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Bounds for the Wallis ratio with two parameters

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Abstract. In this paper, we establish several new bounds for the Wallis ratio $W_n = (2n-1)!!/(2n)!!$ in the form of $1/\sqrt{\pi G_n(a,b)}$, where

$$G_n(a,b) = n + \frac{1}{4} + \frac{b}{n+a}$$

with $a > -1$ and $b > -5(a+1)/4$. In particular, we find that

$$\frac{1}{\sqrt{\pi G_n(1/4, 1/32)}} < W_n < \frac{1}{\sqrt{\pi G_n(a_0, 1/32)}}$$

for $n \in \mathbb{N}$ with the best constants $1/4$ and $a_0 = 5(16-5\pi)/(16\pi)$, where the lower and upper bounds are the sharpest.

1. Introduction

The well-known Wallis ratio is defined by

$$W_n = \frac{(2n-1)!!}{(2n)!} \quad \text{for } n \in \mathbb{N},$$

where $n!!$ denotes the double factorial. It is known that

$$W_n = \frac{(2n-1)!!}{(2n)!!} = \frac{(2n)!}{2^{2n} n!^2} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(n+1/2)}{\Gamma(n+1)}$$

satisfying the recurrence relation

$$W_{n+1} = \frac{n+1/2}{n+1} W_n,$$

where Γ is the classical Euler's gamma function.

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Kazarinoff [7] proved that

$$\frac{1}{\sqrt{\pi\left(n+1/2\right)}}<W_n<\frac{1}{\sqrt{\pi\left(n+1/4\right)}}. \tag{1}$$

The best lower and upper bounds for W_n were obtained by Chen and Qi [4]:

$$\frac{1}{\sqrt{\pi\left(n+4/\pi-1\right)}}<W_n<\frac{1}{\sqrt{\pi\left(n+1/4\right)}}.$$

A simple proof and its generalization were provided in [8] by Koumandos.

Gurland in [6] established a closer approximation to π

$$\frac{4n+3}{(2n+1)^2}\left(\frac{(2n)!!}{(2n-1)!}\right)^2<\pi<\frac{4}{4n+1}\left(\frac{(2n)!!}{(2n-1)!}\right)^2,\ n\in\mathbb{N},$$

which can be rearranged as

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{16}\frac{1}{n+3/4}\right)}}W_n<\frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}},\ n\in\mathbb{N}. \tag{2}$$

In 1962, Chu [5, Theorem 1] demonstrated that

$$\frac{1}{\sqrt{\pi\left(n+(n+1)/(4n+3)\right)}}<W_n<\frac{1}{\sqrt{\pi\left(n+1/4\right)}} \tag{3}$$

for $n\in\mathbb{N}$, which is actually the same as (2). Boyd [1] pointed out that for $n\in\mathbb{N}$ it holds that

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{4}\frac{4n+11}{32n^2+72n+37}\right)}}<W_n<\frac{1}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{32}\frac{1}{n+1}\right)}}. \tag{4}$$

In 2004, Zhao [17] gave an improvement for the double inequality (1) as follows:

$$\frac{1}{\sqrt{\pi n\left(1+1/\left(4n-1/2\right)\right)}}<W_n<\frac{1}{\sqrt{\pi n\left(1+1/\left(4n-1/3\right)\right)}},$$

which is equivalent to

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{32}\frac{1}{n-1/8}\right)}}<W_n<\frac{1}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{48}\frac{1}{n-1/12}\right)}}. \tag{5}$$

In 2007, Cao et al. [2] proved that for $n\in\mathbb{N}$, the double inequality

$$\frac{\sqrt{\pi}}{2\sqrt{n+9\pi/16-1}}<\frac{(2n)!!}{(2n+1)!!}<\frac{\sqrt{\pi}}{2\sqrt{n+3/4}}$$

is valid with the best constants $9\pi/16-1\approx0.76715$ and $3/4$, which can be written as

$$\frac{2\sqrt{n+3/4}}{\sqrt{\pi}\left(2n+1\right)}<W_n<\frac{2\sqrt{n+9\pi/16-1}}{\sqrt{\pi}\left(2n+1\right)},$$

or equivalently,

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{16}\frac{1}{n+3/4}\right)}}<W_n<\frac{1}{\sqrt{\pi\left(n+2-\frac{9}{16}\pi+\frac{9}{256}\frac{(3\pi-8)^2}{n+9\pi/16-1}\right)}}. \tag{6}$$

Zhang [16] showed that

$$\frac{1}{\sqrt{\pi\left(n+1/\left[4-4/\left(8n+3\right)\right]\right)}}<W_n<\frac{1}{\sqrt{\pi\left(n+1/\left[4-1/\left(2n+1\right)\right]\right)}},$$

or equivalently,

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{32}\frac{1}{n+1/4}\right)}}<W_n<\frac{1}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{32}\frac{1}{n+3/8}\right)}}. \tag{7}$$

The left hand side inequality in (7) was generalized in [9] as

$$W_n<\frac{1}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{32}\frac{1}{n+(8\lambda+3)/(32\lambda+8)}\right)}} \tag{8}$$

for $\lambda\in[0,\infty)$ and $n\in\left[\left(72\lambda^2+27\lambda+3\right)/\left(12\lambda+3\right),\infty\right)\cap\mathbb{N}$.

From the inequalities (2), (3), (4), (5), (6), (7) and (8), we find that Wallis ratio W_n has the bounds in the form of $1/\sqrt{\pi G_n\left(a,b\right)}$, where

$$G_n\left(a,b\right)=n+\frac{1}{4}+\frac{b}{n+a}.$$

Comparing the inequalities (2), (3), (4), (5), (6), we easily see that

$$\begin{aligned}&\frac{1}{\sqrt{\pi G_n\left(3/4,1/16\right)}}<\frac{1}{\sqrt{\pi G_n\left(-1/8,1/32\right)}}<\sqrt{\frac{32n^2+72n+37}{4\pi\left(2n+1\right)^2\left(2n+3\right)}}\\&<\frac{1}{\sqrt{\pi G_n\left(1/4,1/32\right)}}<W_n<\frac{1}{\sqrt{\pi G_n\left(3/8,1/32\right)}}\\&<\min\left\{\frac{1}{\sqrt{\pi G_n\left(1,1/32\right)}},\frac{1}{\sqrt{\pi G_n\left(-1/12,1/48\right)}}\right\}<\frac{1}{\sqrt{\pi\left(n+1/4\right)}}\end{aligned}$$

for $n\in\mathbb{N}$. These inequalities show that Zhang's bounds for the Wallis ratio given in (7) is better than Gurland's (2), Chu's (3), Boyd's (4) and Zhao's (5) ones.

It is natural to ask that what are the conditions such that the inequality

$$W_n<(>)\,\frac{1}{\sqrt{\pi G_n\left(a,b\right)}}$$

holds for $n\in\mathbb{N}$? where $a>-1$ and $b>-5\left(a+1\right)/4$. The aim of this paper is to answer this question by considering the monotonicity of the sequence

$$V_n\left(a,b\right)=W_n^2G_n\left(a,b\right)=W_n^2\left(n+\frac{1}{4}+\frac{b}{n+a}\right). \tag{9}$$

2. Monotonicity pattern of $V_n\left(a,b\right)$

In this section, we will investigate the monotonicity of the sequence $\{V_n\left(a,b\right)\}$ defined by (9). First, we claim that

$$\lim_{n\to\infty}V_n\left(a,b\right)=\frac{1}{\pi},$$

which follows from the double inequality

$$\frac{1}{\left(x+c\right)^{1-c}}<\frac{\Gamma\left(x+c\right)}{\Gamma\left(x+1\right)}<\frac{1}{x^{1-c}}.$$

for all $x > 0$ and all $c \in (0, 1)$ (see [13], [10, (2.8)]). Now we observe the monotonicity of the sequence $\{V_n(a,b)\}$. A direct computation yields

$$\frac{V_{n+1}(a,b)}{V_n(a,b)} = \frac{(n+1/2)^2}{(n+1)^2}\frac{n+1+1/4+b/(n+1+a)}{n+1/4+b/(n+a)},$$

$$\frac{V_{n+1}(a,b)}{V_n(a,b)} - 1 = \frac{f_n(a,b)}{4(n+1)^2(n+1+a)(4n^2+(4a+1)n+a+4b)},$$

where

$$f_n(a,b) = (1-32b)n^2 + (2a-44b-16ab+1)n + \left(a-16b-12ab+a^2\right).$$

To obtain the monotonicity of the sequence $\{V_n(a,b)\}$, we write $f_n(a,b)$ as

$$\begin{aligned} f_n(a,b) &= -4\left(8n^2+(11+4a)n+3a+4\right)\left(b - \frac{(n+a)(n+1+a)}{4(8n^2+(11+4a)n+3a+4)}\right) \\ &=: -4\left(8n^2+(11+4a)n+3a+4\right)(b - g_n(a)), \end{aligned}$$

where

$$g_n(a) = \frac{(n+a)(n+1+a)}{4(8n^2+(11+4a)n+3a+4)}.$$

A simple calculation leads us to

$$\begin{aligned} g_{n+1}(a) - g_n(a) &= \frac{(n+1+a)(n+2+a)}{4\left(8(n+1)^2+(4a+11)(n+1)+(3a+4)\right)} - \frac{(n+a)(n+1+a)}{4(8n^2+(4a+11)n+(3a+4))} \\ &= -\frac{n+1+a}{[8n^2+(4a+11)n+(3a+4)][8n^2+(4a+27)n+(7a+23)]}h_n(a), \end{aligned}$$

where

$$h_n(a) = a^2 + \left(3n+\frac{13}{4}\right)a - \left(\frac{3}{4}n+2\right) = (a-a_1(n))(a-a_2(n)), \tag{10}$$

with

$$a_1(n) = \frac{3}{2}\sqrt{n^2+\frac{5}{2}n+\frac{33}{16}} - \frac{3}{2}n - \frac{13}{8}, \quad a_2(n) = -\frac{3}{2}\sqrt{n^2+\frac{5}{2}n+\frac{33}{16}} - \frac{3}{2}n - \frac{13}{8}.$$

It is easy to check that $x \mapsto a_1(x)$ is decreasing on $(0,\infty)$, which implies that so is $a_1(n)$ for $n \in \mathbb{N}$ with

$$a_1(1) = \frac{3\sqrt{89}-25}{8} = 0.412... \text{ and } a_1(\infty) = \lim_{n\to\infty} a_1(n) = \frac{1}{4};$$

while

$$-a_2(n) \ge -a_2(1) = \frac{3\sqrt{89}+25}{8} = 6.662... > 1,$$

which means that $a - a_2(n) > 0$ for $a > -1$. These together with $(a,b) \in \{a > -1, b > -5(a+1)/4\}$ indicate that

$$\operatorname{sgn}\left(\frac{V_{n+1}(a,b)}{V_n(a,b)} - 1\right) = \operatorname{sgn} f_n(a,b) = \operatorname{sgn}(g_n(a) - b), \tag{11}$$

$$\operatorname{sgn}(g_{n+1}(a) - g_n(a)) = -\operatorname{sgn} h_n(a) = -\operatorname{sgn}(a - a_1(n)). \tag{12}$$

We next distinguish three cases to discuss.

Case 1: $-1 < a \leq 1/4$. By (12) the sequence $\{g_n(a)\}$ is strictly increasing for $n \geq 1$. Therefore, we get

$$\frac{(a+1)(a+2)}{4(7a+23)} = g_1(a) < g_n(a) < g_\infty(a) = \frac{1}{32}.$$

Subcase 1.1: $b \geq g_\infty(a) = 1/32$. Then by (11) we see that $f_n(a,b) \leq 0$, and then, the $\{V_n(a,b)\}$ is decreasing for $n \geq 1$.

Subcase 1.2: $b \leq g_1(a) = \frac{(a+1)(a+2)}{4(7a+23)}$. Similarly, we see that $\{V_n(a,b)\}$ is increasing for $n \geq 1$.

Subcase 1.3: $g_1(a) < b < g_\infty(a)$. Since $g_1(a) - b < 0$ and $g_\infty(a) - b > 0$, there is an integer $n_1 > 1$ such that $g_n(a) - b < 0$ for $1 \leq n \leq n_1 - 1$ and $g_n(a) - b > 0$ for $n \geq n_1 + 1$. This, by (11), shows that $\{V_n(a,b)\}$ is decreasing for $1 \leq n \leq n_1$ and increasing for $n \geq n_1$.

Case 2: $a \geq \left(3\sqrt{89} - 25\right)/8$. By (12), this case implies that the sequence $\{g_n(a)\}$ is strictly decreasing for $n \geq 1$. Therefore, we get

$$\frac{1}{32} = g_\infty(a) < g_n(a) < g_1(a) = \frac{(a+1)(a+2)}{4(7a+23)}.$$

Being analogous to Subcases 1.2–1.3, we have

Subcase 2.1: $b \geq g_1(a) = \frac{(a+1)(a+2)}{4(7a+23)}$. The sequence $\{V_n(a,b)\}$ is decreasing for $n \geq 1$.

Subcase 2.2: $b \leq g_\infty(a) = 1/32$. The sequence $\{V_n(a,b)\}$ is increasing for $n \geq 1$.

Subcase 2.3: $g_\infty(a) < b < g_1(a)$. There is an integer $n_2 > 1$ such that the sequence $\{V_n(a,b)\}$ is increasing for $1 \leq n \leq n_2$ and decreasing for $n \geq n_2$.

Case 3: $1/4 < a < \left(3\sqrt{89} - 25\right)/8$. From (10) there is an integer $n_3 \geq 2$ such that $h_n(a) \geq 0$ for $1 \leq n \leq n_3 - 1$ and $h_n(a) < 0$ for $n \geq n_3 + 1$, which, by (12), indicates that $\{g_n(a)\}$ is increasing for $1 \leq n \leq n_3$ and decreasing for $n \geq n_3$. It is obtained that

$$\frac{(a+1)(a+2)}{4(7a+23)} = g_1(a) < g_n(a) \leq g_{n_3}(a) \text{ for } 1 \leq n \leq n_3$$

$$\frac{1}{32} = g_\infty(a) < g_n(a) \leq g_{n_3}(a) \text{ for } n \geq n_3,$$

that is,

$$\min\left\{\frac{1}{32}, \frac{(a+1)(a+2)}{4(7a+23)}\right\} < g_n(a) < g_{n_3}(a),$$

where

$$\min\left\{\frac{1}{32}, \frac{(a+1)(a+2)}{4(7a+23)}\right\} = \begin{cases} \dfrac{(a+1)(a+2)}{4(7a+23)} & \text{if } \dfrac{1}{4} < a < \dfrac{3\sqrt{57} - 17}{16}, \\ \dfrac{1}{32} & \text{if } \dfrac{3\sqrt{57} - 17}{16} \leq a < \dfrac{3\sqrt{89} - 25}{8}. \end{cases}$$

Subcase 3.1: $b \geq g_{n_3}(a)$. Likewise, we see that the sequence $\{V_n(a,b)\}$ is decreasing for $n \geq 1$.

Subcase 3.2: $b \leq \min\{g_1(a), g_\infty(a)\}$. The sequence $\{V_n(a,b)\}$ is increasing for $n \geq 1$.

Subcase 3.3: $\min\{g_1(a), g_\infty(a)\} < b < \max\{g_1(a), g_\infty(a)\} < g_{n_3}(a)$. This subcase can be divided into two subsubcases:

Subsubcase 3.3.1: $g_1(a) < g_\infty(a)$, that is, $1/4 < a < \left(3\sqrt{57} - 17\right)/16$. Then we have $g_1(a) < b \leq g_\infty(a) < g_{n_3}(a)$. Since $\{g_n(a)\}$ is increasing for $1 \leq n \leq n_3$ and decreasing for $n \geq n_3$, so is $\{g_n(a) - b\} := \{g_n^*(a)\}$. This in combination with

$$g_1^*(a) = g_1(a) - b < 0 \text{ and } g_\infty^*(a) = g_\infty(a) - b > 0,$$

yields that there is an integer n_4 with $1 < n_4 \leq n_3$ such that $g_n^*(a) = g_n(a) - b < 0$ for $1 \leq n \leq n_4 - 1$ and $g_n^*(a) = g_n(a) - b > 0$ for $n \geq n_4 + 1$. This, by (11), shows that the sequence $\{V_n(a,b)\}$ is decreasing for $1 \leq n \leq n_4$ and increasing for $n \geq n_4$.

Subsubcase 3.3.2: $g_1(a) > g_\infty(a)$, that is, $\left(3\sqrt{57}-17\right)/16 < a < \left(3\sqrt{89}-25\right)/8$. Then we have $g_\infty(a) < b \le g_1(a) < g_{n_3}(a)$. Since $\{g_n^*(a)\}$ is increasing for $1 \le n \le n_3$ and decreasing for $n \ge n_3$ with

$$g_1^*(a) = g_1(a) - b > 0 \text{ and } g_\infty^*(a) = g_\infty(a) - b < 0,$$

there is an integer $n_5 > n_3$ such that $g_n^*(a) = g_n(a) - b \ge 0$ for $1 \le n \le n_5 - 1$ and $g_n^*(a) = g_n(a) - b \le 0$ for $n \ge n_5 + 1$. This, by (11), shows that the sequence $\{V_n(a,b)\}$ is increasing for $1 \le n \le n_5$ and decreasing for $n \ge n_5$.

Subcase 3.4: $\min\{g_1(a), g_\infty(a)\} \le \max\{g_1(a), g_\infty(a)\} < b < g_{n_3}(a)$. Since $\{g_n^*(a)\}$ is increasing for $1 \le n \le n_3$ and decreasing for $n \ge n_3$ with

$$g_1^*(a) = g_1(a) - b < 0,\ g_{n_3}^*(a) = g_{n_3}(a) - b > 0 \text{ and } g_\infty^*(a) = g_\infty(a) - b < 0,$$

there is two integers n_{31} and n_{32} with $1 < n_{31} < n_3$ and $n_{32} > n_3$ such that

$$g_n^*(a) = g_n(a) - b < 0 \text{ for } 1 \le n \le n_{31} - 1,$$

$$g_n^*(a) = g_n(a) - b > 0 \text{ for } n_{31} + 1 \le n < n_{32} - 1,$$

$$g_n^*(a) = g_n(a) - b < 0 \text{ for } n \ge n_{32} + 1.$$

Thus it can be seen by (11) that the sequence $\{V_n(a,b)\}$ is decreasing for $1 \le n \le n_{31}$ or $n \ge n_{32}$ and increasing for $n_{31} \le n \le n_{32}$.

To sum up, the monotonicity pattern of $V_n(a,b)$ can be listed in Table 1.

Table 1: The monotonicity pattern of $V_n(a,b)$

Cases	Subcases	$V_n(a,b)$
1: $-1 < a \le \frac{1}{4}$	1.1: $b \ge g_\infty(a)$	$\searrow$
	1.2: $b \le g_1(a)$	$\nearrow$
	1.3: $g_1(a) < b < g_\infty(a)$	$\searrow\nearrow$
2: $a \ge \frac{3\sqrt{89}-25}{8}$	2.1: $b \ge g_1(a)$	$\searrow$
	2.2: $b \le g_\infty(a)$	$\nearrow$
	2.3: $g_\infty(a) < b < g_1(a)$	$\nearrow\searrow$
3: $\frac{1}{4} < a < \frac{3\sqrt{89}-25}{8}$	3.1: $b \ge g_{n_3}(a)$	$\searrow$
	3.2: $b \le \min\{g_1(a), g_\infty(a)\}$	$\nearrow$
	3.3: $\min\{g_1(a), g_\infty(a)\} < b \le \max\{g_1(a), g_\infty(a)\}$	
	3.3.1: $g_1(a) < g_\infty(a)$	$\searrow\nearrow$
	3.3.2: $g_1(a) > g_\infty(a)$	$\nearrow\searrow$
	3.4: $\max\{g_1(a), g_\infty(a)\} < b < g_{n_3}(a)$	$\searrow\nearrow\searrow$

Further, the monotonicity results for the sequence $\{V_n(a,b)\}$ and corresponding inequalities can be summarized as follows.

Theorem 2.1. *Let $(a,b) \in \{b > -5(a+1)/4, a > -1\} = E_0$ and let the sequence $\{V_n(a,b)\}$ be defined by (9). The following statements are valid.*

(i) The sequence $\{V_n(a,b)\}$ is decreasing for $n \ge 1$ if $(a,b) \in E_1$, where

$$E_1 = \left\{-1 < a \le \frac{1}{4}, b \ge \frac{1}{32}\right\} \cup \left\{a \ge \frac{3\sqrt{89}-25}{8}, b \ge \frac{(a+1)(a+2)}{4(7a+23)}\right\}$$

$$\cup \left\{\frac{1}{4} < a < \frac{3\sqrt{89}-25}{8}, b \ge \frac{(n_3+a)(n_3+1+a)}{4\left(8n_3^2+(11+4a)n_3+3a+4\right)}\right\},$$

here n_3 is given in Case 3 for $1/4 < a < \left(3\sqrt{89}-25\right)/8$.

Consequently, it holds that

$$\frac{1}{\sqrt{\pi G_n(a,b)}} < W_n < \frac{\sqrt{\pi\lambda(a,b)}}{\sqrt{\pi G_n(a,b)}}$$

for $n \in \mathbb{N}$ *with the best constants* 1 *and*

$$\pi\lambda(a,b) = \pi\frac{5a+4b+5}{16(a+1)}. \tag{13}$$

(ii) The sequence $\{V_n(a,b)\}$ *is increasing for* $n \geq 1$ *if* $(a,b) \in E_2$, *where*

$$E_2 = \left\{-1 < a \leq \frac{1}{4}, b \leq \frac{(a+1)(a+2)}{4(7a+23)}\right\} \cup \left\{a \geq \frac{3\sqrt{89}-25}{8}, b \leq \frac{1}{32}\right\}$$
$$\cup \left\{\frac{1}{4} < a < \frac{3\sqrt{89}-25}{8}, b \leq \min\left\{\frac{(a+1)(a+2)}{4(7a+23)}, \frac{1}{32}\right\}\right\}.$$

Then it holds that

$$\frac{\sqrt{\pi\lambda(a,b)}}{\sqrt{\pi G_n(a,b)}} < W_n < \frac{1}{\sqrt{\pi G_n(a,b)}}$$

for $n \in \mathbb{N}$ *with the best constants* 1 *and* $\pi\lambda(a,b)$ *given in (13).*

(iii) There is an $\check{n}\,(=n_1, n_4) > 1$ *such that* $\{V_n(a,b)\}$ *is decreasing for* $1 \leq n \leq \check{n}$ *and increasing for* $n \geq \check{n}$ *if* $(a,b) \in E_3$, *where*

$$E_3 = \left\{-1 < a \leq \frac{1}{4}, \frac{(a+1)(a+2)}{4(7a+23)} < b < \frac{1}{32}\right\} \cup \left\{\frac{1}{4} < a < \frac{3\sqrt{57}-17}{16}, \frac{(a+1)(a+2)}{4(7a+23)} < b \leq \frac{1}{32}\right\}. \tag{14}$$

Therefore, we have

$$\frac{\sqrt{\pi V_{\check{n}}(a,b)}}{\sqrt{\pi G_n(a,b)}} < W_n < \frac{\sqrt{\max\{1, \pi\lambda(a,b)\}}}{\sqrt{\pi G_n(a,b)}}$$

for $n \in \mathbb{N}$, *where* $\pi V_{\check{n}}(a,b)$ *and* $\max\{1, \pi\lambda(a,b)\}$ *are the best, here* $\pi\lambda(a,b)$ *is given in (13).*

(iv) There is an $\hat{n}\,(=n_2, n_5) > 1$ *such that the sequence* $\{V_n(a,b)\}$ *is increasing for* $1 \leq n \leq \hat{n}$ *and decreasing for* $n \geq \hat{n}$ *if* $(a,b) \in E_4$, *where*

$$E_4 = \left\{a \geq \frac{3\sqrt{89}-25}{8}, \frac{1}{32} < b < \frac{(a+1)(a+2)}{4(7a+23)}\right\} \cup \left\{\frac{3\sqrt{57}-17}{16} < a < \frac{3\sqrt{89}-25}{8}, \frac{1}{32} < b \leq \frac{(a+1)(a+2)}{4(7a+23)}\right\}. \tag{15}$$

Then the inequalities

$$\frac{\sqrt{\min\{1, \pi\lambda(a,b)\}}}{\sqrt{\pi G_n(a,b)}} < W_n < \frac{\sqrt{\pi V_{\check{n}}(a,b)}}{\sqrt{\pi G_n(a,b)}}$$

hold for $n \in \mathbb{N}$, *where* $\pi V_{\hat{n}}(a,b)$ *and* $\min\{1, \pi\lambda(a,b)\}$ *are the best.*

(v) Let n_3 *be given in Case 3 for* $1/4 < a < \left(3\sqrt{89}-25\right)/8$. *Then there are* n_{31} *and* n_{32} *with* $1 < n_{31} < n_3 < n_{32}$ *such that* $\{V_n(a,b)\}$ *is decreasing for* $1 \leq n \leq n_{31}$ *or* $n \geq n_{32}$ *and increasing for* $n_{31} \leq n \leq n_{32}$ *if* $(a,b) \in E_5$, *where*

$$E_5 = \left\{\frac{1}{4} < a < \frac{3\sqrt{89}-25}{8}\right\} \cap \left\{\max\left\{\frac{(a+1)(a+2)}{4(7a+23)}, \frac{1}{32}\right\} < b < \frac{(n_3+a)(n_3+1+a)}{4\left(8n_3^2+(11+4a)n_3+3a+4\right)}\right\}.$$

Thus the double inequality

$$\frac{\sqrt{\pi\alpha(a,b)}}{\sqrt{\pi G_n(a,b)}} < W_n < \frac{\sqrt{\pi\beta(a,b)}}{\sqrt{\pi G_n(a,b)}}.$$

holds for $n \in \mathbb{N}$ with the best constants $\pi\alpha(a,b) = \min\{\pi V_{n_{31}}(a,b), 1\}$ and $\pi\beta(a,b) = \max\{\pi V_{n_{32}}(a,b), \pi\lambda(a,b)\}$, where $\pi\lambda(a,b)$ is given in (13).

Remark 2.2. *It is clear that E_3 and E_4 given in (14) and (15) can be reduced to*

$$E_3 = \left\{-1 < a \le \frac{3\sqrt{57}-17}{16}, \frac{(a+1)(a+2)}{4(7a+23)} < b < \frac{1}{32}\right\} \cup \left\{\frac{1}{4} < a < \frac{3\sqrt{57}-17}{16}, b = \frac{1}{32}\right\}.$$

$$E_4 = \left\{a > \frac{3\sqrt{57}-17}{16}, \frac{1}{32} < b < \frac{(a+1)(a+2)}{4(7a+23)}\right\} \cup \left\{\frac{3\sqrt{57}-17}{16} < a < \frac{3\sqrt{89}-25}{8}, b = \frac{(a+1)(a+2)}{4(7a+23)}\right\}.$$

3. Bounds for W_n

In this section, we will present some bounds for W_n by using Theorem 2.1. First, we give a general result.

3.1. In the general case

Theorem 3.1. *Let $(a,b) \in E_0 = \{b > -5(a+1)/4, a > -1\}$. The inequality*

$$\frac{1}{\sqrt{\pi G_n(a,b)}} < W_n$$

holds for $n \in \mathbb{N}$ if

$$(a,b) \in E_{14<} = E_1 \cup \left(E_4 \cap \left\{\frac{5a+4b+5}{16(a+1)} \le \frac{1}{\pi}\right\}\right).$$

While the inequality

$$W_n < \frac{1}{\sqrt{\pi G_n(a,b)}}$$

holds for $n \in \mathbb{N}$ if

$$(a,b) \in E_{23>} = E_2 \cup \left(E_3 \cap \left\{\frac{5a+4b+5}{16(a+1)} \ge \frac{1}{\pi}\right\}\right).$$

Second, we establish several bounds for W_n in certain special cases.

3.2. In the case of $a = 1/4$

Theorem 3.2. *Let $b > -25/16$ and let the sequence $\{V_n(a,b)\}$ be defined by (9). The following statements are valid.*

(i) If $b \ge 1/32$, then it holds that

$$\frac{1}{\sqrt{\pi G_n(1/4,b)}} < W_n < \frac{\sqrt{\vartheta(b)}}{\sqrt{\pi G_n(1/4,b)}}$$

for $n \in \mathbb{N}$, where 1 and

$$\vartheta(b) = \pi\lambda\left(\frac{1}{4}, b\right) = \frac{\pi(16b+25)}{80}$$

are the best constants. In particular, when $b = 1/32$, we have

$$\frac{1}{\sqrt{\pi G_n(1/4, 1/32)}} < W_n < \frac{\sqrt{51\pi/160}}{\sqrt{\pi G_n(1/4, 1/32)}} \tag{16}$$

(ii) If $b \leq 5/176$, then it holds that

$$\frac{\sqrt{\vartheta(b)}}{\sqrt{\pi G_n(1/4, b)}} < W_n < \frac{1}{\sqrt{\pi G_n(1/4, b)}}$$

for $n \in \mathbb{N}$, where $\vartheta(b) = \pi(16b + 25)/80$ and 1 are the best constants. In particular, when $b = 5/176$, we have

$$\frac{\sqrt{7\pi/22}}{\sqrt{\pi G_n(1/4, 5/176)}} < W_n < \frac{1}{\sqrt{\pi G_n(1/4, 5/176)}} \tag{17}$$

(iii) If $5/176 < b < 1/32$ then there is an $\check{n} > 1$ such that the double inequality

$$\frac{\sqrt{\pi V_{\check{n}}(1/4, b)}}{\sqrt{\pi G_n(1/4, b)}} < W_n < \frac{\sqrt{\max\{1, \vartheta(b)\}}}{\sqrt{\pi G_n(1/4, b)}}$$

holds for $n \in \mathbb{N}$, where $\pi V_{\check{n}}(1/4, b)$ and $\max\{1, \vartheta(b)\}$ are the best, here $\vartheta(b) = \pi(16b + 25)/80$. In particular, when $\vartheta(b) = \pi(16b + 25)/80 = 1$, that is, $b = b_0 = (80 - 25\pi)/(16\pi)$, we have

$$\sqrt{\frac{7\pi + 10}{32}} \frac{1}{\sqrt{\pi G_n(1/4, b_0)}} < W_n < \frac{1}{\sqrt{\pi G_n(1/4, b_0)}}$$

Proof. It is easy to check that when $a = 1/4$,

$$E_1 = \left\{a = \frac{1}{4}, b \geq \frac{1}{32}\right\},\ E_2 = \left\{a = \frac{1}{4}, b \leq \frac{5}{176}\right\},\ E_3 = \left\{a = \frac{1}{4}, \frac{5}{176} < b < \frac{1}{32}\right\},\ E_4 = E_5 = \emptyset.$$

Then the assertions (i), (ii) and the first part of (iii) follow from Theorem 2.1. Finally, when $b = b_0 = (80 - 25\pi)/(16\pi)$, a simple computation leads to

$$V_1\left(\frac{1}{4}, b_0\right) = \frac{1}{\pi} > V_2\left(\frac{1}{4}, b_0\right) = \frac{7\pi + 10}{32\pi} < V_3\left(\frac{1}{4}, b_0\right) = \frac{25(9\pi + 5)}{832\pi},$$

which shows that $\check{n} = 2$. Therefore, $\pi V_2(1/4, b_0) = (7\pi + 10)/32$, and the required double inequality follows. This completes the proof. □

From Theorem 3.2 we immediately obtain the following corollary.

Corollary 3.3. *The double inequality*

$$\frac{1}{\sqrt{\pi G_n(1/4, b_1)}} < W_n < \frac{1}{\sqrt{\pi G_n(1/4, b_0)}} \tag{18}$$

holds for $n \in \mathbb{N}$ with the best constants $b_1 = 1/32 = 0.031...$ and $b_0 = (80 - 25\pi)/(16\pi) = 0.029....$

3.3. In the case of $b = 1/32$

Taking $b = 1/32$ in Theorem 2.1, we have

Theorem 3.4. *Let $a \in (-1, \infty)$. (i) If $-1 < a \le 1/4$, then the double inequality*

$$\frac{1}{\sqrt{\pi G_n(a, 1/32)}} < W_n < \frac{\sqrt{\theta(a)}}{\sqrt{\pi G_n(a, 1/32)}} \tag{19}$$

holds for $n \in \mathbb{N}$, where π and

$$\theta(a) = \pi\lambda\left(a, \frac{1}{32}\right) = \frac{\pi(40a + 41)}{128(a + 1)}$$

are the best possible. In particular, when $a = 1/4$, the double inequality (16) holds for $n \in \mathbb{N}$.

(ii) If $a \ge \left(3\sqrt{57} - 17\right)/16 \approx 0.35309$, then for $n \in \mathbb{N}$, the double inequality (19) is reversed, that is,

$$\frac{\sqrt{\theta(a)}}{\sqrt{\pi G_n(a, 1/32)}} < W_n < \frac{1}{\sqrt{\pi G_n(a, 1/32)}}.$$

In particular, when $a = 3/8$, we have

$$\frac{\sqrt{7\pi/22}}{\sqrt{\pi G_n(3/8, 1/32)}} < W_n < \frac{1}{\sqrt{\pi G_n(3/8, 1/32)}}. \tag{20}$$

(iii) If $1/4 < a < \left(3\sqrt{57} - 17\right)/16 \approx 0.35309$, then there is a unique integer $\check{n} > 1$ such that the double inequality

$$\frac{\sqrt{\pi V_{\check{n}}(a, 1/32)}}{\sqrt{\pi G_n(a, 1/32)}} < W_n < \frac{\sqrt{\max\{1, \theta(a)\}}}{\sqrt{\pi G_n(a, 1/32)}}$$

holds for $n \in \mathbb{N}$. In particular, when

$$a = a_0 = \frac{41\pi - 128}{128 - 40\pi} = 0.344\,690...,$$

we have

$$\frac{\sqrt{\pi V_2(a_0, 1/32)}}{\sqrt{\pi G_n(a_0, 1/32)}} < W_n < \frac{1}{\sqrt{\pi G_n(a_0, 1/32)}} \tag{21}$$

holds for $n \in \mathbb{N}$, where

$$\pi V_2\left(a_0, \frac{1}{32}\right) = \frac{9\pi(292 - 89\pi)}{64(128 - 39\pi)} = 0.999\,907...$$

and 1 *are the best constants.*

Proof. It is easy to check that when $b = 1/32$,

$$E_1 = \left\{-1 < a \le \frac{1}{4}, b = \frac{1}{32}\right\},\ E_2 = \left\{a \ge \frac{3\sqrt{57} - 17}{16}, b = \frac{1}{32}\right\},\ E_3 = \left\{\frac{1}{4} < a < \frac{3\sqrt{57} - 17}{16}, b = \frac{1}{32}\right\}$$

and $E_4 = E_5 = \emptyset$. Then the assertions (i), (ii) and the first part of (iii) follow from Theorem 2.1. When $a = a_0 = (41\pi - 128)/(128 - 40\pi)$, we see that $\theta(a_0) = 1$. By a simple computation we get

$$V_1\left(a_0, \frac{1}{32}\right) = \frac{1}{\pi} > V_2\left(a_0, \frac{1}{32}\right) = \frac{9}{64}\frac{89\pi - 292}{39\pi - 128} < V_3\left(a_0, \frac{1}{32}\right) = \frac{25}{128}\frac{129\pi - 418}{79\pi - 256},$$

which indicates that $\check{n} = 2$. This yields (21), and the proof is done. $\square$

Denote by

$$A_n(a) = \frac{1}{\pi G_n(a, 1/32)} = \frac{1}{\pi} \frac{32(n+a)}{8(4n+1)(n+a)+1}$$

$$B_n(a) = \frac{\theta(a)}{\pi G_n(a, 1/32)} = \frac{40a+41}{128(a+1)} \frac{32(n+a)}{8(4n+1)(n+a)+1}.$$

It is clear that $G_n(a, 1/32)$ is strictly decreasing with respect to the parameter a, and that $A_n(a)$ is increasing in a on $(-1, \infty)$. Also, we claim that $B_n(a)$ is strictly decreasing in a on $(-1, \infty)$. Indeed, differentiation gives

$$B_n'(a) = -\frac{(n-1)}{4(a+1)^2} \frac{32(n+a)^2 + 40(n+2a+1)+1}{(32n^2 + 8(4a+1)n + (8a+1))^2} < 0$$

for $a \in (-1, \infty)$ and $n \in \mathbb{N}$.

Then by Theorem 3.4 together with the decreasing property of $G_n(a, 1/32)$ with respect to the parameter a on $(-1, \infty)$, the following corollary is immediate.

Corollary 3.5. *For $n \in \mathbb{N}$, the double inequality*

$$\frac{1}{\sqrt{\pi G_n(a_1, 1/32)}} < W_n < \frac{1}{\sqrt{\pi G_n(a_0, 1/32)}} \tag{22}$$

holds with the best constant $a_1 = 1/4$ and $a_0 = (41\pi - 128)/(8(16 - 5\pi)) \approx 0.34469$.

Remark 3.6. *Since*

$$G_n\left(\frac{1}{4}, b_0\right) - G_n\left(a_0, \frac{1}{32}\right) = \frac{b_0}{n+1/4} - \frac{1/32}{n+a_0} = -\frac{51\pi - 160}{8\pi} \frac{n-1}{(4n+1)(n+a_0)} < 0,$$

we have

$$\frac{1}{\sqrt{\pi G_n(a_1, 1/32)}} < W_n < \frac{1}{\sqrt{\pi G_n(a_0, 1/32)}} < \frac{1}{\sqrt{\pi G_n(1/4, b_0)}} \tag{23}$$

for $n \in \mathbb{N}$. This shows that the upper bound $1/\sqrt{\pi G_n(a_0, 1/32)}$ is sharper than $1/\sqrt{\pi G_n(1/4, b_0)}$.

Remark 3.7. *Since $a_0 = (41\pi - 128)/(8(16 - 5\pi)) \approx 0.34469 < 3/8$, we have*

$$\frac{1}{\sqrt{\pi G_n(a_1, 1/32)}} < W_n < \frac{1}{\sqrt{\pi G_n(a_0, 1/32)}} < \frac{1}{\sqrt{\pi G_n(3/8, 1/32)}}.$$

This means that the upper bound is better than one in (7). As shown in Introduction, Zhang's bounds for the Wallis ratio given in (7) is better than Gurland's (2), Chu's (3), Boyd's (4) and Zhao's (5) ones, so are our bounds given (22).

Remark 3.8. *It is easy to check that for $-1/8 \le a \le 1/4$, the inequalities*

$$\frac{\sqrt{n+3/4}}{\sqrt{\pi}(n+1/2)} < \frac{1}{\sqrt{\pi G_n(a, 1/32)}} < W_n < \frac{1}{\sqrt{\pi G_n(a_0, 1/32)}} < \frac{1}{\sqrt{\pi G_n(1/4, b_0)}} < \frac{\sqrt{n+9\pi/16-1}}{\sqrt{\pi}(n+1/2)}$$

hold for $n \in \mathbb{N}$. Therefore, our inequalities (22) and (18) improve Cao et al.'s inequalities (6).

3.4. Other special cases

Similarly, using Theorem 2.1 we can obtain the following corollary in the case of $a = 3/4$.

Corollary 3.9. *Let $b, b^* > -35/16$. Then the double inequality*

$$\frac{1}{\sqrt{\pi G_n(3/4, b)}} < W_n < \frac{1}{\sqrt{\pi G_n(3/4, b^*)}} \tag{24}$$

holds for $n \in \mathbb{N}$ if and only if $b \geq b_0 = 7(16 - 5\pi)/(16\pi)$ and $-35/16 < b^* \leq 1/32$.

Remark 3.10. *Taking $b = 1/16$ and $b^* = 0$ in (24) gives (2).*

In the case of $(a, b) = (1/3, 1/33)$, we can deduce the following corollary.

Corollary 3.11. *The double inequality*

$$\frac{\sqrt{7\pi/22}}{\sqrt{\pi G_n(1/3, 1/33)}} < W_n < \frac{1}{\sqrt{\pi G_n(1/3, 1/33)}} \tag{25}$$

holds for $n \in \mathbb{N}$.

4. Conclusions

In this paper, we presented some bounds for the Wallis ratio W_n by considering the monotonicity of the sequence $\{V_n(a, b)\}$ for suitable $(a, b) \in \{b > -5(a+1)/4, a > -1\}$. In particular, we found the sharp bounds for W_n given in (22), that is,

$$\frac{1}{\sqrt{\pi\left(n + \frac{1}{4} + \frac{1}{32}\frac{1}{n+1/4}\right)}} < W_n < \frac{1}{\sqrt{\pi\left(n + \frac{1}{4} + \frac{1}{32}\frac{1}{n+a_0}\right)}}, \tag{26}$$

where $a_0 = (41\pi - 128)/(8(16 - 5\pi)) = 0.344....$ This double inequality is better than those ones introduced in the first section. Moreover, we obtain three interesting double inequalities (17), (20) and (25), which are related to Archimedes' famous approximation 22/7 (also named "yuelu" in China) to the transcendental number π. Comparing them we find that

$$\begin{aligned}\frac{\sqrt{7\pi/22}}{\sqrt{\pi G_n(3/8, 1/32)}} &< \frac{\sqrt{7\pi/22}}{\sqrt{\pi G_n(1/3, 1/33)}} < \frac{\sqrt{7\pi/22}}{\sqrt{\pi G_n(1/4, 5/176)}} < W_n \\ &< \frac{1}{\sqrt{\pi G_n(3/8, 1/32)}} < \frac{1}{\sqrt{\pi G_n(1/3, 1/33)}} < \frac{1}{\sqrt{\pi G_n(1/4, 5/176)}}\end{aligned}$$

for $n \in \mathbb{N}$.

Finally, due to Corollaries 3.3 and 3.5 with the inequalities (23), we claim that the lower and upper bounds in (26) are the sharpest. To this end, let us recall several asymptotic expansions of $\Gamma(x+1)/\Gamma(x+1/2)$. It was proved in [11] (see also [3], [12]) that

$$\ln\frac{\Gamma(x+1)}{\Gamma(x+1/2)} \sim \frac{1}{2}\ln x + \sum_{k=1}^{\infty}\frac{(2 - 2^{1-2n})B_{2n}}{2n(2n-1)x^{2n-1}} \text{ as } x \to \infty,$$

where B_{2n} is the Bernoulli numbers. The following asymptotic expansion appeared in [15, Corollary 3]:

$$2\ln\frac{\Gamma(x+1)}{\Gamma(x+1/2)} \sim \ln\left(x + \frac{1}{4}\right) + \frac{1}{32}\frac{1}{(x+1/4)^2} - \frac{5}{1024}\frac{1}{(x+1/4)^4} + \frac{61}{24\,576}\frac{1}{(x+1/4)^6} + \cdots \text{ as } x \to \infty.$$

Recently, a new asymptotic expansion was established in [14, Eq, (1.15)], which states that

$$\begin{aligned} 2\ln\frac{\Gamma(x+1)}{\Gamma(x+1/2)} &\sim \ln\left(x+\frac{1}{4}\right)+\sum_{k=2}^{\infty}\frac{(k+1)(-1/4)^k+2\left(1-(-1)^k+(-1/2)^k\right)B_{k+1}}{k(k+1)x^k} \\ &= \ln\left(x+\frac{1}{4}\right)+\frac{1}{32x^2}-\frac{1}{64x^3}+\frac{1}{1024x^4}+\cdots \text{ as } x\to\infty. \end{aligned}$$

Now we first show that the coefficient 1/32 is the best. In fact, using the second asymptotic formula, we have

$$\frac{1}{\pi W_n^2}-\left(n+\frac{1}{4}\right)=\left[\frac{\Gamma(n+1)}{\Gamma(n+1/2)}\right]^2-\left(n+\frac{1}{4}\right)\sim\left(n+\frac{1}{4}\right)\left[\exp\left(\frac{1}{32}\frac{1}{(n+1/4)^2}\right)-1\right]\sim\frac{1}{32}\frac{1}{n+1/4}\text{ as } n\to\infty,$$

which yields that, for real a,

$$\xi_n(a):=\left[\frac{1}{\pi W_n^2}-\left(n+\frac{1}{4}\right)\right](n+a)\sim\frac{1}{32}\frac{1}{n+1/4}(n+a)\to\frac{1}{32}\text{ as } n\to\infty.$$

Second, we show that 1/4 and a_0 are the best. Using the asymptotic formula (4) again we obtain

$$\eta_n:=\frac{1}{32}\frac{1}{\frac{1}{\pi W_n^2}-\left(n+\frac{1}{4}\right)}-n\sim n+\frac{1}{4}-n=\frac{1}{4}\text{ as } n\to\infty,$$

and

$$\eta_1=\frac{41\pi-128}{8(16-5\pi)}=a_0.$$

Remark 4.1. *Note that* $\xi_1(1/4)=5(16-5\pi)/(16\pi)=b_0<1/32=\xi_\infty(1/4)$ *and* $\eta_1=0.344...>1/4=\eta_\infty$. *We guess that the sequences* $\{\xi_n(1/4)\}$ *and* $\{\eta_n\}$ *are increasing and decreasing for* $n\in\mathbb{N}$, *respectively.*

References

[1] A. V. Boyd, *Note on a paper by Uppuluri*, Pacific J. Math. **22**(1967), no. 1, 9–10.
[2] J. Cao, D.-W. Niu, and F. Qi, *A Wallis type inequality and a double inequality for probability integral*, Aust. J. Math. Anal. Appl. **4**(2007), no. 1, article 3, 6 pages.
[3] C.-P. Chen, R.B. Paris, *Inequalities, asymptotic expansions and completely monotonic functions related to the gamma function*, Appl. Math. Comput. **250**(2015), 514–529.
[4] C.-P. Chen, F. Qi, *The best bounds in Wallis' inequality*, Proc. Amer. Math. Soc. **133**(2005), no. 2, 397–401.
[5] J. T. Chu, *A modified Wallis product and some applications*, Amer. Math. Monthly **69**(1962), no. 5, 402–404.
[6] J. Gurland, *On Wallis' formula*, Amer. Math. Monthly **63**(1956), 643–645.
[7] D. K. Kazarinoff, *On Wallis formula*, Edinburgh Math. Notes **1956**(1956), no. 40, 19–21.
[8] S. Koumandos, *Remarks on a paper by Chao-Ping Chen and Feng Qi*, Proc. Amer. Math. Soc. **134**(2006), 1365–1367.
[9] E. C. Popa, N.-A. Secelean, *Estimates for the constants of Landau and Lebesgue via some inequalities for the Wallis ratio*, J. Comput. Appl. Math. **269**(2014), 68–74.
[10] F. Qi, *Bounds for the ratio of two gamma functions*, J. Inequal. Appl. **2010** (2010), Art. ID 493058, 84 pages.
[11] D. V. Slavić, *On inequalities for* $\Gamma(x+1)/\Gamma(x+1/2)$, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. **17–20**(1975), 498–541.
[12] J.-F. Tian, Z. Yang, *Asymptotic expansions of Gurland's ratio and sharp bounds for their remainders*, J. Math. Anal. Appl. **493**(2021), Paper No. 124545, 19 pages.
[13] J. Wendel, *Note on the gamma function*, Amer. Math. Monthly **55**(1948), 563–564.
[14] Z. Yang, J.-F. Tian, *Complete monotonicity of the remainder of the asymptotic series for the ratio of two gamma functions*, J. Math. Anal. Appl. **517**(2023), no. 2, Paper No. 126649, 15pages.
[15] Z.-H. Yang, J.-F. Tian and M.-H. Ha, *A new asymptotic expansion of a ratio of two gamma functions and complete monotonicity for its remainder*, Proc. Amer. Math. Soc. **148**(2020), 2163–2178.
[16] G. M. Zhang, *The upper bounds and lower bounds on Wallis' inequality*, Math. Practice Theory **37**(2007), no. 5, 111–116 (in Chinese).
[17] D.-J. Zhao, *On a two-sided inequality involving Wallis' formula*, Math. Practice Theory **34**(2004), 166–168 (in Chinese).