}.\end{aligned}$$

and for $n \geq 4$ we have

$$|u_n(t) - u_{n-1}(t)| \leq n\varepsilon\Big(C^n + C^{n+1} + \cdots + C^{2n} + C^{2n+1}\Big)L^{n-1}\psi(t)\frac{\sigma(t)}{\sigma(0)}. \tag{5.3}$$

Now, by the hypothesis ($H3$), the estimation (3.5) can be expressed as :

$$|u_n(t) - u_{n-1}(t)| \leq \varepsilon(b-0)(CL)^{n-1}\left(1 + C^1 + \cdots + C^{n+1}\right)\psi(t)\frac{\sigma(t)}{\sigma(0)}$$
$$\leq \varepsilon(b-0)\Big(\frac{1-C^{n+1}}{1-C}\Big)(CL)^{n-1}\psi(t)\frac{\sigma(t)}{\sigma(0)}, \forall t \in J.$$

If the assumption

$$|u_n(t) - u_{n-1}(t)| \leq \varepsilon(b-0)\Big(\frac{1-C^{n+1}}{1-C}\Big)(CL)^{n-1}\psi(t)\frac{\sigma(t)}{\sigma(0)}, \forall t \in J, \tag{5.4}$$

is satisfied, then by induction, we also get that :

$$|u_{n+1}(t) - u_n(t) \leq \varepsilon(b-0)\Big(\frac{1-C^{n+2}}{1-C}\Big)(CL)^{n}\psi(t)\frac{\sigma(t)}{\sigma(0)}, \forall t \in J.$$

This yields :

$$\sum_{n=0}^{\infty}|u_{n+1}(t) - u_n(t)| \leq \varepsilon(b-0)\Big(\frac{1}{1-C}\Big)\sum_{n=0}^{\infty}(CL)^n\psi(t)\frac{\sigma(t)}{\sigma(0)}. \tag{5.5}$$

Using the hypothesis (H3), we observe that $\sum_{n=0}^{\infty}(CL)^n \to \frac{1}{1-CL}$ as $n \to \infty$. Hence for every $\varepsilon > 0$ we infer that the series $u_0(t) + \sum_{n=0}^{\infty}[u_{n+1}(t) - u_n(t)]$ is uniformly convergent on J and for all $t \in J$

$$\sum_{n=0}^{\infty}|u_{n+1}(t) - u_n(t)| \leq \varepsilon\frac{b-0}{(1-C)(1-CL}\psi(t)\frac{\sigma(t)}{\sigma(0)}. \tag{5.6}$$

With the same manner as in the proof of theorem (3.1), we can show that $u(\cdot)$ is a solution of (1.2) which has the following form :

$$u(t) = v_0 + \int_0^t \rho(s)d(s), \text{ for all } t \in J,$$

where

$$\rho(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\tau(\alpha)}\sigma(s)d(s) + \frac{vt^{\alpha-1}}{[\tau(\alpha) - A]}I^{2\alpha-1}\sigma(\eta).$$

The following estimate is also satisfied for all $t \in J$:

$$|u(t) - v(t)| \leq \varepsilon\frac{b-0}{(1-C)(1-CL)}\psi(t)\frac{\sigma(t)}{\sigma(0)},$$

which ends the proof. ■

6. Examples

In this section, some examples are presented to illustrate our results.

Example 6.1. Consider the following problem

$$u'(t) = 4 + \int_o^t u(s)ds, \text{ for all } t \in [0,4], \qquad u(0) = 4, \tag{6.1}$$

We see that $v(t) = 4$, for all $t \in [0,4]$ complies with the following inequality

$$\left| v'(t) - 4 - \int_0^4 v(s)ds \right| \leq 5.$$

Now, we can choose $v_0(t) = u(0) = 4$. By using the successive approximation method as in theorem (2.2) , we obtain the following successive solution to (6.1) as

$$v_0(t) = 4,$$

$$u_1(t) = v(0) + \int_o^t \left(4 + \int_o^s u(r)dr\right)ds = 4 + t + \frac{t^2}{2!}$$

Then, it is no difficult to see that $u(t) = 4 + t + \frac{t^2}{2!}$ forms a solution (6.1) and one gets the estimate :

$$|v(t) - u(t)| = \left|4 - (4 + t + \frac{t^2}{2!})\right| \leq \frac{3}{5}.$$

Next, we define the function $u^*(t) = 4 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots$ is also a solution of (6.1) and we also have

$$|v(t) - u^*(t)| = \left|4 - (4 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!}\right| \leq \frac{25}{33}.$$

Therefore, it shows the function $u^*(t)$ is better approximate solution than the function u(t).

Example 6.2. Consider the following problem

$$u'(t) = 2 + \int_0^t \left[\frac{(t-s)}{4} + \frac{t}{5}\right]\sigma(s)ds, \text{for all } t \in [0,2], u(0) = 2. \tag{6.2}$$

where, $\rho(t) = \frac{(t-s)}{4} + \frac{t}{5}$ is continuous and integrable for $t \in [0,2]$ and $\sigma(s) \in C[0,2]$.
Now, we can choose $v_0(t) = u(0) = 2$. By using the successive approximation method as in theorem (3.1) , we obtain the following successive solution to (6.2) as

$$u_1(t) = v(0) + \int_0^t \left[\frac{(t-s)}{4} + \frac{t}{5}\right]\sigma(s)ds = 2 + \left[\frac{t^2}{8} + \frac{t^2}{10}\right] = 2 + \frac{9t^2}{40}.$$

There is no difficult to see that $u(t) = 2 + \frac{9t^2}{40}$ forms a solution (6.2) and one gets the estimate

$$|v(t) - u(t)| = \left|2 - \left(2 + \frac{9t^2}{40}\right)\right| \leq \frac{5}{8}.$$

Next we define the function $u^*(t) = 2 + \frac{9t^2}{40} + \frac{9t^3}{50} + \cdots$ is also a solution of (6.2) and we also have

$$\left|v(t) - u^*(t)| = |2 - \left(2 + \frac{9t^2}{40} + \frac{9t^3}{50}\right)\right| \leq \frac{23}{34}.$$

Therefore , it shows the function $u^*(t)$ is better approximate solution than the function u(t).

Example 6.3. Consider the following problem

$$u'(t) = u(t) + \int_0^t \frac{u(s)}{1+u(s)}ds, \tag{6.3}$$

here $t \in [0,1]$. We set

$$\frac{vt^{\alpha-1}}{[\tau(\alpha)-A]}I^{2\alpha-1}\sigma(\eta) = u(t)$$

and

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\tau(\alpha)}\sigma(s)d(s) = \int_0^t \frac{u(s)}{1+u(s)}ds$$

Then we see that

$$\left|f(t,w_1) - f(t,w_2)\right| = |w_1 - w_2|$$

and

$$\left|\frac{vt^{\alpha-1}}{[\tau(\alpha)-A]}I^{2\alpha-1}w_1 - \frac{vt^{\alpha-1}}{[\tau(\alpha)-A]}I^{2\alpha-1}w_2\right| = |w_1 - w_2|$$

or also

$$\begin{aligned}\left|\frac{(t-s)^{\alpha-1}}{\tau(\alpha)}w_1(s) - \frac{(t-s)^{\alpha-1}}{\tau(\alpha)}w_2(s)\right| &= \left|\frac{w_1(s)}{1+w_1(s)} - \frac{w_2(s)}{1+w_2(s)}\right| \\ &\leq \frac{|w_1 - w_2|}{(1+w_1)(1+w_2)} \\ &\leq |w_1 - w_2|\end{aligned}$$

This yields that the hypotheses of Theorem (2.2) is satisfied. That means (1.2) has unique solution on $[0,1]$. Furthermore, if the function v satisfies

$$\left|v'(t) - v(t) - \int_0^t \frac{v(s)}{1+v(s)}ds\right| \leq \varepsilon$$

then according to Theorem (2.2), there exists a solution u of (1.2) satisfying

$$|u(t) - v(t)| \leq \varepsilon \exp(4), \text{for all } t \in [0,1]$$

This means that the problem (6.3) is Ulam-Hyers stable.

Example 6.4. Consider the functional equation:

$$f(x) = 3f\left(\frac{x}{3}\right), \text{ for all } x \in [0,1] \tag{6.4}$$

To demonstrate Hyers-Ulam-Rassias stability via successive approximation, we set up an iterative scheme starting with an initial guess $f_0(x)$ and refining it by iteration.

Let us choose $f_0(x) = 0$ as an initial approximation.

Define the sequence $(f_n(x))_n$ as follows:

$$f_{n+1}(x) = 3f_n\left(\frac{x}{3}\right).$$

To apply the theorem (4.2), we need to show that the sequence $(f_n(x))_n$ converges uniformly to a solution of the original functional equation.

Start with $f_0(x) = 0$.

$$f_1(x) = 3f_0\left(\frac{x}{3}\right) = 2 \times 0 = 0.$$

$$f_2(x) = 3f_1\left(\frac{x}{3}\right) = 2 \times 0 = 0.$$

$$f_3(x) = 3f_2\left(\frac{x}{3}\right) = 2 \times 0 = 0.$$

so $f_n(x) = 0$, for all $n \geq 0$ and all $x \in [0, 1]$.

To apply the Hyers-Ulam-Rassias stability theorem rigorously, we need to verify uniform convergence of $f_n(x)$ to $f(x) = 0$.

For any $\varepsilon > 0$ choose N such that for all $n \geq N, |f_n(x) - 0| < \varepsilon$ for all $x \in [0, 1]$.

Since $f_n(x) = 0$ for all $n \geq N$ and $x \in [0, 1], |f_n(x) - 0| = 0 < \varepsilon$.

Therefore, $f_n(x)$ converges uniformly to $f(x) = 0$, for all $x \in [0, 1]$.

This means that the problem (6.4) is Hyers-Ulam-Rassias stable.

7. Conclusion

Ulam type Stability results are obtained using the successive approximation method. Also, the outcomes show that the Ulam stability study field finds the successive approximation to be more practical and efficient. The findings prove that the fractional integral differential equations of Riemann-Liouville with non-local boundary conditions have a unique solution and that the approximate solutions may be successfully constrained. A detailed grasp of the differential equation's integral form as well as the effects of non-local boundary conditions on solution and approximation accuracy are necessary for the study of these problems. The results of this investigation confirm pertinent cases.

Conflict of interest

The authors declare that they have no conflict of interest.

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Materials and data availability

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