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New perspectives on Euler-Maclaurin type inequalities via conformable fractional integrals

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Abstract. In this study, we have examined Euler-Maclaurin-type inequalities derived using conformable fractional integrals for various classes of functions. First, we have proved an integral equality that forms the basis of our main results. Then, we have presented Euler-Maclaurin-type inequalities for differentiable convex functions through conformable fractional integrals. Additionally, we have explored and established fractional Euler-Maclaurin-type inequalities for bounded functions and Lipschitzian functions. Our results have extended commonly used Simpson, Milne, and Newton-type integral inequalities, which are widely applied in mathematical analysis. Finally, we have discussed the potential implications of the results we have obtained via fractional integrals for future research.

1. Introduction

Convexity is a fundamental concept in mathematical analysis, with far-reaching implications across numerous fields. Convex functions possess key properties that are indispensable in both theoretical studies and real-world applications [15, 29, 31]. For a function $\aleph : [a, b] \to \mathbb{R} \subset \mathbb{R}$ to be convex, the following inequality must hold:

$$\eta\aleph(a) + (1 - \eta)\aleph(b) \geq \aleph(\eta a + (1 - \eta)b) \tag{1}$$

for all $a, b \in I$, $\eta \in [0, 1]$. If the inequality is reversed, $\aleph$ is classified as concave.

Convexity plays a pivotal role in mathematical analysis, especially in the formulation of inequalities. A function is considered convex if its graph lies below the line segment connecting any two points on the graph. This fundamental property enables the derivation of a wide range of inequalities, including those that establish upper and lower bounds for integrals and sums. Convex functions are often employed to develop

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and extend inequalities such as the Jensen, Hermite-Hadamard, and more recently, Euler-Maclaurin-type inequalities. These inequalities provide powerful tools for estimating integral approximations and error bounds in various numerical methods, thus bridging the theoretical concept of convexity with practical applications in inequality theory.

Integral inequalities, commonly used in fields such as statistical analysis, approximation theory, and distribution theory, establish bounds on distributions, spectral measures, and probability inequalities. They are crucial in addressing complex problems in physics, biology, and control theory [1, 4, 12, 17].

Fractional calculus, a growing field of study, has been widely applied in the analysis of inequalities. Numerous researchers have examined fractional forms of inequalities like the trapezoid-type and Newton-type inequalities [3, 18, 30]. Over the years, a significant body of work has emerged, focusing on the application of fractional integrals in various forms, leading to the derivation of important inequalities such as Simpson-type, Milne-type, and others [2, 5, 6, 11, 13, 36, 37].

Definition 1.1. *[24] Let us consider $\aleph \in L_1[a,b]$. The R-L integrals $\Im^{\lambda}_{\sigma+}\aleph$ and $\Im^{\lambda}_{\rho-}\aleph$, with order $\lambda > 0$, defined as follows:*

$$\Im^{\lambda}_{a+}\aleph(\upsilon) = \frac{1}{\Gamma(\lambda)} \int_a^{\upsilon} (\upsilon-\eta)^{\lambda-1}\aleph(\eta)d\eta, \quad \upsilon > a \tag{2}$$

and

$$\Im^{\lambda}_{b-}\aleph(\upsilon) = \frac{1}{\Gamma(\lambda)} \int_{\upsilon}^{b} (\eta-\upsilon)^{\lambda-1}\aleph(\eta)d\eta, \quad \upsilon < b, \tag{3}$$

respectively. In this context, $\aleph$ is a function belonging to the space $L_1[a,b]$, and $\Gamma(\lambda)$ represents the Gamma function $\Im^{\lambda}_{a+}\aleph(\upsilon) = \Im^{\lambda}_{b-}\aleph(\upsilon) = \aleph(\upsilon)$.

Numerous pioneering conformable fractional operators have arisen to overcome the limitations of classical fractional operators in formulating specific phenomena [38]. Especially, the operator proposed by Jarad et al., as reported in [23], stands out for its versatility to extend various operators, including the R-L and Hadamard operators, offering a more comprehensive framework for obtaining complex systems and phenomena.

Definition 1.2. *[23] Let $\aleph \in L[a,b]$. The left and right-sided conformable fractional integral operator of order $\beta \in \mathbb{C}$ and $\mathbb{R}(\beta) > 0$ and $\lambda \in (0,1]$ are outlined as:*

$${}^{\beta}J^{\lambda}_{a^+}\aleph(\upsilon) = \frac{1}{\Gamma(\beta)} \int_a^{\upsilon} \left(\frac{(\upsilon-a)^{\lambda} - (\eta-a)^{\lambda}}{\lambda}\right)^{\beta-1} \frac{\aleph(\eta)}{(\eta-a)^{1-\lambda}} d\eta, \quad \upsilon > a \tag{4}$$

and

$${}^{\beta}J^{\lambda}_{b^-}\aleph(\upsilon) = \frac{1}{\Gamma(\beta)} \int_{\upsilon}^{b} \left(\frac{(b-\upsilon)^{\lambda} - (b-\eta)^{\lambda}}{\lambda}\right)^{\beta-1} \frac{\aleph(\eta)}{(b-\eta)^{1-\lambda}} d\eta, \quad \upsilon < b. \tag{5}$$

Remark 1.3. *If we set $a = 0$, $b = 0$, and $\lambda = 1$ in (4) and (5), we find the R-L fractional integrals (2) and (3) accordingly.*

On the other hand, Set et al. [32] proposed the one-sided conformable fractional integral operator as follows:

$${}^{\beta}\Im^{\lambda}\aleph(\upsilon) = \frac{1}{\Gamma(\beta)} \int_0^{\upsilon} \left(\frac{\upsilon^{\lambda} - \eta^{\lambda}}{\lambda}\right)^{\beta-1} \frac{\aleph(\eta)}{\eta^{1-\lambda}} d\eta. \tag{6}$$

In recent years, significant advancements have been made in the field of conformable fractional integrals, opening new avenues for exploring various types of inequalities. For instance, [33] introduced a novel identity in this context, which served as a foundation for deriving Ostrowski-type inequalities. Building on this, Celik et al. [7] extended these concepts to Milne-type inequalities, examining their applications to bounded and Lipschitz functions, as well as functions of bounded variation. Hyder et al. [22] further enriched the literature by developing midpoint-type inequalities. Additionally, Kara et al. [25] focused on twice-differentiable convex functions, deriving inequalities of midpoint and trapezoid types. This growing body of work has also spurred research into other well-known inequalities within the conformable fractional calculus framework, including Grüss [26], Chebyshev [27], Minkowski [28], Hermite-Hadamard [35], Hermite-Hadamard-Mercer [34], and Simpson-type inequalities [19].

The subsequent Simpson's rules are applicable to Simpson's inequalities.

I. The subsequent formula represents Simpson's quadrature, alternatively referred to as Simpson's 1/3 rule:

$$\int_a^b \aleph(\eta)\,d\eta \approx \frac{b-a}{6}\left[\aleph(a)+4\aleph\left(\frac{a+b}{2}\right)+\aleph(b)\right]. \tag{7}$$

II. The Newton-Cotes quadrature formula, commonly known as Simpson's second formula (also referred to as Simpson's 3/8 rule; see [8]), can be stated as:

$$\int_a^b \aleph(\eta)\,d\eta \approx \frac{b-a}{8}\left[\aleph(a)+3\aleph\left(\frac{2a+b}{3}\right)+3\aleph\left(\frac{a+2b}{3}\right)+\aleph(b)\right]. \tag{8}$$

III. The Maclaurin rule, derived from the Maclaurin formula (as seen in [8]), is identical to the corresponding dual Simpson's 3/8 formula:

$$\int_a^b \aleph(\eta)\,d\eta \approx \frac{b-a}{8}\left[3\aleph\left(\frac{5a+b}{6}\right)+2\aleph\left(\frac{a+b}{2}\right)+3\aleph\left(\frac{a+5b}{6}\right)\right]. \tag{9}$$

Formulas (7), (8), and (9) hold true for any function f with a continuous fourth derivative over the interval $[a,b]$.

The subsequent Newton-Cotes quadrature, frequently utilized, includes a three-point Simpson's- type inequality:

Theorem 1.4. *Let $\aleph:[a,b]\to\mathbb{R}$ be a four times differentiable and continuous function on (a,b), and let $\left\|\aleph^{(4)}\right\|_\infty = \sup_{\eta\in(a,b)}\left|\aleph^{(4)}(\eta)\right| < \infty$. Then, the subsequent inequality is valid:*

$$\left|\frac{1}{6}\left[\aleph(a)+4\aleph\left(\frac{a+b}{2}\right)+\aleph(b)\right]-\frac{1}{b-a}\int_a^b \aleph(\eta)d\eta\right| \le \frac{1}{2880}\left\|\aleph^{(4)}\right\|_\infty (b-a)^4 .$$

In accordance with the Simpson 3/8 inequality, the Simpson 3/8 rule is a well-known closed-type quadrature rule, and it is expressed as follows:

Theorem 1.5. *Assume that $\aleph:[a,b]\to\mathbb{R}$ is a function that is differentiable four times and continuous on (a,b), and $\left\|\aleph^{(4)}\right\|_\infty = \sup_{\eta\in(a,b)}\left|\aleph^{(4)}(\eta)\right| < \infty$. Then, one observes the subsequent inequality:*

$$\left|\frac{1}{8}\left[\aleph(a)+3\aleph\left(\frac{2a+b}{3}\right)+3\aleph\left(\frac{a+2b}{3}\right)+\aleph(b)\right]-\frac{1}{b-a}\int_a^b \aleph(\eta)d\eta\right| \le \frac{1}{6480}\left\|\aleph^{(4)}\right\|_\infty (b-a)^4 .$$

The Maclaurin rule, originating from the Maclaurin inequality, is equivalent to the corresponding dual Simpson's 3/8 formula:

Theorem 1.6. *Assume $\aleph : [a,b] \to \mathbb{R}$ is a function that is differentiable four times and continuous on (a,b), and let $\left\|\aleph^{(4)}\right\|_\infty = \sup_{\eta\in(a,b)} \left|\aleph^{(4)}(\eta)\right| < \infty$. Then, the subsequent inequality is valid:*

$$\left|\frac{1}{8}\left[3\aleph\left(\frac{5a+b}{6}\right)+2\aleph\left(\frac{a+b}{2}\right)+3\aleph\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b \aleph(\eta)d\eta\right| \leq \frac{7}{51840}\left\|\aleph^{(4)}\right\|_\infty (b-a)^4.$$

In the framework of differentiable convex functions, authors have examined several Euler-Maclaurin-type inequalities in [20]. Dedic et al. [9] reported a sequence of inequalities by employing the Euler-Maclaurin formulas, and these results were utilized to derive error estimates for Maclaurin quadrature rules. In [21], using the Riemann-Liouville fractional integrals, Hezenci has investigated some corrected Euler-Maclaurin-type inequalities. For a deeper understanding of these specific types of inequalities, interested readers are directed to [10, 14, 29] and the citations therein.

In this paper, we will explore Euler-Maclaurin-type inequalities for various classes of functions using conformable fractional integrals. The study is organized into several sections: Section 2 proves an integral equality fundamental to the main results, while Section 3 presents Euler-Maclaurin-type inequalities for convex functions. Section 4 extends the analysis to bounded functions, and Section 5 establishes results for Lipschitzian functions. Finally, we conclude by discussing the broader implications of these inequalities and potential areas for future research.

2. Principal Outcome

In this section, we prove integral equality in order to illustrate the main conclusions of this study.

Lemma 2.1. *Let us consider that $\aleph : [a,b] \to \mathbb{R}$ is an differentiable mapping (a,b) and $\lambda, \beta > 0$ so that $\aleph' \in L_1([a,b])$. Then, the subsequent equality holds:*

$$\frac{1}{8}\left[3\aleph\left(\frac{5a+b}{6}\right)+2\aleph\left(\frac{a+b}{2}\right)+3\aleph\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\lambda\beta-1}\lambda^\beta\Gamma(\beta+1)}{(b-a)^{\lambda\beta}}\left[{}^\beta J^\lambda_{\frac{a+b}{2}+}\aleph(b)+{}^\beta J^\lambda_{\frac{a+b}{2}-}\aleph(a)\right]$$
$$=\frac{\lambda^\beta(b-a)}{4}\left[I_1+I_2\right].$$

Here,

$$\begin{cases} I_1 = \int\limits_0^{\frac{1}{3}} \left(\frac{1-(1-\eta)^\lambda}{\lambda}\right)^\beta \left[\aleph'\left(\frac{\eta}{2}b+\frac{2-\eta}{2}a\right)-\aleph'\left(\frac{\eta}{2}a+\frac{2-\eta}{2}b\right)\right]d\eta, \\ I_2 = \int\limits_{\frac{1}{3}}^1 \left(\left(\frac{1-(1-\eta)^\lambda}{\lambda}\right)^\beta - \frac{3}{4\lambda^\beta}\right)\left[\aleph'\left(\frac{\eta}{2}b+\frac{2-\eta}{2}a\right)-\aleph'\left(\frac{\eta}{2}a+\frac{2-\eta}{2}b\right)\right]d\eta. \end{cases}$$

Proof. Using the integration by parts, we observe

$$\begin{aligned} I_1 &= \int\limits_0^{\frac{1}{3}} \left(\frac{1-(1-\eta)^\lambda}{\lambda}\right)^\beta \left[\aleph'\left(\frac{\eta}{2}b+\frac{2-\eta}{2}a\right)-\aleph'\left(\frac{\eta}{2}a+\frac{2-\eta}{2}b\right)\right]d\eta \qquad (10) \\ &= \frac{2}{b-a}\left(\frac{1-(1-\eta)^\lambda}{\lambda}\right)^\beta \left[\aleph\left(\frac{\eta}{2}b+\frac{2-\eta}{2}a\right)+\aleph\left(\frac{\eta}{2}a+\frac{2-\eta}{2}b\right)\right]\Bigg|_0^{\frac{1}{3}} \\ &\quad - \frac{2\beta}{b-a}\int_0^{\frac{1}{3}} \left(\frac{1-(1-\eta)^\lambda}{\lambda}\right)^{\beta-1}(1-\eta)^{\lambda-1}\left[\aleph\left(\frac{\eta}{2}b+\frac{2-\eta}{2}a\right)+\aleph\left(\frac{\eta}{2}a+\frac{2-\eta}{2}b\right)\right]d\eta \end{aligned}$$

$$= \frac{2}{b-a}\left(\left(\frac{1-\left(\frac{2}{3}\right)^{\lambda}}{\lambda}\right)^{\beta}\right)\left[\aleph\left(\frac{5b+a}{6}\right)+\aleph\left(\frac{5a+b}{6}\right)\right]$$
$$-\frac{2\beta}{(b-a)}\int_0^{\frac{1}{3}}\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta-1}(1-\eta)^{\lambda-1}\left[\aleph\left(\frac{\eta}{2}b+\frac{2-\eta}{2}a\right)+\aleph\left(\frac{\eta}{2}a+\frac{2-\eta}{2}b\right)\right]d\eta.$$

Similarly,

$$I_2 = \int_{\frac{1}{3}}^{1}\left(\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}-\frac{3}{4\lambda^{\beta}}\right)\left[\aleph'\left(\frac{\eta}{2}b+\frac{2-\eta}{2}a\right)-\aleph'\left(\frac{\eta}{2}a+\frac{2-\eta}{2}b\right)\right]d\eta \tag{11}$$
$$= \frac{2}{b-a}\left(\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}-\frac{3}{4\lambda^{\beta}}\right)\left[\aleph\left(\frac{\eta}{2}b+\frac{2-\eta}{2}a\right)+\aleph\left(\frac{\eta}{2}a+\frac{2-\eta}{2}b\right)\right]\Bigg|_{\frac{1}{3}}^{1}$$
$$-\frac{2\beta}{b-a}\int_{\frac{1}{3}}^{1}\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta-1}(1-\eta)^{\lambda-1}\left[\aleph\left(\frac{\eta}{2}b+\frac{2-\eta}{2}a\right)+\aleph\left(\frac{\eta}{2}a+\frac{2-\eta}{2}b\right)\right]d\eta$$
$$= \frac{1}{(b-a)\lambda^{\beta}}\aleph\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\left(\left(\frac{1-\left(\frac{2}{3}\right)^{\lambda}}{\lambda}\right)^{\beta}-\frac{3}{4\lambda^{\beta}}\right)\left[\aleph\left(\frac{5b+a}{6}\right)+\aleph\left(\frac{5a+b}{6}\right)\right]$$
$$= -\frac{2\beta}{b-a}\int_{\frac{1}{3}}^{1}\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta-1}(1-\eta)^{\lambda-1}\left[\aleph\left(\frac{\eta}{2}b+\frac{2-\eta}{2}a\right)+\aleph\left(\frac{\eta}{2}a+\frac{2-\eta}{2}b\right)\right]d\eta.$$

By substituting (10) and (11), then we achieve

$$I_1+I_2 = \frac{3}{2(b-a)\lambda^{\beta}}\left[\aleph\left(\frac{5b+a}{6}\right)+\aleph\left(\frac{5a+b}{6}\right)\right]+\frac{1}{(b-a)\lambda^{\beta}}\aleph\left(\frac{a+b}{2}\right) \tag{12}$$
$$-\frac{2\beta}{(b-a)}\int_0^1\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta-1}(1-\eta)^{\lambda-1}\left[\aleph\left(\frac{\eta}{2}b+\frac{2-\eta}{2}a\right)+\aleph\left(\frac{\eta}{2}a+\frac{2-\eta}{2}b\right)\right]d\eta.$$

If we use the change of the variable $x=\frac{1+\eta}{2}b+\frac{1-\eta}{2}a$ and $y=\frac{1+\eta}{2}a+\frac{1-\eta}{2}b$ for $\eta\in[0,1]$, then the equality (12) can be rewritten as follows

$$I_1+I_2 = \frac{1}{2\lambda^{\beta}(b-a)}\left[3\aleph\left(\frac{5a+b}{6}\right)+2\aleph\left(\frac{a+b}{2}\right)+3\aleph\left(\frac{a+5b}{6}\right)\right] \tag{13}$$
$$-\frac{2^{\lambda\beta+1}\Gamma(\beta+1)}{(b-a)^{\lambda\beta+1}}\left[{}^{\beta}J^{\lambda}_{\frac{a+b}{2}+}\aleph(b)+{}^{\beta}J^{\lambda}_{\frac{a+b}{2}-}\aleph(a)\right].$$

Accordingly, multiplying both sides of (13) by $\frac{\lambda^{\beta}(b-a)}{4}$, concludes the proof of Lemma 2.1. □

3. Convex functions: Fractional Euler-Maclaurin-type inequalities

Theorem 3.1. *Assume the conditions stipulated in Lemma 2.1 holds. If the function $|\aleph'|$ is convex on $[a,b]$, then the subsequent inequality is valid:*

$$\left|\frac{1}{8}\left[3\aleph\left(\frac{5a+b}{6}\right)+2\aleph\left(\frac{a+b}{2}\right)+3\aleph\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\lambda\beta-1}\lambda^{\beta}\Gamma(\beta+1)}{(b-a)^{\lambda\beta}}\left[{}^{\beta}J^{\lambda}_{\frac{a+b}{2}+}\aleph(b)+{}^{\beta}J^{\lambda}_{\frac{a+b}{2}-}\aleph(a)\right]\right| \tag{14}$$

$$\leq \frac{\lambda^{\beta}(b-a)}{4}\left(\mathcal{A}_1(\lambda,\beta)+\mathcal{A}_2(\lambda,\beta)\right)\left[\left|\aleph'(a)\right|+\left|\aleph'(b)\right|\right],$$

where

$$\mathcal{A}_1(\lambda,\beta)=\int_0^{\frac{1}{3}}\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}d\eta$$

and

$$\mathcal{A}_2(\lambda,\beta)=\int_{\frac{1}{3}}^{1}\left|\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}-\frac{3}{4\lambda^{\beta}}\right|d\eta.$$

Proof. If we consider the absolute value in Lemma 2.1, we can directly get

$$\left|\frac{1}{8}\left[3\aleph\left(\frac{5a+b}{6}\right)+2\aleph\left(\frac{a+b}{2}\right)+3\aleph\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\lambda\beta-1}\lambda^{\beta}\Gamma(\beta+1)}{(b-a)^{\lambda\beta}}\left[{}^{\beta}J^{\lambda}_{\frac{a+b}{2}+}\aleph(b)+{}^{\beta}J^{\lambda}_{\frac{a+b}{2}-}\aleph(a)\right]\right| \tag{15}$$

$$\leq \frac{\lambda^{\beta}(b-a)}{4}\left\{\int_0^{\frac{1}{3}}\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}\left[\left|\aleph'\left(\frac{\eta}{2}b+\frac{2-\eta}{2}a\right)\right|+\left|\aleph'\left(\frac{\eta}{2}a+\frac{2-\eta}{2}b\right)\right|\right]d\eta\right.$$

$$\left.+\int_{\frac{1}{3}}^{1}\left|\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}-\frac{3}{4\lambda^{\beta}}\right|\left[\left|\aleph'\left(\frac{\eta}{2}b+\frac{2-\eta}{2}a\right)\right|+\left|\aleph'\left(\frac{\eta}{2}a+\frac{2-\eta}{2}b\right)\right|\right]d\eta\right\}.$$

Given that $|\aleph'|$ is convex, it becomes

$$\left|\frac{1}{8}\left[3\aleph\left(\frac{5a+b}{6}\right)+2\aleph\left(\frac{a+b}{2}\right)+3\aleph\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\lambda\beta-1}\lambda^{\beta}\Gamma(\beta+1)}{(b-a)^{\lambda\beta}}\left[{}^{\beta}J^{\lambda}_{\frac{a+b}{2}+}\aleph(b)+{}^{\beta}J^{\lambda}_{\frac{a+b}{2}-}\aleph(a)\right]\right|$$

$$\leq \frac{\lambda^{\beta}(b-a)}{4}\left\{\int_0^{\frac{1}{3}}\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}\left[\frac{\eta}{2}\left|\aleph'(b)\right|+\frac{2-\eta}{2}\left|\aleph'(a)\right|+\frac{\eta}{2}\left|\aleph'(a)\right|+\frac{2-\eta}{2}\left|\aleph'(b)\right|\right]d\eta\right.$$

$$\left.+\int_{\frac{1}{3}}^{1}\left|\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}-\frac{3}{4\lambda^{\beta}}\right|\left[\frac{\eta}{2}\left|\aleph'(b)\right|+\frac{2-\eta}{2}\left|\aleph'(a)\right|+\frac{\eta}{2}\left|\aleph'(a)\right|+\frac{2-\eta}{2}\left|\aleph'(b)\right|\right]d\eta\right\}$$

$$=\frac{\lambda^{\beta}(b-a)}{4}\left(\mathcal{A}_1(\lambda)+\mathcal{A}_2(\lambda)\right)\left[\left|\aleph'(a)\right|+\left|\aleph'(b)\right|\right].$$

Therefore, the proof is completed. □

Theorem 3.2. *Assume the conditions stipulated in Lemma 2.1 holds. If the function $|\aleph'|^{q}$, $q>1$ is convex on $[a,b]$, then the subsequent inequality is valid:*

$$\left|\frac{1}{8}\left[3\aleph\left(\frac{5a+b}{6}\right)+2\aleph\left(\frac{a+b}{2}\right)+3\aleph\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\lambda\beta-1}\lambda^{\beta}\Gamma(\beta+1)}{(b-a)^{\lambda\beta}}\left[{}^{\beta}J^{\lambda}_{\frac{a+b}{2}+}\aleph(b)+{}^{\beta}J^{\lambda}_{\frac{a+b}{2}-}\aleph(a)\right]\right| \tag{16}$$

$$\leq \frac{\lambda^{\beta}(b-a)}{4}\left\{\left(\int_0^{\frac{1}{3}}\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{p\beta}d\eta\right)^{\frac{1}{p}}\left[\left(\frac{\left|\aleph'(b)\right|^{q}+11\left|\aleph'(a)\right|^{q}}{36}\right)^{\frac{1}{q}}+\left(\frac{11\left|\aleph'(b)\right|^{q}+\left|\aleph'(a)\right|^{q}}{36}\right)^{\frac{1}{q}}\right]\right.$$

$$+\left(\int_{\frac{1}{3}}^{1}\left|\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}-\frac{3}{4\lambda^{\beta}}\right|^{p}d\eta\right)^{\frac{1}{p}}\left[\left(\frac{2\left|\aleph'(b)\right|^{q}+4\left|\aleph'(a)\right|^{q}}{9}\right)^{\frac{1}{q}}+\left(\frac{4\left|\aleph'(b)\right|^{q}+2\left|\aleph'(a)\right|^{q}}{9}\right)^{\frac{1}{q}}\right]\Bigg\}.$$

Here, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Through utilizing Hölder's inequality to (15), we obtain

$$\left|\frac{1}{8}\left[3\aleph\left(\frac{5a+b}{6}\right)+2\aleph\left(\frac{a+b}{2}\right)+3\aleph\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\lambda\beta-1}\lambda^{\beta}\Gamma(\beta+1)}{(b-a)^{\lambda\beta}}\left[{}^{\beta}J^{\lambda}_{\frac{a+b}{2}+}\aleph(b)+{}^{\beta}J^{\lambda}_{\frac{a+b}{2}-}\aleph(a)\right]\right|$$

$$\leq\frac{\lambda^{\beta}(b-a)}{4}\left\{\left(\int_{0}^{\frac{1}{3}}\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{p\beta}d\eta\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{1}{3}}\left|\aleph'\left(\frac{\eta}{2}b+\frac{2-\eta}{2}a\right)\right|^{q}d\eta\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_{0}^{\frac{1}{3}}\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{p\beta}d\eta\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{1}{3}}\left|\aleph'\left(\frac{\eta}{2}a+\frac{2-\eta}{2}b\right)\right|^{q}d\eta\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{3}}^{1}\left|\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}-\frac{3}{4\lambda^{\beta}}\right|^{p}d\eta\right)^{\frac{1}{p}}\left(\int_{\frac{1}{3}}^{1}\left|\aleph'\left(\frac{\eta}{2}b+\frac{2-\eta}{2}a\right)\right|^{q}d\eta\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_{\frac{1}{3}}^{1}\left|\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}-\frac{3}{4\lambda^{\beta}}\right|^{p}d\eta\right)^{\frac{1}{p}}\left(\int_{\frac{1}{3}}^{1}\left|\aleph'\left(\frac{\eta}{2}a+\frac{2-\eta}{2}b\right)\right|^{q}d\eta\right)^{\frac{1}{q}}\right\}.$$

By employing the convexity $|\aleph'|^{q}$, we get

$$\left|\frac{1}{8}\left[3\aleph\left(\frac{5a+b}{6}\right)+2\aleph\left(\frac{a+b}{2}\right)+3\aleph\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\lambda\beta-1}\lambda^{\beta}\Gamma(\beta+1)}{(b-a)^{\lambda\beta}}\left[{}^{\beta}J^{\lambda}_{\frac{a+b}{2}+}\aleph(b)+{}^{\beta}J^{\lambda}_{\frac{a+b}{2}-}\aleph(a)\right]\right|$$

$$\leq\frac{\lambda^{\beta}(b-a)}{4}\left\{\left(\int_{0}^{\frac{1}{3}}\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{p\beta}d\eta\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{1}{3}}\left[\frac{\eta}{2}\left|\aleph'(b)\right|^{q}+\frac{2-\eta}{2}\left|\aleph'(a)\right|^{q}\right]d\eta\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_{0}^{\frac{1}{3}}\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{p\beta}d\eta\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{1}{3}}\left[\frac{\eta}{2}\left|\aleph'(a)\right|^{q}+\frac{2-\eta}{2}\left|\aleph'(b)\right|^{q}\right]d\eta\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{3}}^{1}\left|\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}-\frac{3}{4\lambda^{\beta}}\right|^{p}d\eta\right)^{\frac{1}{p}}\left(\int_{\frac{1}{3}}^{1}\left[\frac{\eta}{2}\left|\aleph'(b)\right|^{q}+\frac{2-\eta}{2}\left|\aleph'(a)\right|^{q}\right]d\eta\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_{\frac{1}{3}}^{1}\left|\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}-\frac{3}{4\lambda^{\beta}}\right|^{p}d\eta\right)^{\frac{1}{p}}\left(\int_{\frac{1}{3}}^{1}\left[\frac{\eta}{2}\left|\aleph'(a)\right|^{q}+\frac{2-\eta}{2}\left|\aleph'(b)\right|^{q}\right]d\eta\right)^{\frac{1}{q}}\right\}$$

$$= \frac{\lambda^{\beta}(b-a)}{4}\left\{\left(\int_0^{\frac{1}{3}}\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{p\beta}d\eta\right)^{\frac{1}{p}}\left[\left(\frac{|\aleph'(b)|^q+11|\aleph'(a)|^q}{36}\right)^{\frac{1}{q}}+\left(\frac{11|\aleph'(b)|^q+|\aleph'(a)|^q}{36}\right)^{\frac{1}{q}}\right]\right.$$

$$\left.+\left(\int_{\frac{1}{3}}^{1}\left|\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}-\frac{1}{\lambda^{\beta}}\right|^{p}d\eta\right)^{\frac{1}{p}}\left[\left(\frac{2|\aleph'(b)|^q+4|\aleph'(a)|^q}{9}\right)^{\frac{1}{q}}+\left(\frac{4|\aleph'(b)|^q+2|\aleph'(a)|^q}{9}\right)^{\frac{1}{q}}\right]\right\}.$$

Hence, the proof is finished. □

Theorem 3.3. *Assume the conditions stipulated in Lemma 2.1 holds. If the function $|\aleph'|^q$, $q \geq 1$ is convex on $[a,b]$, then the subsequent inequality is valid:*

$$\left|\frac{1}{8}\left[3\aleph\left(\frac{5a+b}{6}\right)+2\aleph\left(\frac{a+b}{2}\right)+3\aleph\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\lambda\beta-1}\lambda^{\beta}\Gamma(\beta+1)}{(b-a)^{\lambda\beta}}\left[{}^{\beta}J^{\lambda}_{\frac{a+b}{2}+}\aleph(b)+{}^{\beta}J^{\lambda}_{\frac{a+b}{2}-}\aleph(a)\right]\right| \tag{17}$$

$$\leq \frac{\lambda^{\beta}(b-a)}{4}\left\{(\mathcal{A}_1(\lambda,\beta))^{1-\frac{1}{q}}\left([\mathcal{A}_3(\lambda,\beta)|\aleph'(b)|^q+\mathcal{A}_4(\lambda,\beta)|\aleph'(a)|^q]\right)^{\frac{1}{q}}\right.$$

$$+(\mathcal{A}_1(\lambda,\beta))^{1-\frac{1}{q}}\left([\mathcal{A}_3(\lambda,\beta)|\aleph'(a)|^q+\mathcal{A}_4(\lambda,\beta)|\aleph'(b)|^q]\right)^{\frac{1}{q}}$$

$$+(\mathcal{A}_2(\lambda,\beta))^{1-\frac{1}{q}}\left([\mathcal{A}_5(\lambda,\beta)|\aleph'(b)|^q+\mathcal{A}_6(\lambda,\beta)|\aleph'(a)|^q]\right)^{\frac{1}{q}}$$

$$\left.+(\mathcal{A}_2(\lambda,\beta))^{1-\frac{1}{q}}\left([\mathcal{A}_5(\lambda,\beta)|\aleph'(a)|^q+\mathcal{A}_6(\lambda,\beta)|\aleph'(b)|^q]\right)^{\frac{1}{q}}\right\}.$$

Here, $\mathcal{A}_1(\lambda,\beta)$ and $\mathcal{A}_2(\lambda,\beta)$ are defined in Theorem 3.1 and

$$\mathcal{A}_3(\lambda,\beta)=\int_0^{\frac{1}{3}}\frac{\eta}{2}\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}d\eta,$$

$$\mathcal{A}_4(\lambda,\beta)=\int_0^{\frac{1}{3}}\left(\frac{2-\eta}{2}\right)\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}d\eta,$$

$$\mathcal{A}_5(\lambda,\beta)=\int_{\frac{1}{3}}^{1}\frac{\eta}{2}\left|\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}-\frac{3}{4\lambda^{\beta}}\right|d\eta,$$

$$\mathcal{A}_6(\lambda,\beta)=\int_{\frac{1}{3}}^{1}\left(\frac{2-\eta}{2}\right)\left|\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}-\frac{3}{4\lambda^{\beta}}\right|d\eta.$$

Proof. With help of the power-mean inequality in (15), we have

$$\left|\frac{1}{8}\left[3\aleph\left(\frac{5a+b}{6}\right)+2\aleph\left(\frac{a+b}{2}\right)+3\aleph\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\lambda\beta-1}\lambda^{\beta}\Gamma(\beta+1)}{(b-a)^{\lambda\beta}}\left[{}^{\beta}J^{\lambda}_{\frac{a+b}{2}+}\aleph(b)+{}^{\beta}J^{\lambda}_{\frac{a+b}{2}-}\aleph(a)\right]\right|$$

$$\leq \frac{\lambda^{\beta}(b-a)}{4}\left\{\left(\int_0^{\frac{1}{3}}\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}d\eta\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}\left|\aleph'\left(\frac{\eta}{2}b+\frac{2-\eta}{2}a\right)\right|^q d\eta\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^{\frac{1}{3}}\left(\frac{1-(1-\eta)^\lambda}{\lambda}\right)^\beta d\eta\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}\left(\frac{1-(1-\eta)^\lambda}{\lambda}\right)^\beta\left|\aleph'\left(\frac{\eta}{2}a+\frac{2-\eta}{2}b\right)\right|^q d\eta\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{3}}^1\left|\left(\frac{1-(1-\eta)^\lambda}{\lambda}\right)^\beta-\frac{3}{4\lambda^\beta}\right|d\eta\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^1\left|\left(\frac{1-(1-\eta)^\lambda}{\lambda}\right)^\beta-\frac{3}{4\lambda^\beta}\right|\left|\aleph'\left(\frac{\eta}{2}b+\frac{2-\eta}{2}a\right)\right|^q d\eta\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{3}}^1\left|\left(\frac{1-(1-\eta)^\lambda}{\lambda}\right)^\beta-\frac{3}{4\lambda^\beta}\right|d\eta\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^1\left|\left(\frac{1-(1-\eta)^\lambda}{\lambda}\right)^\beta-\frac{3}{4\lambda^\beta}\right|\left|\aleph'\left(\frac{\eta}{2}a+\frac{2-\eta}{2}b\right)\right|^q d\eta\right)^{\frac{1}{q}}\Bigg\}.$$

Utilizing the convexity of $|\aleph'|^q$, we get

$$\left|\frac{1}{8}\left[3\aleph\left(\frac{5a+b}{6}\right)+2\aleph\left(\frac{a+b}{2}\right)+3\aleph\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\lambda\beta-1}\lambda^\beta\Gamma(\beta+1)}{(b-a)^{\lambda\beta}}\left[{}^\beta J^\lambda_{\frac{a+b}{2}+}\aleph(b)+{}^\beta J^\lambda_{\frac{a+b}{2}-}\aleph(a)\right]\right|$$

$$\leq\frac{\lambda^\beta(b-a)}{4}\left\{\left(\int_0^{\frac{1}{3}}\left(\frac{1-(1-\eta)^\lambda}{\lambda}\right)^\beta d\eta\right)^{1-\frac{1}{q}}\right.$$

$$\times\left(\int_0^{\frac{1}{3}}\left(\frac{1-(1-\eta)^\lambda}{\lambda}\right)^\beta\left[\frac{\eta}{2}|\aleph'(b)|^q+\frac{2-\eta}{2}|\aleph'(a)|^q\right]d\eta\right)^{\frac{1}{q}}$$

$$+\left(\int_0^{\frac{1}{3}}\left(\frac{1-(1-\eta)^\lambda}{\lambda}\right)^\beta d\eta\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^{\frac{1}{3}}\left(\frac{1-(1-\eta)^\lambda}{\lambda}\right)^\beta\left[\frac{\eta}{2}|\aleph'(a)|^q+\frac{2-\eta}{2}|\aleph'(b)|^q\right]d\eta\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{3}}^1\left|\left(\frac{1-(1-\eta)^\lambda}{\lambda}\right)^\beta-\frac{3}{4\lambda^\beta}\right|d\eta\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_{\frac{1}{3}}^1\left|\left(\frac{1-(1-\eta)^\lambda}{\lambda}\right)^\beta-\frac{3}{4\lambda^\beta}\right|\left[\frac{\eta}{2}|\aleph'(b)|^q+\frac{2-\eta}{2}|\aleph'(a)|^q\right]d\eta\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{3}}^1\left|\left(\frac{1-(1-\eta)^\lambda}{\lambda}\right)^\beta-\frac{3}{4\lambda^\beta}\right|d\eta\right)^{1-\frac{1}{q}}$$

$$\left.\times\left(\int_{\frac{1}{3}}^1\left|\left(\frac{1-(1-\eta)^\lambda}{\lambda}\right)^\beta-\frac{3}{4\lambda^\beta}\right|\left[\frac{\eta}{2}|\aleph'(a)|^q+\frac{2-\eta}{2}|\aleph'(b)|^q\right]d\eta\right)^{\frac{1}{q}}\right\}.$$

This completes the proof. □

4. Bounded functions: Euler-Maclaurin-type inequalities with conformable fractional integrals

In this section, we establish certain Euler-Maclaurin-type inequalities for bounded functions using conformable fractional integrals.

Theorem 4.1. *Assume the conditions stipulated in Lemma 2.1 holds. If there exist $m, M \in \mathbb{R}$ such that $m \leq \aleph'(\eta) \leq M$ for $\eta \in [a,b]$, then it follows:*

$$\left|\frac{1}{8}\left[3\aleph\left(\frac{5a+b}{6}\right)+2\aleph\left(\frac{a+b}{2}\right)+3\aleph\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\lambda\beta-1}\lambda^{\beta}\Gamma(\beta+1)}{(b-a)^{\lambda\beta}}\left[{}^{\beta}J^{\lambda}_{\frac{a+b}{2}+}\aleph(b)+{}^{\beta}J^{\lambda}_{\frac{a+b}{2}-}\aleph(a)\right]\right| \tag{18}$$
$$\leq \frac{\lambda^{\beta}(b-a)}{4}\left[\mathcal{A}_1(\lambda,\beta)+\mathcal{A}_2(\lambda,\beta)\right](M-m),$$

where $\mathcal{A}_1(\lambda,\beta)$ and $\mathcal{A}_2(\lambda,\beta)$ are defined in Theorem 3.1.

Proof. By using Lemma 2.1, we have

$$\frac{1}{8}\left[3\aleph\left(\frac{5a+b}{6}\right)+2\aleph\left(\frac{a+b}{2}\right)+3\aleph\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\lambda\beta-1}\lambda^{\beta}\Gamma(\beta+1)}{(b-a)^{\lambda\beta}}\left[{}^{\beta}J^{\lambda}_{\frac{a+b}{2}+}\aleph(b)+{}^{\beta}J^{\lambda}_{\frac{a+b}{2}-}\aleph(a)\right] \tag{19}$$
$$\leq \frac{\lambda^{\beta}(b-a)}{4}\left\{\int_0^{\frac{1}{3}}\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}\left[\aleph'\left(\frac{\eta}{2}b+\frac{2-\eta}{2}a\right)-\frac{m+M}{2}\right]d\eta\right.$$
$$+\int_0^{\frac{1}{3}}\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}\left[\frac{m+M}{2}-\aleph'\left(\frac{\eta}{2}a+\frac{2-\eta}{2}b\right)\right]d\eta$$
$$+\int_{\frac{1}{3}}^{1}\left[\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}-\frac{3}{4\lambda^{\beta}}\right]\left[\aleph'\left(\frac{\eta}{2}b+\frac{2-\eta}{2}a\right)-\frac{m+M}{2}\right]d\eta$$
$$\left.+\int_{\frac{1}{3}}^{1}\left[\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}-\frac{3}{4\lambda^{\beta}}\right]\left[\aleph'\left(\frac{\eta}{2}b+\frac{2-\eta}{2}a\right)-\frac{m+M}{2}\right]d\eta\right\}.$$

Through the absolute value of (19), we obtain

$$\left|\frac{1}{8}\left[3\aleph\left(\frac{5a+b}{6}\right)+2\aleph\left(\frac{a+b}{2}\right)+3\aleph\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\lambda\beta-1}\lambda^{\beta}\Gamma(\beta+1)}{(b-a)^{\lambda\beta}}\left[{}^{\beta}J^{\lambda}_{\frac{a+b}{2}+}\aleph(b)+{}^{\beta}J^{\lambda}_{\frac{a+b}{2}-}\aleph(a)\right]\right| \tag{20}$$
$$\leq \frac{\lambda^{\beta}(b-a)}{4}\left\{\int_0^{\frac{1}{3}}\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}\left|\aleph'\left(\frac{\eta}{2}b+\frac{2-\eta}{2}a\right)-\frac{m+M}{2}\right|d\eta\right.$$
$$+\int_0^{\frac{1}{3}}\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}\left|\frac{m+M}{2}-\aleph'\left(\frac{\eta}{2}a+\frac{2-\eta}{2}b\right)\right|d\eta$$
$$+\int_{\frac{1}{3}}^{1}\left(\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}-\frac{3}{4\lambda^{\beta}}\right)\left|\aleph'\left(\frac{\eta}{2}b+\frac{2-\eta}{2}a\right)-\frac{m+M}{2}\right|d\eta$$

$$+\int_{\frac{1}{3}}^{1}\left(\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}-\frac{3}{4\lambda^{\beta}}\right)\left|\frac{m+M}{2}-\aleph'\left(\frac{\eta}{2}a+\frac{2-\eta}{2}b\right)\right|d\eta\Bigg\}.$$

It is known that $m \leq \aleph'(\eta) \leq M$ for $\eta \in [a,b]$. Then, we have

$$\left|\aleph'\left(\frac{\eta}{2}b+\frac{2-\eta}{2}a\right)-\frac{m+M}{2}\right| \leq \frac{M-m}{2}, \tag{21}$$

$$\left|\frac{m+M}{2}-\aleph'\left(\frac{\eta}{2}a+\frac{2-\eta}{2}b\right)\right| \leq \frac{M-m}{2}. \tag{22}$$

With the help of the (21) and (22), we achieve

$$\begin{aligned}
&\left|\frac{1}{8}\left[3\aleph\left(\frac{5a+b}{6}\right)+2\aleph\left(\frac{a+b}{2}\right)+3\aleph\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\lambda\beta-1}\lambda^{\beta}\Gamma(\beta+1)}{(b-a)^{\lambda\beta}}\left[{}^{\beta}J^{\lambda}_{\frac{a+b}{2}+}\aleph(b)+{}^{\beta}J^{\lambda}_{\frac{a+b}{2}-}\aleph(a)\right]\right| \\
&\leq \frac{\lambda^{\beta}(b-a)}{4}(M-m)\left\{\int_{0}^{\frac{1}{3}}\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}d\eta+\int_{\frac{1}{3}}^{1}\left(\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}-\frac{3}{4\lambda^{\beta}}\right)d\eta\right\} \\
&= \frac{\lambda^{\beta}(b-a)}{4}\left[\mathcal{A}_1(\lambda,\beta)+\mathcal{A}_2(\lambda,\beta)\right](M-m).
\end{aligned}$$

□

5. Lipschitzian functions: Fractional Euler-Maclaurin-type inequalities

In this section, we give some fractional Euler-Maclaurin-type inequalities for Lipschitzian functions.

Theorem 5.1. *Assume the conditions stipulated in Lemma 2.1 holds. If $\aleph'$ is an L-Lipschitzian function on $[a,b]$, then the subsequent inequality is valid:*

$$\begin{aligned}
&\left|\frac{1}{8}\left[3\aleph\left(\frac{5a+b}{6}\right)+2\aleph\left(\frac{a+b}{2}\right)+3\aleph\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\lambda\beta-1}\lambda^{\beta}\Gamma(\beta+1)}{(b-a)^{\lambda\beta}}\left[{}^{\beta}J^{\lambda}_{\frac{a+b}{2}+}\aleph(b)+{}^{\beta}J^{\lambda}_{\frac{a+b}{2}-}\aleph(a)\right]\right| \\
&\leq \frac{\lambda^{\beta}(b-a)^2}{4}\left[\mathcal{A}_7(\lambda,\beta)+\mathcal{A}_8(\lambda,\beta)\right]L,
\end{aligned} \tag{23}$$

where

$$\mathcal{A}_7(\lambda,\beta)=\int_{0}^{\frac{1}{3}}(1-\eta)\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}d\eta$$

and

$$\mathcal{A}_8(\lambda,\beta)=\int_{\frac{1}{3}}^{1}(1-\eta)\left|\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}-\frac{3}{4\lambda^{\beta}}\right|d\eta.$$

Proof. By utilizing Lemma 2.1, we get

$$\left|\frac{1}{8}\left[3\aleph\left(\frac{5a+b}{6}\right)+2\aleph\left(\frac{a+b}{2}\right)+3\aleph\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\lambda\beta-1}\lambda^{\beta}\Gamma(\beta+1)}{(b-a)^{\lambda\beta}}\left[{}^{\beta}J^{\lambda}_{\frac{a+b}{2}+}\aleph(b)+{}^{\beta}J^{\lambda}_{\frac{a+b}{2}-}\aleph(a)\right]\right|$$

$$\leq \frac{\lambda^{\beta}(b-a)}{4}\left\{\int_{0}^{\frac{1}{3}}\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}\left|\aleph'\left(\frac{\eta}{2}b+\frac{2-\eta}{2}a\right)-\aleph'\left(\frac{\eta}{2}a+\frac{2-\eta}{2}b\right)\right|d\eta\right.$$
$$\left.+\int_{\frac{1}{3}}^{1}\left|\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}-\frac{3}{4\lambda^{\beta}}\right|\left|\aleph'\left(\frac{\eta}{2}b+\frac{2-\eta}{2}a\right)-\aleph'\left(\frac{\eta}{2}a+\frac{2-\eta}{2}b\right)\right|d\eta\right\}.$$

As $|\aleph'|$ is L-Lipschitzian function, we can conclude

$$\left|\frac{1}{8}\left[3\aleph\left(\frac{5a+b}{6}\right)+2\aleph\left(\frac{a+b}{2}\right)+3\aleph\left(\frac{a+5b}{6}\right)\right]-\frac{2^{\lambda\beta-1}\lambda^{\beta}\Gamma(\beta+1)}{(b-a)^{\lambda\beta}}\left[{}^{\beta}J^{\lambda}_{\frac{a+b}{2}+}\aleph(b)+{}^{\beta}J^{\lambda}_{\frac{a+b}{2}-}\aleph(a)\right]\right|$$
$$\leq \frac{\lambda^{\beta}(b-a)}{4}\left\{\int_{0}^{\frac{1}{3}}\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}L(1-\eta)\,d\eta+\int_{\frac{1}{3}}^{1}\left|\left(\frac{1-(1-\eta)^{\lambda}}{\lambda}\right)^{\beta}-\frac{3}{4\lambda^{\beta}}\right|L(1-\eta)\,d\eta\right\}$$
$$\leq \frac{\lambda^{\beta}(b-a)^{2}}{4}\left[\mathcal{A}_7(\lambda,\beta)+\mathcal{A}_8(\lambda,\beta)\right]L.$$

□

6. Conclusion

In this research, we developed Euler-Maclaurin type inequalities applicable to diverse function classes by employing conformable fractional integrals. Initially, we established an integral identity that underpins the primary results. We investigated Euler-Maclaurin type inequalities for differentiable convex functions and extended these findings to bounded and Lipschitz functions using fractional integrals. Furthermore, we confirmed the applicability of these inequalities to functions of bounded variation. In the special case $\lambda = 1$, the results of this paper are reduced to Euler-Maclaurin type results for Riemann-Liouville fractional integrals in [16]. These inequalities are versatile, applicable to various fractional integral forms, and provide a robust foundation for future studies.

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